



SAPIENZA
UNIVERSITÀ DI ROMA

Network Optimization

Exercises 2 — Trees and paths

Worked exercises on the chapters of Volume 2, trees and paths: each exercise is followed by its complete solution.

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Online version and downloadable material:
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Minimum spanning trees

Rules used in these exercises

The definitions and the algorithms of the chapter «[Minimum spanning trees](#)», for a connected undirected graph $G = (V, E)$ with $n = |V|$, $m = |E|$ and weights w_e .

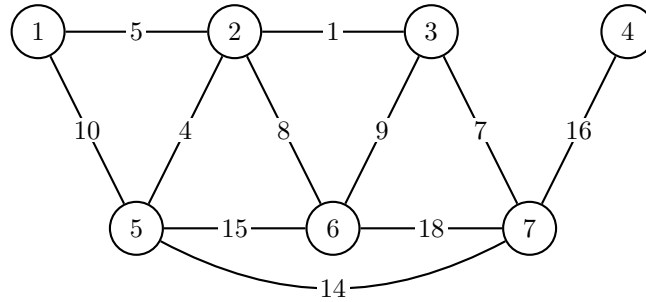
- (M1) **Cut property.** If no edge of F crosses $\delta(S)$ and F is contained in a minimum spanning tree, adding a lightest edge of $\delta(S)$ keeps this property; a unique lightest edge of $\delta(S)$ is in every minimum spanning tree.
- (M2) **Cycle property.** A heaviest edge of a cycle can be left out; an edge strictly heavier than the other edges of a cycle is in no minimum spanning tree.
- (M3) **Optimality.** A spanning tree T is optimal if and only if $w_e \geq w_f$ for every $e \notin T$ and every f of its fundamental cycle $C(T, e)$; it is the unique optimum if and only if all these inequalities are strict.
- (M4) **Kruskal.** The edges are scanned by non-decreasing weight, ties in lexicographic order of $\{u, v\}$, $u < v$; an edge is accepted if its endpoints are in different components of (V, F) .
- (M5) **Prim.** From $s = 1$: select $v \notin S$ with minimum $d(v)$, ties by the smallest index, add $\{pred(v), v\}$; then $d(u) := w_{vu}$, $pred(u) := v$ for every $u \notin S$ adjacent to v with $w_{vu} < d(u)$.

Exercise 2.1 — Kruskal's algorithm

We consider the undirected graph $G = (V, E)$ with $V = \{1, 2, \dots, 7\}$ and the 11 edges with the weights

edge	$\{1, 2\}$	$\{1, 5\}$	$\{2, 3\}$	$\{2, 5\}$	$\{2, 6\}$	$\{3, 6\}$
w_e	5	10	1	4	8	9
edge	$\{3, 7\}$	$\{4, 7\}$	$\{5, 6\}$	$\{5, 7\}$	$\{6, 7\}$	
w_e	7	16	15	14	18	

drawn below.



Questions

1. Sort the edges as in (M4).
2. Run Kruskal's algorithm: for every scanned edge, say whether it is accepted or rejected and give the components of (V, F) .
3. Draw the minimum spanning tree and give its weight.
4. For every rejected edge, give a cycle of which it is the heaviest edge.

Solution

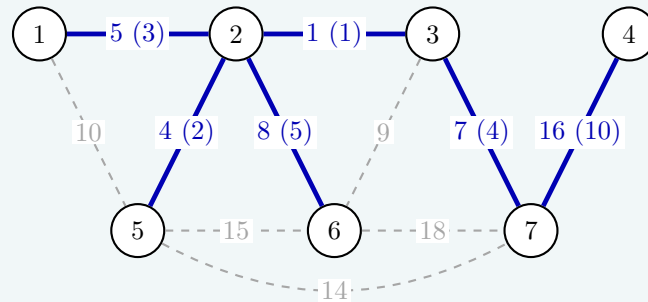
We use the rules (M1)–(M5) of the box at the beginning of these exercises.

1. By non-decreasing weight: $\{2, 3\}$, $\{2, 5\}$, $\{1, 2\}$, $\{3, 7\}$, $\{2, 6\}$, $\{3, 6\}$, $\{1, 5\}$, $\{5, 7\}$, $\{5, 6\}$, $\{4, 7\}$, $\{6, 7\}$ with weights 1, 4, 5, 7, 8, 9, 10, 14, 15, 16, 18.
2. Each row is one scanned edge:

k	edge e_k	w_{e_k}	decision and components of (V, F)
1	$\{2, 3\}$	1	accept: $\{1\}, \{2, 3\}, \{4\}, \{5\}, \{6\}, \{7\}$
2	$\{2, 5\}$	4	accept: $\{1\}, \{2, 3, 5\}, \{4\}, \{6\}, \{7\}$
3	$\{1, 2\}$	5	accept: $\{1, 2, 3, 5\}, \{4\}, \{6\}, \{7\}$
4	$\{3, 7\}$	7	accept: $\{1, 2, 3, 5, 7\}, \{4\}, \{6\}$
5	$\{2, 6\}$	8	accept: $\{1, 2, 3, 5, 6, 7\}, \{4\}$
6	$\{3, 6\}$	9	reject (3 and 6 are both in $\{1, 2, 3, 5, 6, 7\}$)
7	$\{1, 5\}$	10	reject (1 and 5 are both in $\{1, 2, 3, 5, 6, 7\}$)
8	$\{5, 7\}$	14	reject (5 and 7 are both in $\{1, 2, 3, 5, 6, 7\}$)
9	$\{5, 6\}$	15	reject (5 and 6 are both in $\{1, 2, 3, 5, 6, 7\}$)
10	$\{4, 7\}$	16	accept: $\{1, 2, 3, 4, 5, 6, 7\}$

The algorithm stops after 10 of the 11 edges, when $|F| = 6$. The edges $\{6, 7\}$ are never scanned.

