Production planning

Outline

The family

Lot sizing with a fixed set-up cost

Production and workforce: two equivalent formulations

Vehicles: minimum lot and a bonus for variety

Learning objectives (1/2)

At the end of this chapter you will be able to:

- write inventory and workforce balances as equality constraints;
- link quantities and binaries with a fixed cost and a big-M read off the data;

Learning objectives (2/2)

At the end of this chapter you will be able to:

- model a minimum lot and a bonus for variety;
- write the dual of the relaxation with free variables for the balances.

Prerequisites

- From this module, Chapter 6 «Links between variables»: fixed cost, minimum lot, integer counts and counting the types
- From this module, Chapter 4 «Constructive heuristics»: the least unit cost heuristic for lot sizing
- From the module *Combinatorial Optimization Problems*, the page «Uncapacitated lot sizing problem (ULS)»: the single-item lot sizing problem

The family

Introduction: The family

- Three problems in which one decides **how much** to produce, not only **whether** to do something
- The quantity variables are continuous or integer; on top of them sit **binaries** that switch on, limit or reward
- Problem 1: the production run with its **fixed cost**
- Problem 2: the **workforce** with the hirings
- Problem 3: the **minimum lot** per type with a bonus for variety

The common feature: the balance

- A quantity that comes in, one that goes out and one that stays
- In problems 1 and 2: the **inventory balance**

$$x_t + s_{t-1} - s_t = d_t$$

- In problem 2 there is also the **workforce balance**

$$y_t - y_{t-1} - z_t = 0$$

- These are equality constraints: their duals are **free variables**
- It is the first family of the course in which this happens systematically

The links between variables that come back here

- **Fixed cost** (problem 1): $x_t \leq M_t\, y_t$, with M_t read off the data (the residual demand) and not picked at random
- **Integer counts** (problem 2): the workforce y_t and the hirings z_t are numbers of people, and the hours available are $r\, y_t$
- **Minimum lot** (problem 3): the semi-integer variable $q_j\, y_j \leq x_j \leq M_j\, y_j$, which is zero or at least q_j
- **Counting the types** and **if and only if** (problem 3): the bonus is collected only if the count of active types reaches two; the missing direction is imposed by optimality

The scheme of every problem

For each of the three problems we shall see:

- model and proof of the links
- instance and constructive heuristic (primal bound)
- dual of the LP relaxation with a hand-built dual solution (dual bound)
- solver solution and additional considerations
- a modelling question and a worked variant

All the numbers come from the scripts `fam09_1_lotsizing.py`, `fam09_2_workforce.py` and `fam09_3_vehicles.py`

Notation of the family (1/3): periods and costs

Symbol	Type	Meaning
n	$\in \mathbb{Z}_{\geq 1}$	number of periods, $t \in \{1, 2, \ldots, n\}$
d_t	$\in \mathbb{Q}_{\geq 0}$	demand of period t
p_t	$\in \mathbb{Q}_{> 0}$	unit production cost
q_t	$\in \mathbb{Q}_{\geq 0}$	fixed set-up cost of a run
h_t	$\in \mathbb{Q}_{\geq 0}$	storage cost at the end of period t
$r_0,\ r_n$	$\in \mathbb{Q}_{\geq 0}$	initial and required final inventory
M_t	$\in \mathbb{Q}_{> 0}$	largest useful production: $\sum_{\tau \geq t} d_\tau + r_n$

Notation of the family (2/3): workforce

Symbol	Type	Meaning
m_0	$\in \mathbb{Z}_{\geq 0}$	workers on duty at the start
w	$\in \mathbb{Q}_{>0}$	wage of one worker per period
u	$\in \mathbb{Q}_{\geq 0}$	cost of hiring one worker
r	$\in \mathbb{Q}_{>0}$	hours worked by one worker in one period
g	$\in \mathbb{Q}_{>0}$	labour hours needed for one unit of product

Notation of the family (3/3): types and resources

Symbol	Type	Meaning
s	$\in \mathbb{Z}_{\geq 1}$	number of product types, indexed by j
k	$\in \mathbb{Z}_{\geq 1}$	number of resources, indexed by i
a_{ij}	$\in \mathbb{Q}_{\geq 0}$	amount of resource i per unit of type j
b_i	$\in \mathbb{Q}_{>0}$	availability of resource i
$\bar{p}_j$	$\in \mathbb{Q}_{>0}$	unit profit of type j
$\bar{q}_j$	$\in \mathbb{Z}_{\geq 1}$	minimum lot of type j
$\bar{r}$	$\in \mathbb{Q}_{>0}$	bonus if at least two types are produced

Summary: The family

- One decides **how much** to produce: quantity variables with binaries that switch on, limit or reward
- The common feature is the **balance**, written with equality constraints and therefore with free duals
- Links: fixed cost, integer counts, minimum lot (semi-integer variable), counting the types
- For every problem: primal bound, dual bound and solver optimum

Lot sizing with a fixed set-up cost

Introduction: Lot sizing with a fixed set-up cost

- The first problem: a single product to be planned over several periods
- Every time production is started a **fixed cost** is paid; keeping stock costs
- We shall see the link $x_t \leq M_t\, y_t$ and how the **big-M is read off the data**
- We shall build a primal bound with two heuristics and a dual bound by hand

Problem 9.1: lot sizing with a fixed cost (1/2)

- A company must plan the production of a single product over $n \in \mathbb{Z}_{\geq 1}$ periods
- For every period $t \in \{1, 2, \ldots, n\}$:
  - $d_t \in \mathbb{Q}_{\geq 0}$ is the demand to be met
  - $p_t \in \mathbb{Q}_{> 0}$ the cost of producing one unit
  - $q_t \in \mathbb{Q}_{\geq 0}$ the fixed cost to be paid if production is started in that period
- For every $t \in \{1, 2, \ldots, n-1\}$, $h_t \in \mathbb{Q}_{\geq 0}$ is the cost of keeping one unit in stock from the end of period t to the start of the next one

Problem 9.1: lot sizing with a fixed cost (2/2)

- The initial inventory is $r_0 \in \mathbb{Q}_{\geq 0}$
- The one required at the end of the horizon is $r_n \in \mathbb{Q}_{\geq 0}$
- The company wants to decide how much to produce in every period so as to meet all the demand at **minimum total cost**

The problem in words

- We **decide** how much to produce in every period, and therefore how much stays in stock
- **The objective**: minimum total cost (production, set-ups and storage)
- **The constraints**:
  - the demand of every period must be met exactly, with what is produced and with what has been kept aside
  - one cannot produce without paying the fixed set-up cost
- It is the **lot sizing** problem with set-up

Data (input of the model)

Symbol	Type	Meaning
n	$\in \mathbb{Z}_{\geq 1}$	number of periods, $t \in \{1, 2, \ldots, n\}$
d_t	$\in \mathbb{Q}_{\geq 0}$	demand of period t
p_t	$\in \mathbb{Q}_{> 0}$	unit production cost in period t
q_t	$\in \mathbb{Q}_{\geq 0}$	fixed set-up cost in period t
h_t	$\in \mathbb{Q}_{\geq 0}$	storage cost at the end of period t, $t \leq n-1$
$r_0,\ r_n$	$\in \mathbb{Q}_{\geq 0}$	initial and required final inventory

Decision variables

n non-negative continuous variables:

$$x_t = \text{units produced in period } t, \qquad \forall t \in \{1, 2, \ldots, n\}$$

$n - 1$ non-negative continuous variables:

$s_t = \text{units in stock at the end of period } t, \qquad \forall t \in \{1, 2, \ldots, n-1\}$

n binary variables:

$$y_t = \begin{cases} 1 & \text{if production is started in period } t \\ 0 & \text{otherwise} \end{cases}$$

Model 9.1

Model 9.1 — lot sizing with a fixed cost

$$\min \quad \sum_{t=1}^{n} p_t\, x_t + \sum_{t=1}^{n} q_t\, y_t + \sum_{t=1}^{n-1} h_t\, s_t \tag{9.1a}$$

$$\text{subject to} \quad x_1 - s_1 = d_1 - r_0, \tag{9.1b}$$

$$x_t + s_{t-1} - s_t = d_t, \qquad \forall t \in \{2, 3, \ldots, n-1\}, \tag{9.1c}$$

$$x_n + s_{n-1} = d_n + r_n, \tag{9.1d}$$

$$-x_t + M_t\, y_t \geq 0, \qquad \forall t \in \{1, 2, \ldots, n\}, \tag{9.1e}$$

$$x_t \geq 0, \qquad \forall t \in \{1, 2, \ldots, n\}, \tag{9.1f}$$

$$s_t \geq 0, \qquad \forall t \in \{1, 2, \ldots, n-1\}, \tag{9.1g}$$

$$y_t \in \{0, 1\}, \qquad \forall t \in \{1, 2, \ldots, n\}. \tag{9.1h}$$

Description of the model

- The **objective function** adds up the production cost, the cost of the set-ups and the storage cost
- The **inventory balance** constraints are one per period (n equality constraints): what one had plus what one produces minus what one sells is what is left
- The **set-up** constraints link quantity and set-up (n linear constraints)

