Pricing and revenue management

Outline

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Learning objectives (1/2)

At the end of this chapter you will be able to:

- model demand as a function of the price and understand when profit is concave;
- handle a non-convex bilinear objective;

Learning objectives (2/2)

At the end of this chapter you will be able to:

- evaluate capacity correctly: an additional seat is not worth the full margin;
- recognize corner optima, where no derivative vanishes.

Prerequisites

- From this module, Chapter 2 «Background: convexity, quadratic programming and KKT conditions»: convex quadratic programs, the KKT conditions and the multipliers as shadow prices
- From this module, Chapter 3 «The solver for nonlinear models»: bilinear constraints and `NonConvex = 2`; shadow prices estimated by perturbation

Managerial motivation

Introduction: Managerial motivation

- Which price maximizes the profit of a concert, of a hotel, of a subscription, when demand **decreases as the price grows**?
- Here the price is not a datum but a **variable**: demand becomes endogenous
- The profit $p \cdot q$ introduces a **bilinear term**
- This is the chapter in which the laboratory meets **non-convexity** for the first time — and learns how to handle it

Why the price matters

- Setting the price is perhaps the business decision with the highest impact on the margin
- One percentage point of price is often worth more than many points of volume
- **Revenue management** — born in the airlines and today standard in hotels, events and rentals — adds the **capacity constraint**
- The seats left unsold tonight cannot be recovered tomorrow

Summary

- In pricing the price p is a **decision variable** and demand depends on p
- The profit $p\,q$ is **bilinear**: first encounter with non-convexity
- Revenue management adds a limited and perishable **capacity**

Guided construction of the model

Introduction: Guided construction of the model

- We translate the pricing problem into a mathematical model
- We fix data, decision variables, objective function and constraints
- Then we see **where non-convexity comes from** and how Gurobi handles it

The problem in words

- **We decide** the selling price and the quantity to sell
- **The objective** is to maximize the profit, that is revenue minus variable costs
- **The constraints**: we cannot sell
  - more than the market asks for at that price
  - nor more than the available capacity

Data (input of the model)

Symbol	Type	Meaning
$d(\cdot)$	function $\mathbb{Q}_{\geq 0} \to \mathbb{Q}_{\geq 0}$	expected demand at price p, decreasing in p
c	$\in \mathbb{Q}_{\geq 0}$	unit marginal cost (€)
k	$\in \mathbb{Q}_{>0}$	available capacity (seats, rooms, units)

Decision variables

We introduce the following 2 non-negative variables:

$$\begin{cases} p = \text{selling price decided by the firm (€)} \\ q = \text{quantity sold (units)} \end{cases}$$

Using these variables we build a pricing model for the problem

The model

Pricing model with a capacity constraint

$$\begin{aligned}
\max \quad & p\,q - c\,q && (7.1a)\\
\text{subject to} \quad & q \le d(p), && (7.1b)\\
& q \le k, && (7.1c)\\
& p \ge 0, && (7.1d)\\
& q \ge 0. && (7.1e)
\end{aligned}$$

The objective function

$$\max \quad p\,q - c\,q = (p - c)\,q$$

- It maximizes the **profit**: revenue minus variable cost
- It contains the product of the two variables p and q: a **bilinear** term, non-convex
- This is where all the difficulty of the chapter comes from

The constraints

- $q \leq d(p)$: we do not sell more than the market asks for at price p (one constraint, linear if d is linear)
- $q \leq k$: we do not sell more than the available capacity; when it is active, it is this constraint that determines the optimal price (one linear constraint)
- $p \geq 0$, $q \geq 0$: define the variables of the model

Demand functions of the case study

All data are positive rationals:

- **Linear**: $d(p) = a - b\,p$
- **Constant elasticity**: $d(p) = \theta\, p^{-\varepsilon}$ with $\varepsilon > 1$
- **Logistic**: $d(p) = \dfrac{m}{1 + e^{-\alpha+\beta p}}$

Why $q \leq d(p)$ and not $q = d(p)$?