3. The tree is $F = \{\{1, 2\}, \{2, 3\}, \{2, 5\}, \{2, 6\}, \{3, 7\}, \{4, 7\}\}$, of weight $5 + 1 + 4 + 8 + 7 + 16 = 41$; on each tree edge, the weight and in brackets the step of its acceptance:



4. A rejected edge joins two vertices already connected by F , and the path of F between them uses only edges scanned before it, so lighter ones: $\{3, 6\}$, of weight 9, closes the cycle 3, 2, 6, 3 with tree edges of weights 1, 8; $\{1, 5\}$, of weight 10, closes the cycle 1, 2, 5, 1 with tree edges of weights 5, 4; $\{5, 7\}$, of weight 14, closes the cycle 5, 2, 3, 7, 5 with tree edges of weights 4, 1, 7; $\{5, 6\}$, of weight 15, closes the cycle 5, 2, 6, 5 with tree edges of weights 4, 8. By (M2) none of these edges is in a minimum spanning tree.

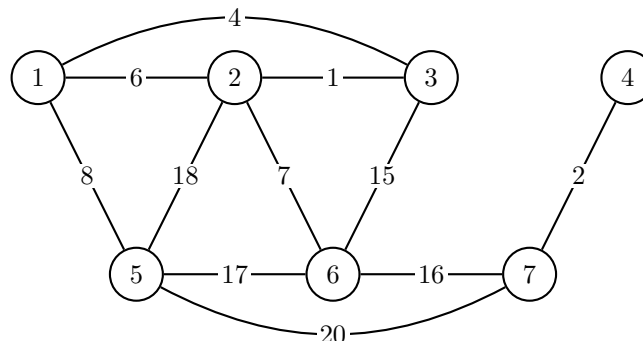
Kruskal's algorithm accepts $\{2, 3\}, \{2, 5\}, \{1, 2\}, \{3, 7\}, \{2, 6\}, \{4, 7\}$ in this order and finds the minimum spanning tree of weight 41; it is unique, since the weights are distinct.

Exercise 2.2 — Prim's algorithm

We consider the undirected graph $G = (V, E)$ with $V = \{1, 2, \dots, 7\}$ and the 11 edges with the weights

edge	$\{1, 2\}$	$\{1, 3\}$	$\{1, 5\}$	$\{2, 3\}$	$\{2, 5\}$	$\{2, 6\}$
w_e	6	4	8	1	18	7
edge	$\{3, 6\}$	$\{4, 7\}$	$\{5, 6\}$	$\{5, 7\}$	$\{6, 7\}$	
w_e	15	2	17	20	16	

drawn below.



Questions

1. Run Prim's algorithm from $s = 1$: after every iteration, give the selected vertex, the added edge and the labels $d(v)$ and $pred(v)$ of the vertices outside S .
2. Which labels decrease during the execution, and why?
3. Draw the minimum spanning tree and give its weight.
4. In which order does Kruskal's algorithm accept the same edges?

Solution

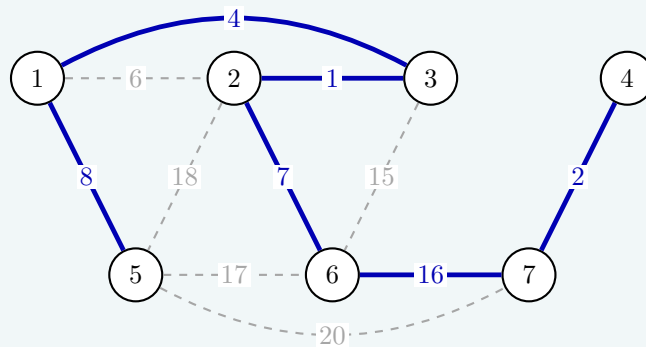
We use the rules (M1)–(M5) of the box at the beginning of these exercises.

1. Each row shows $d(v)$ after the update, with $pred(v)$ in brackets; a dash marks the vertices in S :

it.	selected	added edge	$d(2)$	$d(3)$	$d(4)$	$d(5)$	$d(6)$	$d(7)$
0	1	–	6 (1)	4 (1)	∞	8 (1)	∞	∞
1	3	{1, 3}	1 (3)	–	∞	8 (1)	15 (3)	∞
2	2	{2, 3}	–	–	∞	8 (1)	7 (2)	∞
3	6	{2, 6}	–	–	∞	8 (1)	–	16 (6)
4	5	{1, 5}	–	–	∞	–	–	16 (6)
5	7	{6, 7}	–	–	2 (7)	–	–	–
6	4	{4, 7}	–	–	–	–	–	–

The vertices enter S in the order 1, 3, 2, 6, 5, 7, 4.

2. A label decreases when the vertex just inserted has a lighter edge towards u : at iteration 1, $d(2)$ drops from 6 to 1 (edge {2, 3}); at iteration 1, $d(6)$ drops from ∞ to 15 (edge {3, 6}); at iteration 2, $d(6)$ drops from 15 to 7 (edge {2, 6}); at iteration 3, $d(7)$ drops from ∞ to 16 (edge {6, 7}); at iteration 5, $d(4)$ drops from ∞ to 2 (edge {4, 7}).
3. The tree is $F = \{\{1, 3\}, \{1, 5\}, \{2, 3\}, \{2, 6\}, \{4, 7\}, \{6, 7\}\}$, of weight 38:



4. Kruskal's algorithm accepts the edges by non-decreasing weight: {2, 3}, {4, 7}, {1, 3}, {2, 6}, {1, 5}, {6, 7}. The tree is the same, because the weights are distinct and the minimum spanning tree is unique; only the order changes.

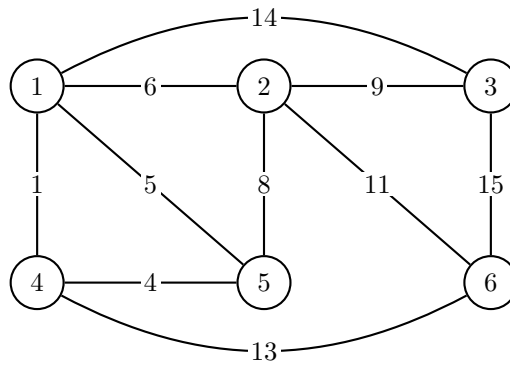
Prim's algorithm from 1 adds $\{1, 3\}, \{2, 3\}, \{2, 6\}, \{1, 5\}, \{6, 7\}, \{4, 7\}$; the minimum spanning tree weighs 38, as found by Kruskal's algorithm.