The link between the variables (1/2)

The set-up constraint reads

$$x_t \leq M_t \, y_t$$

- If $y_t = 0$ then $x_t \leq 0$ and, together with $x_t \geq 0$, one gets $x_t = 0$: **without a set-up nothing is produced**
- If $y_t = 1$ the constraint becomes $x_t \leq M_t$ and is not restrictive, provided M_t is chosen large enough

The link between the variables (2/2)

- The other direction, *if* $x_t = 0$ *then* $y_t = 0$, is not imposed by any constraint
- It follows from **optimality**: if $x_t = 0$ and $y_t = 1$, setting $y_t = 0$ stays feasible and saves q_t
- If $q_t > 0$ every optimal solution has $y_t = 0$
- If $q_t = 0$ the saving is nil: one can only say that there *exists* an optimal solution with $y_t = 0$

The truth table of the link $x_t \leq M_t\, y_t$

The link must be checked on all the values of y_t, keeping in mind that $x_t \geq 0$:

y_t	constraint	x_t allowed	reading
0	$x_t \leq 0$	only $x_t = 0$	no set-up, no production
1	$x_t \leq M_t$	$0 \leq x_t \leq M_t$	with the set-up one produces

The combination $x_t = 0$ with $y_t = 1$ is left uncovered: it is feasible, but it appears in no optimal solution, because

$y_t = 1$ and $x_t = 0 \implies$ setting $y_t = 0$ stays feasible and saves $q_t \geq 0$

The big-M is read off the data (1/2)

Observation 1: The big-M is read off the data

A valid M_t must be at least the largest quantity that it is worth producing in period t. In an optimal solution one never produces more than the demand still to be covered, final inventory included:

$$M_t = \sum_{\tau=t}^{n} d_\tau + r_n$$

In the first period the initial inventory, which is a datum, can be subtracted: $M_1 = \sum_{\tau=1}^{n} d_\tau + r_n - r_0$.

The big-M is read off the data (2/2)

- From the second period on the incoming inventory is not known in advance, and the bound stays $\sum_{\tau \geq t} d_\tau + r_n$
- On the instance $r_0 = 0$, so the two values coincide
- Any larger value is still valid but **weakens the LP relaxation**; any smaller value may **cut off optimal solutions**
- On the instance $M = (110, 90, 80, 50, 10)$: the big-M of the last period is 10, not 110

The two bounds we shall build, the primal one (heuristic) and the dual one (relaxation), are the two sides of the sandwich that a MIP solver tries to close.

The model in gurobipy (1/2)

```
m = gp.Model("lot_sizing")
x = m.addVars(n, name="x")
s = m.addVars(n - 1, name="s")
y = m.addVars(n, vtype=GRB.BINARY, name="y")
m.setObjective(
    gp.quicksum(p[t] * x[t] for t in range(n))
    + gp.quicksum(q[t] * y[t] for t in range(n))
    + gp.quicksum(h[t] * s[t]
                  for t in range(n - 1)),
    GRB.MINIMIZE)
```

The model in gurobipy (2/2)

```
m.addConstr(x[0] - s[0] == d[0] - r0,
            name="balance[0]")
m.addConstrs((x[t] + s[t - 1] - s[t] == d[t]
              for t in range(1, n - 1)),
             name="balance")
m.addConstr(x[n - 1] + s[n - 2] == d[n - 1] + rn,
            name=f"balance[{n␣-␣1}]")
m.addConstrs((-x[t] + M[t] * y[t] >= 0
              for t in range(n)), name="run")
```

The instance

$n = 5$ periods (days), $r_0 = r_n = 0$:

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
d_t	20	10	30	40	10
p_t	2	3	2	3	2
q_t	50	50	50	50	50
M_t	110	90	80	50	10
h_t	1	1	1	1	—

The total demand is 110 units.

Model 9.1 on the instance

Model 9.1 on the instance

$$
\begin{aligned}
\min\ & 2x_1+3x_2+2x_3+3x_4+2x_5+s_1+s_2+s_3+s_4 \\
& +50y_1+50y_2+50y_3+50y_4+50y_5 \\
\text{subject to}\ & x_1 - s_1 = 20 \\
& x_2 + s_1 - s_2 = 10 \\
& x_3 + s_2 - s_3 = 30 \\
& x_4 + s_3 - s_4 = 40 \\
& x_5 + s_4 = 10 \\
& -x_1 + 110y_1 \geq 0 \\
& -x_2 + 90y_2 \geq 0 \\
& -x_3 + 80y_3 \geq 0 \\
& -x_4 + 50y_4 \geq 0 \\
& -x_5 + 10y_5 \geq 0 \\
& x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0 \\
& s_1,\ s_2,\ s_3,\ s_4 \geq 0 \\
& y_1,\ y_2,\ y_3,\ y_4,\ y_5 \in \{0,1\}
\end{aligned}
$$

Dual on the instance

Dual on the instance

$$
\begin{aligned}
& \max 20\mu_1 + 10\mu_2 + 30\mu_3 + 40\mu_4 + 10\mu_5 \\
& \text{subject to} \quad \mu_1 - \pi_1 \le 2 \\
& \mu_2 - \pi_2 \le 3 \\
& \mu_3 - \pi_3 \le 2 \\
& \mu_4 - \pi_4 \le 3 \\
& \mu_5 - \pi_5 \le 2 \\
& 110\pi_1 \le 50 \\
& 90\pi_2 \le 50 \\
& 80\pi_3 \le 50 \\
& 50\pi_4 \le 50 \\
& 10\pi_5 \le 50 \\
& -\mu_1 + \mu_2 \le 1 \\
& -\mu_2 + \mu_3 \le 1 \\
& -\mu_3 + \mu_4 \le 1 \\
& -\mu_4 + \mu_5 \le 1 \\
& \mu_1, \ \mu_2, \ \mu_3, \ \mu_4, \ \mu_5 \gtreqless 0 \\
& \pi_1, \ \pi_2, \ \pi_3, \ \pi_4, \ \pi_5 \ge 0
\end{aligned}
$$

Two things to notice in this dual

- The constraints on the columns of the stocks, $-\mu_t + \mu_{t+1} \leq 1$, say that the price of one unit cannot grow by more than the storage cost from one period to the next
- The constraint $10\pi_5 \leq 50$ is much looser than the others, because the big-M of the last period is 10 and not 110
- A small big-M in the primal leaves more room to the dual variable: it is another face of the fact that small big-Ms give **tighter relaxations**

Constructive heuristics: (a) lot-for-lot

Two classic rules, both feasible by construction.

(a) Lot-for-lot: every day one produces exactly the demand of the day

- no inventory, but a set-up every day

The cost is

$$\sum_{t=1}^{5} p_t \, d_t + \sum_{t=1}^{5} q_t = 270 + 250 = 520$$

Constructive heuristics: (b) least unit cost (1/2)

One starts from the first uncovered period and covers with a single run the k periods that minimise the average cost per unit

$$\frac{q + \sum_{i=1}^{k-1} h \cdot (\text{units kept } i \text{ periods})}{\sum_{i=0}^{k-1} d_{t+i}}$$

then one restarts from the first period that is still uncovered.