- At the optimum the inequality $q \leq d(p)$ is active by itself
- Selling less than what is possible at that price is never worthwhile if $p > c$
- Writing it as $\leq$ keeps the feasible region simpler
- And keeps the model feasible even when $d(p) > k$

Where is the non-convexity?

- In the product $p \cdot q$ of the objective
- With linear demand, substituting $q = a - b\,p$, the profit $(p - c)(a - b\,p)$ is a **concave parabola**: the *reduced* problem is easy
- Gurobi nevertheless solves the bilinear version to global optimality (`NonConvex=2`)
- For general non-linear demands (constant elasticity, logistic) the function constraints of Gurobi are enough (`addGenConstrPow`, `addGenConstrExp`): same solver, still a global optimum

Summary

- Variables: price p and quantity q; objective: $\max (p - c)\, q$
- Constraints: demand $q \leq d(p)$ and capacity $q \leq k$
- Non-convexity lies in the bilinear term $p\, q$
- Gurobi finds the global optimum (`NonConvex=2` or function constraints)

Worked example by hand

Introduction: Worked example by hand

- We solve by hand a case with linear demand
- First without capacity, then with capacity that "bites"
- Finally we compute **how much an additional seat is worth**: an often surprising result

Example: linear demand, with and without capacity (1/3)

Concert: $d(p) = 1200 - 5p$ (hence $a = 1200$, $b = 5$), marginal cost $c = 20$ €, theatre with $k = 400$ seats

Step 1 — without capacity. Profit

$$\Pi(p) = (p - 20)(1200 - 5p), \qquad \Pi'(p) = 1200 - 10p + 100 = 0$$

gives

$$p^\circ = \frac{a/b + c}{2} = \frac{240 + 20}{2} = 130 \text{ €}, \qquad q^\circ = 1200 - 650 = 550$$

tickets

Example: linear demand, with and without capacity (2/3)

Step 2 — capacity bites.

- $q^\circ = 550 > 400$: the constraint is active, hence exactly k units are sold
- The price that empties the theatre is

$$p^* = \frac{a - k}{b} = \frac{1200 - 400}{5} = 160 \text{ €}$$

- Profit: $(160 - 20) \cdot 400 = 56{,}000$ €

Example: linear demand, with and without capacity (3/3)

Step 3 — how much is an additional seat worth? As k grows, the optimal price falls by $1/b$:

$$\frac{d\Pi}{dk} = \underbrace{(p^* - c)}_{140} + k \underbrace{\frac{dp^*}{dk}}_{-1/5} = 140 - 80 = 60 \text{ €}$$

- An additional seat is *not* worth the full margin (140 €): in order to fill it the price must be lowered for everybody
- A classic mistake in the evaluation of capacity investments

Summary

- Without capacity: $p^\circ = (a/b + c)/2 = 130$ €, $q^\circ = 550$
- With $k = 400$ capacity is active: $p^* = (a - k)/b = 160$ €, profit 56,000 €
- Value of an additional seat: $p^* - c - k/b = 60$ €, **not** the full margin of 140 €

Case study and implementation

Introduction: Case study and implementation

- We implement the model in Python with Gurobi
- We compare three demand functions: linear, constant elasticity, logistic
- We will discover an optimum that **makes no derivative vanish**: the corner optimum

Data of the case study

- The same data as in the example: the rare case in which the realistic instance is already small
- Plus the variants:
  - constant elasticity: $\theta = 6 \cdot 10^6$, $\varepsilon = 2.2$
  - logistic: $m = 900$, $\alpha = 6$, $\beta = 0.045$
  - multi-product (next section)

