Continuous location of a service

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Learning objectives (1/2)

At the end of this chapter you will be able to:

- formulate three location objectives and understand when they coincide;
- write Euclidean distances with conic constraints and solve them with Gurobi from good starting points;

Learning objectives (2/2)

At the end of this chapter you will be able to:

- handle convex geographic constraints;
- build an efficiency-equity frontier with the ε-constraint method.

Prerequisites

- From the module *Convex Optimization*, Chapter 1 «Convex sets»: convex sets, on which the geographic constraints are built
- From this module, Chapter 2 «Background: convexity, quadratic programming and KKT conditions»: convex quadratic programs, the KKT conditions and the multipliers as shadow prices
- From this module, Chapter 3 «The solver for nonlinear models»: functional constraints and tolerances in `gurobipy`

Managerial motivation

Introduction: Managerial motivation

- In this chapter we decide **where to place** a new facility on the plane
- The model is a **convex NLP**: the coordinates are continuous variables
- We will see why the choice of the objective is the real managerial decision

Where to place a service?

- A fast-charging station, a logistics micro-hub, a health-care post: we want to be "close" to demand
- But what does close mean?
  - *mean* distance: **efficiency**
  - *maximum* distance: **equity** towards the most unlucky user
  - *squared* distance: leads to the centroid
- Three objectives, three different points on the map: geometry makes the trade-off visible as in no other model

Efficiency versus equity

- Location decisions are among the most expensive ones to correct: a hub in the wrong place stays in the wrong place for years
- The continuous version (free coordinates on the plane) is the ideal laboratory for a conflict that reappears in all public and private decisions:
  - *efficiency*: serving well on average
  - *equity*: leaving nobody behind

Summary: Managerial motivation

- Locating a facility means choosing its coordinates (x, y)
- "Close" may mean mean, maximum or squared distance
- The central theme is the conflict between **efficiency** and **equity**

Guided construction of the model

Introduction: Guided construction of the model

- We describe the problem in words, then the data and the variables
- We write two models: **Weber** (mean distance) and **minimax** (maximum distance)
- We add the quadratic variant and the geographic constraints
- Finally we see what characterizes the optimum of each objective

The problem in words

- *We decide* the coordinates at which the new facility is placed
- *The objective* — and this is the real managerial choice — may be to minimize:
  - the mean distance weighted by the users
  - the distance of the farthest user
  - the squared distance
- *The constraints*, when present, delimit the feasible area (a rectangular zone, a maximum radius from an infrastructure)

Data (input of the model)

Symbol	Type	Meaning
n	$\in \mathbb{Z}_{\geq 1}$	number of demand points (districts), indexed by $i \in \{1, 2, \ldots, n\}$
(a_i, b_i)	$\in \mathbb{Q}^2$	coordinates of district i (km)
w_i	$\in \mathbb{Q}_{\geq 0}$	weight of district i: population, demand or priority
(a_0, b_0), r	$\in \mathbb{Q}^2$, $\in \mathbb{Q}_{>0}$	centre and radius of the possible geographic constraint (maximum distance from an infrastructure)

Decision variables

We introduce 2 free variables (they may take any real value) and, for the minimax version, a non-negative auxiliary variable:

$$\begin{cases} x = \text{east coordinate of the new facility (km)} \\ y = \text{north coordinate of the new facility (km)} \\ z = \text{distance from the farthest district (minimax only)} \end{cases}$$

Weber model

Weber model (weighted mean distance)

$$\min \quad \sum_{i=1}^{n} w_i \sqrt{(x - a_i)^2 + (y - b_i)^2} \tag{9.1a}$$

$$\text{subject to} \quad x \gtreqless 0, \tag{9.1b}$$

$$y \gtreqless 0. \tag{9.1c}$$

- The convex objective function minimizes the total distance weighted by the districts (**efficiency**)
- The constraints define the variables, which are free

Minimax model

Minimax model (maximum distance)

$$\min \quad z \tag{9.2a}$$

$$\text{subject to} \quad \sqrt{(x - a_i)^2 + (y - b_i)^2} \leq z, \quad \forall i \in \{1, 2, \ldots, n\}, \tag{9.2b}$$

$$x \gtreqless 0, \tag{9.2c}$$

$$y \gtreqless 0, \tag{9.2d}$$

$$z \geq 0. \tag{9.2e}$$

- The objective minimizes the distance of the farthest district (**equity**)
- The convex distance constraints impose that every district is at most z away from the facility (n constraints)
- The last constraints define the variables

Quadratic variant and geographic constraints

A third variant, quadratic: minimize

$$\sum_{i=1}^{n} w_i \left[(x - a_i)^2 + (y - b_i)^2 \right],$$

which has a closed-form solution (the weighted centroid).

