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Network Optimization

Volume 4 — The network simplex

The network simplex: bases as spanning trees, the pivot in the uncapacitated and capacitated cases, degeneracy.

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The network simplex

Prerequisites

- From this module, [Chapter 1 «Graphs»](#): spanning trees, leaves, fundamental cycles and the rank of the node–arc incidence matrix.
- From this module, [Chapter 4 «Network flow models»](#): the minimum cost flow model, the removal of the lower bounds, the potentials, the reduced costs and the transshipment instance on the reference network.
- From the module Linear Programming, [Chapter 6 «Simplex algorithm»](#), [Chapter 7 «Optimality conditions and pivot rules»](#), [Chapter 8 «Phase 0 of the simplex algorithm»](#) and [Chapter 11 «Revised simplex algorithm»](#): bases, basic solutions, reduced costs, pivoting operations, degeneracy and Phase 0.

Objectives

At the end of this chapter you will be able to:

- prove that the bases of the minimum cost flow model are the spanning trees, and compute a tree solution from the leaves;
- compute the potentials of a spanning tree and the reduced costs, and test the optimality of a tree solution with arcs at the lower and at the upper bound;
- perform a pivot: entering arc, oriented cycle, step, leaving arc, and the update of the flows, of the tree and of the potentials;
- recognize a degenerate pivot, check whether a tree structure is strongly feasible, and apply the leaving arc rule that keeps it strongly feasible and prevents cycling;
- find an initial tree structure with Phase 0 on the auxiliary network, or prove that no feasible flow exists.

8.1 The simplex algorithm on the minimum cost flow model

Definition 8.1: basis and basic solution

Let $G = (N, A)$ be a network with n nodes and m arcs, whose underlying undirected graph is connected, with balances b_i such that $\sum_{i \in N} b_i = 0$, costs c_{ij} , zero lower bounds and capacities $0 < u_{ij} \leq +\infty$. Let $\mathbf{M}$ be its node–arc incidence matrix $\mathbf{B}$ without the row of node n , with the column $\mathbf{M}_a$ for the arc a , and let $\mathbf{h} = (b_1, b_2, \dots, b_{n-1})'$.

- A **basis** is a set $T \subseteq A$ of $n - 1$ arcs whose columns $\mathbf{M}_a$, $a \in T$, are linearly independent. The arcs of T are **basic**, the other arcs are **nonbasic**.
- A **bound status** of the nonbasic arcs is a partition of $A \setminus T$ into two sets L and U , where every arc of U has a finite capacity.
- The **basic solution** of (T, L, U) is the vector $\mathbf{x}$ with $x_{ij} = 0$ on the arcs of L , $x_{ij} = u_{ij}$ on the arcs of U , and $\mathbf{M}\mathbf{x} = \mathbf{h}$. It is **feasible** if $0 \leq x_{ij} \leq u_{ij}$ on the arcs of T .

- In the whole chapter the network is as in the definition. The lower bounds are removed as in [Chapter 4 «Network flow models»](#), an arc with $u_{ij} = 0$ is deleted, and the MCF model is

$$\min \{ \mathbf{c}'\mathbf{x} : \mathbf{B}\mathbf{x} = \mathbf{b}, \mathbf{0} \leq \mathbf{x} \leq \mathbf{u} \}.$$

- The basis is a set of columns; L and U say at which bound each nonbasic variable is fixed. The values of the basic variables solve the square system $\mathbf{M}_T \mathbf{x}_T = \mathbf{h} - \sum_{a \in U} \mathbf{M}_a u_a$, which has a unique solution because $\mathbf{M}_T$ is nonsingular.

Lemma 8.2: the redundant equation

Let the underlying undirected graph of G be connected and $\sum_{i \in N} b_i = 0$. Then $\mathbf{M}$ has rank $n - 1$, and a vector $\mathbf{x} \in \mathbb{R}^m$ satisfies $\mathbf{B}\mathbf{x} = \mathbf{b}$ if and only if it satisfies $\mathbf{M}\mathbf{x} = \mathbf{h}$.

Proof. The rank is $n - 1$ by [Chapter 1, Corollary «bases of the incidence matrix»](#): deleting one row of $\mathbf{B}$ leaves $n - 1$ linearly independent rows.

- If $\mathbf{B}\mathbf{x} = \mathbf{b}$, then in particular $\mathbf{M}\mathbf{x} = \mathbf{h}$.
- Conversely, every column of $\mathbf{B}$ has one $+1$ and one -1 , so the rows of $\mathbf{B}$ add up to the zero vector. If $\mathbf{M}\mathbf{x} = \mathbf{h}$, row n of $\mathbf{B}\mathbf{x}$ is $-\sum_{i=1}^{n-1} b_i = b_n$. $\square$

- Hence a basic solution satisfies the conservation constraints of all the nodes, node n included.
- The definition is the one of the simplex algorithm with one difference: in [Chapter 6 «Simplex algorithm»](#) of the module Linear Programming every nonbasic variable is zero, while here a nonbasic variable can also sit at its upper bound u_{ij} .

8.1.1 The simplex algorithm with bounds

- As in [Chapter 11 «Revised simplex algorithm»](#), at a basis T the **simplex multipliers** $\pi \in \mathbb{R}^{n-1}$ solve the first system $\pi' M_T = c'_T$, and the reduced cost of the variable x_a is $c_a - \pi' M_a$.
- The MCF model is a minimization. A nonbasic variable at its lower bound lowers the cost when it increases if its reduced cost is negative; a nonbasic variable at its upper bound lowers the cost when it decreases if its reduced cost is positive.
- If no nonbasic variable can lower the cost, the basic solution is optimal. Otherwise one of them enters the basis, the ratio test gives the largest step that keeps the basic variables within their bounds, and a basic variable that reaches a bound leaves the basis.
- The **network simplex algorithm** performs exactly these operations, without matrices: the bases are spanning trees, the multipliers are potentials computed along the tree, and the ratio test runs along a cycle.

Example 8.1: bases of the reference network

In the reference network of [Chapter 4 «Network flow models»](#) the matrix M is the incidence matrix without the row of node 6; a basis has $n - 1 = 5$ arcs.

- The arcs (1, 2), (2, 4), (3, 2), (4, 6), (5, 4) give a 5×5 submatrix of M with determinant 1: they form a basis.
- The arcs (1, 2), (2, 4), (3, 2), (4, 3), (5, 6) touch every node, but they contain the cycle 2, 4, 3, 2: the columns of (2, 4), (4, 3), (3, 2) add up to zero, and the determinant is 0.
- Of the 126 sets of 5 arcs, exactly 55 give a nonsingular submatrix, and they are the 55 spanning trees of the reference graph. All these determinants are 1 or -1 .

In a connected network the equation of node n is redundant. A basis is a set of $n - 1$ arcs with linearly independent columns in M ; the nonbasic arcs are fixed at 0 (set L) or at their capacity (set U), and the basic solution follows.

8.2 Bases and spanning trees

Theorem 8.3: the bases are the spanning trees

Let the underlying undirected graph of $G = (N, A)$ be connected. A set T of $n - 1$ arcs is a basis if and only if it is a spanning tree. Moreover, for a spanning tree T the rows and the columns of $\mathbf{M}_T$ can be ordered so that the matrix is lower triangular with diagonal entries ± 1 , and $\det(\mathbf{M}_T) = \pm 1$.

Proof. We use the results of Chapter 1 «Graphs».

1. *The columns of T in $\mathbf{M}$ and in $\mathbf{B}$ are independent together.* The entry of node n of $\sum_{a \in T} \lambda_a \mathbf{B}_a$ is minus the sum of the other entries, since the rows of $\mathbf{B}$ add up to zero. Hence $\sum_{a \in T} \lambda_a \mathbf{M}_a = \mathbf{0}$ if and only if $\sum_{a \in T} \lambda_a \mathbf{B}_a = \mathbf{0}$.
2. *A spanning tree is a basis.* Its columns in $\mathbf{B}$ are linearly independent by Lemma «columns of a forest».
3. *A basis is a spanning tree.* If the $n - 1$ arcs of T were acyclic, they would form a spanning tree by Theorem «characterizations of a tree»(c). Otherwise T contains a cycle, whose columns are linearly dependent by Lemma «columns of a cycle», and T is not a basis.
4. *Triangular form.* By induction on n ; for $n = 2$ the matrix is 1×1 with entry ± 1 . For $n \geq 3$ the tree has at least two leaves (Lemma «leaves»), so it has a leaf $\ell \neq n$, with its only tree arc a_ℓ . Row ℓ of $\mathbf{M}_T$ has the single nonzero entry ± 1 , in the column of a_ℓ .
5. Put row ℓ and the column of a_ℓ first. Deleting them leaves the matrix built in the same way for the spanning tree $T \setminus \{a_\ell\}$ of the nodes $N \setminus \{\ell\}$, which still contain n : by induction it can be ordered in lower triangular form. Expanding along the first row, $\det(\mathbf{M}_T) = \pm 1$. $\square$

Corollary 8.4: integer tree solutions

If the balances and the finite capacities are integer, every basic solution of the MCF model is integer.