Implementation in Gurobi

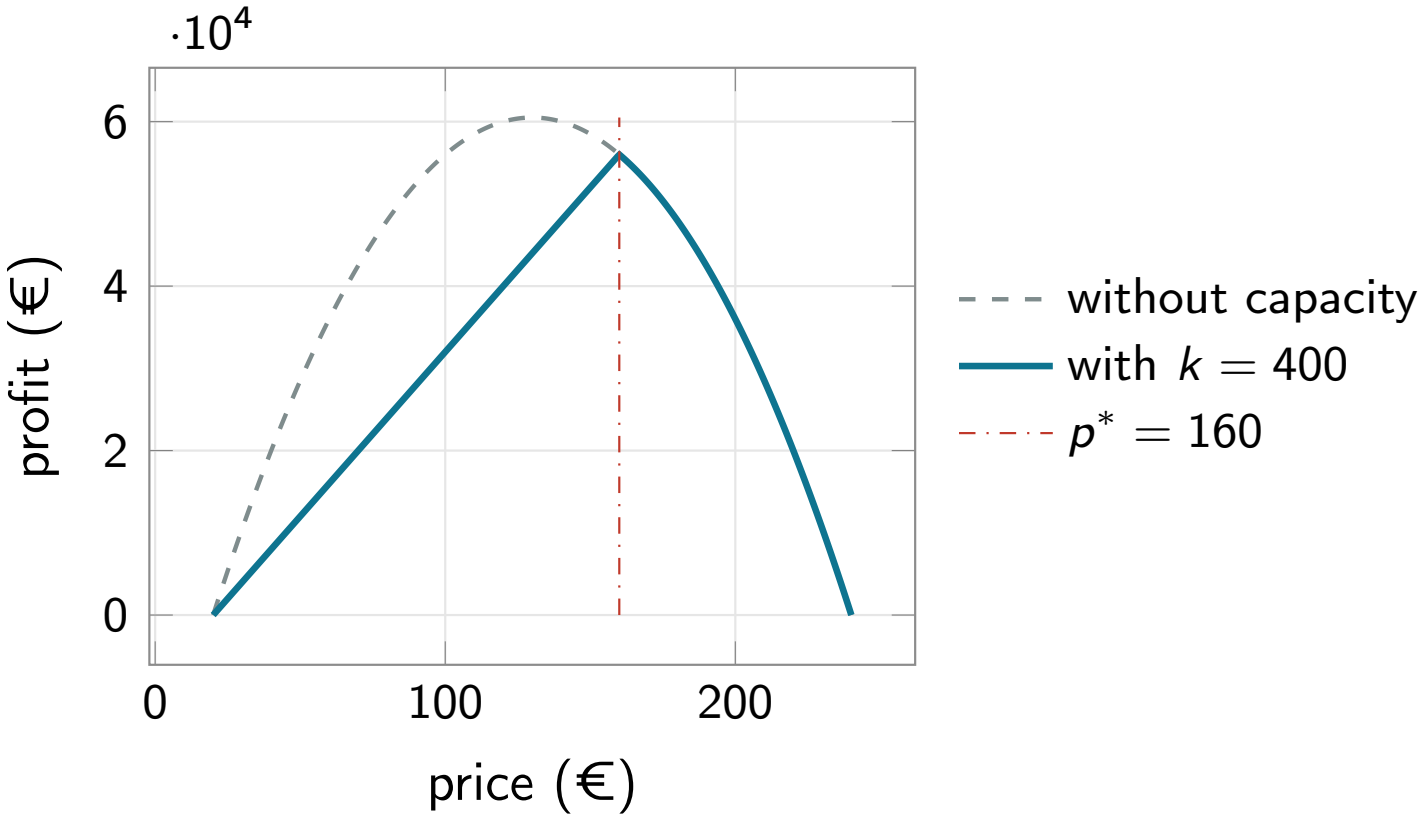
```
m = gp.Model("pricing_lineare")
m.Params.NonConvex = 2              # bilinear objective p*q
p = m.addVar(ub=a/b, name="p")
q = m.addVar(name="q")
m.addConstr(q <= a - b*p, name="domanda")
v_cap = m.addConstr(q <= K, name="capienza")
m.setObjective(p*q - c*q, GRB.MAXIMIZE)
m.optimize()
# no duals in non-convex QPs: sensitivity by perturbation
v_cap.RHS = K + 1; m.optimize()
```