Exercise 2.3 — Cut and cycle properties

We consider the undirected graph $G = (V, E)$ with $V = \{1, 2, \dots, 6\}$ and the 10 edges with the weights

edge	$\{1, 2\}$	$\{1, 3\}$	$\{1, 4\}$	$\{1, 5\}$	$\{2, 3\}$	$\{2, 5\}$
w_e	6	14	1	5	9	8
edge	$\{2, 6\}$	$\{3, 6\}$	$\{4, 5\}$	$\{4, 6\}$		
w_e	11	15	4	13		

drawn below. We also consider the spanning tree $T_0 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{3, 6\}\}$.



Questions

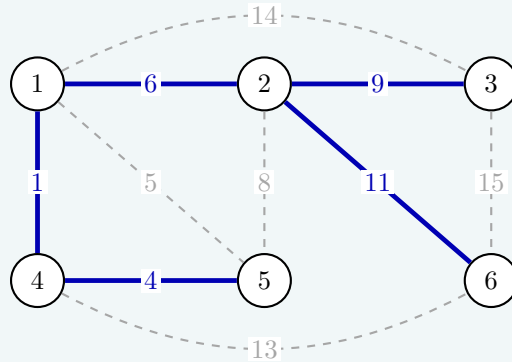
- Find the cut $\delta(\{1, 2\})$. Which of its edges belongs to every minimum spanning tree?
- Find a cycle that contains the edge $\{2, 5\}$ and use it to show that $\{2, 5\}$ is in no minimum spanning tree.
- Is T_0 a minimum spanning tree? List the pairs (e, f) , $e \notin T_0$, $f \in C(T_0, e)$, that violate the cycle condition (M3).
- Improve T_0 with the exchange of the violating pair with the largest difference $w_f - w_e$, and give the new weight. What is the weight of a minimum spanning tree?

Solution

We use the rules (M1)–(M5) of the box at the beginning of these exercises.

- $\delta(\{1, 2\}) = \{\{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 5\}, \{2, 6\}\}$, with weights 14, 1, 5, 9, 8, 11. The edge $\{1, 4\}$ is the unique lightest one, so it belongs to every minimum spanning tree by (M1).

2. The cycle $2, 1, 4, 5, 2$ uses the edges $\{1, 2\}, \{1, 4\}, \{4, 5\}, \{2, 5\}$, with weights 6, 1, 4, 8. The edge $\{2, 5\}$, of weight 8, is strictly the heaviest, so it is in no minimum spanning tree by (M2).
3. T_0 weighs 41. For every edge e outside T_0 we compare w_e with the edges of its fundamental cycle; the violating pairs, with $w_f > w_e$, are $(\{2, 3\}, \{1, 3\})$, $(\{2, 6\}, \{1, 3\})$, $(\{2, 6\}, \{3, 6\})$, $(\{4, 5\}, \{1, 5\})$, $(\{4, 6\}, \{1, 3\})$, $(\{4, 6\}, \{3, 6\})$. By (M3), T_0 is not optimal.
4. The largest difference is given by $e = \{2, 3\}$ and $f = \{1, 3\}$, with $w_f - w_e = 14 - 9 = 5$: the edge f lies on $C(T_0, e)$, whose edges are $\{1, 2\}, \{1, 3\}, \{2, 3\}$. The exchange gives the spanning tree $\{\{1, 2\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{3, 6\}\}$ of weight $41 - 5 = 36$. Kruskal's algorithm gives the minimum spanning tree $\{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 6\}, \{4, 5\}\}$, of weight 31:



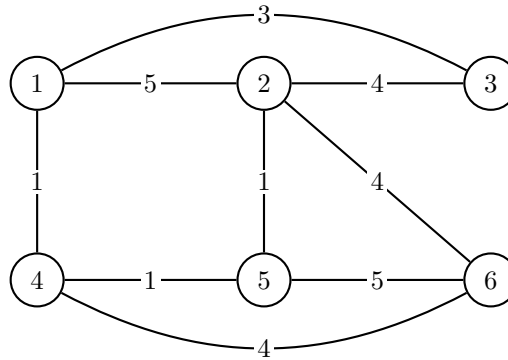
$\{1, 4\}$ is forced by a cut, $\{2, 5\}$ is excluded by a cycle; T_0 (weight 41) violates the cycle condition, and the optimum weighs 31.

Exercise 2.4 — Uniqueness and ties

We consider the undirected graph $G = (V, E)$ with $V = \{1, 2, \dots, 6\}$ and the 9 edges with the weights

edge	$\{1, 2\}$	$\{1, 3\}$	$\{1, 4\}$	$\{2, 3\}$	$\{2, 5\}$	$\{2, 6\}$
w_e	5	3	1	4	1	4
edge	$\{4, 5\}$	$\{4, 6\}$	$\{5, 6\}$			
w_e	1	4	5			

drawn below.



Questions

1. Find a minimum spanning tree T with Kruskal's algorithm, and its weight.
2. Using the fundamental cuts of T , find the edges that belong to every minimum spanning tree.
3. Using the fundamental cycles of T , find the edges that belong to no minimum spanning tree.
4. Is the minimum spanning tree unique? Give another one, if it exists, and the number of minimum spanning trees.

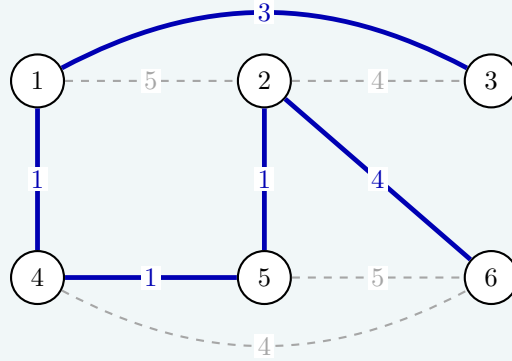
Solution

We use the rules (M1)–(M5) of the box at the beginning of these exercises.

1. Kruskal's algorithm, with ties in lexicographic order:

k	edge e_k	w_{e_k}	decision and components of (V, F)
1	$\{1, 4\}$	1	accept: $\{1, 4\}, \{2\}, \{3\}, \{5\}, \{6\}$
2	$\{2, 5\}$	1	accept: $\{1, 4\}, \{2, 5\}, \{3\}, \{6\}$
3	$\{4, 5\}$	1	accept: $\{1, 2, 4, 5\}, \{3\}, \{6\}$
4	$\{1, 3\}$	3	accept: $\{1, 2, 3, 4, 5\}, \{6\}$
5	$\{2, 3\}$	4	reject (2 and 3 are both in $\{1, 2, 3, 4, 5\}$)
6	$\{2, 6\}$	4	accept: $\{1, 2, 3, 4, 5, 6\}$

$T = \{\{1, 3\}, \{1, 4\}, \{2, 5\}, \{2, 6\}, \{4, 5\}\}$, of weight 10.