Proof. The right-hand side $\mathbf{h} - \sum_{a \in U} \mathbf{M}_a u_a$ is integer, and $\det(\mathbf{M}_T) = \pm 1$ by Theorem 8.3: by Cramer's rule every basic variable is an integer. This is the integrality of Chapter 4 «Network flow models», seen on the bases. $\square$

Definition 8.5: tree structure and tree solution

Let T be a spanning tree of G . A **tree structure** is a triple (T, L, U) , where L and U partition $A \setminus T$ and every arc of U has a finite capacity. Its **tree solution** is its basic solution: $x_{ij} = 0$ on L , $x_{ij} = u_{ij}$ on U , and $\mathbf{B}\mathbf{x} = \mathbf{b}$.

- The tree structure is **feasible** if $0 \leq x_{ij} \leq u_{ij}$ on every arc of T .
- A feasible tree structure is **degenerate** if some arc of T has $x_{ij} = 0$ or $x_{ij} = u_{ij}$.

Basis and bound status

- The tree T is the basis. The sets L and U only say at which bound each nonbasic arc sits.
- An arc of U is nonbasic, although its flow u_{ij} is positive. An arc of T can have flow 0 or u_{ij} : the tree solution is then degenerate, but the arc is still basic.

Proposition 8.6: the tree solution from the leaves

Let (T, L, U) be a tree structure, and let the **residual balances** be

$$\hat{b}_i = b_i - \sum_{(i,j) \in U} u_{ij} + \sum_{(j,i) \in U} u_{ji}, \quad \forall i \in N.$$

Repeat until no tree arc is left: take a leaf $\ell \neq n$ of the remaining tree, with its tree arc a and its neighbour j ; set $x_a = \hat{b}_\ell$ if $a = (\ell, j)$ and $x_a = -\hat{b}_\ell$ if $a = (j, \ell)$; add $\hat{b}_\ell$ to $\hat{b}_j$; delete ℓ and a . The values obtained, with $x = 0$ on L and $x = u$ on U , are the tree solution.

Proof. The residual balance $\hat{b}_i$ is the balance of node i once the flows on the arcs of U are moved to the right-hand side. We show that each step keeps this invariant: the remaining tree arcs must satisfy conservation with the current residual balances.

1. At the leaf ℓ the only remaining unknown is x_a , so the conservation constraint of ℓ reads $x_a = \hat{b}_\ell$ if ℓ is the tail of a , and $-x_a = \hat{b}_\ell$ if ℓ is its head.
2. In the constraint of j the arc a appears with $-x_a$ if $a = (\ell, j)$ and with $+x_a$ if $a = (j, \ell)$. Moving the known term to the right-hand side adds $x_a = \hat{b}_\ell$ or $-x_a = \hat{b}_\ell$ to $\hat{b}_j$.
3. A tree with at least two nodes has a leaf other than n ([Chapter 1, Lemma «leaves»](#)), so the procedure ends with node n alone. Its constraint holds by [Lemma 8.2](#), and the solution is unique by [Theorem 8.3](#). $\square$

- In the examples the leaf is always the one with the **smallest index** different from n .
- Each leaf passes its residual balance to its neighbour. Hence the flow on a tree arc equals the total residual balance of the nodes that the arc separates from n , as in [Chapter 4, Lemma «flow across a cut»](#).

Example 8.2: the flow $\bar{x}$ of Chapter 4 as a tree solution

We use the reference network of [Chapter 4](#), with $\mathbf{b} = (5, 2, 0, -3, 0, -4)$ and

arc	(1, 2)	(1, 3)	(2, 4)	(3, 2)	(3, 5)	(4, 3)	(4, 6)	(5, 4)	(5, 6)
c_{ij}	2	2	5	2	3	3	5	5	3
u_{ij}	4	4	4	2	4	3	5	3	2

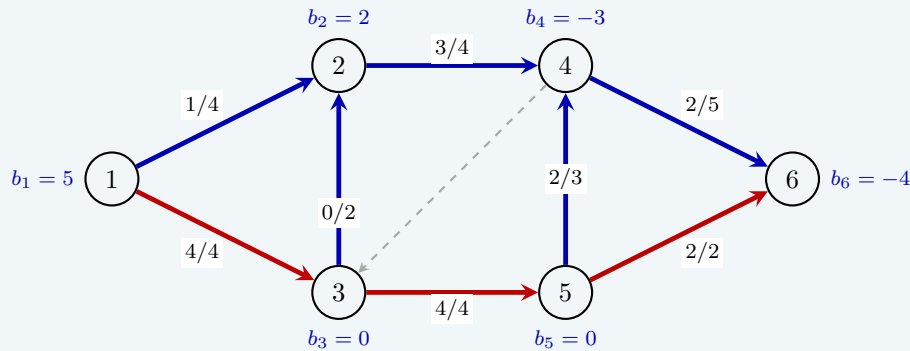
Take $T = \{(1, 2), (2, 4), (3, 2), (4, 6), (5, 4)\}$, $U = \{(1, 3), (3, 5), (5, 6)\}$ and $L = \{(4, 3)\}$. The residual balances are

$$\hat{\mathbf{b}} = (5 - 4, 2, 0 + 4 - 4, -3, 0 + 4 - 2, -4 + 2) = (1, 2, 0, -3, 2, -2).$$

The leaves are processed in the order 1, 3, 2, 5, 4:

leaf ℓ	tree arc	flow	residual balance of the neighbour
1	(1, 2)	$x_{12} = \hat{b}_1 = 1$	$\hat{b}_2 = 2 + 1 = 3$
3	(3, 2)	$x_{32} = \hat{b}_3 = 0$	$\hat{b}_2 = 3 + 0 = 3$
2	(2, 4)	$x_{24} = \hat{b}_2 = 3$	$\hat{b}_4 = -3 + 3 = 0$
5	(5, 4)	$x_{54} = \hat{b}_5 = 2$	$\hat{b}_4 = 0 + 2 = 2$
4	(4, 6)	$x_{46} = \hat{b}_4 = 2$	$\hat{b}_6 = -2 + 2 = 0$

The tree solution is the flow $\bar{\mathbf{x}} = (1, 4, 3, 0, 4, 0, 2, 2, 2)$ of Chapter 4, with cost 63. In the figure the tree arcs are blue, the arcs of U red and the arc of L grey and dashed; each arc carries x_{ij}/u_{ij} .



The tree arc (3, 2) has flow 0: the tree structure is feasible and degenerate.

The bases of the MCF model are the spanning trees, and their matrices are triangular with determinant ± 1 . A tree solution fixes the arcs of L at 0 and those of U at their capacity, and the flows on the tree arcs follow from the leaves.

8.3 Potentials and optimality

Definition 8.7: potentials of a tree

Let T be a spanning tree of G . Node n is the **root** of T . The **potentials** of T are the values π_i , $i \in N$, such that

$$\bar{c}_{ij} = c_{ij} - \pi_i + \pi_j = 0, \quad \forall (i, j) \in T, \quad \pi_n = 0.$$

- The potentials are computed from the root, as in the certificate of [Chapter 4](#): $\pi_n = 0$, and for a tree arc (i, j) with one known endpoint, $\pi_j = \pi_i - c_{ij}$ or $\pi_i = \pi_j + c_{ij}$.
- The normalization $\pi_n = 0$ corresponds to the deleted row of node n in M .

Lemma 8.8: potentials and simplex multipliers

The potentials of a spanning tree T exist and are unique. The vector $(\pi_1, \pi_2, \dots, \pi_{n-1})$ is the vector of the simplex multipliers of the basis T , and $\bar{c}_{ij}$ is the reduced cost of x_{ij} . Moreover, π_i is the cost of the tree path from i to n , where an arc traversed from tail to head counts $+c_{ij}$ and an arc traversed from head to tail counts $-c_{ij}$.

Proof. We write $\pi = (\pi_1, \pi_2, \dots, \pi_{n-1})$ for the first $n - 1$ potentials, and $\pi_n = 0$.

