Supply chain with congestion and sustainability

SAPIENZA UNIVERSITÀ DI ROMA

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Learning objectives (1/2)

At the end of this chapter you will be able to:

- formulate a minimum-cost flow problem with conservation constraints;
- model congestion with convex quadratic terms;

Learning objectives (2/2)

At the end of this chapter you will be able to:

- draw a cost-emissions frontier with an internal CO_2 price;
- understand why the solutions of LPs jump from one vertex to another.

Prerequisites

- From the module *LP Modelling*, Chapter 2 «Background: linear programming and sensitivity analysis»: shadow prices, reduced costs and their ranges
- From this module, Chapter 2 «Background: convexity, quadratic programming and KKT conditions»: convex quadratic programs, the KKT conditions and the multipliers as shadow prices

Managerial motivation

Introduction: Managerial motivation

- Products cross a network of **plants**, **hubs** and **markets**
- Every leg has a **cost**, a **capacity** and a **CO_2 footprint**
- The effective cost grows rapidly when a leg approaches **saturation**
- Problem class: **LP / convex NLP**

Three managerial questions

1. How should the flows be routed at **minimum cost**?
2. What changes if we penalize **congestion**?
3. How much must $\mathbf{CO_2}$ be worth for the "clean" routes to become convenient?

An everyday problem in logistics

- This is the everyday problem of whoever manages distribution and logistics:
  - choosing the routes
  - deciding how much to load on each leg
  - evaluating investments in capacity
  - accounting for the carbon footprint of the shipments
- The mathematical structure — the **network flow** — is among the most reusable ones in the whole of operations research

Summary

- A network of plants, hubs and markets: every leg has a cost, a capacity and emissions
- Three objectives to compare: **minimum cost**, **congestion**, **CO_2**
- The reference model is the **network flow**

Guided construction of the model

Introduction: Guided construction of the model

- We describe the problem in words, then we fix **data** and **variables**
- We write the **minimum-cost flow LP model** and explain every constraint
- Finally we see three **variants of the objective**: congestion, CO_2 price, minimax

The problem in words

- **We decide** how many units travel on each leg of the network
- **The objective** is to minimize the total transport cost (in the variants: plus congestion or CO_2)
- **The constraints** impose that the flows add up at every node:
  - the plants do not ship more than they produce
  - the hubs retain nothing
  - the markets receive exactly their demand

  and that no leg exceeds its own capacity

Data (input of the model) (1/2)

Symbol	Type	Meaning
$G = (N, A)$	directed graph	nodes N and arcs $A \subseteq N \times N$; we write $\|N\|$ and $\|A\|$ for the number of nodes and of arcs
b_i	$\in \mathbb{Q}$	balance of node $i \in N$: supply if > 0, demand if < 0, transit if $= 0$

Data (input of the model) (2/2)

Symbol	Type	Meaning
u_{ij}	$\in \mathbb{Q}_{>0}$	capacity of arc $(i,j) \in A$ (units)
c_{ij}	$\in \mathbb{Q}_{\geq 0}$	transport cost of one unit on arc (i,j) (€)
e_{ij}	$\in \mathbb{Q}_{\geq 0}$	emissions per unit transported on (i,j) (kg CO_2)

Decision variables

- We introduce one non-negative variable for each of the $|A|$ arcs:

$$x_{ij} = \text{units of product sent on arc } (i,j), \qquad \forall (i,j) \in A$$

- Using these variables, we write an LP model for the problem

LP model

Minimum-cost flow LP model

$$\min \sum_{(i,j)\in A} c_{ij}\, x_{ij} \tag{5.1a}$$

$$\text{subject to} \sum_{j:\,(i,j)\in A} x_{ij} - \sum_{j:\,(j,i)\in A} x_{ji} = b_i, \qquad \forall i \in N, \tag{5.1b}$$

$$x_{ij} \le u_{ij}, \quad \forall (i,j) \in A, \tag{5.1c}$$

$$x_{ij} \ge 0, \quad \forall (i,j) \in A. \tag{5.1d}$$

Objective function and flow conservation

- The linear **objective function** minimizes the total transport cost: every unit that crosses arc (i, j) pays the unit cost c_{ij}
- The **flow conservation constraints**: at every node, what goes out minus what comes in equals the balance b_i
  - positive at the plants, negative at the markets, zero at the hubs
  - $|N|$ linear constraints
- The sum of the balances over all the nodes must be ≥ 0, otherwise the model is **infeasible**