2. An edge of T is in every minimum spanning tree if and only if it is strictly the lightest of its fundamental cut (the component S_e of $T \setminus \{e\}$ with the smaller endpoint):

$e \in T$	w_e	S_e	weights of the other edges of $\delta(S_e)$	strictly lightest?
$\{1, 3\}$	3	$\{1, 2, 4, 5, 6\}$	4	yes
$\{1, 4\}$	1	$\{1, 3\}$	5, 4	yes
$\{2, 5\}$	1	$\{2, 6\}$	5, 4, 4, 5	yes
$\{2, 6\}$	4	$\{1, 2, 3, 4, 5\}$	4, 5	no: $\{4, 6\}$
$\{4, 5\}$	1	$\{1, 3, 4\}$	5, 4, 4	yes

The edges in every minimum spanning tree are $\{1, 3\}, \{1, 4\}, \{2, 5\}, \{4, 5\}$.

3. An edge outside T is in no minimum spanning tree if and only if it is strictly the heaviest of its fundamental cycle:

$e \notin T$	w_e	tree edges of $C(T, e)$	their weights	strictly heaviest?
$\{1, 2\}$	5	$\{1, 4\}, \{4, 5\}, \{2, 5\}$	1, 1, 1	yes
$\{2, 3\}$	4	$\{2, 5\}, \{4, 5\}, \{1, 4\}, \{1, 3\}$	1, 1, 1, 3	yes
$\{4, 6\}$	4	$\{4, 5\}, \{2, 5\}, \{2, 6\}$	1, 1, 4	no: $\{2, 6\}$
$\{5, 6\}$	5	$\{2, 5\}, \{2, 6\}$	1, 4	yes

The edges in no minimum spanning tree are $\{1, 2\}, \{2, 3\}, \{5, 6\}$.

4. Some inequality is not strict, so by (M3) the optimum is not unique. For example $\{1, 3\}, \{1, 4\}, \{2, 5\}, \{4, 5\}, \{4, 6\}$ also weighs 10. Enumerating the 64 spanning trees, exactly 2 have weight 10.

The minimum weight is 10, attained by 2 spanning trees; the ties in the fundamental cuts and cycles of one optimal tree tell which edges are forced and which are excluded.

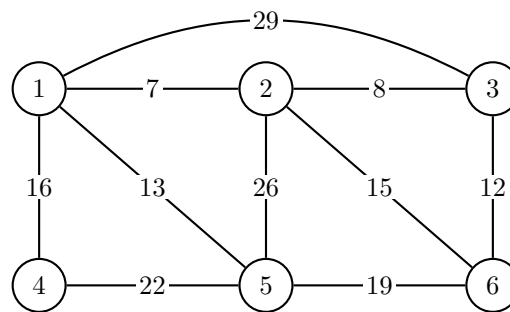
Exercise 2.5 — Modelling with minimum spanning trees

A company wants to connect its 6 buildings $1, 2, \dots, 6$ with a fibre network. A direct link can be built between the pairs below, and w_e is its cost (in thousands of euros):

edge	$\{1, 2\}$	$\{1, 3\}$	$\{1, 4\}$	$\{1, 5\}$	$\{2, 3\}$	$\{2, 5\}$
w_e	7	29	16	13	8	26

edge	$\{2, 6\}$	$\{3, 6\}$	$\{4, 5\}$	$\{5, 6\}$
w_e	15	12	22	19

Every building must be able to reach every other one, possibly through other buildings.



Each link e also works with probability p_e , independently of the others:

link	$\{1, 2\}$	$\{1, 3\}$	$\{1, 4\}$	$\{1, 5\}$	$\{2, 3\}$
p_e	0.97	0.90	0.91	0.84	0.94

link	$\{2, 5\}$	$\{2, 6\}$	$\{3, 6\}$	$\{4, 5\}$	$\{5, 6\}$
p_e	0.96	0.93	0.85	0.81	0.98

Questions

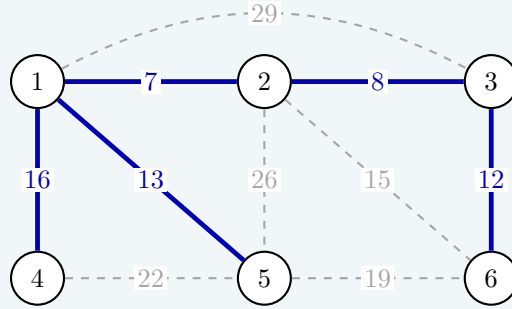
1. Model the cheapest network as a graph problem and solve it.
2. For security reasons, buildings 2 and 6 must be linked directly. How do you find the cheapest network, and what does it cost?
3. A network with $n - 1$ links works if all its links work. Model the most reliable network as a minimum spanning tree problem, solve it and give its probability of working.

Solution

We use the rules (M1)–(M5) of the box at the beginning of these exercises.

1. The buildings are the vertices and the possible links the edges of G , with weights w_e . A cheapest network is connected, and it has no cycle: removing an edge of a cycle keeps it connected and lowers the cost, since the costs are positive. Hence it is a minimum spanning tree. Kruskal's algorithm gives

$$T = \{\{1, 2\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{3, 6\}\}, \quad w(T) = 56.$$



2. Every spanning tree that contains $\{2, 6\}$ changes its weight by the same amount if w_{26} changes. Setting $w_{26} = 0$, below every other weight, Kruskal's algorithm accepts $\{2, 6\}$ first and returns the cheapest tree containing it: $\{\{1, 2\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 6\}\}$, with the original costs $w = 59$, that is 3 more than T .
3. The network works with probability $\prod_{e \in T} p_e$. Maximizing it is the same as minimizing $\sum_{e \in T} (-\log p_e)$, a minimum spanning tree problem with weights $-\log p_e$. These weights have the order of $-p_e$, and Kruskal's algorithm uses only the order: it scans the links by non-increasing reliability and gives

$$\{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 5\}, \{5, 6\}\}, \quad \prod_e p_e = 0.97 \cdot 0.91 \cdot 0.94 \cdot 0.96 \cdot 0.98 \approx 0.7806.$$

Cheapest network: a minimum spanning tree of cost 56; with the forced link $\{2, 6\}$ it costs 59; the most reliable network is a maximum spanning tree for p_e .