1. The column of (i, j) in M has $+1$ in row i and -1 in row j , the row of n being deleted. Hence $\pi' M_{(i,j)} = \pi_i - \pi_j$, also when $i = n$ or $j = n$.
2. The system $\pi' M_T = c'_T$ reads $\pi_i - \pi_j = c_{ij}$ on the tree arcs, that is $\bar{c}_{ij} = 0$. Its matrix is nonsingular by [Theorem 8.3](#), so the potentials exist and are unique, and $c_{ij} - \pi' M_{(i,j)} = c_{ij} - \pi_i + \pi_j = \bar{c}_{ij}$.
3. Let $i = v_0, v_1, \dots, v_r = n$ be the tree path. An arc traversed from tail to head gives $\pi_{v_{t-1}} - \pi_{v_t} = c_{v_{t-1}v_t}$; an arc traversed from head to tail gives $\pi_{v_{t-1}} - \pi_{v_t} = -c_{v_t v_{t-1}}$. Adding up, the sum telescopes to $\pi_i - \pi_n = \pi_i$. $\square$

- A **cycle** C of G is traversed in one of its two directions. Its arcs traversed from tail to head are **forward**, the others **backward**, and its cost is

$$c(C) = \sum_{(i,j) \text{ forward}} c_{ij} - \sum_{(i,j) \text{ backward}} c_{ij}.$$

Lemma 8.9: the cost of a cycle

For every vector of potentials π and every cycle C traversed in one direction,

$$c(C) = \sum_{(i,j) \text{ forward}} \bar{c}_{ij} - \sum_{(i,j) \text{ backward}} \bar{c}_{ij}.$$

In particular, if π are the potentials of a spanning tree T and C is the fundamental cycle of an arc $(k, l) \notin T$, traversed in the direction of (k, l) , then $c(C) = \bar{c}_{kl}$.

Proof. Let C visit $v_0, v_1, \dots, v_r = v_0$, and let the arc a_t join v_{t-1} and v_t .

1. If $a_t = (v_{t-1}, v_t)$ is forward, $\bar{c}_{a_t} = c_{a_t} - (\pi_{v_{t-1}} - \pi_{v_t})$. If $a_t = (v_t, v_{t-1})$ is backward, $-\bar{c}_{a_t} = -c_{a_t} - (\pi_{v_{t-1}} - \pi_{v_t})$.
2. Adding up over the cycle, the terms $\pi_{v_{t-1}} - \pi_{v_t}$ telescope to $\pi_{v_0} - \pi_{v_r} = 0$, which gives the formula.
3. For the fundamental cycle of (k, l) , every tree arc has $\bar{c}_{ij} = 0$, and (k, l) is forward: the sum is $\bar{c}_{kl}$. $\square$

Theorem 8.10: optimality of a tree solution

Let (T, L, U) be a feasible tree structure, with tree solution $\mathbf{x}$ and potentials $\boldsymbol{\pi}$. If

$$\bar{c}_{ij} \geq 0, \quad \forall (i, j) \in L, \quad \bar{c}_{ij} \leq 0, \quad \forall (i, j) \in U,$$

then $\mathbf{x}$ is an optimal solution of the MCF model. If the inequalities are strict, $\mathbf{x}$ is the unique optimal solution.

Proof. Let $\mathbf{y}$ be a feasible flow. By Chapter 4, Lemma «cost and reduced cost», $\mathbf{c}'\mathbf{y} = \mathbf{c}'\mathbf{y} + \mathbf{b}'\boldsymbol{\pi}$ and $\mathbf{c}'\mathbf{x} = \bar{\mathbf{c}}'\mathbf{x} + \mathbf{b}'\boldsymbol{\pi}$, hence

$$\mathbf{c}'\mathbf{y} - \mathbf{c}'\mathbf{x} = \sum_{(i,j) \in A} \bar{c}_{ij} (y_{ij} - x_{ij}).$$

- On the arcs of T the term is zero, since $\bar{c}_{ij} = 0$.
- On the arcs of L , $x_{ij} = 0 \leq y_{ij}$ and $\bar{c}_{ij} \geq 0$: the term is $\bar{c}_{ij}y_{ij} \geq 0$.
- On the arcs of U , $x_{ij} = u_{ij} \geq y_{ij}$ and $\bar{c}_{ij} \leq 0$: the term is $\bar{c}_{ij}(y_{ij} - u_{ij}) \geq 0$.

Hence $\mathbf{c}'\mathbf{y} \geq \mathbf{c}'\mathbf{x}$. With strict inequalities, $\mathbf{c}'\mathbf{y} = \mathbf{c}'\mathbf{x}$ forces $y_{ij} = 0$ on L and $y_{ij} = u_{ij}$ on U ; the flows on T are then those of the tree solution, which is unique (Theorem 8.3), and $\mathbf{y} = \mathbf{x}$. $\square$

- These are the conditions of Chapter 4, Corollary «reduced cost certificate»: an arc with positive reduced cost is empty, an arc with negative reduced cost is saturated, and every tree arc has reduced cost zero.
- An arc of L with $\bar{c}_{ij} < 0$, or of U with $\bar{c}_{ij} > 0$, violates the conditions: by Lemma 8.9, sending flow around its fundamental cycle in the right direction lowers the cost by $|\bar{c}_{ij}|$ per unit.

Example 8.3: potentials and reduced costs of the tree of $\bar{\mathbf{x}}$

For the tree structure of Example 8.2 we start from the root 6:

$$\begin{aligned} \pi_6 &= 0, & \pi_4 &= \pi_6 + c_{46} = 5, & \pi_2 &= \pi_4 + c_{24} = 10, \\ \pi_5 &= \pi_4 + c_{54} = 10, & \pi_1 &= \pi_2 + c_{12} = 12, & \pi_3 &= \pi_2 + c_{32} = 12. \end{aligned}$$

For example, $\pi_1 = c_{12} + c_{24} + c_{46} = 12$ is the cost of the tree path 1, 2, 4, 6. The reduced costs of the nonbasic arcs are

$$\begin{aligned}\bar{c}_{13} &= 2 - 12 + 12 = 2, & \bar{c}_{35} &= 3 - 12 + 10 = 1, \\ \bar{c}_{43} &= 3 - 5 + 12 = 10, & \bar{c}_{56} &= 3 - 10 + 0 = -7.\end{aligned}$$

- The arc $(4, 3) \in L$ has $\bar{c}_{43} \geq 0$, and the arc $(5, 6) \in U$ has $\bar{c}_{56} \leq 0$.
- The arcs $(1, 3)$ and $(3, 5)$ are in U with positive reduced cost: the conditions of [Theorem 8.10](#) fail.
- The fundamental cycle of $(1, 3)$, in its direction, is 1, 3, 2, 1: it uses $(1, 3)$ and $(3, 2)$ forward and $(1, 2)$ backward, and its cost is $c_{13} + c_{32} - c_{12} = 2 + 2 - 2 = 2 = \bar{c}_{13}$. Decreasing x_{13} and sending the flow around the cycle in the opposite direction lowers the cost by 2 per unit.

The potentials of a tree vanish at the root and give reduced cost zero on the tree arcs; they are the simplex multipliers. The reduced cost of a nonbasic arc is the cost of its fundamental cycle, and a tree solution with $\bar{c} \geq 0$ on L and $\bar{c} \leq 0$ on U is optimal.

8.4 The pivot

Definition 8.11: the pivot of the network simplex

Let (T, L, U) be a feasible tree structure, with tree solution $\mathbf{x}$ and potentials $\boldsymbol{\pi}$.

- An arc is **eligible** if it is in L with $\bar{c}_{ij} < 0$, or in U with $\bar{c}_{ij} > 0$. The **entering arc** $e = (k, l)$ is the eligible arc with the largest $|\bar{c}_{ij}|$; if more than one, the first in lexicographic order.
- The **cycle of the pivot** C is the fundamental cycle of e , traversed in the direction of e if $e \in L$, and in the opposite direction if $e \in U$.
- The **residual capacity** of an arc of C is $u_{ij} - x_{ij}$ if the arc is forward ($+\infty$ if $u_{ij} = +\infty$), and x_{ij} if it is backward. The **step** θ is the minimum residual capacity along C , and the arcs that attain it are **blocking**.
- The **apex** w of C is the first node shared by the tree paths from k and from l to the root n . The **leaving arc** f is the last blocking arc met when C is traversed in its direction starting from w .

A pivot with $\theta = 0$ is **degenerate**.

- Sending θ units along C increases the flow on the forward arcs and decreases it on the backward arcs. The entering arc is forward if it is in L , whose flow must increase, and backward if it is in U , whose flow must decrease.
- Let p and q be the endpoints of e in the direction of C : $p = k$, $q = l$ if $e \in L$, and $p = l$, $q = k$ if $e \in U$. Starting from the apex, C follows the tree path from w down to p , the

arc e from p to q , and the tree path from q up to w .

- The entering arc is a candidate to leave: its residual capacity is u_{kl} , so $\theta \leq u_{kl}$.

Proposition 8.12: the pivot

In the setting of Definition 8.11, let $\theta < +\infty$, and let $\mathbf{x}'$ be obtained from $\mathbf{x}$ by adding θ on the forward arcs of C and subtracting θ on the backward arcs.

- $\mathbf{x}'$ is a feasible flow, and $\mathbf{c}'\mathbf{x}' = \mathbf{c}'\mathbf{x} - \theta |\bar{c}_{kl}|$.
- $T' = T \cup \{e\} \setminus \{f\}$ is a spanning tree, and $\mathbf{x}'$ is the tree solution of the structure (T', L', U') , where f goes to U if it is forward in C and to L if it is backward. If $f = e$, then $T' = T$ and e moves from L to U or from U to L .

