Production with quadratic smoothing

Sapienza Università di Roma

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Learning objectives

At the end of this chapter you will be able to:

- add a convex quadratic term to a linear model;
- read the multipliers of a QP and verify them by perturbation;
- trace the cost-smoothness frontier as the weight varies.

Prerequisites

- From the module *LP Modelling*, Chapter 4 «Multi-period production and inventory»: the linear production and inventory model, with its data
- From this module, Chapter 2 «Background: convexity, quadratic programming and KKT conditions»: convex quadratic programs, the KKT conditions and the multipliers as shadow prices
- From this module, Chapter 3 «The solver for nonlinear models»: quadratic objectives and the multipliers of a QP in `gurobipy`

What changes

The case and its limit

- Case, data and linear model: LP Modelling, "Production and inventory"
- The minimum-cost plan changes pace abruptly: total production from 219.5 to 389.0 units, jumps of up to 81.4 units in a month
- Every jump costs: overtime, shifts, suppliers to realign
- Problem class: **convex QP**

The idea

- A **quadratic** penalty on the jumps of total production
- A jump twice as large costs four times as much: many small adjustments are better
- A single new datum: $\gamma \in \mathbb{Q}_{\geq 0}$, weight of the jumps (€ per unit^2)
- With $\gamma = 0$ we recover the linear model

The model

QP model of production with smoothing

QP model of production with smoothing

$$\min \sum_{i=1}^{n}\sum_{t=1}^{m}\left(c_{it}x_{it} + h_{it}s_{it}\right)$$

$$+\gamma\sum_{t=2}^{m}\left(\sum_{i=1}^{n}x_{it} - \sum_{i=1}^{n}x_{i,t-1}\right)^2 \qquad (4.1a)$$

$$\text{subject to} \quad s_{i,t-1} + x_{it} = d_{it} + s_{it}, \qquad \forall i, \forall t, \qquad (4.1b)$$

$$\sum_{i=1}^{n} a_i\, x_{it} \le b_t, \qquad \forall t \in \{1,\dots,m\}, \qquad (4.1c)$$

$$x_{it} \ge 0, \qquad \forall i, \forall t, \qquad (4.1d)$$

$$s_{it} \ge 0, \qquad \forall i, \forall t. \qquad (4.1e)$$

Why the QP is convex

- Every jump $\sum_i x_{it} - \sum_i x_{i,t-1}$ is linear in the variables
- The square of a linear function is convex; the sum with weights $\gamma \geq 0$ stays convex
- Linear constraints $\Rightarrow$ every local minimum is global
- The KKT conditions are necessary and sufficient: multipliers are read as in linear models, but only **at the margin**

Worked example

One product, two months

Instance

$d_1 = 60$, $d_2 = 100$, $c = 10$, $h = 1$, zero initial stock, ample capacity

Linear model: $x = (60, 100)$, cost 1600, jump 40

- Step 1, with $\gamma = 0.5$: $\tilde{x} = (79.75;\ 80.25)$, $\tilde{s}_1 = 19.75$, $\tilde{z} = 1619.875$, $\tilde{\pi} = (9.5;\ 10.5)$
- Step 2: $x_1 = 60 + s_1$, $x_2 = 100 - s_1 \Rightarrow$ $f(s_1) = 1600 + s_1 + \gamma(40 - 2s_1)^2$