Typical **convex geographic constraints**, to be added to any variant:

- rectangular zone: $x^L \leq x \leq x^U, \;\; y^L \leq y \leq y^U$
- maximum distance from an infrastructure: $(x - a_0)^2 + (y - b_0)^2 \leq r^2$

Characterization of the optima (1/2)

Quadratic $\Rightarrow$ centroid. Setting the gradient to zero:

$$x^* = \frac{\sum_{i=1}^{n} w_i \, a_i}{\sum_{i=1}^{n} w_i} \qquad \text{(and analogously for } y\text{)}$$

- It is the only case with a closed formula
- Useful as a starting point for the other models

Characterization of the optima (2/2)

Weber. It has no closed form; the function is not differentiable at the points (a_i, b_i).

- At the optimum the weighted unit "pulls" of the districts cancel out: $\sum_{i=1}^{n} w_i \, \boldsymbol{u}_i = 0$
- $\boldsymbol{u}_i$ is the unit vector from the facility towards district i

Minimax. The optimum is the centre of the *smallest enclosing circle* of all the points:

- it depends only on the extreme districts and ignores the weights
- this is why it protects the outskirts

Summary: Guided construction of the model

- **Weber**: minimizes the weighted sum of Euclidean distances (efficiency)
- **Minimax**: minimizes the maximum distance with a variable z (equity)
- **Quadratic**: closed-form solution, the weighted centroid
- Geographic constraints (rectangle, circle) are convex and can be added to any variant

Worked example by hand

Introduction: Worked example by hand

- A one-dimensional example, solvable by hand
- We compute the optimum of the three objectives on the same data
- We will see that the three objectives give three different answers

Example: Three customers on a road (1/2)

Customers at km 0, 4 and 10 with weights 1, 1 and 3.

Weber in 1D = weighted median. The cost is

$$f(x) = |x| + |x - 4| + 3|x - 10|.$$

- Moving by δ to the left from $x = 10$, the cost of the first two customers decreases by 2δ but that of the third increases by 3δ: it gets worse
- From the right, symmetrically, it gets worse again
- Optimum: $x^* = 10$, the point that leaves *at least half of the total weight* ($5/2 = 2.5$) on both sides: the weighted median

Example: Three customers on a road (2/2)

Centroid.

$$x = \frac{1 \cdot 0 + 1 \cdot 4 + 3 \cdot 10}{5} = 6.8$$

sensitive to *how far* the points are, not only to which side they are on.

Minimax. Midpoint of the extremes: $(0 + 10)/2 = 5$, weights ignored.

Three reasonable objectives, three different answers: 10, 6.8, 5. The choice of the objective **is** the managerial decision.

Summary: Worked example by hand

- In 1D Weber coincides with the **weighted median**
- The centroid is the **weighted mean**: it depends on how far the points are
- The minimax is the **midpoint of the extremes**: it ignores the weights

Case study

Introduction: Case study

- We apply the models to a city with twelve districts
- We see the modelling trick that makes Euclidean distances tractable for Gurobi: conic constraints

The data of the case

- Twelve districts with weights between 5 and 25 thousand inhabitants (file `localizzazione_quartieri.csv` in the folder `data/`)
- Technical constraint: the station must lie within $r = 2$ km of the substation at $(7, 6)$

The modelling trick: conic constraints

- A variable $d_k \geq$ the Euclidean distance from district k, imposed with the (convex) conic constraint

$$d_x^2 + d_y^2 \leq d_k^2$$

- Weber minimizes $\sum_k w_k d_k$
- The minimax minimizes a variable z with $d_k \leq z$

The model in Gurobi

```
d = m.addVars(n) # d_k >= dist. from district k
for k in range(n):
    m.addConstr(dx[k] == px - coord[k, 0])
    m.addConstr(dy[k] == py - coord[k, 1])
    m.addQConstr(dx[k]**2 + dy[k]**2 <= d[k]*d[k]) # cone
# Weber: min sum w_k d_k
# minimax: min z with d_k <= z
# geographic constraint: (px-7)^2 + (py-6)^2 <= R^2
```