Proof. Let $\varepsilon_a = +1$ on the forward arcs and -1 on the backward arcs of C , and let $\mathbf{x}^C$ be the signed incidence vector of C , with $x_a^C = \varepsilon_a$ on C and 0 elsewhere: then $\mathbf{x}' = \mathbf{x} + \theta \mathbf{x}^C$.

1. *Conservation.* $\mathbf{B}\mathbf{x}^C = \sum_{a \in C} \varepsilon_a \mathbf{B}_a = \mathbf{0}$ by Chapter 1, Lemma «columns of a cycle», so $\mathbf{B}\mathbf{x}' = \mathbf{b}$.
2. *Bounds.* A forward arc increases by $\theta \leq u_{ij} - x_{ij}$, a backward arc decreases by $\theta \leq x_{ij}$, the other arcs do not change.
3. *Cost.* $\mathbf{c}'\mathbf{x}' - \mathbf{c}'\mathbf{x} = \theta c(C) = \theta \varepsilon_e \bar{c}_{kl}$ by Lemma 8.9, since the tree arcs have $\bar{c}_{ij} = 0$. If $e \in L$, then $\varepsilon_e = +1$ and $\bar{c}_{kl} < 0$; if $e \in U$, then $\varepsilon_e = -1$ and $\bar{c}_{kl} > 0$. In both cases $\varepsilon_e \bar{c}_{kl} = -|\bar{c}_{kl}|$.
4. *The new tree.* C is the only cycle of $T \cup \{e\}$ (Chapter 1, Proposition «fundamental cycle»). Removing $f \in C$ leaves a connected graph, since the rest of C still joins the endpoints of f ; it has $n - 1$ arcs, so it is a spanning tree (Theorem «characterizations of a tree»(b)).
5. *The new tree solution.* The nonbasic arcs other than e and f keep their flow and their bound. The arc f is blocking, so after the step $x'_f = u_f$ if it is forward and $x'_f = 0$ if it is backward. Hence $\mathbf{x}'$ satisfies $\mathbf{B}\mathbf{x}' = \mathbf{b}$ and the bounds fixed by (T', L', U') , and it is the tree solution by Theorem 8.3. $\square$

Proposition 8.13: the update of the potentials

In the pivot of Proposition 8.12, let $f \neq e$, and let R and S be the node sets of the two components of $T \setminus \{f\}$, with $n \in R$. Then e has one endpoint in R and one in S , and the potentials of T' are

$$\pi'_i = \pi_i, \quad \forall i \in R, \quad \pi'_i = \begin{cases} \pi_i - \bar{c}_{kl} & \text{if } l \in S, \\ \pi_i + \bar{c}_{kl} & \text{if } k \in S, \end{cases} \quad \forall i \in S.$$

Proof. The tree T' is connected, and $T' \setminus \{e\} = T \setminus \{f\}$ has no arc between R and S : hence e joins R and S . We check that the values π' of the statement are the potentials of T' , which are unique by Lemma 8.8.

1. The arcs of $T' \setminus \{e\}$ have both endpoints in R or both in S , where all the potentials change by the same amount: their reduced cost stays 0.
2. If $l \in S$, then $c_{kl} - \pi'_k + \pi'_l = c_{kl} - \pi_k + \pi_l - \bar{c}_{kl} = 0$. If $k \in S$, then $c_{kl} - \pi'_k + \pi'_l = c_{kl} - \pi_k - \bar{c}_{kl} + \pi_l = 0$.
3. The root n is in R , so $\pi'_n = \pi_n = 0$. □

- Only the potentials of the subtree S cut off by f change, all by the same amount. If $f = e$, the tree and the potentials do not change.
- The reduced costs change only on the arcs with one endpoint in R and the other in S .

Proposition 8.14: unbounded problem

If $\theta = +\infty$, then $e \in L$, the cycle C is a directed cycle whose arcs have infinite capacity, its cost is $-|\bar{c}_{kl}| < 0$, and the MCF model is unbounded.

Proof. A backward arc has the finite residual capacity x_{ij} , and a forward arc with finite capacity has the finite residual capacity $u_{ij} - x_{ij}$. Hence every arc of C is forward with $u_{ij} = +\infty$; in particular e is forward, so $e \in L$. As in Proposition 8.12, $\mathbf{x} + t\mathbf{x}^C$ is feasible for every $t \geq 0$, with cost $\mathbf{c}'\mathbf{x} - t|\bar{c}_{kl}|$, which tends to $-\infty$. □

Example 8.4: a pivot from the tree of $\bar{\mathbf{x}}$

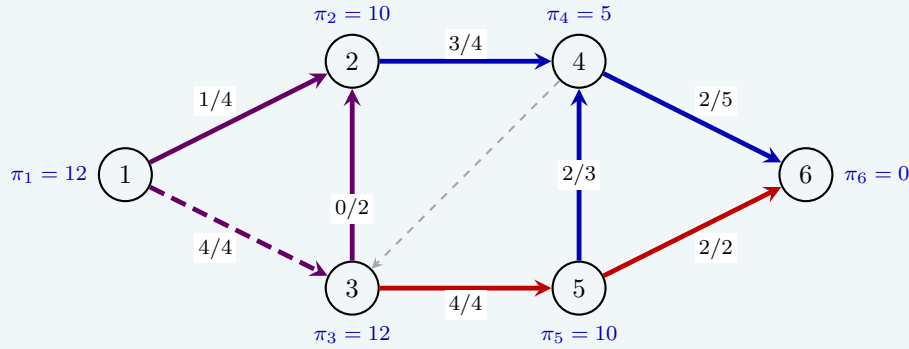
We continue Example 8.3. The eligible arcs are $(1,3)$ and $(3,5)$, both in U , with $\bar{c}_{13} = 2$ and $\bar{c}_{35} = 1$.

- *Entering arc.* The largest $|\bar{c}_{ij}|$ is 2: $e = (1,3)$. It is in U , so C is traversed against $(1,3)$: $p = 3$ and $q = 1$.
- *Apex.* The tree paths to the root are 1,2,4,6 and 3,2,4,6: the apex is $w = 2$. From w , the cycle goes $2 \rightarrow 3$ along $(3,2)$, $3 \rightarrow 1$ along $(1,3)$, $1 \rightarrow 2$ along $(1,2)$:

arc of C	(3,2)	(1,3)	(1,2)
direction	−	−	+
residual	0	4	3

- *Step and leaving arc.* $\theta = \min\{0, 4, 3\} = 0$, and the only blocking arc is $(3,2)$: it leaves the tree, backward, so it goes to L . The pivot is degenerate.
- *Update.* The flow does not change and the cost stays 63. The new tree is $T' = \{(1,2), (1,3), (2,4), (4,6), (5,4)\}$, where $(1,3)$ is a basic arc at its capacity. Removing $(3,2)$ from T cuts off $S = \{3\}$, which contains $l = 3$: $\pi'_3 = \pi_3 - \bar{c}_{13} = 12 - 2 = 10$.

In the figure the arcs of C are violet, the entering arc dashed; next to each node its potential.



A pivot sends θ units around the fundamental cycle of the entering arc, oriented so that its flow moves away from its bound; the cost decreases by $\theta|\bar{c}_{kl}|$. The leaving arc is the last blocking arc from the apex, and only the potentials of the subtree it cuts off change.

8.5 The network simplex algorithm

Algorithm

Algorithm: Network simplex algorithm

Input: a network $G = (N, A)$ with a feasible tree structure (T, L, U)

Output: an optimal tree solution, or a cycle that proves that the problem is unbounded

- 1 **Step 0 (initialization).** Compute the tree solution $\mathbf{x}$ from the leaves,
- 2 and the potentials $\boldsymbol{\pi}$ from the root n , with $\pi_n = 0$.
- 3 **Step 1 (optimality check).** Compute the reduced costs $\bar{c}_{ij} = c_{ij} - \pi_i + \pi_j$.
- 4 If $\bar{c}_{ij} \geq 0$ on L and $\bar{c}_{ij} \leq 0$ on U , **STOP**: $\mathbf{x}$ is optimal.
- 5 **Step 2 (entering arc).** Choose the eligible arc e with the largest $|\bar{c}_{ij}|$.
- 6 Orient its cycle C as e if $e \in L$, and against e if $e \in U$.
- 7 **Step 3 (ratio test).** Compute θ , the minimum residual capacity along C .
- 8 If $\theta = +\infty$, **STOP**: the problem is unbounded.
- 9 Otherwise the leaving arc f is the last blocking arc along C from the apex.
- 10 **Step 4 (update).** Send θ units along C , and set $T := T \cup \{e\} \setminus \{f\}$.
- 11 The arc f goes to U or to L .
- 12 Update the potentials of the subtree cut off by f , and go to Step 1.

- Ties in Step 2 are broken by the lexicographic order of the arcs. Step 0 uses the procedure of [Proposition 8.6](#), Step 4 the updates of [Proposition 8.12](#) and [Proposition 8.13](#).
- The stop in Step 1 is correct by [Theorem 8.10](#), the stop in Step 3 by [Proposition 8.14](#).
- Every structure met is feasible by [Proposition 8.12](#). With integer balances and capacities all the flows are integer ([Corollary 8.4](#)), and with integer costs all the potentials are integer.