Shortest paths

Rules used in these exercises

The definitions and the algorithms of the chapter «Shortest paths», for a directed graph $G = (N, A)$ with lengths d_{ij} , $n = |N|$, $m = |A|$ and source $s = 1$.

(S1) **Dijkstra** ($d_{ij} \geq 0$). The temporary node with the smallest label becomes permanent (smallest index on ties); for each arc (i, j) with j temporary, if $d(i) + d_{ij} < d(j)$ then $d(j) := d(i) + d_{ij}$ and $pred(j) := i$.

(S2) **Bellman–Ford**. Pass k computes

$$d^k(j) = \min \left\{ d^{k-1}(j), \min_{(i,j) \in A} (d^{k-1}(i) + d_{ij}) \right\};$$

the predecessor changes only if the label strictly decreases (smallest i on ties). Stop when $d^k = d^{k-1}$; a change in pass n means a negative cycle reachable from s .

(S3) **Negative cycle**. If $d^n(j) < d^{n-1}(j)$, the predecessors read backwards from j , one row per pass from row n , give a walk with n arcs that contains a negative cycle.

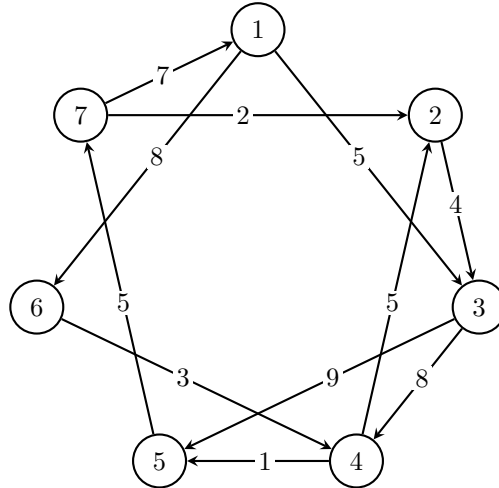
(S4) **Acyclic graphs**. Topological order by removing a node without entering arcs (smallest index); the nodes are processed in this order, each one relaxing its leaving arcs. Lengths $-d_{ij}$ give longest paths.

(S5) **Potentials**. $\bar{c}_{ij} = d_{ij} - \pi_i + \pi_j$; with labels, $\pi_i = -d(i)$. Feasible potentials ($\bar{c} \geq 0$) and a path from s to t of tight arcs ($\bar{c} = 0$) prove that the path is shortest, of length $\pi_s - \pi_t$. The dual of SP(s, t) is $\max\{\pi_s - \pi_t : \pi_i - \pi_j \leq d_{ij}\}$.

Exercise 3.1 — Dijkstra's algorithm

We consider the directed graph $G = (N, A)$ with $N = \{1, 2, \dots, 7\}$, the 11 arcs and the lengths d_{ij} of the table, drawn below; the source is $s = 1$.

arc	(1, 3)	(1, 6)	(2, 3)	(3, 4)	(3, 5)	(4, 2)	(4, 5)	(5, 7)	(6, 4)	(7, 1)	(7, 2)
d_{ij}	5	8	4	8	9	5	1	5	3	7	2



Questions

1. Run Dijkstra's algorithm from $s = 1$: give the labels after every iteration and the order in which the nodes become permanent.
2. Draw the shortest path tree and give a shortest path from 1 to 7 with its length.
3. With the potentials $\pi_i = -\tilde{d}(i)$, compute the reduced costs and list the tight arcs.

Solution

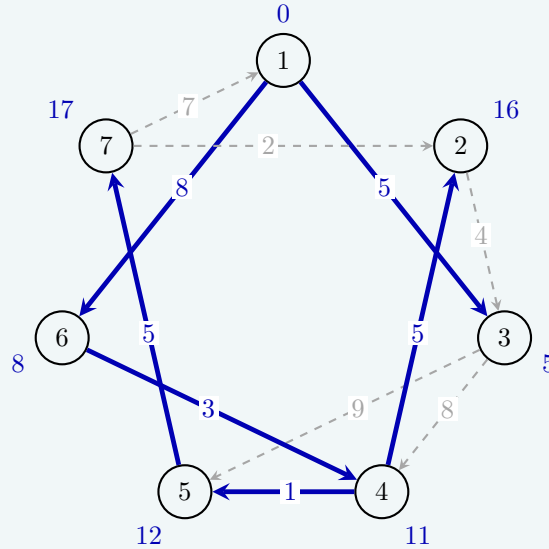
We use the rules (S1)–(S5) of the box at the beginning of these exercises.

1. Each row is an iteration; permanent labels in bold, a label that decreased with the new predecessor in brackets:

it.	i	$d(1)$	$d(2)$	$d(3)$	$d(4)$	$d(5)$	$d(6)$	$d(7)$
1	1	0	∞	5 (1)	∞	∞	8 (1)	∞
2	3	0	∞	5	13 (3)	14 (3)	8	∞
3	6	0	∞	5	11 (6)	14	8	∞
4	4	0	16 (4)	5	11	12 (4)	8	∞
5	5	0	16	5	11	12	8	17 (5)
6	2	0	16	5	11	12	8	17
7	7	0	16	5	11	12	8	17

The nodes become permanent in the order 1, 3, 6, 4, 5, 2, 7, and $\tilde{\mathbf{d}} = (0, 16, 5, 11, 12, 8, 17)$.

2. The arcs $(pred(j), j)$ are (4, 2), (1, 3), (6, 4), (4, 5), (1, 6), (5, 7) (blue); next to each node its distance:



The shortest path from 1 to 7 is 1, 6, 4, 5, 7, of length $8 + 3 + 1 + 5 = 17$.