- The cost p does not appear: the alternatives start in the same period and cover the same demands, so they pay the same p
- The average costs are **incremental** costs of set-up plus inventory

Constructive heuristics: (b) least unit cost (2/2)

It is a constructive heuristic: a single solution is built, one element at a time, never backtracking. Running time $O(n^2)$. On the instance:

- period 1: covers up to period 2, quantity 30, average cost 2
- period 3: covers up to period 4, quantity 70, average cost ≈ 1.286
- period 5: covers period 5 alone, quantity 10, average cost 5

Runs are started on days $1, 3, 5$; the total cost is 420. Keeping the better of the two heuristics,

$$z(MILP) \leq UB = 420$$

LP relaxation and dual: the dual variables

- One **free** dual variable μ_t for each balance constraint (equalities)
- One **non-negative** dual variable π_t for each set-up constraint ($\geq$ direction)
- Right-hand sides of the balances:

$$b_1 = d_1 - r_0, \qquad b_t = d_t \ (2 \leq t \leq n-1), \qquad b_n = d_n + r_n$$

Dual of the LP relaxation of model 9.1

Dual of the LP relaxation of model 9.1

$$\max \quad \sum_{t=1}^{n} b_t \, \mu_t \tag{9.2a}$$

$$\text{subject to} \quad \mu_t - \pi_t \leq p_t, \qquad \forall t \in \{1, 2, \ldots, n\}, \tag{9.2b}$$

$$M_t \, \pi_t \leq q_t, \qquad \forall t \in \{1, 2, \ldots, n\}, \tag{9.2c}$$

$$-\mu_t + \mu_{t+1} \leq h_t, \quad \forall t \in \{1, 2, \ldots, n-1\}, \tag{9.2d}$$

$$\mu_t \gtreqless 0, \qquad \forall t \in \{1, 2, \ldots, n\}, \tag{9.2e}$$

$$\pi_t \geq 0, \qquad \forall t \in \{1, 2, \ldots, n\}. \tag{9.2f}$$

Description of the dual (1/2)

- μ_t is the value attached to one unit available in period t; π_t the price of one unit of set-up capacity
- The **objective** prices the demand of each period at those values
- **Columns of the** x_t: producing one unit in period t is worth μ_t and consumes set-up capacity at price π_t; the balance cannot exceed the unit cost p_t

Description of the dual (2/2)

- **Columns of the y_t**: all the capacity M_t that a set-up makes available, priced at π_t, cannot be worth more than the fixed cost q_t
- **Columns of the s_t**: carrying one unit from period t to $t+1$ earns $\mu_{t+1} - \mu_t$, which cannot exceed h_t
- The μ_t are free because the balances are equalities; the π_t are non-negative because the set-up constraints are of type $\geq$ in a minimisation

A hand-built dual solution (1/2)

- Set $\bar{\pi}_t = 0$ for every t: the constraints $M_t \pi_t \leq q_t$ are satisfied because $q_t \geq 0$, and the set-ups become free
- What is left is $\mu_t \leq p_t$ and $\mu_{t+1} \leq \mu_t + h_t$: the largest feasible value is built forward

$$\bar{\mu}_1 = p_1, \qquad \bar{\mu}_t = \min(\bar{\mu}_{t-1} + h_{t-1},\ p_t) \quad \text{for } t \geq 2$$

- $\bar{\mu}_t$ is the lowest unit cost of having one unit available in period t: either by producing it then, or by producing it earlier and keeping it in stock

A hand-built dual solution (2/2)

On the instance $h = 1$ is low and the production costs are $2, 3, 2, 3, 2$:

$$\bar{\mu} = (2,\ 3,\ 2,\ 3,\ 2)$$

$$LB = \sum_{t=1}^{5} d_t\, \bar{\mu}_t = 2 \cdot 20 + 3 \cdot 10 + 2 \cdot 30 + 3 \cdot 40 + 2 \cdot 10 = 270$$

- It is the production cost if the set-ups were free: a valid bound, and deliberately optimistic
- The LP relaxation pays for the set-ups in fractions ($\pi_t = q_t / M_t$ is feasible) and goes higher:

$$z(LP) = z(LP^+) = \frac{3890}{11} \approx 353.64$$

Optimal solution

The optimum is obtained by solving the model: the two bounds just built are what helps the solver get there.

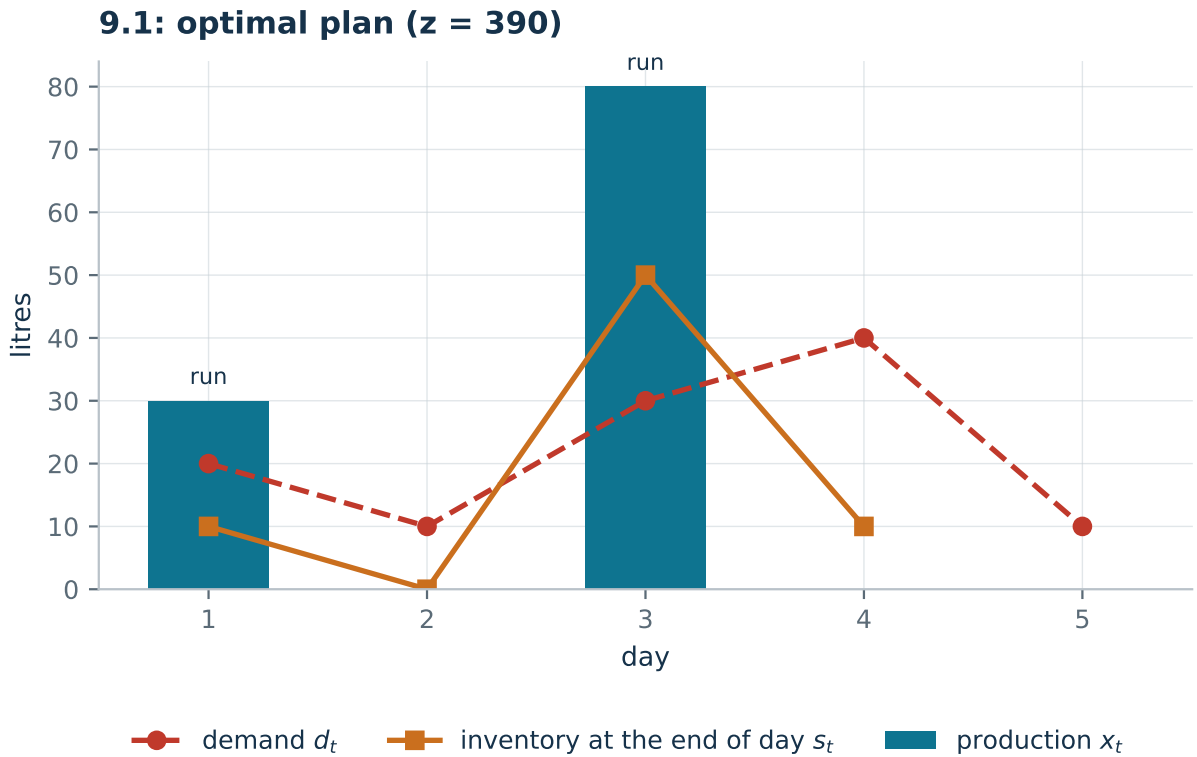
	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
run y_t	1	0	1	0	0
production x_t	30	0	80	0	0
inventory s_t	10	0	50	10	—

The cost is $z(MILP) = 390$, and the chain of bounds is

$$270 = LB \leq z(LP) = \frac{3890}{11} \leq z(MILP) = 390 \leq UB = 420$$

Optimal solution: the gaps and the plan

- Heuristic gap: $(420 - 390)/390 = 7.7\%$
- Certified gap, before solving the MILP: $(420 - 270)/390 = 38.5\%$



The two runs cover, the first one, days 1–2 and, the second one, days 3–5.

Additional considerations (1/2)

- The model is a special case of the problem solved in $O(n^2)$ time by the **Wagner–Whitin** dynamic program
- Least unit cost is *not* that algorithm: it is a myopic rule that looks at one run at a time and in general does not find the optimum (on the instance 420 against 390)
- The valid inequality $x_t \leq M_t$, obtained from the set-up constraint with $y_t \leq 1$, adds nothing: M_t is already the largest useful production. Indeed $z(LP) = z(LP^+)$

Additional considerations (2/2)

- An alternative formulation, much tighter, uses the variables

$$x_{t\tau} = \text{units produced in } t \text{ and sold in } \tau$$

(the **facility location** formulation of lot sizing)

- Its LP relaxation is integral, but it has $O(n^2)$ variables
- It is the typical trade-off between the **size of the model** and the **quality of the relaxation**

Modelling question: minimum lot

If one produces on a given day, at least 25 units must be produced (below that threshold the set-up is not justified).

How does the model change? What is the new optimum?