Capacity and domain of the variables

- The **capacity constraints** $x_{ij} \leq u_{ij}$ ensure that no leg carries more than its own capacity
- They are $|A|$ linear constraints
- The constraints $x_{ij} \geq 0$ define the variables of the model

Variants on the objective (1/2)

The constraints do not change; only the objective function changes.

Congestion (convex):

$$\sum_{(i,j)\in A} \left(c_{ij}\, x_{ij} + \alpha\, c_{ij}\, x_{ij}^2 / u_{ij} \right)$$

Internal CO_2 price:

$$\sum_{(i,j)\in A} \left(c_{ij} + \tau\, e_{ij} \right) x_{ij}$$

Variants on the objective (2/2)

Minimax utilization:

$$\min z \ : \ x_{ij}/u_{ij} \leq z, \ \ \forall (i,j) \in A$$

where:

- $\alpha \in \mathbb{Q}_{\geq 0}$ weighs congestion
- $\tau \in \mathbb{Q}_{\geq 0}$ (€/kg) is the internal price of CO_2

Congestion

- The term $\alpha\, c_{ij}\, x_{ij}^2 / u_{ij}$ is **convex** and grows with the utilization of the arc
- It represents overtime, queues at the terminals, insurance premiums
- Its effect is to **spread** the flows over several routes instead of saturating the cheapest one

The price of CO_2

- No emissions constraint is needed: it is enough to add them to the cost with the internal price τ
- By varying τ one draws the whole **cost-emissions frontier** ("weight" approach)
- The "constraint" alternative imposes

$$\sum_{(i,j)\in A} e_{ij}\, x_{ij} \leq \bar{e}$$

and reads the dual

Summary

- Variables x_{ij}: units sent on each arc
- Constraints: **flow conservation** at the nodes and **capacity** on the arcs
- Variants of the objective: convex quadratic **congestion**, **internal CO_2 price**, **minimax** utilization
- The price τ draws the cost-emissions frontier without additional constraints

Worked example by hand

Introduction: Worked example by hand

- A minimal example with **two competing routes**
- First we solve the pure LP and read the **shadow prices**
- Then we add **congestion** and see how the solution changes

Example: Two competing routes (1/3)

A market asks for 100 units that can be served by two routes:

- route 1 with $c_1 = 2$ € and $u_1 = 80$
- route 2 with $c_2 = 5$ € and abundant capacity

Pure LP. The cheap route is filled and the rest overflows onto the other one:

$$x_1 = 80, \qquad x_2 = 20, \qquad \text{cost } 2 \cdot 80 + 5 \cdot 20 = 260 \text{ €}$$

Example: Two competing routes (2/3)

Shadow prices of the pure LP:

- **Demand**: serving the 101st unit would cost 5 € (route 1 is saturated)
- **Capacity** u_1: $+1$ unit of capacity saves $5 - 2 = 3$ €

Example: Two competing routes (3/3)

With congestion ($\alpha = 1$): cost of route 1 $= 2x_1 + 2x_1^2/80$

- The marginal cost of route 1, $2 + 4x_1/80$, has to be equalized with that of route 2 (otherwise it is worth shifting flow):

$$2 + x_1/20 = 5 \;\Rightarrow\; x_1 = 60, \quad x_2 = 40$$

- The cheap route **is no longer saturated**: the model leaves a safety margin, exactly as a prudent planner would do

Summary

- The pure LP **saturates** the cheap route and overflows onto the expensive one
- The shadow prices measure the value of one more unit of demand or of capacity
- With congestion the **marginal costs are equalized**: the flows are spread and a safety margin remains

Case study

Introduction: Case study

- We apply the model to a realistic network, with costs and emissions that are **not proportional**
- We see how the model is written in **Python with Gurobi**, with the congestion and CO_2 variants