3. The potentials are $\pi = (0, -16, -5, -11, -12, -8, -17)$, and $\bar{c}_{ij} = d_{ij} - \pi_i + \pi_j$:

arc	(1, 3)	(1, 6)	(2, 3)	(3, 4)	(3, 5)	(4, 2)	(4, 5)	(5, 7)	(6, 4)	(7, 1)	(7, 2)
d_{ij}	5	8	4	8	9	5	1	5	3	7	2
$\bar{c}_{ij}$	0	0	15	2	2	0	0	0	0	24	3

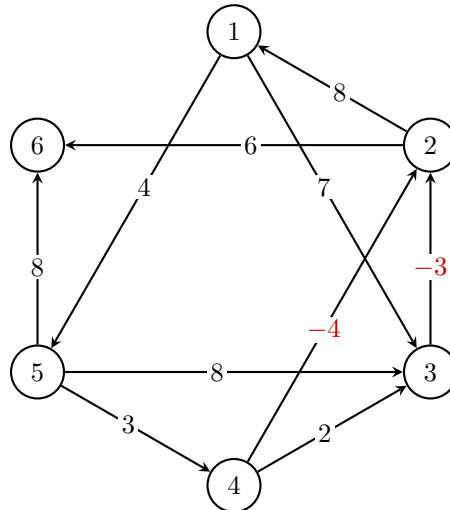
All reduced costs are non-negative, and the tight arcs are (1, 3), (1, 6), (4, 2), (4, 5), (5, 7), (6, 4): exactly the arcs of the shortest path tree.

Dijkstra's algorithm gives $\tilde{d} = (0, 16, 5, 11, 12, 8, 17)$; the potentials $\pi_i = -\tilde{d}(i)$ are feasible, and their tight arcs form the shortest path tree.

Exercise 3.2 — Bellman–Ford with negative arcs

We consider the directed graph $G = (N, A)$ with $N = \{1, 2, \dots, 6\}$, the 10 arcs and the lengths d_{ij} of the table, drawn below; the source is $s = 1$.

arc	(1, 3)	(1, 5)	(2, 1)	(2, 6)	(3, 2)	(4, 2)	(4, 3)	(5, 3)	(5, 4)	(5, 6)
d_{ij}	7	4	8	6	-3	-4	2	8	3	8



Questions

1. Run the Bellman–Ford algorithm from $s = 1$: give the labels d^k of every pass and the number of passes.
2. Draw the shortest path tree and give a shortest path from 1 to 6.
3. Run Dijkstra's algorithm on the same lengths: which labels are wrong, and why?

Solution

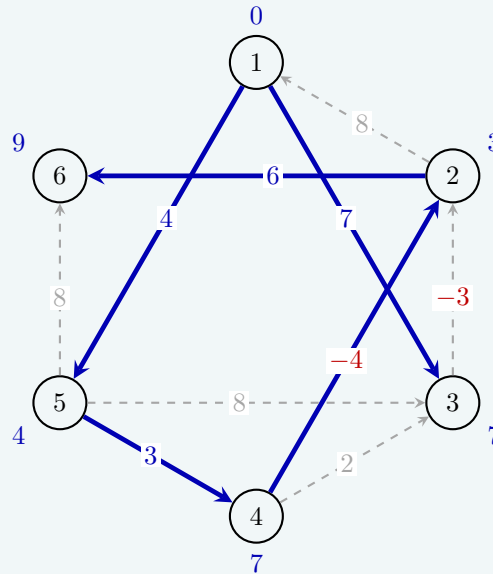
We use the rules (S1)–(S5) of the box at the beginning of these exercises.

1. Each row is a pass; next to a label that decreased, the new predecessor in brackets:

k	$d^k(1)$	$d^k(2)$	$d^k(3)$	$d^k(4)$	$d^k(5)$	$d^k(6)$
0	0	∞	∞	∞	∞	∞
1	0	∞	7 (1)	∞	4 (1)	∞
2	0	4 (3)	7	7 (5)	4	12 (5)
3	0	3 (4)	7	7	4	10 (2)
4	0	3	7	7	4	9 (2)
5	0	3	7	7	4	9

Pass 5 repeats the labels of pass 4: the algorithm stops after 5 passes with $\tilde{\mathbf{d}} = (0, 3, 7, 7, 4, 9)$.

2. The last predecessors give the arcs (4, 2), (1, 3), (5, 4), (1, 5), (2, 6) (blue):



The shortest path from 1 to 6 is 1, 5, 4, 2, 6, of length 9.

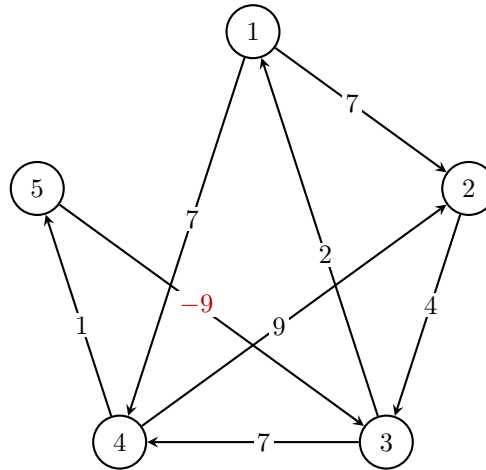
- Dijkstra's algorithm makes the nodes permanent in the order 1, 5, 3, 2, 4, 6 and returns $(0, 4, 7, 7, 4, 10)$: the labels of nodes 2, 6 are wrong. The negative arc $(4, 2)$ ends in a node that is already permanent when its tail is selected, so (S1) never uses it, although $\tilde{d}(4) + d_{42} = 7 - 4 = 3 < 4$.

With negative arcs and no negative cycle, the Bellman–Ford algorithm stops after 5 passes with $\tilde{\mathbf{d}} = (0, 3, 7, 7, 4, 9)$, while Dijkstra's algorithm fails.

Exercise 3.3 — Detecting a negative cycle

We consider the directed graph $G = (N, A)$ with $N = \{1, 2, \dots, 5\}$, the 8 arcs and the lengths d_{ij} of the table, drawn below; the source is $s = 1$.

arc	(1, 2)	(1, 4)	(2, 3)	(3, 1)	(3, 4)	(4, 2)	(4, 5)	(5, 3)
d_{ij}	7	7	4	2	7	9	1	−9



Questions

1. Run the Bellman–Ford algorithm from $s = 1$ for 5 passes. Does G have a negative cycle reachable from 1?
2. Find a negative cycle by reading the predecessors backwards, and give its length.
3. What is the optimal value of the single-source LP model $SP(1)$?