Summary: Case study

- The Euclidean distance is modelled with a variable d_k and a convex **conic constraint**
- Weber and minimax differ only in the objective function
- The substation constraint is a circle: a convex quadratic constraint

Results

Introduction: Results

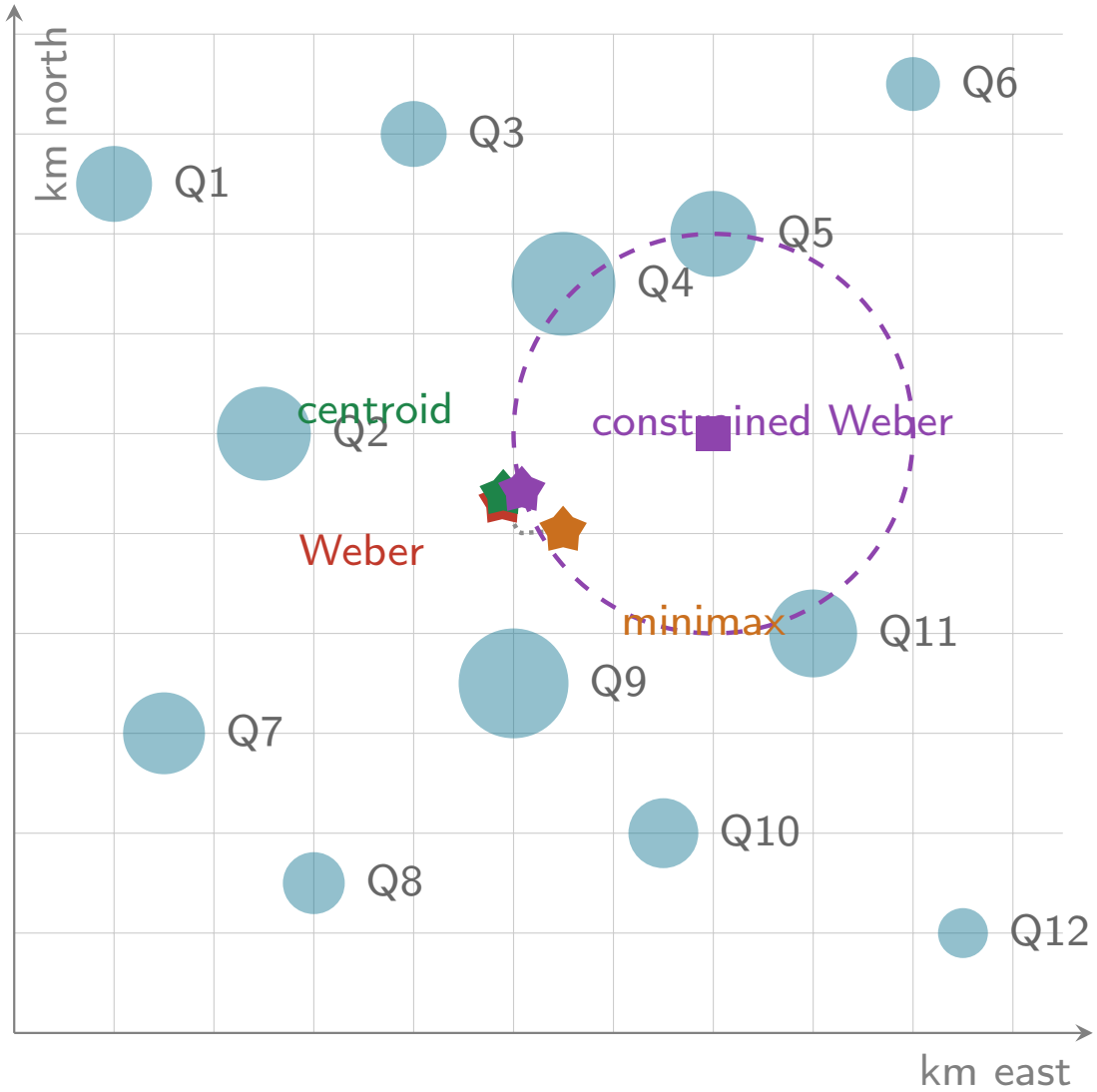
- We compare the four optimal locations: centroid, Weber, minimax, constrained Weber
- We see them on the map of the city
- We draw a managerial lesson on the cost of constraints