Stationarity, threshold and multipliers

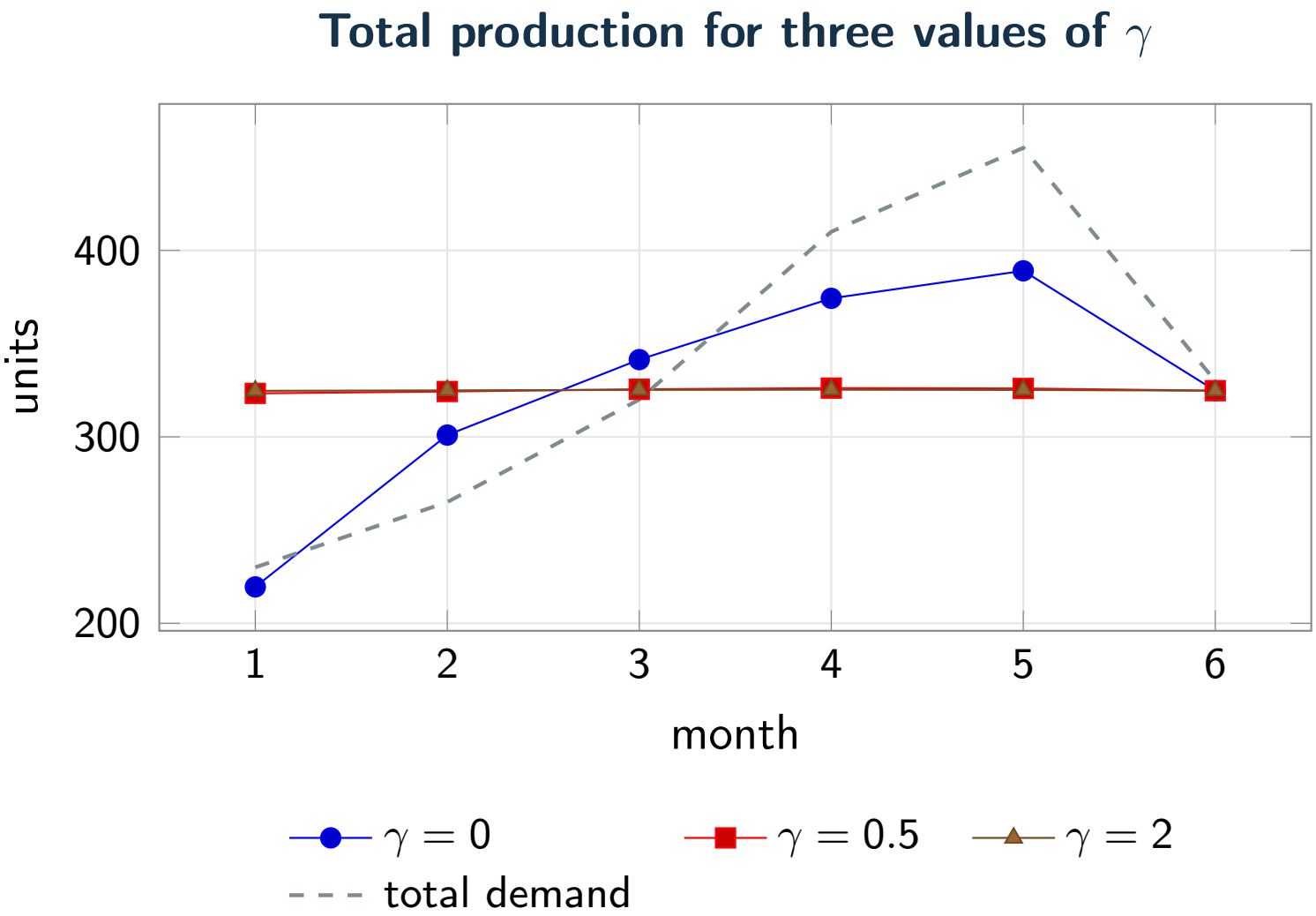
- Step 3: $f'(s_1) = 1 - 4\gamma(40 - 2s_1) = 0 \Rightarrow$ $s_1 = 20 - 1/(8\gamma) = 19.75$
- Step 4: $f'(0) = 1 - 160\gamma \Rightarrow$ stock only if $\gamma > 1/160$; with $\gamma = 0.004$: $\tilde{s}_1 = 0$, $\tilde{z} = 1606.4$
- Step 5: with positive stock $\tilde{\pi}_2 - \tilde{\pi}_1 = h = 1$; with $\gamma = 0.004$ the difference is $0.64 < h$

Implementation and results

Only the objective changes

```
tot = {t: gp.quicksum(x[i, t] for i in prodotti)
       for t in mesi}
salti = gp.quicksum((tot[t] - tot[t-1])**2
                    for t in mesi[1:])
m.setObjective(costo + gamma * salti, GRB.MINIMIZE)
m.optimize()
pi = [v_cap[t].Pi for t in mesi]
```

Three values of γ



The cost of smoothness

```
gamma = 0.0: linear cost 32,889.33  largest jump 81.4
gamma = 0.5: linear cost 33,000.96  largest jump  1.2
gamma = 2.0: linear cost 33,004.14  largest jump  0.4
```

- An almost flat plan (about 325 units per month) for € 111.63 more (+0.34%)
- Mostly product 1 goes to stock (up to 212.6 units): the penalty counts **units**, not hours