Solution

We use the rules (S1)–(S5) of the box at the beginning of these exercises.

1. Each row is a pass; next to a label that decreased, the new predecessor in brackets:

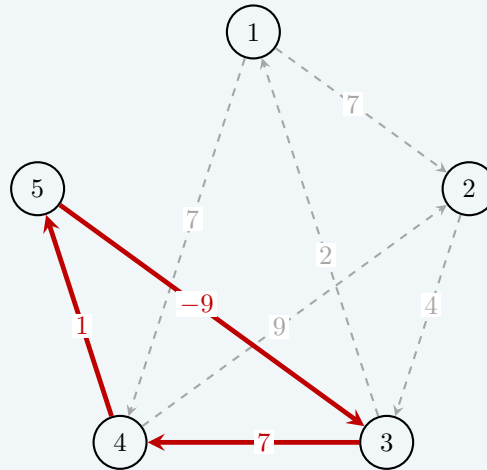
k	$d^k(1)$	$d^k(2)$	$d^k(3)$	$d^k(4)$	$d^k(5)$
0	0	∞	∞	∞	∞
1	0	7 (1)	∞	7 (1)	∞
2	0	7	11 (2)	7	8 (4)
3	0	7	-1 (5)	7	8
4	0	7	-1	6 (3)	8
5	0	7	-1	6	7 (4)

In pass $n = 5$ the label of node 5 decreases from 8 to 7: $d^5 \neq d^4$, and a negative cycle is reachable from 1.

2. Reading backwards from node 5 (bold entries): row 5 gives 4, row 4 gives 3, row 3 gives 5, row 2 gives 4, row 1 gives 1. The walk is

$$1, 4, 5, 3, 4, 5, \quad \text{of length } 7 = d^5(5),$$

and its first repeated node closes the cycle 4, 5, 3, 4, of length $1 - 9 + 7 = -1$ (red):



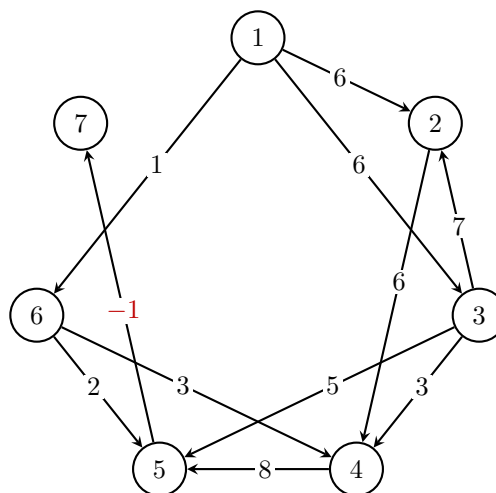
3. The model is feasible, since every node is reachable from 1, and G has a negative cycle: adding λ units of flow around it keeps the balances and lowers the value by λ . The model SP(1) is unbounded (Gurobi confirms it).

A change in pass $n = 5$ reveals a negative cycle: the walk read backwards in the table contains the cycle 4, 5, 3, 4 of length -1 , and the LP model is unbounded.

Exercise 3.4 — Shortest paths in an acyclic graph

We consider the directed graph $G = (N, A)$ with $N = \{1, 2, \dots, 7\}$, the 11 arcs and the lengths d_{ij} of the table, drawn below; the source is $s = 1$.

arc	(1, 2)	(1, 3)	(1, 6)	(2, 4)	(3, 2)	(3, 4)	(3, 5)	(4, 5)	(5, 7)	(6, 4)	(6, 5)
d_{ij}	6	6	1	6	7	3	5	8	-1	3	2



Questions

1. Show that G is acyclic by finding its topological order with (S4).
2. Compute the shortest paths from $s = 1$ with the algorithm for acyclic graphs, and draw the shortest path tree.
3. Compute a longest path from 1 to 7 and its length.

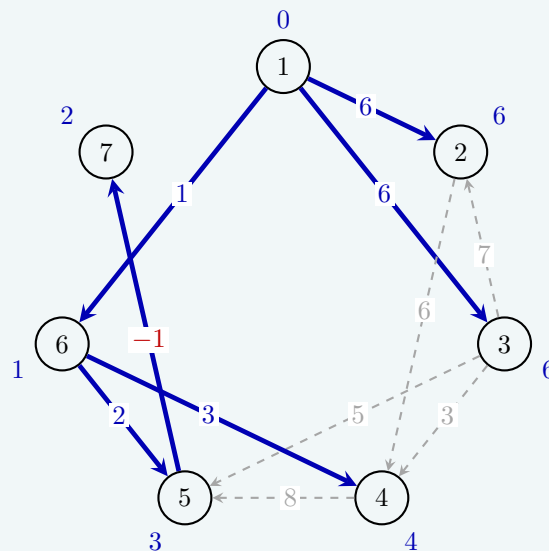
Solution

We use the rules (S1)–(S5) of the box at the beginning of these exercises.

1. Removing each time the node without entering arcs of smallest index gives the order 1, 3, 2, 6, 4, 5, 7. Every arc goes forward in this order, so G has no directed cycle.
2. Each row processes one node and shows the labels after it:

step	node i	$d(1)$	$d(2)$	$d(3)$	$d(4)$	$d(5)$	$d(6)$	$d(7)$
1	1	0	6	6	∞	∞	1	∞
2	3	0	6	6	9	11	1	∞
3	2	0	6	6	9	11	1	∞
4	6	0	6	6	4	3	1	∞
5	4	0	6	6	4	3	1	∞
6	5	0	6	6	4	3	1	2
7	7	0	6	6	4	3	1	2

Hence $\tilde{\mathbf{d}} = (0, 6, 6, 4, 3, 1, 2)$; the predecessors give the arcs (1, 2), (1, 3), (6, 4), (6, 5), (1, 6), (5, 7) (blue):



The shortest path from 1 to 7 is 1, 6, 5, 7, of length 2.

3. With the lengths $-d_{ij}$ the same algorithm gives the longest path lengths

(0, 13, 6, 19, 27, 1, 26).