Results

```
Gurobi:  p* = 160.00 EUR, q* = 400,
         profit = 56,000.00 EUR
Marginal value of a seat: 59.80 EUR (theory: 60.00)
Constant elasticity: p* = 79.11 EUR
           (optimum at the point where d(p) = k)
Logistic: p* = 138.29 EUR, profit = 47,316.83 EUR
```

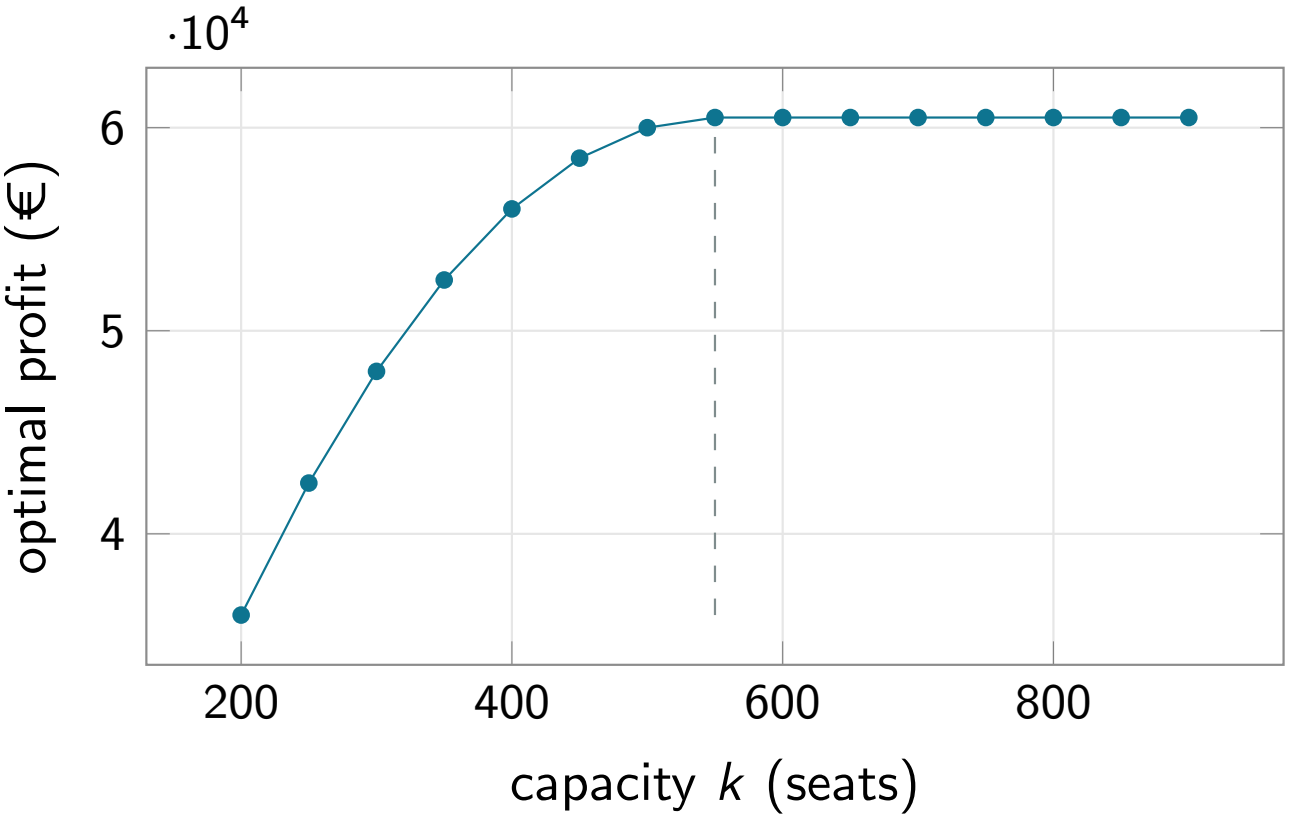
- In non-convex QPs there are no duals: the value of a seat is estimated **by perturbation** ($k \to k+1$)