Proposition 8.15: termination without degenerate pivots

If every pivot of the algorithm has $\theta > 0$, the algorithm stops after a finite number of pivots.

Proof. A tree structure determines its tree solution, hence its cost. By Proposition 8.12 every pivot with $\theta > 0$ decreases the cost by $\theta|\bar{c}_{kl}| > 0$, so no tree structure is met twice. There are finitely many tree structures: at most $\binom{m}{n-1}$ trees, each with at most 2^{m-n+1} bound statuses. $\square$

8.5.1 The algebra of the pivot

- In the language of Chapter 11 «Revised simplex algorithm», the potentials solve the first system $\pi' M_T = c'_T$.
- The second system $M_T d = M_e$ is solved by the tree path from k to l : by Chapter 1, Lemma «columns of a cycle», $d_a = +1$ on the arcs of the path traversed from tail to head, $d_a = -1$ on the others, and $d_a = 0$ off the path. Increasing x_e by t changes the basic variables by $-t d$: this is the flow sent around the fundamental cycle.
- The potential method for the transportation problem in Chapter 7 «Transportation and assignment» is the network simplex algorithm on the bipartite graph of origins and destinations: the basic cells form a spanning tree, and the potentials and reduced costs are those of this chapter.

The network simplex algorithm moves from tree to tree: potentials from the root, reduced costs, entering arc, cycle, step, leaving arc. Without degenerate pivots the cost decreases at every pivot, and the algorithm stops.

8.6 An uncapacitated example

- We use the reference network with the balances $\mathbf{b} = (5, 2, 0, -3, 0, -4)$ and the costs of Example 8.2, but without capacities: $u_{ij} = +\infty$ for every arc, so $U = \emptyset$.
- The initial tree structure sends the 5 units of node 1 along the path 1, 3, 5, 4 and the 2 units of node 2 along (2, 4); node 4 keeps 3 units and forwards 4 units to node 6:
$$T = \{(1, 3), (2, 4), (3, 5), (4, 6), (5, 4)\}, \quad L = \{(1, 2), (3, 2), (4, 3), (5, 6)\}, \quad U = \emptyset.$$
- Every tree solution of the example is nondegenerate. Each table gives the state of the arcs (T, L, U), the flows and the reduced costs; each cycle table lists the arcs of C from the apex.

Example 8.5: the network simplex without capacities

Iteration 1

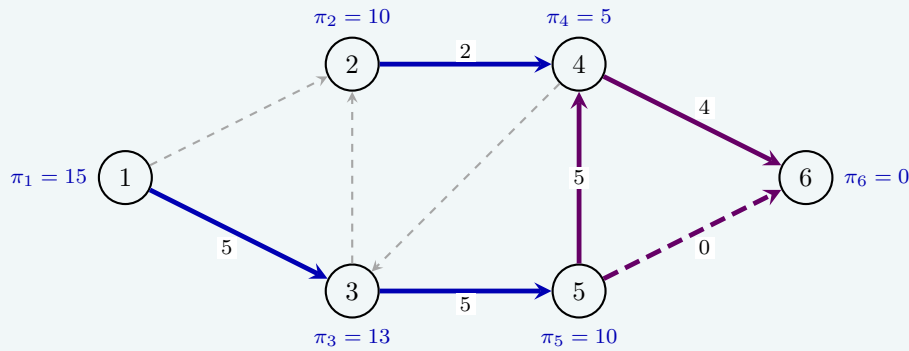
The potentials are $\pi = (15, 10, 13, 5, 10, 0)$ and the cost is 80.

arc	(1, 2)	(1, 3)	(2, 4)	(3, 2)	(3, 5)	(4, 3)	(4, 6)	(5, 4)	(5, 6)
state	L	T	T	L	T	L	T	T	L
x_{ij}	0	5	2	0	5	0	4	5	0
$\bar{c}_{ij}$	-3	0	0	-1	0	11	0	0	-7

- The eligible arcs are (1, 2), (3, 2) and (5, 6); the entering arc is $e = (5, 6)$, with $\bar{c}_{56} = -7$. It is in L , so C follows (5, 6).
- The tree paths to the root are 5, 4, 6 and 6: the apex is $w = 6$, and C is 6, 4, 5, 6.

arc of C	(4, 6)	(5, 4)	(5, 6)
direction	-	-	+
residual	4	5	$+\infty$

- $\theta = 4$, and (4, 6) leaves the tree, to L . The 4 units for node 6 now travel on (5, 6) instead of 5, 4, 6, and the cost becomes $80 - 4 \cdot 7 = 52$.
- Removing (4, 6) cuts off $S = \{1, 2, 3, 4, 5\}$, which contains $k = 5$: these potentials change by $\bar{c}_{56} = -7$.



Iteration 2

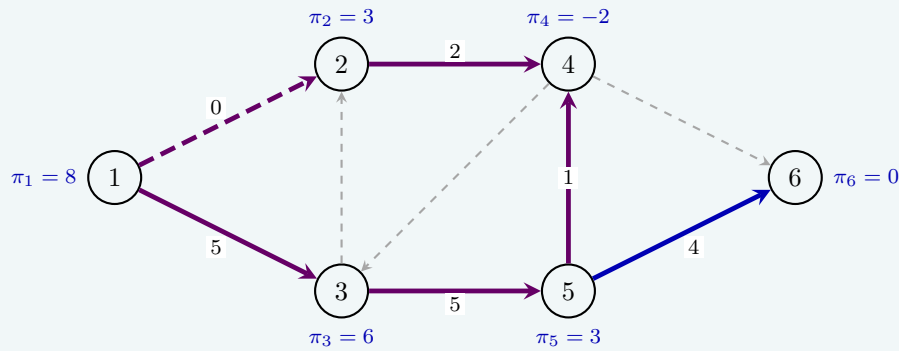
The potentials are $\pi = (8, 3, 6, -2, 3, 0)$ and the cost is 52.

arc	(1, 2)	(1, 3)	(2, 4)	(3, 2)	(3, 5)	(4, 3)	(4, 6)	(5, 4)	(5, 6)
state	L	T	T	L	T	L	L	T	T
x_{ij}	0	5	2	0	5	0	0	1	4
$\bar{c}_{ij}$	-3	0	0	-1	0	11	7	0	0

- The eligible arcs are (1, 2) and (3, 2); the entering arc is $e = (1, 2)$, with $\bar{c}_{12} = -3$, in L .
- The tree paths to the root are 1, 3, 5, 6 and 2, 4, 5, 6: the apex is $w = 5$, and C is 5, 3, 1, 2, 4, 5.

arc of C	(3, 5)	(1, 3)	(1, 2)	(2, 4)	(5, 4)
direction	—	—	+	+	—
residual	5	5	$+\infty$	$+\infty$	1

- $\theta = 1$, and (5, 4) leaves the tree, to L . The unit for node 4 that travelled along 1, 3, 5, 4, at cost 10, now travels along 1, 2, 4, at cost 7: the cost becomes $52 - 1 \cdot 3 = 49$.
- Removing (5, 4) cuts off $S = \{2, 4\}$, which contains $l = 2$: these potentials change by $-\bar{c}_{12} = 3$.



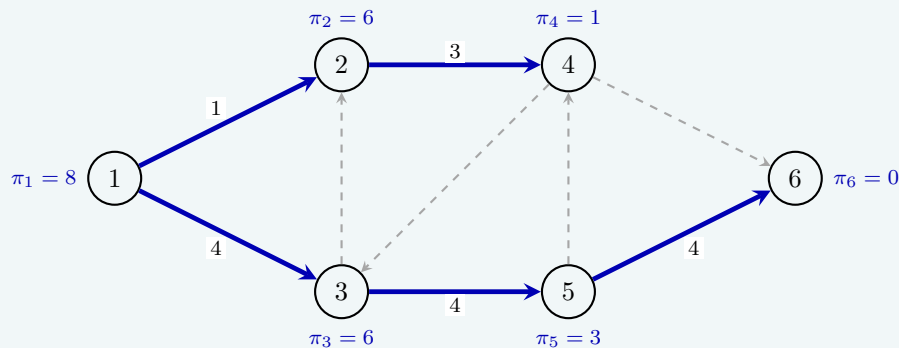
Iteration 3

The potentials are $\pi = (8, 6, 6, 1, 3, 0)$ and the cost is 49.

arc	(1, 2)	(1, 3)	(2, 4)	(3, 2)	(3, 5)	(4, 3)	(4, 6)	(5, 4)	(5, 6)
state	T	T	T	L	T	L	L	L	T
x_{ij}	1	4	3	0	4	0	0	0	4
$\bar{c}_{ij}$	0	0	0	2	0	8	4	3	0

All the arcs of L have $\bar{c}_{ij} > 0$: STOP. By Theorem 8.10 the flow $x_{12} = 1$, $x_{13} = 4$, $x_{24} = 3$, $x_{35} = 4$, $x_{56} = 4$ is the unique optimal flow, with value

$$2 \cdot 1 + 2 \cdot 4 + 5 \cdot 3 + 3 \cdot 4 + 3 \cdot 4 = 49.$$



- Gurobi (gurobipy) confirms the optimal value 49 and the optimal flow; enumerating all the feasible tree structures gives the same unique optimum.