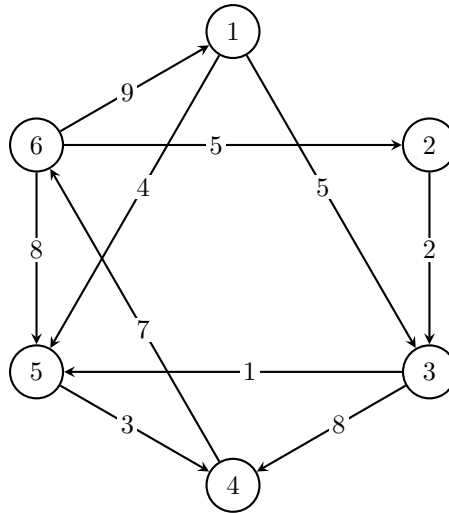
A longest path from 1 to 7 is 1, 3, 2, 4, 5, 7, of length 26.

In the topological order 1, 3, 2, 6, 4, 5, 7 every label is final when its node is processed: shortest path to 7 of length 2, longest path of length 26.

Exercise 3.5 — Potentials as optimality certificate

We consider the directed graph $G = (N, A)$ with $N = \{1, 2, \dots, 6\}$, the 10 arcs and the lengths d_{ij} of the table, drawn below; the source is $s = 1$.

arc	(1, 3)	(1, 5)	(2, 3)	(3, 4)	(3, 5)	(4, 6)	(5, 4)	(6, 1)	(6, 2)	(6, 5)
d_{ij}	5	4	2	8	1	7	3	9	5	8



The tree $\bar{T}$ with the arcs (6, 2), (1, 3), (3, 4), (1, 5), (4, 6) gives the labels $\bar{d} = (0, 25, 5, 13, 4, 20)$, the lengths of its paths from 1.

Questions

1. Write the dual model $D(\text{SP}(1, 6))$ of the LP model of the shortest path from 1 to 6.
2. Compute the reduced costs of the potentials $\bar{\pi}_i = -\bar{d}(i)$. Is the path of $\bar{T}$ from 1 to 6 a shortest path?
3. The potentials $\tilde{\pi} = (0, -19, -5, -7, -4, -14)$ are given. Show that they are feasible, and find a path from 1 to 6 that they certify as shortest.
4. What are the optimal values of $\text{SP}(1, 6)$ and of its dual?

Solution

We use the rules (S1)–(S5) of the box at the beginning of these exercises.

1. There is one free variable π_i per node and one constraint $\pi_i - \pi_j \leq d_{ij}$ per arc:

$$\begin{aligned}
 & \max \quad \pi_1 - \pi_6 \\
 & \text{subject to} \quad \pi_1 - \pi_3 \leq 5 \quad \text{arc } (1, 3) \\
 & \quad \pi_1 - \pi_5 \leq 4 \quad \text{arc } (1, 5) \\
 & \quad \pi_2 - \pi_3 \leq 2 \quad \text{arc } (2, 3) \\
 & \quad \pi_3 - \pi_4 \leq 8 \quad \text{arc } (3, 4) \\
 & \quad \pi_3 - \pi_5 \leq 1 \quad \text{arc } (3, 5) \\
 & \quad \pi_4 - \pi_6 \leq 7 \quad \text{arc } (4, 6) \\
 & \quad \pi_5 - \pi_4 \leq 3 \quad \text{arc } (5, 4) \\
 & \quad \pi_6 - \pi_1 \leq 9 \quad \text{arc } (6, 1) \\
 & \quad \pi_6 - \pi_2 \leq 5 \quad \text{arc } (6, 2) \\
 & \quad \pi_6 - \pi_5 \leq 8 \quad \text{arc } (6, 5) \\
 & \quad \pi_i \in \mathbb{R}, \quad i \in \{1, \dots, 6\}.
 \end{aligned}$$

2. With $\bar{\pi} = (0, -25, -5, -13, -4, -20)$ the reduced costs are:

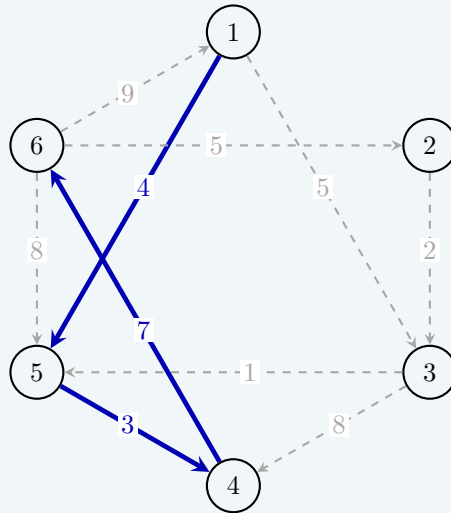
arc	(1, 3)	(1, 5)	(2, 3)	(3, 4)	(3, 5)	(4, 6)	(5, 4)	(6, 1)	(6, 2)	(6, 5)
d_{ij}	5	4	2	8	1	7	3	9	5	8
$\bar{c}_{ij}$	0	0	22	0	2	0	-6	29	0	24

The arc (5, 4) has $\bar{c}_{54} = 3 - (-4) + (-13) = -6 < 0$: the potentials are not feasible, and entering node 4 through (5, 4) gives a path of length $7 < 13 = \bar{d}(4)$. The path 1, 3, 4, 6 of $\bar{T}$ has length 20, larger than the distance 14: it is not shortest.

3. The reduced costs of $\tilde{\pi}$ are all non-negative:

arc	(1, 3)	(1, 5)	(2, 3)	(3, 4)	(3, 5)	(4, 6)	(5, 4)	(6, 1)	(6, 2)	(6, 5)
d_{ij}	5	4	2	8	1	7	3	9	5	8
$\tilde{c}_{ij}$	0	0	16	6	2	0	0	23	0	18

The path 1, 5, 4, 6 uses only tight arcs, so by (S5) it is a shortest path, of length $\tilde{\pi}_1 - \tilde{\pi}_6 = 14$.



4. By weak duality the incidence vector of 1, 5, 4, 6, of value 14, and $\tilde{\pi}$, of value $\tilde{\pi}_1 - \tilde{\pi}_6 = 14$, are both optimal: $z(\text{SP}(1, 6)) = z(D(\text{SP}(1, 6))) = 14$ (Gurobi confirms it).

Potentials with a negative reduced cost point to a shorter path; feasible potentials with a tight path certify it: the distance from 1 to 6 is 14.