The final comparison

The three numbers that every problem of the course produces:

	value	what it is
UB	420	heuristic solution
LB	270	certified bound
$z(MILP)$	390	MILP optimum

- Gap of the heuristic: $\dfrac{|UB - z(MILP)|}{|z(MILP)|} = 7.7\%$
- Certified gap, known *before* solving the MILP (it cannot have $z(MILP)$ as its denominator): $UB - LB = 150$, that is $\dfrac{UB - LB}{|UB|} = 35.7\%$

A worked variant: daily capacity (1/3)

The plant cannot produce more than 35 units a day. One adds the n linear constraints

$$x_t \leq 35, \qquad \forall t \in \{1, 2, \ldots, n\}$$

- Alternatively one replaces the big-M with $M'_t = \min(M_t, 35)$: the constraint then acts both as the activation link and as the capacity constraint, exactly as in capacitated facility location
- The optimal plan with only two runs is no longer feasible ($80 > 35$): the optimum rises to 470

A worked variant: daily capacity (2/3)

- The cap $x_t \leq 35$ adds to the dual a family $\nu_t \geq 0$ with right-hand side -35: it enters the objective with a negative sign and loosens the column of x_t
- Raising ν_t by one unit lifts the cap on μ_t and collects d_t, but costs 35: it only pays where the demand exceeds the capacity
- On this instance not even the busiest day justifies it: the certificate stays $LB = 270$
- What changes is the heuristic: on day four lot-for-lot would exceed the cap, and the 5 excess units are anticipated to day three

A worked variant: daily capacity (3/3)

	value	what it is
UB	520	heuristic solution
LB	270	dual certificate built by hand
$z(LP)$	$\frac{305365}{792}$	relaxation without the bounds
$z(LP^+)$	$\frac{305365}{792}$	relaxation with the bounds
$z(MILP)$	470	optimum of the MILP

Summary: Lot sizing with a fixed set-up cost

- Inventory balance (equalities) and the link $x_t \leq M_t\, y_t$; the missing direction is imposed by optimality
- The big-M is read off the data: $M_t = \sum_{\tau \geq t} d_\tau + r_n$
- Bounds: $LB = 270$ (free set-ups), $z(LP) \approx 353.64$, $z(MILP) = 390$, $UB = 420$ (least unit cost)
- The facility location formulation has an integral relaxation but $O(n^2)$ variables

Production and workforce: two equivalent formulations

Introduction: Production and workforce

- The second problem adds to the production the decision on the **workforce**
- The staff variables are **integer counts**: numbers of people
- We shall see two formulations, with the hirings and with the workforce, and prove that they describe the same set of plans
- It is the first case in the course of two apparently different but equivalent models

Problem 9.2: production and workforce (1/2)

- A shoe factory must plan its production over $n \in \mathbb{Z}_{\geq 1}$ months
- For every month t: $d_t \in \mathbb{Z}_{\geq 0}$ is the demand in pairs and $p_t \in \mathbb{Q}_{>0}$ the cost of the raw materials for one pair
- For every month $t \leq n-1$: $h_t \in \mathbb{Q}_{\geq 0}$ is the cost of keeping one pair in stock at the end of the month
- At the start the company has $m_0 \in \mathbb{Z}_{\geq 0}$ workers; every worker works r hours a month and costs w euros a month; one pair requires g labour hours

Problem 9.2: production and workforce (2/2)

- At the start of every month workers may be hired, at a cost of $u \in \mathbb{Q}_{\geq 0}$ euros each
- Nobody is laid off
- The company wants to decide how much to produce and how many workers to employ in every month, at **minimum total cost**

The problem in words

- We **decide** how much to produce, how much to keep in stock and how many workers to hire
- **The objective**: minimum total cost (raw materials, storage, wages and hirings)
- **The constraints**:
  - the demand must be met exactly
  - the production of a month cannot require more hours than the workers on duty can do

Data (input of the model)

Symbol	Type	Meaning
n	$\in \mathbb{Z}_{\geq 1}$	number of months, $t \in \{1, 2, \ldots, n\}$
d_t	$\in \mathbb{Z}_{\geq 0}$	demand (pairs) of month t
p_t	$\in \mathbb{Q}_{>0}$	cost of the raw materials for one pair
h_t	$\in \mathbb{Q}_{\geq 0}$	storage cost at the end of month t
m_0	$\in \mathbb{Z}_{\geq 0}$	workers on duty at the start
w	$\in \mathbb{Q}_{>0}$	monthly wage of one worker
u	$\in \mathbb{Q}_{\geq 0}$	cost of hiring one worker
r	$\in \mathbb{Q}_{>0}$	hours worked by one worker in one month
g	$\in \mathbb{Q}_{>0}$	labour hours for one pair

Formulation A: the hirings

The staff variables, integer and non-negative, are

$$z_t = \text{workers hired at the start of month } t, \qquad \forall t \in \{1, 2, \ldots, n\}$$

- A worker hired in month t stays on duty until the end of the horizon: they cost u once plus w for each of the remaining $n - t + 1$ months
- The wage of the m_0 initial workers, $m_0\, w\, n$, is a **constant term**: it is left out of the model and added to the final value

Formulation B: the workforce

The staff variables, integer and non-negative, are

$$y_t = \text{workers on duty in month } t, \qquad \forall t \in \{1, 2, \ldots, n\}$$

- $y_t \geq y_{t-1}$ because nobody is laid off
- The cost is $w\, y_t$ every month, plus u for every hiring, that is $u\,(y_n - m_0)$ in total
- The hirings are the increments of the workforce: the telescoping sum leaves only the endpoints

Model 9.2A

Model 9.2A — with the hirings

$$\min \quad \sum_{t=1}^{n} p_t\, x_t + \sum_{t=1}^{n-1} h_t\, s_t + \sum_{t=1}^{n} \big(u + w\,(n-t+1)\big) z_t \tag{9.3a}$$

$$\text{subject to} \quad x_1 - s_1 = d_1, \tag{9.3b}$$

$$x_t + s_{t-1} - s_t = d_t, \qquad \forall t \in \{2, 3, \ldots, n-1\}, \tag{9.3c}$$

$$x_n + s_{n-1} = d_n, \tag{9.3d}$$

$$-g\, x_t + r \sum_{j=1}^{t} z_j \geq -r\, m_0, \qquad \forall t \in \{1, 2, \ldots, n\}, \tag{9.3e}$$

$$x_t \in \mathbb{Z}_{\geq 0}, \qquad \forall t \in \{1, 2, \ldots, n\}, \tag{9.3f}$$

$$s_t \in \mathbb{Z}_{\geq 0}, \qquad \forall t \in \{1, 2, \ldots, n-1\}, \tag{9.3g}$$

$$z_t \in \mathbb{Z}_{\geq 0}, \qquad \forall t \in \{1, 2, \ldots, n\}. \tag{9.3h}$$

Model 9.2B

Model 9.2B — with the workforce

$$\min \quad \sum_{t=1}^{n} p_t\, x_t + \sum_{t=1}^{n-1} h_t\, s_t + \sum_{t=1}^{n} w\, y_t + u\,(y_n - m_0) \tag{9.4a}$$

$$\text{subject to} \quad x_1 - s_1 = d_1, \tag{9.4b}$$

$$x_t + s_{t-1} - s_t = d_t, \qquad \forall t \in \{2, 3, \ldots, n-1\}, \tag{9.4c}$$

$$x_n + s_{n-1} = d_n, \tag{9.4d}$$

$$-g\, x_t + r\, y_t \geq 0, \qquad \forall t \in \{1, 2, \ldots, n\}, \tag{9.4e}$$

$$y_1 \geq m_0, \tag{9.4f}$$

$$-y_{t-1} + y_t \geq 0, \qquad \forall t \in \{2, 3, \ldots, n\}, \tag{9.4g}$$

$$x_t \in \mathbb{Z}_{\geq 0}, \qquad \forall t \in \{1, 2, \ldots, n\}, \tag{9.4h}$$

$$s_t \in \mathbb{Z}_{\geq 0}, \qquad \forall t \in \{1, 2, \ldots, n-1\}, \tag{9.4i}$$

$$y_t \in \mathbb{Z}_{\geq 0}, \qquad \forall t \in \{1, 2, \ldots, n\}. \tag{9.4j}$$

Description of the two formulations

- They share the **stock balance** constraints: one per month, what is produced plus what is in stock covers the demand exactly
- The **available hours** constraints, one per month: the production cannot require more hours than the workers on the payroll can do
- B has in addition $y_1 \geq m_0$, which starts from the initial workers, and the **monotonicity** constraints, which forbid layoffs
- In A those same facts are hidden in the domain of the z_t, which are non-negative

The two formulations describe the same problem (1/2)

The correspondence is

$$y_t = m_0 + \sum_{j=1}^{t} z_j, \qquad \text{that is} \qquad z_t = y_t - y_{t-1} \quad (\text{with } y_0 = m_0)$$

Feasibility:

- the hours constraint of A becomes exactly the one of B
- $z_t \geq 0$ becomes $y_t \geq y_{t-1}$ (monotonicity) and $y_1 \geq m_0$
- the balances contain no staff variables and stay identical

The correspondence is a **bijection** between the feasible plans of the two models.

The two formulations describe the same problem (2/2)

Cost: the staff cost in B is

$$\sum_{t=1}^{n} w\, y_t + u\,(y_n - m_0) = \sum_{t=1}^{n} w\left(m_0 + \sum_{j \le t} z_j\right) + u \sum_{t=1}^{n} z_t$$

The first term equals $m_0\, w\, n + w \sum_t (n - t + 1)\, z_t$, because z_j appears in every month from j onwards. Adding the second one:

$$m_0\, w\, n + \sum_{t=1}^{n} \big(u + w\,(n - t + 1)\big) z_t$$

The two objective functions differ by the constant $m_0\, w\, n$ alone: same optima.