The four solutions

Solution	(x, y)	mean dist.	max dist.
Weighted centroid	$(4.897, 5.397)$	–	–
Weber	$(4.886, 5.310)$	3.344 km	6.314 km
Minimax	$(5.496, 5.029)$	3.391 km	5.680 km
Constrained Weber	$(5.083, 5.429)$	cost $+0.2\%$	

- Constrained Weber: the constraint is active, the optimum lies on the boundary of the circle

Map of the city



How to read the map

- The teal circles are the 12 districts (area proportional to the population)
- The stars indicate the four optimal locations: Weber in red, minimax in orange, centroid in green, constrained Weber in purple
- The dashed purple circle is the constraint of 2 km from the electrical substation
- The dotted line is the trajectory of the compromise solutions between minimax and Weber

Managerial insight

- Weber and the centroid almost coincide here (the weights are evenly distributed)
- The minimax moves more than half a kilometre towards the south-east in order to protect the peripheral districts
- The constraint of the substation costs very little (+0.2% of total distance)
- Discovering that a feared constraint is almost free is a managerial result as important as the optimum itself

Summary: Results

- Weber $\approx$ centroid when the weights are balanced
- The minimax moves towards the outskirts
- The geographic constraint is active but costs only 0.2%

Sensitivity analysis: the efficiency-equity frontier

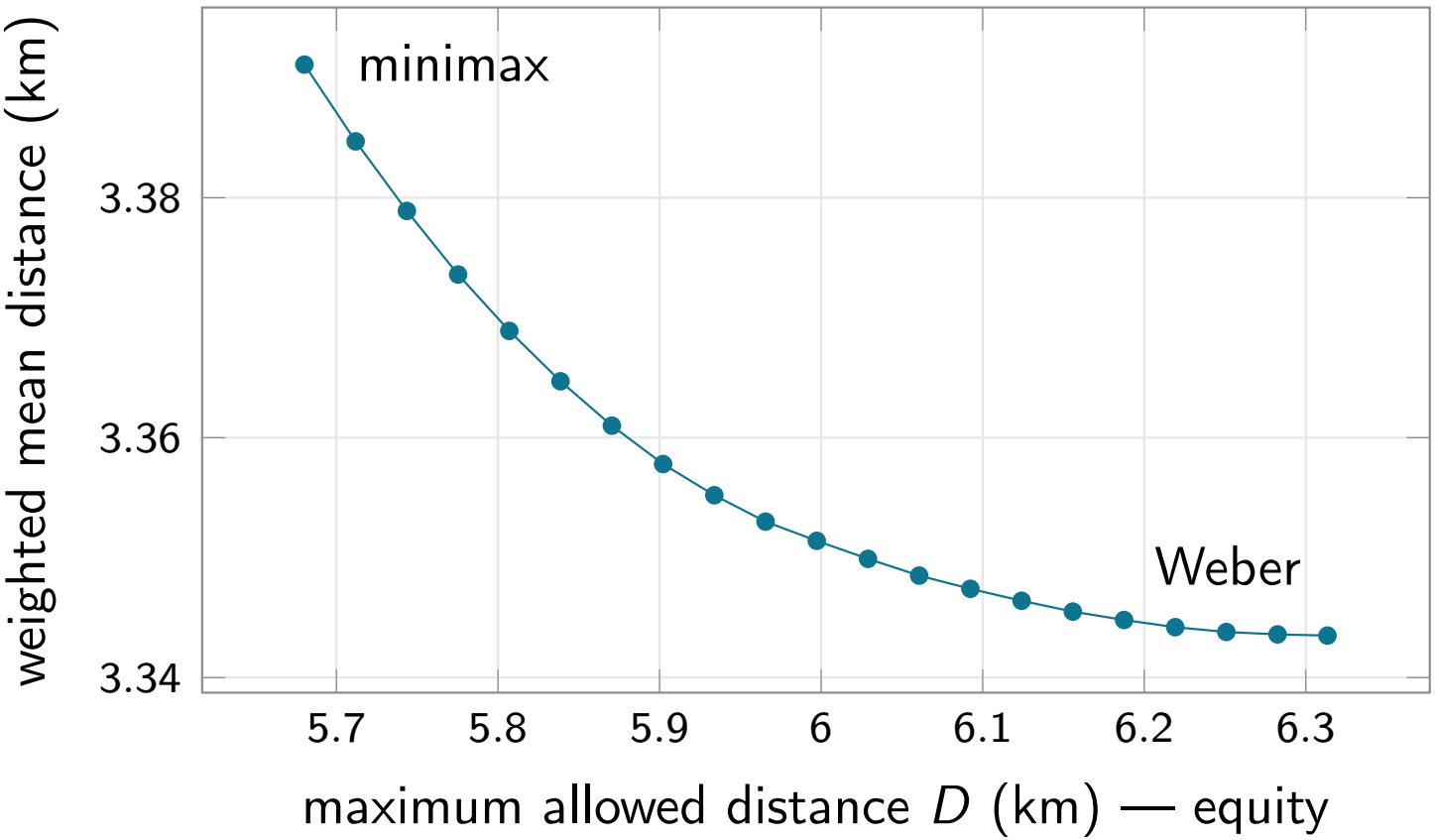
Introduction: the efficiency-equity frontier