- The optimal flow sends 4 units on $(5, 6)$, whose capacity in Chapter 4 is $u_{56} = 2$: with the capacities this flow is not feasible, and the optimal value rises from 49 to 60.

Without capacities U is empty and only backward arcs can block. On the reference network two pivots lower the cost from 80 to 52 and to 49; the reduced costs of the last tree are positive on all the nonbasic arcs.

8.7 Degeneracy and strongly feasible trees

Definition 8.16: strongly feasible tree structure

Let (T, L, U) be a feasible tree structure with tree solution $\mathbf{x}$ and root n . For a node $i \neq n$, the tree arc between i and the next node of its path to the root **points towards the root** if i is its tail, and **away from the root** if i is its head. The tree structure is **strongly feasible** if

- every tree arc with $x_{ij} = 0$ points towards the root, and
 - every tree arc with $x_{ij} = u_{ij}$ points away from the root.
- Equivalently, a positive amount of flow can be sent from every node to the root along its tree path. An arc pointing towards the root is traversed forward and needs $x_{ij} < u_{ij}$; an arc pointing away is traversed backward and needs $x_{ij} > 0$.
 - A nondegenerate feasible tree structure is strongly feasible.
 - In a degenerate pivot the flow and the cost do not change, but the tree does. [Proposition 8.15](#) does not exclude that a sequence of degenerate pivots comes back to a tree structure already met: the algorithm would then **cycle**, as the simplex algorithm of [Chapter 7 «Optimality conditions and pivot rules»](#).

Example 8.6: the four tree structures of $\bar{\mathbf{x}}$

In the flow $\bar{\mathbf{x}}$ of [Example 8.2](#) the arcs $(1, 2)$, $(2, 4)$, $(4, 6)$, $(5, 4)$ are strictly between their bounds, and they connect all the nodes except 3. A spanning tree containing them adds one arc at a bound incident to node 3, and each choice gives a tree structure with the same tree solution $\bar{\mathbf{x}}$:

added arc	flow	path from 3 to the root	strongly feasible
$(3, 2)$	0	3, 2, 4, 6: the arc points towards the root	yes
$(1, 3)$	$4 = u_{13}$	3, 1, 2, 4, 6: the arc points away from the root	yes
$(3, 5)$	$4 = u_{35}$	3, 5, 4, 6: the arc points towards the root	no
$(4, 3)$	0	3, 4, 6: the arc points away from the root	no

With $(3, 5)$ in the tree, no flow can leave node 3 towards the root, because $(3, 5)$ is saturated; with $(4, 3)$, no flow can reach 4 from 3, because $(4, 3)$ is empty. The pivot of [Example 8.4](#) goes from the first structure to the second.

Lemma 8.17: degenerate pivots

Let (T, L, U) be strongly feasible, and let C be the cycle of a pivot, traversed from the apex w : the tree path P from w to p , the arc e , the tree path Q from q to w . Then every arc of Q and the arc e have positive residual capacity. In particular, if $\theta = 0$, all the blocking arcs lie on P .

Proof. The residual capacity of e is $u_{kl} > 0$: it is $u_{kl} - 0$ if $e \in L$ and $x_{kl} = u_{kl}$ if $e \in U$. The apex w lies on the tree path from q to the root, so Q is the first part of this path, traversed towards the root. By strong feasibility a positive amount of flow can be sent from q to the root along it: every arc of Q has positive residual capacity in the direction of C . $\square$

Theorem 8.18: the leaving arc rule keeps strong feasibility

If (T, L, U) is strongly feasible and $\theta < +\infty$, the tree structure obtained by the pivot, with the leaving arc of [Definition 8.11](#), is strongly feasible.

Proof. Let C visit $w = v_0, v_1, \dots, v_r = w$ in its direction, with f between v_s and v_{s+1} , and let x' be the new flow. Let i be a node and y the first node of C on its tree path to the root in T .

1. *The new path.* If the path of i does not meet C , it is the same in T' . Otherwise the path of i in T' goes from i to y as in T , then from y to w inside $C \setminus \{f\}$, then from w to the root as in T . The two outer parts use no arc of C , so their flows do not change and they still carry positive flow towards the root.
2. *The part inside C .* If $y = v_t$ with $t \leq s$, the path goes $v_t, v_{t-1}, \dots, v_0 = w$, against the direction of C . If $t \geq s+1$, it goes $v_t, v_{t+1}, \dots, v_r = w$, in the direction of C .
3. *Case $\theta > 0$.* Against C , the residual capacity of every arc grows by $\theta > 0$. In the direction of C , the arcs after f are not blocking, because f is the last blocking arc: their residual capacity was larger than θ , and it stays positive.
4. *Case $\theta = 0$.* The flow does not change, and f lies on P by [Lemma 8.17](#). For $t \leq s$, the path from v_t to w goes up P : it is part of the tree path of v_t in T , which carries positive flow towards the root. For $t \geq s+1$, the arcs of P after f are not blocking, and e and the arcs of Q have positive residual capacity by the same lemma.

In all cases every node can send a positive amount of flow to the root in T' . $\square$

Theorem 8.19: finiteness of the network simplex algorithm

If the initial tree structure is strongly feasible and every pivot uses the leaving arc of [Definition 8.11](#), then every tree structure met is strongly feasible, no tree structure is met twice, and the algorithm stops after a finite number of pivots. This holds for any rule that chooses the entering arc among the eligible arcs.

Proof. Strong feasibility follows by induction from [Theorem 8.18](#).

1. A pivot with $\theta > 0$ decreases the cost ([Proposition 8.12](#)).
2. In a degenerate pivot, f lies on P ([Lemma 8.17](#)), so the path from p to the root passes through f , and $p \in S$. If $e \in L$, then $p = k$ and the potentials of S change by $\bar{c}_{kl} < 0$; if $e \in U$, then $p = l$ and they change by $-\bar{c}_{kl} < 0$ ([Proposition 8.13](#)). Hence $\sum_{i \in N} \pi_i$ strictly decreases, while the cost does not change.
3. A tree structure determines its cost and its potentials. At every pivot the pair $(\mathbf{c}'\mathbf{x}, \sum_{i \in N} \pi_i)$ strictly decreases in lexicographic order, so no tree structure is met twice; there are finitely many of them. $\square$

- The proof uses the rule in two points: in a degenerate pivot the leaving arc lies between the apex and p , and the new tree is strongly feasible. A different choice among the blocking arcs can produce a tree structure that is not strongly feasible.
- The theorem needs a strongly feasible initial tree structure: when none is known, it is built with an auxiliary network.

In a strongly feasible tree every node can push flow to the root. Choosing as leaving arc the last blocking arc from the apex keeps this property; then a degenerate pivot lowers the potentials of a subtree, and no tree structure repeats: the algorithm cannot cycle.

8.8 A capacitated example with a degenerate pivot

- We solve the transshipment instance of [Chapter 4](#), the reference network with capacities, starting from the flow $\bar{\mathbf{x}}$ of cost 63 with the strongly feasible tree structure of [Example 8.2](#):

$$T = \{(1, 2), (2, 4), (3, 2), (4, 6), (5, 4)\}, \quad L = \{(4, 3)\}, \quad U = \{(1, 3), (3, 5), (5, 6)\}.$$

Example 8.7: the network simplex with capacities

Iteration 1

The potentials are $\boldsymbol{\pi} = (12, 10, 12, 5, 10, 0)$ and the cost is 63.

arc	(1, 2)	(1, 3)	(2, 4)	(3, 2)	(3, 5)	(4, 3)	(4, 6)	(5, 4)	(5, 6)
state	T	U	T	T	U	L	T	T	U
x_{ij}	1	4	3	0	4	0	2	2	2
$\bar{c}_{ij}$	0	2	0	0	1	10	0	0	-7

- This is the pivot of [Example 8.4](#): $e = (1, 3) \in U$, apex $w = 2$, $C = 2, 3, 1, 2$, residual capacities 0, 4, 3.
- $\theta = 0$: the pivot is degenerate. The arc (3, 2) leaves the tree, to L , and (1, 3) enters it at its capacity. The cost stays 63, and π_3 drops from 12 to 10.