Why keep both

- A has fewer constraints (no monotonicity) but cost coefficients that depend on the period
- B has uniform coefficients and extends better if layoffs are added: one only needs $\ell_t \geq 0$ and the balance $y_t = y_{t-1} + z_t - \ell_t$
- On the instance even the **relaxations** coincide: $z(LP) = 15\,960$ for both
- Two formulations that are equivalent on the integers are not always equivalent on the relaxation: the check must be made, not assumed

The model in gurobipy (1/2)

```
m = gp.Model("workforce_B")
x = m.addVars(n, vtype=GRB.INTEGER, name="x")
s = m.addVars(n - 1, vtype=GRB.INTEGER, name="s")
y = m.addVars(n, vtype=GRB.INTEGER, name="y")
m.setObjective(
    gp.quicksum(p[t] * x[t] for t in range(n))
    + gp.quicksum(h[t] * s[t]
                  for t in range(n - 1))
    + gp.quicksum(w * y[t] for t in range(n))
    + u * (y[n - 1] - m0), GRB.MINIMIZE)
```

The model in gurobipy (2/2)

```
m.addConstr(x[0] - s[0] == d[0],
            name="balance[0]")
m.addConstrs((x[t] + s[t - 1] - s[t] == d[t]
              for t in range(1, n - 1)),
             name="balance")
m.addConstr(x[n - 1] + s[n - 2] == d[n - 1],
            name=f"balance[{n␣-␣1}]")
m.addConstrs((-g * x[t] + r * y[t] >= 0
              for t in range(n)), name="hours")
m.addConstr(y[0] >= m0, name="initial_workforce")
m.addConstrs((-y[t - 1] + y[t] >= 0
              for t in range(1, n)),
             name="no_layoffs")
```

The instance

$n = 3$ months, $m_0 = 2$ workers, $w = 1500$, $u = 100$, $r = 160$ hours, $g = 4$ hours per pair, $h_t = 3$:

	$t = 1$	$t = 2$	$t = 3$
d_t	60	100	140
p_t	15	15	15

With two workers the initial capacity is $2 \cdot 160/4 = 80$ pairs a month: enough for the first month, not for the other two.

Model 9.2B on the instance

Model 9.2 on the instance

$$
\begin{array}{lrrrrrrrrrl}
\min & 15x_1 & +15x_2 & +15x_3 & +3s_1 & +3s_2 & +1500y_1 & +1500y_2 & +1600y_3 & \\
\text{subject to} & x_1 & & & -s_1 & & & & & = 60 \\
& & x_2 & & +s_1 & -s_2 & & & & = 100 \\
& & & x_3 & & +s_2 & & & & = 140 \\
& -4x_1 & & & & & +160y_1 & & & \geq 0 \\
& & -4x_2 & & & & & +160y_2 & & \geq 0 \\
& & & -4x_3 & & & & & +160y_3 & \geq 0 \\
& & & & & & y_1 & & & \geq 2 \\
& & & & & & -y_1 & +y_2 & & \geq 0 \\
& & & & & & & -y_2 & +y_3 & \geq 0 \\
& x_1, & x_2, & x_3 & & & & & & \in \mathbb{Z}_{\geq 0} \\
& & & & s_1, & s_2 & & & & \in \mathbb{Z}_{\geq 0} \\
& & & & & & y_1, & y_2, & y_3 & \in \mathbb{Z}_{\geq 0}
\end{array}
$$

Dual of formulation 9.2A on the instance

Dual on the instance

$$
\begin{aligned}
\max\ & 60\mu_1 + 100\mu_2 + 140\mu_3 - 320\nu_1 - 320\nu_2 - 320\nu_3 \\
\text{subject to}\ & \mu_1 - 4\nu_1 \le 15 \\
& \mu_2 - 4\nu_2 \le 15 \\
& \mu_3 - 4\nu_3 \le 15 \\
& -\mu_1 + \mu_2 \le 3 \\
& -\mu_2 + \mu_3 \le 3 \\
& 160\nu_1 + 160\nu_2 + 160\nu_3 \le 4600 \\
& 160\nu_2 + 160\nu_3 \le 3100 \\
& 160\nu_3 \le 1600 \\
& \mu_1, \mu_2, \mu_3 \gtrless 0 \\
& \nu_1, \nu_2, \nu_3 \ge 0
\end{aligned}
$$

Constructive heuristic: the primal bound

Just in time production: every month one produces exactly the demand and hires the minimum number of workers needed.

On the instance:

- month 1: $\lceil 4 \cdot 60/160 \rceil = 2$ workers, no hiring
- month 2: $\lceil 4 \cdot 100/160 \rceil = 3$, one hiring
- month 3: $\lceil 4 \cdot 140/160 \rceil = 4$, one more hiring

The cost, including the constant term $m_0\, w\, n = 9000$, is

$$z(MILP) \leq UB = 18\,200$$

Dual of the LP relaxation of model 9.2A

One free variable μ_t per balance, one non-negative ν_t per hours constraint:

Dual of the LP relaxation of model 9.2A

$$\max \quad \sum_{t=1}^{n} d_t \, \mu_t - r \, m_0 \sum_{t=1}^{n} \nu_t \tag{9.5a}$$

$$\text{subject to} \quad \mu_t - g \, \nu_t \leq p_t, \qquad \forall t \in \{1, 2, \ldots, n\}, \tag{9.5b}$$

$$-\mu_t + \mu_{t+1} \leq h_t, \qquad \forall t \in \{1, 2, \ldots, n-1\}, \tag{9.5c}$$

$$r \sum_{t=j}^{n} \nu_t \leq u + w \, (n - j + 1), \qquad \forall j \in \{1, 2, \ldots, n\}, \tag{9.5d}$$

$$\mu_t \gtreqless 0, \qquad \forall t \in \{1, 2, \ldots, n\}, \tag{9.5e}$$

$$\nu_t \geq 0, \qquad \forall t \in \{1, 2, \ldots, n\}. \tag{9.5f}$$

Description of the dual (1/2)

- μ_t is the value of one pair available in month t; ν_t the price of one working hour in month t
- The **objective** prices the demand at those values and subtracts $r\, m_0 \sum_t \nu_t$: the hours the m_0 initial workers supply for free
- **Columns of the** x_t: producing one pair is worth μ_t and consumes g hours at price ν_t; the balance cannot exceed p_t

Description of the dual (2/2)

- **Columns of the s_t**: holding one pair in stock earns $\mu_{t+1} - \mu_t$, which cannot exceed h_t
- **Columns of the z_j**: a worker hired at the beginning of month j supplies r hours in each month from j onwards
- Their total value cannot exceed what that hire costs, namely $u + w\,(n - j + 1)$

A hand-built dual solution

- Set $\bar{\nu}_t = 0$: the working hours are not priced, the constraints of the z_j are satisfied; what is left is $\mu_t \leq p_t$ and $\mu_{t+1} \leq \mu_t + h_t$
- As in problem 9.1: $\bar{\mu}_1 = p_1$, $\bar{\mu}_t = \min(\bar{\mu}_{t-1} + h_{t-1},\ p_t)$

On the instance $\bar{\mu} = (15, 15, 15)$ and

$$\sum_{t=1}^{n} d_t\, \bar{\mu}_t = 15 \cdot 300 = 4500, \qquad LB = 4500 + 9000 = 13\,500$$

- Raw materials plus the wage of the workers already on duty: the extra labour is given away
- The LP relaxation pays for it in fractions of a person: $z(LP) = z(LP^+) = 15\,960$