The network of the case study

- 2 plants (S1, S2), 2 hubs (H1, H2) and 4 markets (M1–M4)
- 12 arcs with capacities, costs and emissions that are **not** proportional to the costs:
  - the cheap legs are "by road" (polluting)
  - the expensive ones "by rail" (clean)
- Data in `supplychain_archi.csv`; supply $(260, 240)$, total demand 450

The model in Gurobi (1/2)

Variables and conservation constraints:

```
m = gp.Model("supply_chain")
# ub directly on the arc
x = m.addVars(A, name="x", ub=U)
m.addConstrs((x.sum(s, "*") <= offerta[s]
              for s in offerta))
m.addConstrs((x.sum("*", h) == x.sum(h, "*")
              for h in hub))
v_dom = m.addConstrs((x.sum("*", k) == domanda[k]
                      for k in domanda),
                     name="domanda")
```

The model in Gurobi (2/2)

Objective with CO_2 price and congestion:

```
obj = gp.quicksum((c[a] + tau * e[a]) * x[a]
                  for a in A)
if congestione > 0:  # convex quadratic term
    obj += gp.quicksum(
        congestione * c[a] * x[a] * x[a] / U[a]
        for a in A)
m.setObjective(obj, GRB.MINIMIZE)
```

Summary

- A network with 8 nodes and 12 arcs: road cheap and polluting, rail expensive and clean
- The capacities enter as **upper bounds** of the variables
- A single script covers LP, congestion (`congestione` > 0) and CO_2 price (`tau`)

Results

Introduction: Results

- We compare three models: **minimum-cost LP**, **congestion**, **minimax**
- We read the **shadow prices** of demand and the **reduced costs** of the unused arcs
- We look at the optimal flows drawn on the network

Results of the three models

Model	Cost (€)	Emissions (kg)	Max utilization
Minimum-cost LP	3,385.00	2,730	100%
Congestion $\alpha = 1$	3,702.81	2,530	90%
Minimax	–	–	58.4%

- LP: saturated arcs S1→H1, S2→H2, H2→M3
- Minimax: minimum possible maximum utilization 58.4%

LP: shadow prices and reduced costs

Shadow prices of demand (€/unit):