- Weber and minimax are two extremes: are there intermediate solutions?
- We build them with the ε-constraint method
- The shape of the frontier tells how much equity costs

The ε-constraint method

- We minimize the mean distance while imposing a cap D on the maximum distance
- We vary D between the two extremes:
  - $D = 5.680$ km: minimax solution
  - $D = 6.314$ km: Weber solution
- For every D we obtain the minimum attainable weighted mean distance

Efficiency-equity frontier



Some points of the frontier

Cap D	Mean distance	
5.680 km	3.391 km	minimax solution
5.997 km	3.351 km	
6.314 km	3.344 km	Weber solution

- The extremes of the curve are the minimax solution (on the left) and the Weber point (on the right)
- The almost flat curve indicates that equity costs very little in terms of efficiency

An almost flat frontier

- Guaranteeing 0.63 km less to the farthest district costs only 47 metres of mean distance
- When the frontier is this flat, the equitable solution is "almost free": a very strong argument in a public discussion
- The opposite case (a steep frontier) would signal a genuine efficiency-equity conflict

Summary: the efficiency-equity frontier

- The ε-constraint method optimizes one objective and caps the other
- The frontier connects minimax and Weber
- Flat frontier: equity is almost free; steep frontier: genuine conflict

Exercises

Introduction: Exercises

- Five exercises to test the models of the chapter
- Checks by hand, sensitivity to the weights, extensions of the model (two facilities, Manhattan distance, multipliers)

Exercise: check

- Compute by hand the weighted centroid of the 12 districts from the CSV data
- Compare it with the output of the script

Exercise: a weight that changes everything

- Raise the weight of Q12 (the district at the bottom right) from 5 to 40
- How far do Weber and minimax move?
- Which of the two is more sensitive to the weights, and why?

Exercise: two facilities

- With *two* stations and the assignment of every district to the nearest one, does the problem remain convex?
- Try an alternating approach (assign–optimize)
- Discuss why it only finds local optima

Exercise: Manhattan distance

- Replace the Euclidean distance with $|x - a_i| + |y - b_i|$
- Show that the Weber problem becomes an LP (introduce auxiliary variables for the absolute values)
- Solve it with Gurobi and compare the optimum

Exercise: multiplier of the radius

- Estimate by perturbation the multiplier of the constraint $r = 2$ km (solve with $r = 2.1$)
- How much is every additional 100 m of radius worth, in km of total distance?

Summary: Exercises

- The exercises check the centroid, the sensitivity to the weights and the value of the geographic constraint
- The extensions show when the problem loses convexity (two facilities) or becomes an LP (Manhattan)

Summary (1/2)

Models:

- Weber (weighted mean distance), minimax (maximum distance), quadratic (centroid)
- Euclidean distances are written with convex conic constraints

Optima:

- Centroid: closed formula; Weber: the weighted pulls cancel out; minimax: centre of the smallest enclosing circle

Summary (2/2)

Managerial lessons:

- The choice of the objective is the managerial decision
- A feared constraint can be almost free (+0.2%)
- The efficiency-equity frontier measures how much equity costs: here almost nothing