Optimal solution

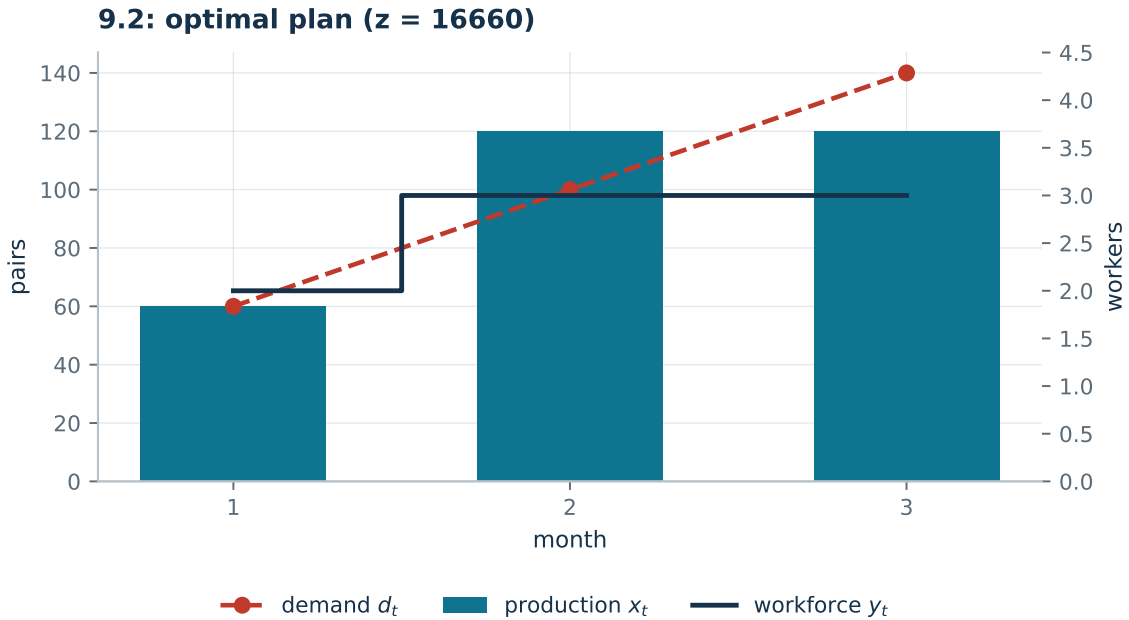
	$t = 1$	$t = 2$	$t = 3$
production x_t	60	120	120
workforce y_t	2	3	3
hirings z_t	0	1	0
inventory s_t	0	20	—

The cost is $z(MILP) = 16\,660$, with the chain

$$13\,500 = LB \leq z(LP) = 15\,960 \leq z(MILP) = 16\,660 \leq UB = 18\,200$$

The heuristic gap is 9.2%.

Optimal solution: the plan



- The optimal plan hires *one* worker instead of two and moves twenty pairs from the third month to the second
- 3 euros of storage for twenty pairs cost 60 euros, against the 1600 of a hiring in the third month

Additional considerations (1/2)

- **Monotonicity** is what ties the months together on the staffing side: if one could lay off at no cost, every month would pick on its own the smallest staff that is enough, and that family would disappear
- The problem would not split into n independent problems, though: the stocks s_t still tie each month to the next
- The constant term $m_0\, w\, n$ must be remembered in every comparison: forgetting it makes A look much cheaper than B

Additional considerations (2/2)

- x_t and s_t are integer because pairs of shoes cannot be split; here they could be left continuous without changing the optimum
- *Not* because the matrix is totally unimodular: it is not ($z(LP) = 15\,960$ against $16\,660$)
- The reason is more local: once y is fixed at integer values, what is left in (x, s) is a balance problem with an interval matrix, totally unimodular, and integer right-hand sides (d_t and $r\,y_t/g = 40\,y_t$)
- It is an argument about the subproblem, not about the whole model

Modelling question: overtime

Every worker may do up to 40 hours of overtime a month, paid 25 euros an hour.

How does the model change? Is it worth using them?

The final comparison

	value	what it is
UB	18200	heuristic solution
LB	13500	certified bound
$z(MILP)$	16660	MILP optimum

- Gap of the heuristic: $\frac{|UB - z(MILP)|}{|z(MILP)|} = 9.2\%$
- Certified gap, known *before* solving the MILP:
  $UB - LB = 4700$, that is $\frac{UB - LB}{|UB|} = 25.8\%$

A worked variant: very expensive hirings (1/2)

The hiring cost goes up from 100 to 3000 euros (selection and training).

- Only the datum u changes: the new optimum is 19 560, that is 2900 euros more
- The plan does not change: one worker is hired anyway, because without that hiring the problem is infeasible ($4 \cdot 120/160 = 3$ workers)
- The difference is exactly $3000 - 100 = 2900$: the extra cost of an unavoidable hiring
- It is an example of sensitivity analysis carried out by re-solving the model

A worked variant: very expensive hirings (2/2)

- Model, dual and recipe stay those of the base problem; the certificate does not notice the change: the hiring cost does not appear in μ_t, and the bound stays 13500
- A limiting case: a correct dual bound can be insensitive to the datum that moves the problem; here it is the heuristic that follows it

	value	what it is
UB	24000	heuristic solution
LB	13500	dual certificate built by hand
$z(LP) = z(LP^+)$	17490	relaxation without / with the bounds
$z(MILP)$	19560	optimum of the MILP

Summary: Production and workforce

- Two equivalent formulations: with the hirings z_t (A) and with the workforce y_t (B), linked by $z_t = y_t - y_{t-1}$
- The objectives differ by the constant $m_0\, w\, n$: it must be remembered in the comparisons
- On the instance the relaxations coincide too, but the check must be made
- Bounds: $LB = 13\,500$, $z(LP) = 15\,960$, $z(MILP) = 16\,660$, $UB = 18\,200$

Vehicles: minimum lot and a bonus for variety

Introduction: Vehicles

- The third problem is a **maximisation**: profit with limited resources
- Every type is produced in quantity zero or at least equal to a **minimum lot**: it is the **semi-integer** variable
- A **bonus** is collected if at least two types are produced: we shall learn to count the active types with one binary and one constraint
- We shall see that here the relaxation with the bounds beats the hand-built dual

Problem 9.3: vehicles with minimum lot and bonus (1/2)

- A company produces $s \in \mathbb{Z}_{\geq 1}$ types of vehicle using $k \in \mathbb{Z}_{\geq 1}$ resources
- $a_{ij} \in \mathbb{Q}_{\geq 0}$ is the amount of resource i needed for one unit of type j, and $b_i \in \mathbb{Q}_{>0}$ is the availability of resource i
- For every type j: $\bar{p}_j \in \mathbb{Q}_{>0}$ is the profit of one unit and $\bar{q}_j \in \mathbb{Z}_{\geq 1}$ the minimum quantity to be produced if that type is chosen

Problem 9.3: vehicles with minimum lot and bonus (2/2)

- If the production includes at least two different types, the company collects a bonus $\bar{r} \in \mathbb{Q}_{>0}$ (a grant for diversification)
- The company wants to **maximise the total profit**

The problem in words

- We **decide** how many units of each type to produce
- **The objective**: maximum profit, bonus included
- **The constraints**: the resources must not be exceeded; of a given type one produces either zero units or at least $\bar{q}_j$

Data (input of the model)

Symbol	Type	Meaning
s	$\in \mathbb{Z}_{\geq 1}$	number of types, $j \in \{1, 2, \ldots, s\}$
k	$\in \mathbb{Z}_{\geq 1}$	number of resources, $i \in \{1, 2, \ldots, k\}$
a_{ij}	$\in \mathbb{Q}_{\geq 0}$	resource i for one unit of type j
b_i	$\in \mathbb{Q}_{>0}$	availability of resource i
$\bar{p}_j$	$\in \mathbb{Q}_{>0}$	unit profit of type j
$\bar{q}_j$	$\in \mathbb{Z}_{\geq 1}$	minimum lot of type j
$\bar{r}$	$\in \mathbb{Q}_{>0}$	variety bonus
M_j	$\in \mathbb{Z}_{\geq 1}$	largest producible amount of type j alone

where $M_j = \min\{\lfloor b_i / a_{ij} \rfloor : a_{ij} > 0\}$.

Decision variables

s non-negative integer variables, s binaries and one binary:

$$x_j = \text{units produced of type } j, \qquad \forall j \in \{1, 2, \ldots, s\}$$

$$y_j = \begin{cases} 1 & \text{if type } j \text{ is produced} \\ 0 & \text{otherwise} \end{cases} \qquad z = \begin{cases} 1 & \text{if the bonus is collected} \\ 0 & \text{otherwise} \end{cases}$$

Model 9.3

Model 9.3 — vehicles with minimum lot and bonus

$$\max \quad \sum_{j=1}^{s} \bar{p}_j \, x_j + \bar{r} \, z \tag{9.6a}$$

$$\text{subject to} \quad \sum_{j=1}^{s} a_{ij} \, x_j \leq b_i, \qquad \forall i \in \{1, 2, \ldots, k\}, \tag{9.6b}$$

$$x_j - \bar{q}_j \, y_j \geq 0, \qquad \forall j \in \{1, 2, \ldots, s\}, \tag{9.6c}$$

$$x_j - M_j \, y_j \leq 0, \qquad \forall j \in \{1, 2, \ldots, s\}, \tag{9.6d}$$

$$-\sum_{j=1}^{s} y_j + 2 \, z \leq 0, \tag{9.6e}$$

$$x_j \in \mathbb{Z}_{\geq 0}, \qquad \forall j \in \{1, 2, \ldots, s\}, \tag{9.6f}$$

$$y_j \in \{0, 1\}, \qquad \forall j \in \{1, 2, \ldots, s\}, \tag{9.6g}$$

$$z \in \{0, 1\}. \tag{9.6h}$$

Description of the model

- The **objective** adds up the profits of the vehicles produced and the bonus for variety
- The **resource** constraints, one per resource, are the availabilities
- The **minimum lot** and **activation** constraints, one per type each, make x_j semi-integer: either zero, or at least $\bar{q}_j$ and at most M_j
- The **bonus** constraint, a single one, says that the bonus is collected only if at least two types are active

The link between the variables: semicontinuity

The minimum lot and activation constraints together say

$$\bar{q}_j \, y_j \;\leq\; x_j \;\leq\; M_j \, y_j$$

- If $y_j = 0$ both give $x_j = 0$
- If $y_j = 1$ one gets $\bar{q}_j \leq x_j \leq M_j$
- x_j is zero or between the minimum lot and the maximum: it is a **semi-integer** variable

Neither constraint alone is enough: without the activation one a type could be produced with $y_j = 0$; without the minimum lot one the minimum lot would be empty.