Profit and capacity



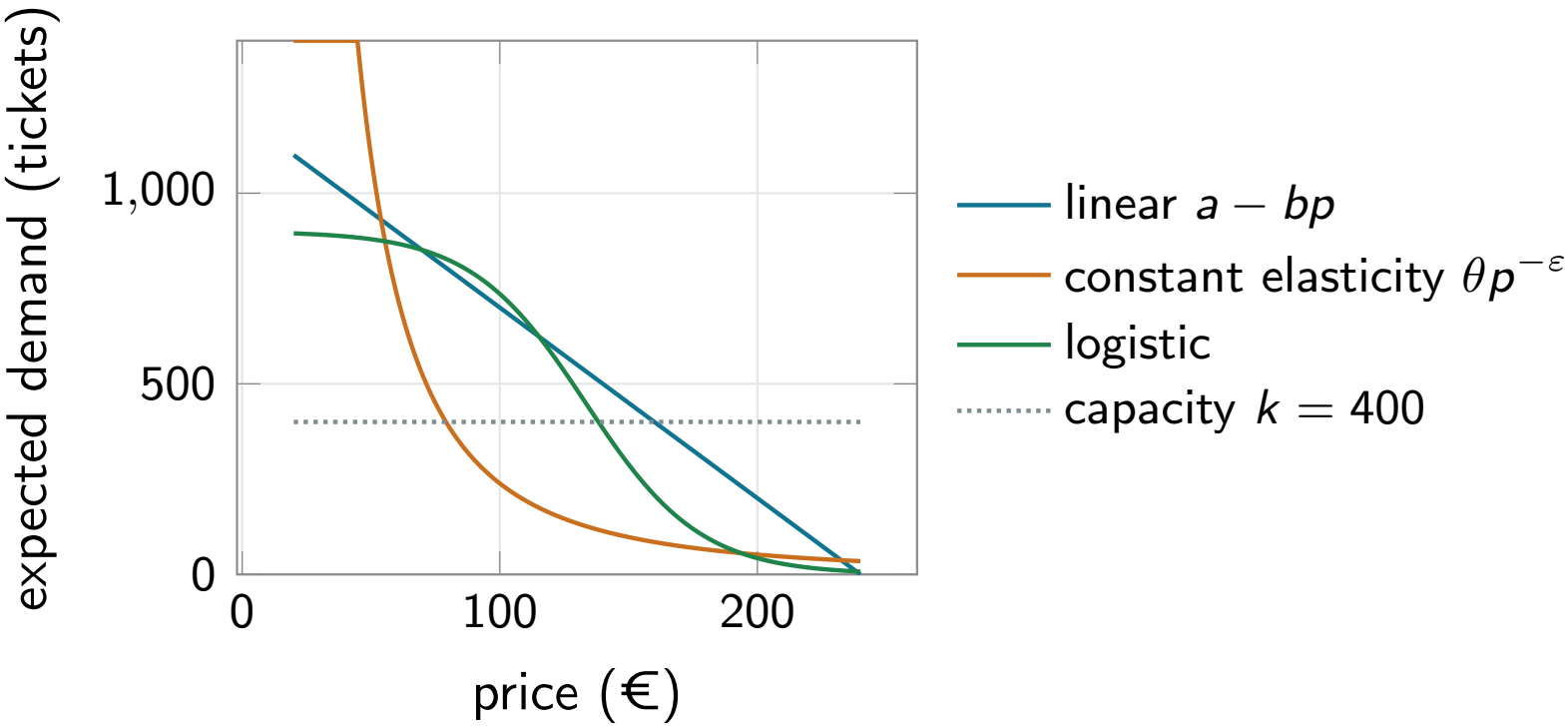
The constraint cuts the parabola and moves the optimum to $p^* = 160$ €