M1	M2	M3	M4
8.00	9.50	10.00	10.50

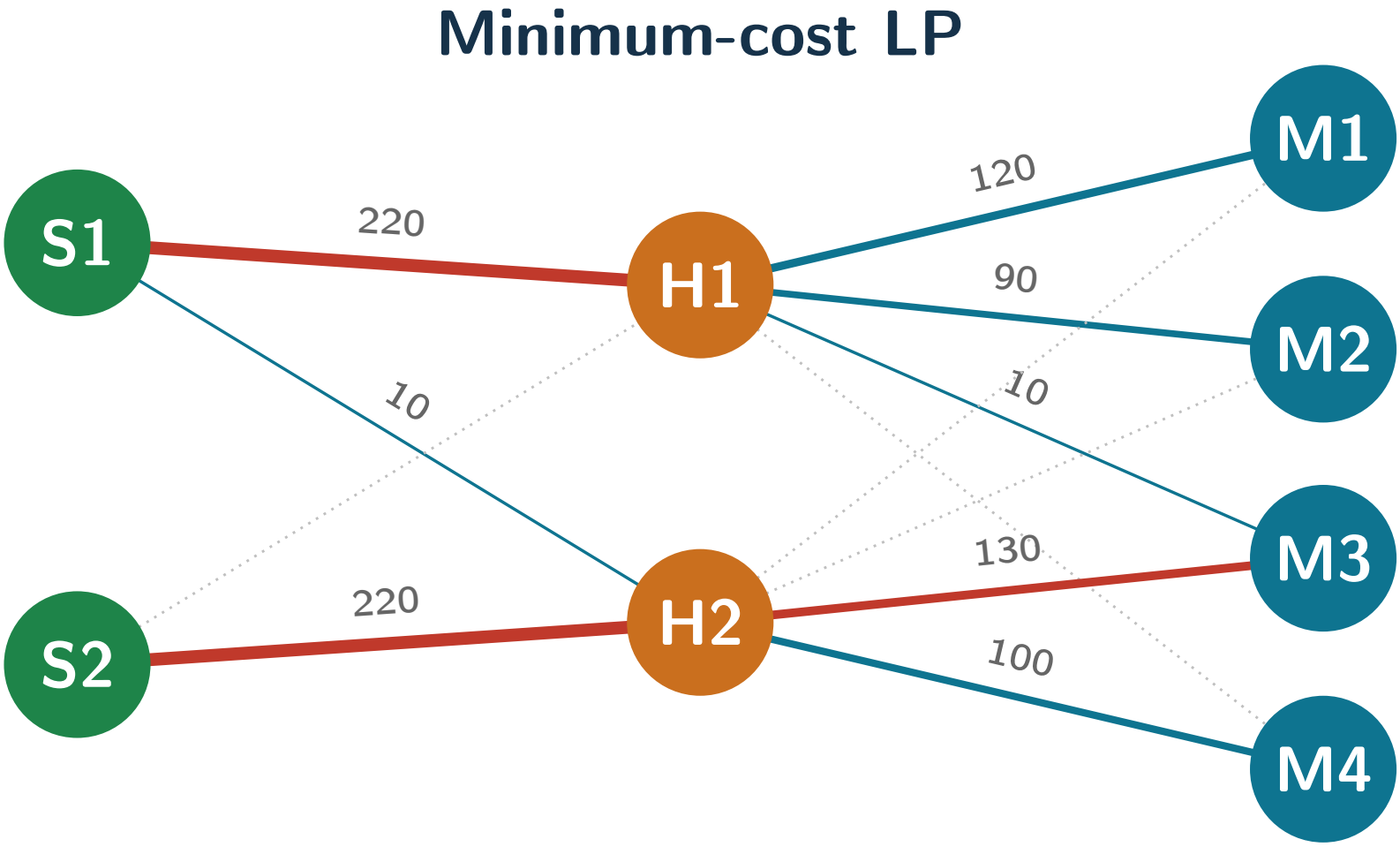
Reduced costs of arcs at zero (threshold = `SAObjLow`, €/unit):

Arc	Reduced cost	Threshold
S2→H1	+2.00	5.00
H1→M4	+0.50	5.50
H2→M1	+4.50	1.50
H2→M2	+1.00	3.00