The bonus is collected only with at least two types

The bonus constraint reads $2z \leq \sum_{j=1}^{s} y_j$.

- If $z = 1$ then $\sum_j y_j \geq 2$: at least two types are activated and, by semicontinuity, actually produced (in quantity $\geq \bar{q}_j > 0$)
- The other direction is not imposed by the constraint, but follows from **optimality**: if $\sum_j y_j \geq 2$ and $z = 0$, setting $z = 1$ stays feasible and increases the objective by $\bar{r} > 0$
- Hence in every optimal solution z is a faithful indicator
- With $\bar{r} = 0$ this argument breaks down, and z stops being faithful

The model in gurobipy (1/2)

```
mm = gp.Model("vehicles")
x = mm.addVars(n, vtype=GRB.INTEGER, name="x")
y = mm.addVars(n, vtype=GRB.BINARY, name="y")
z = mm.addVar(vtype=GRB.BINARY, name="z")
mm.setObjective(
    gp.quicksum(p[j] * x[j] for j in range(n))
    + r * z, GRB.MAXIMIZE)
```

The model in gurobipy (2/2)

```
mm.addConstrs(
    (gp.quicksum(a[i][j] * x[j] for j in range(n))
     <= b[i] for i in range(m)), name="resource")
mm.addConstrs((x[j] - q[j] * y[j] >= 0
               for j in range(n)), name="minimum_lot")
mm.addConstrs((x[j] - M[j] * y[j] <= 0
               for j in range(n)), name="activate")
mm.addConstr(
    -gp.quicksum(y[j] for j in range(n)) + 2 * z <= 0,
    name="bonus")
```

The instance

$s = 3$ types, $k = 2$ resources (steel in tons and labour hours), $\bar{q}_j = 10$ for every type, $\bar{r} = 500$:

	$j = 1$	$j = 2$	$j = 3$	b_i
a_{1j} (steel)	2	3	5	100
a_{2j} (hours)	30	25	40	1200
$\bar{p}_j$	200	250	300	
M_j	40	33	20	

The big-Ms are computed from the data:
$M_1 = \min(\lfloor 100/2 \rfloor, \lfloor 1200/30 \rfloor) = 40$, and likewise $M_2 = 33$, $M_3 = 20$.

The model for the instance

The instance model

$$
\begin{array}{rrrrrrrrrrr}
\max & 200x_1 & +250x_2 & +300x_3 & & & & & +500z & & \\
\text{subject to} & 2x_1 & +3x_2 & +5x_3 & & & & & & \leq & 100 \\
& 30x_1 & +25x_2 & +40x_3 & & & & & & \leq & 1200 \\
& x_1 & & & -10y_1 & & & & & \geq & 0 \\
& & x_2 & & & -10y_2 & & & & \geq & 0 \\
& & & x_3 & & & -10y_3 & & & \geq & 0 \\
& x_1 & & & -40y_1 & & & & & \leq & 0 \\
& & x_2 & & & -33y_2 & & & & \leq & 0 \\
& & & x_3 & & & -20y_3 & & & \leq & 0 \\
& & & & -y_1 & -y_2 & -y_3 & & +2z & \leq & 0 \\
& x_1, & & x_2, & & x_3 & & & & \in & \mathbb{Z}_{\geq 0} \\
& & & & y_1, & & y_2, & & y_3 & \in & \{0,1\} \\
& & & & & & & & z & \in & \{0,1\}
\end{array}
$$

Constructive heuristic: the primal bound

The problem is a maximisation: the heuristic gives the **primal** bound, which sits below the optimum.

- The types are sorted by profit relative to the consumption of the tightest resource
- The two best ones are activated at their minimum lot (so the bonus is guaranteed)
- Then one fills up with the most profitable among those activated

On the instance types 2 and 1 are activated, then one fills up: the production is $(11, 26, 0)$, the steel runs out and 220 hours are left. The profit is 8700 plus the bonus of 500:

$$z(MILP) \geq LB = 9200$$

LP relaxation and dual: the dual variables

The primal is a maximisation with $\leq$ and $\geq$ constraints; following the conversion table we associate

- $\pi_i \geq 0$ with the resource constraints
- $\lambda_j \leq 0$ with the minimum lot constraints
- $\beta_j \geq 0$ with the activation constraints
- $\gamma \geq 0$ with the bonus constraint

Writing $\lambda_j = -\ell_j$ with $\ell_j \geq 0$, one gets the dual.

Dual of the LP relaxation of model 9.3

Dual of the LP relaxation of model 9.3

$$\min \quad \sum_{i=1}^{k} b_i \, \pi_i \tag{9.7a}$$

$$\text{subject to} \quad \sum_{i=1}^{k} a_{ij} \, \pi_i - \ell_j + \beta_j \geq \bar{p}_j, \quad \forall j \in \{1, 2, \ldots, s\}, \tag{9.7b}$$

$$\bar{q}_j \, \ell_j - M_j \, \beta_j - \gamma \geq 0, \quad \forall j \in \{1, 2, \ldots, s\}, \tag{9.7c}$$

$$2 \, \gamma \geq \bar{r}, \tag{9.7d}$$

$$\pi_i \geq 0, \quad \forall i \in \{1, 2, \ldots, k\}, \tag{9.7e}$$

$$\ell_j \geq 0, \quad \forall j \in \{1, 2, \ldots, s\}, \tag{9.7f}$$

$$\beta_j \geq 0, \quad \forall j \in \{1, 2, \ldots, s\}, \tag{9.7g}$$

$$\gamma \geq 0. \tag{9.7h}$$

Dual for the instance

The instance model

$$
\begin{array}{rrrrrrrrrrrr}
\min & 100\pi_1 & +1200\pi_2 & & & & & & & & & \\
\text{subject to} & 2\pi_1 & +30\pi_2 & +\alpha_1 & & & +\beta_1 & & & & \geq & 200 \\
& 3\pi_1 & +25\pi_2 & & +\alpha_2 & & & +\beta_2 & & & \geq & 250 \\
& 5\pi_1 & +40\pi_2 & & & +\alpha_3 & & & +\beta_3 & & \geq & 300 \\
& & & -10\alpha_1 & & & -40\beta_1 & & & -\gamma & \geq & 0 \\
& & & & -10\alpha_2 & & & -33\beta_2 & & -\gamma & \geq & 0 \\
& & & & & -10\alpha_3 & & & -20\beta_3 & -\gamma & \geq & 0 \\
& & & & & & & & & 2\gamma & \geq & 500 \\
& \pi_1, & & \pi_2 & & & & & & & \geq & 0 \\
& & & \alpha_1, & \alpha_2, & \alpha_3 & & & & & \leq & 0 \\
& & & & & & \beta_1, & \beta_2, & \beta_3 & & \geq & 0 \\
& & & & & & & & & \gamma & \geq & 0
\end{array}
$$

Description of the dual (1/2)

- π_i is the price of one unit of resource i; ℓ_j and β_j the prices of the two semicontinuity constraints of type j; γ the price of the bonus
- The **objective** prices all the available resources at those values
- **Columns of the** x_j: the resources one unit of type j consumes, priced at π_i and corrected by the two semicontinuity constraints, must cover the profit $\bar{p}_j$

Description of the dual (2/2)

- **Columns of the** y_j: switching on type j forces at least $\bar{q}_j$ units and allows at most M_j; the balance of the two prices must cover the bonus γ
- **Column of** z: the bonus $\bar{r}$ is collected only with two active types, and indeed it has to be covered by 2γ

A hand-built dual solution (1/2)

One proceeds in three steps.