Value of capacity



Beyond $q^\circ = 550$ seats the curve becomes flat: additional capacity is worth nothing more

Three demand functions compared



The linear one cuts the axis, the constant-elasticity one has a long tail, the logistic one describes the gradual transition between high and low demand

The optimum at a corner point (1/2)

With constant elasticity:

- The *unconstrained* optimum would be $p = c\,\varepsilon/(\varepsilon - 1) = 36.67$ €
- But at that price demand would exceed capacity by far
- For p below 79.11 € we sell k anyway: so it is worthwhile to raise the price
- Above 79.11 €, the profit $(p - c)\,\theta p^{-\varepsilon}$ decreases

The optimum at a corner point (2/2)

- The optimum

$$p^* = \left(\frac{\theta}{k}\right)^{1/\varepsilon} = 79.11 \text{ €}$$

  lies at the **corner point** where $d(p) = k$

- It makes no derivative vanish

- **The moral**: never look for the optimum among stationary points only when there are constraints

Summary

- Gurobi solves the bilinear model to global optimality with `NonConvex=2`
- Without duals, the value of a seat is estimated by perturbation: 59.80 € (theory: 60)
- Capacity is worth something only while it is active ($k < 550$)
- With constant elasticity the optimum $p^* = 79.11$ € is a **corner point**

Multi-product version

Introduction: Multi-product version

- We move to two products whose demands influence each other
- We will see why the prices of substitute products must be decided **together**
- And which precautions are needed when the profit is not concave

Two ticket categories with substitution

- Raising the price of the stalls pushes part of the demand towards the balcony
- Demands:

$$d_1(p_1, p_2) = 500 - 2p_1 + 0.6\, p_2$$

$$d_2(p_1, p_2) = 900 + 0.8\, p_1 - 4p_2$$

- Costs $(30, 15)$ €, capacities $(150, 300)$ seats

Multi-product results

```
stalls   : p1* = 234.04 EUR   q1* = 150/150 (full)
balcony  : p2* = 196.81 EUR   q2* = 300/300 (full)
total profit: 85,148.94 EUR
```

Prices decided jointly

- Both categories fill up
- But the prices are not "the ones that empty" each hall taken on its own
- The model exploits substitution: it keeps the stalls expensive in order to push demand towards the balcony
- With substitute products prices must be decided **jointly**: optimizing them one at a time leaves money on the table

Warning

Observation 1: Non-concave profit and local optima

With general demand functions the profit may fail to be concave: a local solver may stop at a local optimum.

Practice of the laboratory:

- *study the domain and the derivatives of the function before trusting the solver*
- *try several starting points*
- *always plot the profit over the relevant interval*

Summary

- With substitute products the demands depend on all prices
- At the optimum both halls are full, with prices 234.04 and 196.81 €
- Prices must be decided **jointly**
- With non-concave profit: study the function, several starting points, plot

Exercises

Introduction: Exercises

- We test what we have seen with five exercises
- They revisit the example by hand, the corner optimum and the multi-product model
- For each one: first the statement, then the solution

Exercise: check

- Verify with Gurobi the marginal value of a seat for $k = 300$ (from the table: 100 €)
- Obtain it also with the formula $p^* - c - k/b$

Solution

- With $k = 300$ capacity is active ($300 < 550$):

$$p^* = \frac{a - k}{b} = \frac{1200 - 300}{5} = 180 \text{ €}$$

- Formula:

$$p^* - c - \frac{k}{b} = 180 - 20 - 60 = 100 \text{ €}$$

- In Gurobi: solve with $k = 300$ and $k = 301$ and take the difference of the profits