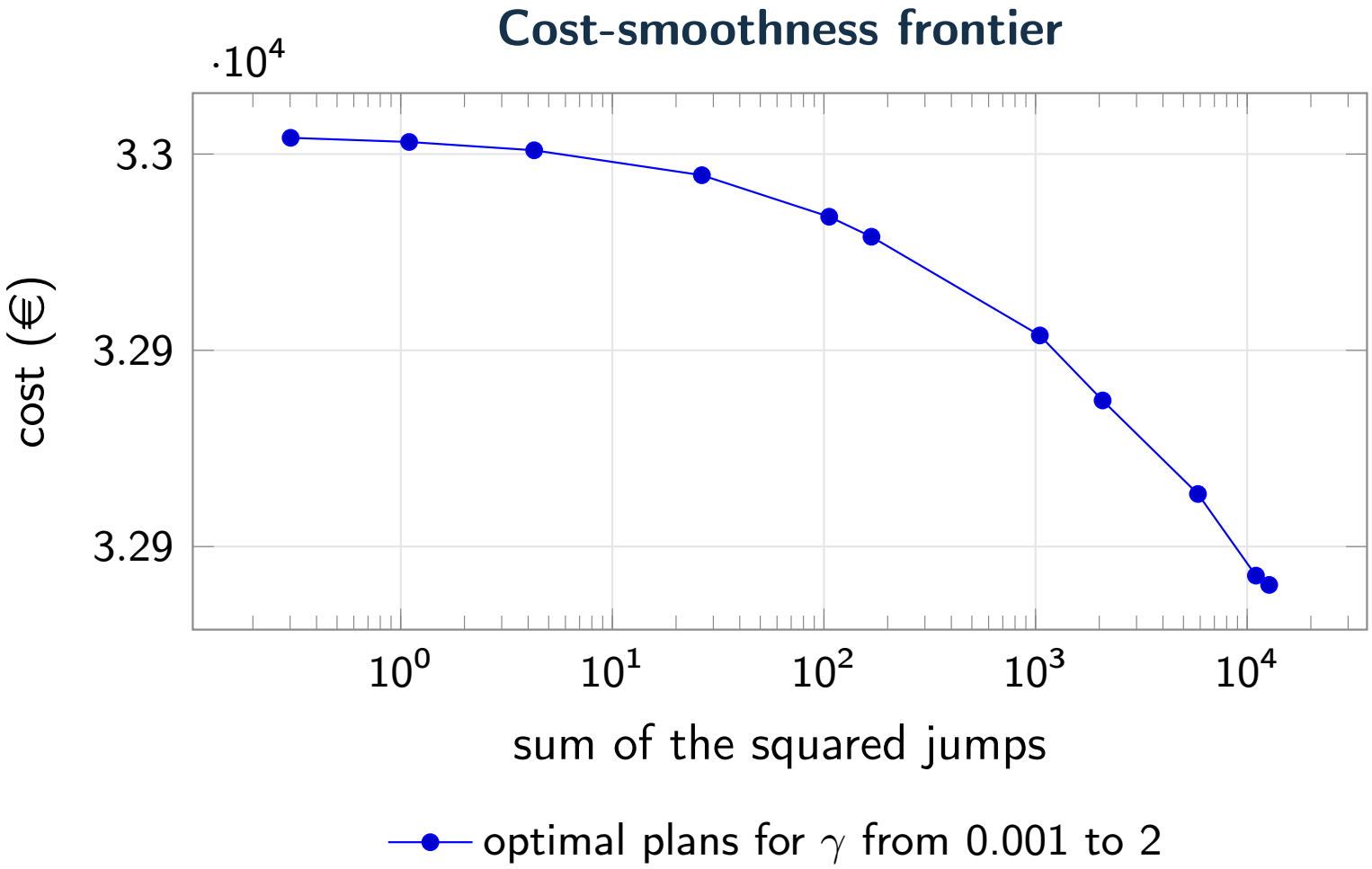
Sensitivity analysis

Capacity multipliers

```
month 4: Pi(gamma=0) = -2.286   Pi(gamma=0.5) = -0.667
month 5: Pi(gamma=0) = -3.048   Pi(gamma=0.5) = -1.333
month 6: Pi(gamma=0) = -3.810   Pi(gamma=0.5) = -3.134
+1    hour  in month 6 -> difference quotient -3.060
+0.01 hours in month 6 -> difference quotient -3.133
```

- The multiplier of a QP is exact only **at the margin**: the optimal value has curvature
- With smoothing the capacity of months 1–3 is no longer active

The cost-smoothness frontier



- With $\gamma = 0.01$ the sum of the squared jumps falls from 13,699 to 2,075 for € 48

Exercises

Exercises

1. **By hand**: example with $\gamma = 0.05$, compute s_1, x, value and multipliers
2. **Check**: capacity multiplier of month 5 with increments of 1 and 0.01 hours
3. **Smoothing by product**: $\gamma \sum_i \sum_t (x_{it} - x_{i,t-1})^2$, $\gamma = 0.5$
4. **Linear smoothing**: absolute value of the jumps as an LP, $\gamma = 5$
5. **Target level**: $\gamma \sum_t (v_t - 325)^2$, $\gamma = 0.5$

Summary

- A convex quadratic term added to an LP gives a convex QP
- The weight γ traces the cost-smoothness frontier: the first steps cost little
- QP multipliers are read with `Pi`, but hold at the margin
- A new term can change the structure of the solution