1. The constraint $2\gamma \geq \bar{r}$ forces $\gamma \geq \bar{r}/2$: take the minimum, $\bar{\gamma} = \bar{r}/2 = 250$
2. Set $\bar{\beta}_j = 0$ and read from the constraint of the y_j the smallest feasible ℓ_j:

$$\bar{\ell}_j = \frac{\bar{\gamma}}{\bar{q}_j} = \frac{250}{10} = 25 \qquad \text{for every } j$$

Every activated type "carries" its share of the bonus

A hand-built dual solution (2/2)

3. What is left is $\sum_i a_{ij}\,\pi_i \geq \bar{p}_j + \bar{\ell}_j$. One prices *a single* resource, so that it covers all the types, and keeps the one giving the smaller bound:

$$\bar{\pi}_i = \max_j \frac{\bar{p}_j + \bar{\ell}_j}{a_{ij}}, \qquad \text{bound} = b_i\,\bar{\pi}_i$$

On the instance:

$$\bar{\pi}_1 = \max\left(\tfrac{225}{2}, \tfrac{275}{3}, \tfrac{325}{5}\right) = \tfrac{225}{2}, \quad b_1\,\bar{\pi}_1 = 11\,250$$

$$\bar{\pi}_2 = \max\left(\tfrac{225}{30}, \tfrac{275}{25}, \tfrac{325}{40}\right) = 11, \quad b_2\,\bar{\pi}_2 = 13\,200$$

The better bound is the one from the steel:
$z(MILP) \leq UB = 11\,250$.

Optimal solution

- The optimal production is $(26, 16, 0)$: types 1 and 2 are activated and the bonus is collected
- All 100 tons of steel are consumed and 1180 hours out of 1200

$$z(MILP) = 200 \cdot 26 + 250 \cdot 16 + 500 = 9700$$

The chain of bounds is

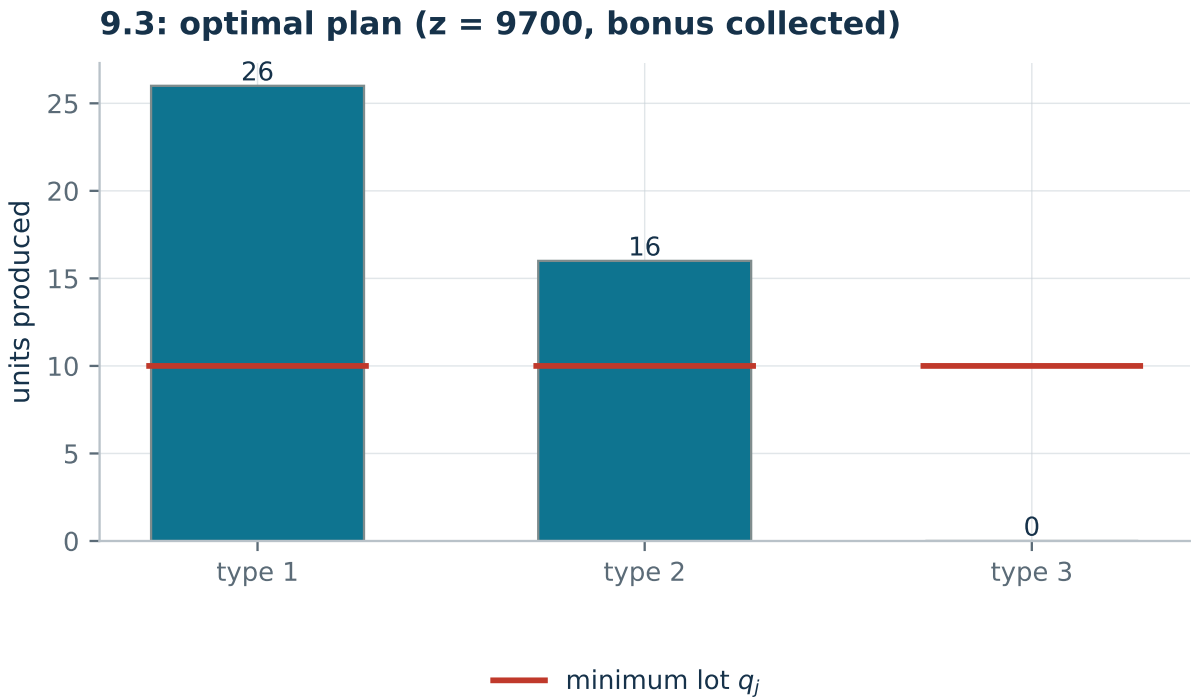
$$\begin{aligned} 9200 = LB \;\leq\; z(MILP) = 9700 \;&\leq\; z(LP^{+}) = 9750 \\ &\leq\; z(LP) = \tfrac{20625}{2} \;\leq\; UB = 11\,250 \end{aligned}$$

The heuristic gap is 5.2%.

Here the relaxation with the bounds beats the hand-built dual

- It is the only problem in the chapter in which $z(LP^+) = 9750$ is **better** than the hand-built dual bound (11 250), and only slightly worse than the optimum (9700)
- The relaxation without the bounds lets y_j and z grow above 1, and with them the bonus: $z(LP) = 20625/2 \approx 10\,312$
- Adding $y_j \leq 1$ and $z \leq 1$ removes exactly that freedom
- When a model contains rewarded indicators, the relaxation with the bounds is not a detail

Optimal solution: the plan



Units produced per type, with the minimum lot $\bar{q}_j$ marked in red. Type 3 stays at zero.

Additional considerations

- The big-M M_j is the largest producible amount of type j *alone*, not of the whole plan: already much tighter than an arbitrary constant ($M_3 = 20$: the steel alone would allow 20 units, the hours 30)
- If $\bar{q}_j > M_j$, type j is impossible to produce and is removed by a check on the data: a pre-processing step always worth doing
- The bonus is modelled with *one* binary and *one* constraint; with a threshold of "at least f types" one replaces 2 with f; with bonuses in brackets one needs one variable per bracket and a chain of constraints

Modelling question: zero bonus

The diversification grant is abolished, that is $\bar{r} = 0$.

What happens to the variable z?

The final comparison

	value	what it is
LB	9200	heuristic solution
UB	11250	certified bound
$z(MILP)$	9700	MILP optimum

- Gap of the heuristic: $\dfrac{|LB - z(MILP)|}{|z(MILP)|} = 5.2\%$
- Certified gap, known *before* solving the MILP:
  $UB - LB = 2050$, that is $\dfrac{UB - LB}{|LB|} = 22.3\%$

A worked variant: a more demanding bonus (1/2)

The bonus is collected only if at least **three** different types are produced. Only one coefficient changes:

$$-\sum_{j=1}^{s} y_j + 3\,z \leq 0$$

- The new optimum is 9200, that is 500 less: activating the third type would cost more than the bonus, and at the optimum $z = 0$
- The value coincides, by coincidence, with that of the heuristic solution of the original problem

A worked variant: a more demanding bonus (2/2)

- In the dual only the column of z changes: $\gamma = \bar{r}/3 = 500/3$, and the bound drops from 11250 to $32500/3$
- Three types instead of two spread the bonus over more activations: each type carries less of it
- The heuristic does not manage to switch on a third type: it stays at 8700

	value	what it is
UB	$\frac{32500}{3}$	dual certificate built by hand
LB	8700	heuristic solution
$z(LP)$ / $z(LP^+)$	$\frac{29875}{3}$ / $\frac{28750}{3}$	relaxation without / with the bounds
$z(MILP)$	9200	optimum of the MILP

Summary: Vehicles

- **Semi-integer** variable: $\bar{q}_j\, y_j \le x_j \le M_j\, y_j$, with M_j computed from the data
- Counting the types: $2z \le \sum_j y_j$; the missing direction is imposed by optimality because $\bar{r} > 0$
- The problem is a maximisation: the heuristic gives $LB = 9200$, the hand-built dual $UB = 11\,250$
- The relaxation with the bounds ($z(LP^+) = 9750$) beats the hand-built dual: with rewarded indicators the bounds $y_j \le 1$, $z \le 1$ matter

Final summary (1/2)

Modelling production:

- quantity variables with binaries that switch on (fixed cost), limit (minimum lot) or reward (variety)
- the inventory and workforce balances are written with equalities, whose duals are free

Big-M:

- it is read off the data (residual demand, largest producible amount): larger values weaken the relaxation

Final summary (2/2)

Formulations:

- two different models can describe the same plans (hirings against workforce); the equivalence on the relaxation must be checked

Bounds:

- constructive heuristics for the primal bound, hand-built dual solutions for the dual one
- a dual bound can be insensitive to the datum that moves the problem
- with rewarded indicators the relaxation with the bounds can beat the hand-built dual