Exercise: marginal cost

- How do p^*, q^* and the value of a seat change if the marginal cost rises to $c = 60$ €?
- Redo the three steps of the example by hand

Solution

- **Step 1**: $p^\circ = \dfrac{240 + 60}{2} = 150$ €, $q^\circ = 1200 - 750 = 450$
- **Step 2**: $450 > 400$, capacity remains active: $p^* = 160$ €, $q^* = 400$, profit $(160 - 60) \cdot 400 = 40{,}000$ €
- **Step 3**: value of a seat $= 100 - 80 = 20$ €
- The price does not change (capacity fixes it), but the additional seat is worth much less

Exercise: reference price

- Add the reputational penalty $-\gamma\,(p-120)^2$ with $\gamma = 2$
- Which price balances profit and reputation?
- Does the problem remain concave?

Solution (1/2)

With $q = \min\{1200 - 5p,\ 400\}$ we study the two pieces:

- $p \leq 160$ (capacity active): $\Pi(p) = 400(p - 20) - 2(p - 120)^2$,
  $\Pi'(p) = 400 - 4(p - 120) > 0$ for $p < 220$: **increasing**
- $p \geq 160$ (demand active):
  $\Pi(p) = (p - 20)(1200 - 5p) - 2(p - 120)^2$,
  $\Pi'(p) = 1780 - 14p < 0$ for $p > 127.1$: **decreasing**

Hence the optimum stays at $p^* = 160$ €, profit
$56{,}000 - 3{,}200 = 52{,}800$ €

Solution (2/2)

- The problem **remains concave**
- For $p \geq c$, $(p - 20)\min\{1200 - 5p, 400\}$ is the minimum of two concave functions, hence concave
- Subtracting the convex $2(p - 120)^2$ preserves concavity
- The penalty lowers the profit but does not move the price: the corner $d(p) = k$ remains optimal

Exercise: elasticity

- For the constant-elasticity demand, plot p^* as a function of $\varepsilon \in [1.3; 3]$ with $k = 400$
- For which ε does the capacity constraint remain active?

Solution

- Two candidates: the corner $(\theta/k)^{1/\varepsilon}$ and the unconstrained optimum $c\,\varepsilon/(\varepsilon-1)$

$$p^*(\varepsilon) = \max\left\{\left(\frac{\theta}{k}\right)^{1/\varepsilon},\ \frac{c\,\varepsilon}{\varepsilon-1}\right\}$$

- Capacity is active when the corner is the larger of the two
- With $\theta/k = 15{,}000$ and $c = 20$ this happens for $\varepsilon \lesssim 2.80$; beyond, $p^* = c\,\varepsilon/(\varepsilon-1)$ and the hall does not fill up

Exercise: multi-product

- In the two-category model, by how much does the profit increase with 50 additional stalls seats?
- And with 50 additional balcony seats?
- Which expansion would you recommend, at equal cost?

Solution

Both capacities remain active: the prices follow from $d_1 = k_1$, $d_2 = k_2$

- **Stalls 200**: $p_1^* = 207.45$, $p_2^* = 191.49$ €, profit $\approx$ 88,436 € (+3,287)
- **Balcony 350**: $p_1^* = 230.05$, $p_2^* = 183.51$ €, profit $\approx$ 88,987 € (+3,838)
- At equal cost it is better to **expand the balcony**

Final summary

- In pricing the price is a variable: the profit $(p - c)\, q$ is **bilinear**
- With linear demand the reduced problem is concave; Gurobi finds the global optimum anyway
- An additional seat is **not** worth the full margin: $p^* - c - k/b$
- With constraints the optimum may lie at a **corner point**
- Substitute products: prices decided **jointly**