Iteration 2

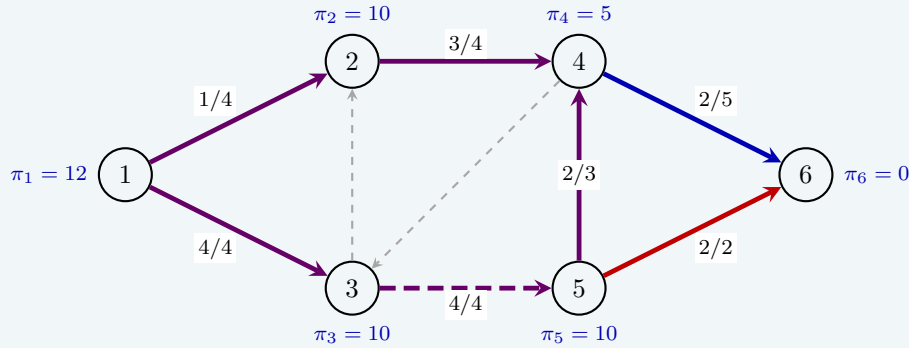
The potentials are $\pi = (12, 10, 10, 5, 10, 0)$ and the cost is 63.

arc	(1, 2)	(1, 3)	(2, 4)	(3, 2)	(3, 5)	(4, 3)	(4, 6)	(5, 4)	(5, 6)
state	T	T	T	L	U	L	T	T	U
x_{ij}	1	4	3	0	4	0	2	2	2
$\bar{c}_{ij}$	0	0	0	2	3	8	0	0	-7

- The only eligible arc is $e = (3, 5) \in U$, with $\bar{c}_{35} = 3$: C is traversed against it, with $p = 5$ and $q = 3$.
- The tree paths to the root are 3, 1, 2, 4, 6 and 5, 4, 6: the apex is $w = 4$, and C is 4, 5, 3, 1, 2, 4.

arc of C	(5, 4)	(3, 5)	(1, 3)	(1, 2)	(2, 4)
direction	—	—	—	+	+
residual	2	4	4	3	1

- $\theta = 1$, and the blocking arc (2, 4), forward, leaves the tree and goes to U . One unit moves from the path 1, 3, 5, 4 to the path 1, 2, 4, and the cost becomes $63 - 1 \cdot 3 = 60$.
- Removing (2, 4) cuts off $S = \{1, 2, 3\}$, which contains $k = 3$: these potentials change by $\bar{c}_{35} = 3$.

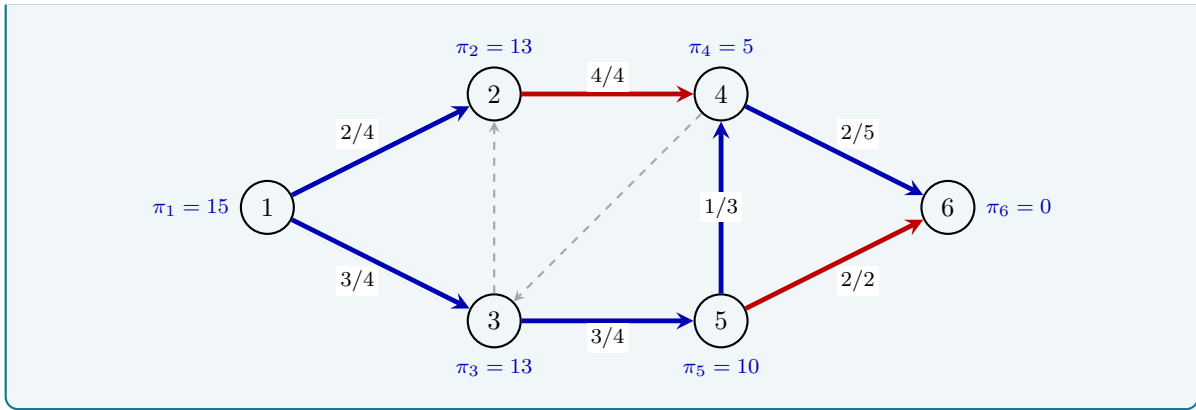


Iteration 3

The potentials are $\pi = (15, 13, 13, 5, 10, 0)$ and the cost is 60.

arc	(1, 2)	(1, 3)	(2, 4)	(3, 2)	(3, 5)	(4, 3)	(4, 6)	(5, 4)	(5, 6)
state	T	T	U	L	T	L	T	T	U
x_{ij}	2	3	4	0	3	0	2	1	2
$\bar{c}_{ij}$	0	0	-3	2	0	11	0	0	-7

The arcs of L have $\bar{c}_{32} = 2 > 0$ and $\bar{c}_{43} = 11 > 0$, the arcs of U have $\bar{c}_{24} = -3 < 0$ and $\bar{c}_{56} = -7 < 0$: **STOP**. The flow is the unique optimal flow of Chapter 4, with value 60, and the potentials are those of its certificate.



- The three tree structures are strongly feasible. In the second one the basic arc $(1, 3)$ is at its capacity and points away from the root, as the definition requires.
- The degenerate pivot does not lower the cost: it only replaces $(3, 2)$ with $(1, 3)$ in the basis. It comes first because the entering rule prefers $|\bar{c}_{13}| = 2$ to $|\bar{c}_{35}| = 1$.
- Gurobi confirms the optimal value 60, and enumerating the feasible tree structures gives the same unique optimum.

With capacities, arcs of U enter by decreasing their flow, and a saturated arc can leave the tree to U . From the flow $\bar{x}$ a degenerate pivot and a pivot with $\theta = 1$ reach the optimal flow of Chapter 4, of value 60.

8.9 The initial tree: Phase 0

Definition 8.20: auxiliary network

Let $G = (N, A)$ be a network with $\sum_{i \in N} b_i = 0$. The **auxiliary network** G^0 adds to G the **artificial node** $r = n + 1$, with $b_r = 0$, and one **artificial arc** for every node $i \in N$:

$$(i, r) \text{ if } b_i \geq 0, \quad (r, i) \text{ if } b_i < 0,$$

with capacity $+\infty$. The costs of Phase 0 are $c_{ij}^0 = 1$ on the artificial arcs and $c_{ij}^0 = 0$ on the arcs of A . **Phase 0** applies the network simplex algorithm to G^0 with the costs c_{ij}^0 , starting from the tree of the artificial arcs, with $L = A$ and $U = \emptyset$; the root is the last node, r .

- The value of Phase 0 is the total flow on the artificial arcs: it measures how far the flow is from being a flow of G . This is the auxiliary problem of Chapter 8 «Phase 0 of the simplex algorithm», written on the network.

Proposition 8.21: the initial tree of Phase 0

The artificial arcs form a spanning tree of G^0 . Its tree solution has $x_{ir} = b_i$ if $b_i \geq 0$, $x_{ri} = -b_i$ if $b_i < 0$, and $x_{ij} = 0$ on A ; its cost is $\sum_{i \in N} |b_i|$, and the tree structure is strongly feasible.

Proof. The n artificial arcs join every node of N to r : they form a connected graph with $n + 1$ nodes and n arcs, a spanning tree.

- At a node $i \in N$ the only arc with flow is its artificial arc, so conservation holds. At r the outflow minus the inflow is $\sum_{i: b_i < 0} (-b_i) - \sum_{i: b_i \geq 0} b_i = -\sum_{i \in N} b_i = 0 = b_r$.
- The tree arcs with zero flow are the arcs (i, r) with $b_i = 0$: their tail is i , so they point towards the root. No tree arc is at its capacity, which is $+\infty$. $\square$

Theorem 8.22: Phase 0

Phase 0 stops after a finite number of pivots with an optimal value $z_0 \geq 0$. The network G has a feasible flow if and only if $z_0 = 0$, and in that case the final flow of Phase 0, restricted to A , is a feasible flow of G .

Proof. All the capacities of G^0 are positive and the initial tree structure is strongly feasible (Proposition 8.21), so Phase 0 stops by Theorem 8.19.

1. It cannot stop in Step 3: the costs c_{ij}^0 are non-negative, so no cycle has negative cost (Proposition 8.14). It stops in Step 1 with an optimal flow, of value $z_0 \geq 0$.
2. If G has a feasible flow, the same flow with zero on the artificial arcs is feasible for G^0 with cost 0, so $z_0 = 0$.
3. If $z_0 = 0$, every artificial arc carries zero flow, so the conservation constraints of the nodes of N involve only arcs of A : the restriction to A is a feasible flow of G . $\square$

- When $z_0 = 0$ the final tree of Phase 0 still contains at least one artificial arc, since the root r must be connected to the other nodes. The next proposition shows how to continue from this tree.

Proposition 8.23: from Phase 0 to the network simplex

Let Phase 0 end with $z_0 = 0$, with flow $\tilde{x}^0$, tree structure (T, L, U) and reduced costs $\bar{c}_{ij}^0$. Apply the network simplex algorithm to G^0 from (T, L, U) , with the costs c_{ij} on A and 0 on the artificial arcs, letting enter only the arcs with $\bar{c}_{ij}^0 = 0$. Then every flow met has zero flow on the artificial arcs, and the algorithm ends with an optimal flow of G , or proves that the MCF model is unbounded.

Proof. Call *frozen* the arcs with $\bar{c}_{ij}^0 \neq 0$. They are not in T , and by the optimality of Phase 0 those with $\bar{c}_{ij}^0 > 0$ are in L , with flow 0, and those with $\bar{c}_{ij}^0 < 0$ are in U , with flow u_{ij} .