How to read the flow figures

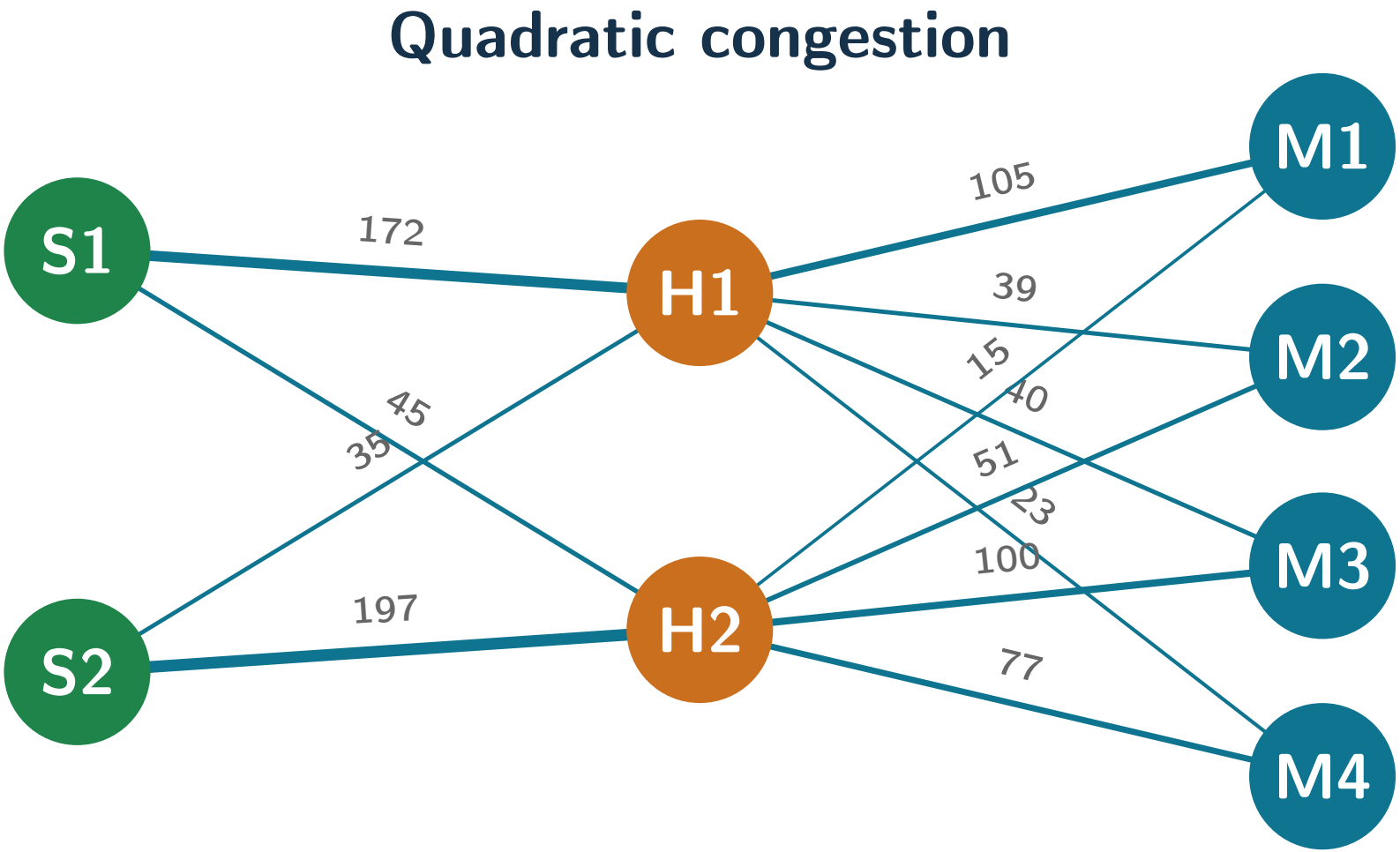
- Plants S in **green**, hubs H in **orange**, markets M in **teal**
- The thickness of the arcs is proportional to the flow
- **Red** indicates a saturated arc
- The number on the arc is the flow in units

Optimal flows: minimum-cost LP



The minimum-cost LP **saturates three arcs**.

Optimal flows: model with congestion



Quadratic congestion spreads the flows over several routes and **saturates no arc**.

The shadow prices of demand

- The shadow prices of demand (8–10.50 €/unit) are the **marginal costs of service** of the four markets
- This is the number to compare with the commercial margin before accepting new orders
- It is the correct basis for an internal transfer price
- M4 is the most expensive market to serve: +31% with respect to M1

The reduced costs of the unused arcs, with their range (1/2)

- Four arcs stay at zero and their reduced cost (attribute `x[a].RC`) is their **convenience threshold**
- Route S2 $\rightarrow$ H1 (cost 7 €/unit, `RC` $= +2$) would enter the optimal solution only if its cost dropped below

$$7 - 2 = 5 \text{ €/unit}$$

- This is the number to bring to the table in a negotiation with the carrier

The reduced costs of the unused arcs, with their range (2/2)

- Next to the reduced cost we quote its range of validity `SAObjLow`–`SAObjUp`
- For a variable at zero it is [threshold, $+\infty$): the cost of the arc may grow as much as it likes (it stays unused) and the threshold itself is `SAObjLow`
- Route H1 $\rightarrow$ M4 is one step away from being used (`RC` $= +0.50$): a discount of 50 cents is enough
- The positive variables all have zero reduced cost