1. *A key identity.* Let y be a flow of G^0 that satisfies conservation and the bounds.

By Chapter 4, Lemma «cost and reduced cost», and since the cost of $\tilde{\mathbf{x}}^0$ is $z_0 = 0$,

$$\sum_a c_a^0 y_a = \sum_a \bar{c}_a^0 (y_a - \tilde{x}_a^0).$$

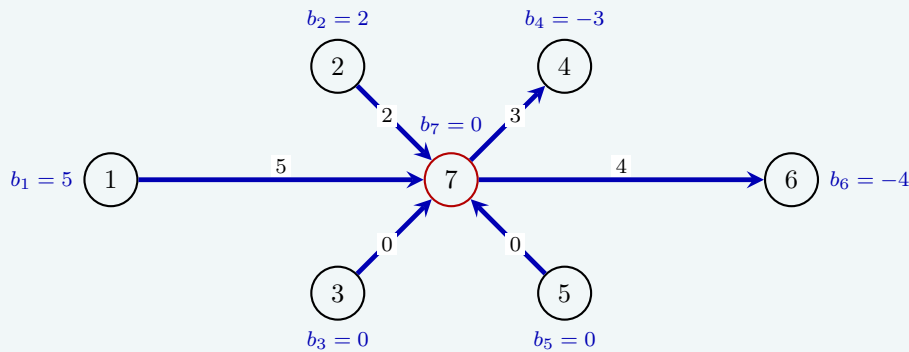
Each term is $\bar{c}_a^0 y_a \geq 0$ on the frozen arcs of L , $\bar{c}_a^0 (y_a - u_a) \geq 0$ on the frozen arcs of U , and 0 elsewhere.

2. *No artificial flow.* The frozen arcs never enter the tree, so every flow met keeps their values, and every term of the identity is 0. Hence the total artificial flow $\sum_a c_a^0 y_a$ is 0, and every artificial arc carries zero flow.
3. *The feasible flows of G keep the frozen values.* A feasible flow $\mathbf{y}$ of G , with zero on the artificial arcs, has $\sum_a c_a^0 y_a = 0$: every term of the identity vanishes, so $y_a = 0$ on the frozen arcs of L and $y_a = u_a$ on those of U .
4. *Optimality.* Let the algorithm stop with flow $\mathbf{x}$ and potentials $\boldsymbol{\pi}$. As in Theorem 8.10, $\mathbf{c}'\mathbf{y} - \mathbf{c}'\mathbf{x} = \sum_a \bar{c}_a (y_a - x_a)$. The frozen arcs give 0 by step 3, the tree arcs give 0, and the other arcs of L and U satisfy the stopping conditions and give a non-negative term. Hence $\mathbf{c}'\mathbf{y} \geq \mathbf{c}'\mathbf{x}$.
5. *Unboundedness.* If the algorithm stops in Step 3, the flows $\mathbf{x} + t\mathbf{x}^C$ of Proposition 8.14 keep the frozen values: by the identity of step 1 they have zero artificial flow, so they are flows of G with cost tending to $-\infty$. $\square$

The structures met stay strongly feasible, and the algorithm stops, by Theorem 8.19.

Example 8.8: Phase 0 on the reference network

For the reference network with capacities, the auxiliary network has the artificial node 7 and the artificial arcs $(1, 7)$, $(2, 7)$, $(3, 7)$, $(5, 7)$, since $b_1, b_2, b_3, b_5 \geq 0$, and $(7, 4)$, $(7, 6)$. The initial tree (blue) carries 5, 2, 0, 0, 3, 4 units on these arcs, all the arcs of A are empty (they are not drawn), and $z_0 = 14$. The initial potentials are $\pi_i = 1$ for the nodes with an arc $(i, 7)$ and $\pi_i = -1$ for those with an arc $(7, i)$.



Phase 0 makes six pivots; every entering arc has $\bar{c}_{ij}^0 = -2$, and every apex is the root 7.

pivot	entering	$\bar{c}_{ij}^0$	apex	θ	blocking arcs	leaving	z_0 after
1	(2, 4)	-2	7	2	(2, 7)	(2, 7)	10
2	(1, 2)	-2	7	1	(7, 4)	(7, 4)	8
3	(4, 6)	-2	7	1	(2, 4)	(2, 4)	6
4	(5, 4)	-2	7	0	(5, 7)	(5, 7)	6
5	(3, 5)	-2	7	0	(3, 7)	(3, 7)	6
6	(1, 3)	-2	7	3	(1, 7), (5, 4), (7, 6)	(7, 6)	0

- Pivots 4 and 5 are degenerate. In pivot 3 the arc (2, 4) reaches its capacity and goes to U .
- In pivot 6 the cycle is 7, 1, 3, 5, 4, 6, 7, with three blocking arcs. The rule chooses (7, 6), the last one from the apex. With (1, 7) the tree would contain (5, 4) saturated pointing towards the root and (7, 6) empty pointing away from it; with (5, 4), the empty arc (7, 6) pointing away: neither tree would be strongly feasible.
- Phase 0 ends with $z_0 = 0$, the tree (1, 2), (1, 3), (1, 7), (3, 5), (4, 6), (5, 4) and $U = \{(2, 4)\}$. All the potentials of N are 1, so $\bar{c}_{ij}^0 = 0$ on every arc except (7, 4) and (7, 6), which have $\bar{c}_{ij}^0 = 2$ and are frozen.

With the costs c_{ij} the tree solution costs 74. The arcs (7, 4) and (7, 6) have reduced costs -10 and -15, but they are frozen: the only arc that may enter is (5, 6), with $\bar{c}_{56} = -7$. Its residual capacity $u_{56} = 2$ is the smallest along its cycle, so (5, 6) itself is the leaving arc and moves to U , and the cost becomes $74 - 2 \cdot 7 = 60$: the optimum of [Example 8.7](#). If (7, 6) could enter, 3 units would travel from 1 to 6 through the artificial node, at cost 29: not a flow of G .

Phase 0 starts from the strongly feasible star of the artificial arcs and minimizes the artificial flow: $z_0 > 0$ proves that no feasible flow exists. If $z_0 = 0$, the network simplex continues from the final tree, with the original costs, and only the arcs with zero reduced cost in Phase 0 may enter.

8.10 Typical mistakes

Warning

- **Calling basic the arcs with positive flow.** The basis is the tree T : an arc of U has positive flow but is nonbasic, and a tree arc can have flow 0 or u_{ij} .
- **Using the sign test of a maximization.** In the MCF model, a minimization, an arc of L is eligible if $\bar{c}_{ij} < 0$ and an arc of U if $\bar{c}_{ij} > 0$.
- **Orienting the cycle as the entering arc when it is in U .** Its flow must decrease, so the cycle goes against it, and the entering arc is backward.
- **Forgetting the entering arc in the ratio test.** Its residual capacity is u_{kl} ; if it is the only blocking arc, the tree does not change and the arc moves between L and U .
- **Updating all the potentials.** Only the potentials of the subtree cut off by the leaving arc change, all by $\bar{c}_{kl}$ or $-\bar{c}_{kl}$.
- **Choosing any blocking arc.** The last blocking arc from the apex keeps the tree strongly feasible; another choice can lose this property and the guarantee against cycling.
- **Letting the frozen arcs enter after Phase 0.** An arc with $\bar{c}_{ij}^0 \neq 0$ can put flow on the artificial arcs; only the arcs with $\bar{c}_{ij}^0 = 0$ may enter.

8.11 In summary

Summary

- In a connected network the equation of node n is redundant, and the bases of the MCF model are the spanning trees; their matrices are triangular with determinant ± 1 .
- A tree structure (T, L, U) fixes the nonbasic arcs at 0 or at their capacity; the tree solution follows from the leaves, and the potentials from the root, with $\pi_n = 0$.
- The reduced cost of a nonbasic arc is the cost of its fundamental cycle; $\bar{c} \geq 0$ on L and $\bar{c} \leq 0$ on U certify optimality.
- A pivot sends θ units around the cycle of the entering arc and lowers the cost by $\theta|\bar{c}_{kl}|$; the leaving arc is the last blocking arc from the apex, and only the potentials of one subtree change.
- Strongly feasible trees and this leaving arc rule prevent cycling. Phase 0 on the auxiliary network gives a strongly feasible initial tree, or proves that no feasible flow exists.

The reference network at a glance.

without capacities	costs 80, 52, 49; optimal tree (1, 2), (1, 3), (2, 4), (3, 5), (5, 6)
with capacities, from $\bar{x}$	costs 63, 63 (degenerate), 60; $U = \{(2, 4), (5, 6)\}$
optimal potentials ($\pi_6 = 0$)	(8, 6, 6, 1, 3, 0) without capacities, (15, 13, 13, 5, 10, 0) with
Phase 0	z_0 from 14 to 0 in six pivots, then one pivot from 74 to 60