Summary

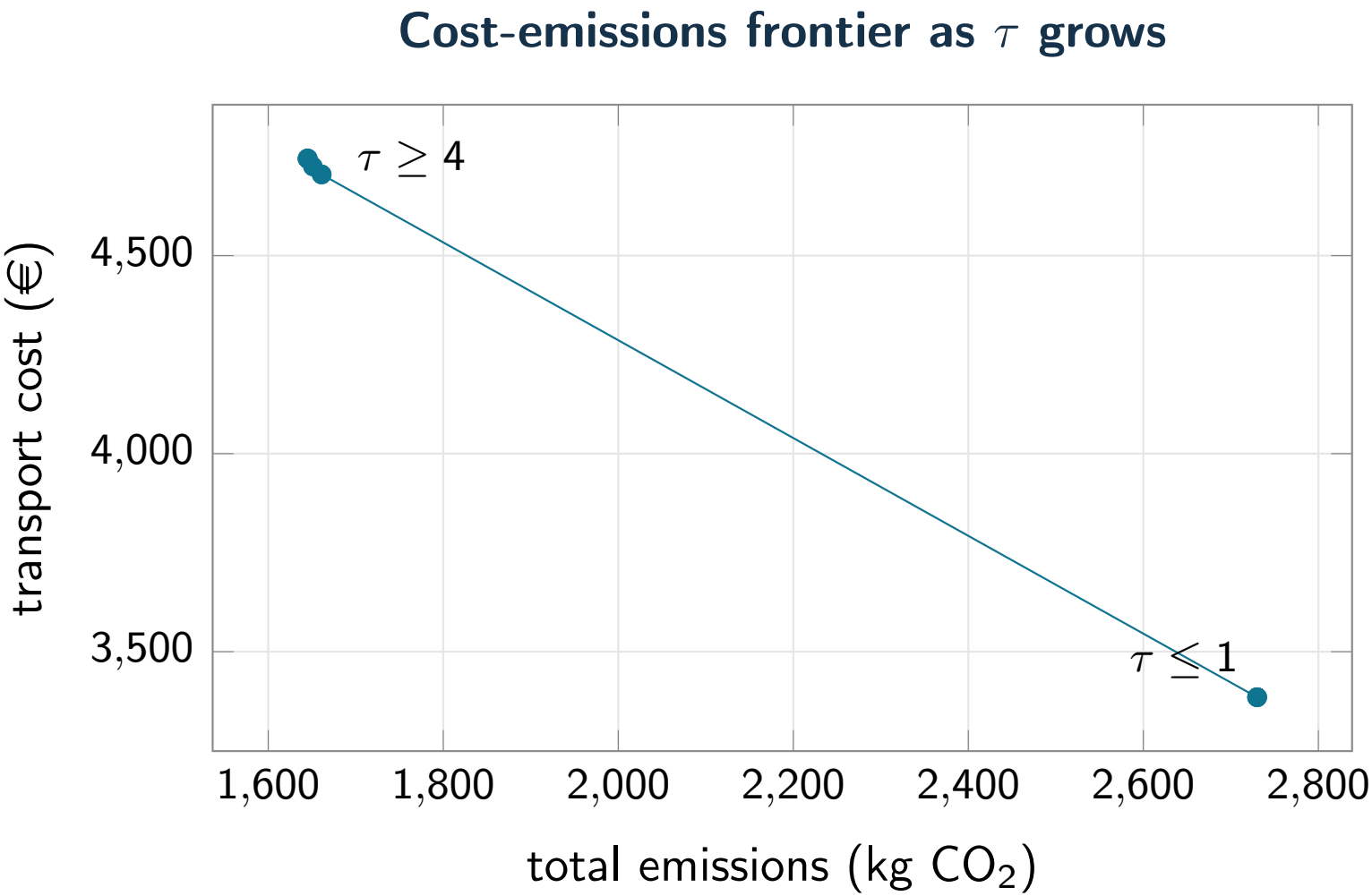
- The LP saturates three arcs; congestion lowers the maximum utilization to 90% at a higher cost (+9%)
- **Shadow prices of demand** = marginal costs of service of the markets
- **Reduced costs** of the arcs at zero = convenience thresholds, with range `SAObjLow`–`SAObjUp`

Sensitivity analysis: the price of CO_2

Introduction: the price of CO_2

- We vary the internal CO_2 price τ and solve the LP for every value
- We obtain the **cost-emissions frontier**
- We will understand why the frontier proceeds **by jumps** and why pure minimax is degenerate

Cost-emissions frontier



How to read the frontier

- Every point is the optimal solution for an internal price of CO_2 (τ)
- At the bottom right the "by road" configuration (cheap, 2730 kg)
- At the top left the "by rail" configuration (expensive, 1645 kg)
- The frontier proceeds by **jumps** between vertices of the LP

Results as τ varies

τ (€/kg)	Cost (€)	Emissions (kg)	
0.0 . . . 1.0	3385	2730	routes by road
1.24	4025	2173	intermediate vertex
1.26	4265	1981	
1.5	4705	1661	"rail" configuration
4.0 and up	4745	1645	stable

Why the frontier proceeds by jumps (1/2)

- The solutions of an LP lie at the **vertices** of the feasible polyhedron
- As τ grows the optimum stays put on one vertex, then it **jumps** to the next one
- A bisection on the model shows that:
  - up to $\tau = 1$ nothing changes
  - between $\tau = 1.0$ and $\tau = 1.5$ the network crosses intermediate vertices (at $\tau = 1.24$: 2173 kg)
  - the main jump happens exactly at $\tau = 1.25$ €/kg

Why the frontier proceeds by jumps (2/2)

- Managerially: an internal price of CO_2 **below the threshold** does not change the routes
- Just **above** it, it restructures the whole network at once
- The model with congestion, being strictly convex, would instead move **continuously**

Beware of minimax

Observation 1: Pure minimax is degenerate

The "pure" minimax ($\min z$) is degenerate: any routing with utilization $\leq z^*$ is optimal, even an extremely expensive one.

In practice one always uses a combination (cost $+\rho \cdot z$) or one constrains z and minimizes the cost.

Summary

- By varying τ we draw the cost-emissions frontier: from 2730 kg (road) to 1645 kg (rail)
- In an LP the optimum **jumps between vertices**: main threshold at $\tau = 1.25$ €/kg
- A strictly convex model moves continuously instead
- Pure minimax is degenerate: it must be combined with the cost

Exercises

Introduction: Exercises

- Five exercises to be solved with the script of the case study
- They check the shadow prices, simulate a failure, reproduce the CO_2 threshold and compare alternative formulations

Exercise: check

- Verify by perturbation the shadow price of the demand of M4 (raise it to 101)
- Verify in the same way the shadow price of the capacity of arc H2→M3

Exercise: failure of a hub

- Reduce by 50% the capacities of the arcs entering and leaving H2
- How much does the failure cost?
- Which markets suffer the most? (Compare the shadow prices before/after.)

Exercise: the rail threshold

- Reproduce with a bisection on τ the threshold $\tau = 1.25$ €/kg found in the text
- List which arcs change flow when crossing it
- Compare the threshold with the current ETS price of CO_2

Exercise: emissions constraint

- Reformulate with the constraint

$$\sum_{(i,j)\in A} e_{ij}\, x_{ij} \leq \bar{e}, \qquad \bar{e} = 2000 \text{ kg}$$

- Verify that the dual of the constraint coincides with the "weight" τ that produces the same emissions

Exercise: barrier

- Replace the congestion term with the barrier

$$\alpha \frac{x_{ij}}{u_{ij} - x_{ij}}$$

- Compare: which formulation keeps the flows further away from saturation?

Summary

- Checking shadow prices by perturbation
- Scenario analysis: failure of a hub
- CO_2 price: bisection on τ and the "constraint" approach with its dual
- Alternative models of congestion: quadratic term versus barrier

Final summary (1/2)

Model:

- Minimum-cost flow: flow conservation at the nodes, capacity on the arcs
- Variants: convex quadratic congestion, internal CO_2 price, minimax

Economic reading:

- Shadow prices of demand = marginal costs of service
- Reduced costs = convenience thresholds of the unused arcs

Final summary (2/2)

Congestion:

- Spreads the flows and leaves a safety margin on the arcs

Sustainability:

- The cost-emissions frontier of an LP proceeds by jumps between vertices
- Below the threshold of τ nothing changes; above it, the network is restructured at once