Location and coverage

Outline

The family

Capacitated facility location

p-median: at most k locations

Signal coverage with interference

Hub location with maximum cost

Learning objectives (1/2)

At the end of this chapter you will be able to:

- model the opening of a facility and the variables that depend on it;
- choose between aggregated and disaggregated activation and justify the choice;

Learning objectives (2/2)

At the end of this chapter you will be able to:

- add a budget on the number of open locations;
- build the primal and dual bounds of a location problem by hand.

Prerequisites

- From this module, Chapter 6 «Links between variables»: aggregated and disaggregated activation, the maximum variable, the *if and only if*
- From this module, Chapter 7 «Assignment and scheduling»: problems 7.4 and 7.5, whose links are reused here

The family

Introduction: The family

- Four problems in which we decide **where** to open a facility: a location, a hub
- The opening decision constrains the variables that depend on it:
  - where to ship goods (problem 1)
  - which clients to serve (problem 2)
  - which clients to cover (problem 3)
  - which hub each terminal connects to (problem 4)
- What changes is the nature of the **link** between the opening variable and the variables that depend on it, and how the budget on openings enters the model

The links between variables (1/2)

- **Aggregated activation** (problem 1): a single family of constraints acts as both the link and the capacity constraint,

$$\sum_c y_{lc} \leq u_l x_l$$

- **Disaggregated activation** (problem 2): $x_l \geq y_{lc}$ for every location and client, derived from the CNF of a Boolean implication as in problem 7.5, here with a budget k on the number of open locations instead of a time budget

The links between variables (2/2)

- **If and only if** (problem 3): one direction imposed by two families of constraints (signal threshold and interference), the other following from the objective
- **Aggregated activation and a maximum variable** (problem 4): the same activation as problem 1 together with the maximum link

$$z_j \geq c_{ij} x_{ij}$$

already seen in problem 7.4

The scheme of every problem

For each of the four problems:

- model and proof of the links
- instance
- constructive heuristic: a bound from the feasible-solutions side
- dual of the LP relaxation with a hand-built dual solution: a bound from the other side
- solver solution
- additional considerations and modelling questions

Family notation (1/2)

Symbol	Type	Meaning
m	$\in \mathbb{Z}_{\geq 1}$	number of candidate locations/hubs, $l \in \{1, 2, \ldots, m\}$
n	$\in \mathbb{Z}_{\geq 1}$	number of clients/terminals, $c \in \{1, 2, \ldots, n\}$
t_{lc}	$\in \mathbb{Q}_{>0}$	transport/connection cost from location/hub l to client/terminal c
$i_l,\ f_l$	$\in \mathbb{Q}_{\geq 0}$	opening/activation cost of location/hub l
u_l	$\in \mathbb{Q}_{>0}$	capacity of location l

Family notation (2/2)

Symbol	Type	Meaning
d_c	$\in \mathbb{Q}_{>0}$	demand of client c
k	$\in \mathbb{Z}_{\geq 1}$	maximum number of open locations / capacity of each hub
s_{lc}	$\in \mathbb{Q}_{\geq 0}$	signal strength from location l to client c
p_c	$\in \mathbb{Q}_{>0}$	profit if client c is covered
$t,\ b$	$\in \mathbb{Q}_{>0}$	signal threshold and interference limit

Summary: The family

- In every problem an **opening** variable constrains other variables
- The link can be **aggregated** (one constraint per location) or **disaggregated** (one constraint per location–client pair)
- Some directions of the links are imposed by the constraints, others follow from **optimality**
- For every problem we build two bounds: a **primal** one (heuristic) and a **dual** one (LP relaxation)

Capacitated facility location

Introduction: Capacitated facility location

- The first problem of the family: where to install the facilities and how to serve the clients
- We will see an **aggregated activation** link: a single family of constraints acts as both the link and the capacity constraint
- We build a primal bound with a heuristic and a dual bound with a hand-built dual solution, and compare them with the solver optimum

Problem 8.1: capacitated facility location (1/2)

- A company must serve $n \in \mathbb{Z}_{\geq 1}$ clients with its product and has identified $m \in \mathbb{Z}_{\geq 1}$ candidate locations where a facility can be installed
- For each client c, the value $d_c \in \mathbb{Q}_{>0}$ is the demand, in liters of product, of that client
- For each location l and each client c, the value $t_{lc} \in \mathbb{Q}_{>0}$ is the cost, in euros, of transporting one liter of product from location l to client c

Problem 8.1: capacitated facility location (2/2)

- For each location l, the value $u_l \in \mathbb{Q}_{>0}$ is the capacity, in liters, of the facility installed there, and $i_l \in \mathbb{Q}_{>0}$ is the installation cost
- The company wants to decide where to install facilities and how to serve the clients, at **minimum total cost**

The problem in words

- **We decide** where to install facilities and how much product to transport from each location to each client
- **The objective**: minimum total cost (installation plus transport)
- **The constraints**: a location that is not installed ships nothing, and an installed location does not exceed its capacity; every client's demand must be satisfied exactly
- This is the **capacitated facility location** problem

Data (model input)

Symbol	Type	Meaning
m	$\in \mathbb{Z}_{\geq 1}$	number of candidate locations, $l \in \{1, 2, \ldots, m\}$
n	$\in \mathbb{Z}_{\geq 1}$	number of clients, $c \in \{1, 2, \ldots, n\}$
t_{lc}	$\in \mathbb{Q}_{>0}$	transport cost (per liter) from location l to client c
u_l	$\in \mathbb{Q}_{>0}$	capacity (liters) of location l
i_l	$\in \mathbb{Q}_{>0}$	installation cost of location l
d_c	$\in \mathbb{Q}_{>0}$	demand (liters) of client c

Decision variables

We introduce the following m binary variables:

$$x_l = \begin{cases} 1 & \text{if a facility is installed at location } l, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall l \in \{1, 2, \ldots, m\},$$

and the following $m\,n$ non-negative continuous variables:

$$y_{lc} = \text{liters transported from location } l \text{ to client } c$$

$$\forall l \in \{1, 2, \ldots, m\},\ \forall c \in \{1, 2, \ldots, n\}.$$

The model

Model 8.1 — capacitated facility location

$$\min \quad \sum_{l=1}^{m} i_l\, x_l + \sum_{l=1}^{m}\sum_{c=1}^{n} t_{lc}\, y_{lc} \tag{8.1a}$$

$$\text{subject to} \quad u_l\, x_l - \sum_{c=1}^{n} y_{lc} \geq 0, \qquad \forall l \in \{1, 2, \ldots, m\}, \tag{8.1b}$$

$$\sum_{l=1}^{m} y_{lc} = d_c, \qquad \forall c \in \{1, 2, \ldots, n\}, \tag{8.1c}$$

$$x_l \in \{0, 1\}, \qquad \forall l \in \{1, 2, \ldots, m\}, \tag{8.1d}$$

$$y_{lc} \geq 0, \qquad \forall l \in \{1, 2, \ldots, m\},\ \forall c \in \{1, 2, \ldots, n\}. \tag{8.1e}$$

Description of the objective function and constraints

- The objective function minimizes the total cost: installation costs of the open locations plus transport costs
- Constraints (8.1b) link the transport variables to the installation variables and impose the capacity: an uninstalled location ships nothing, and an installed location ships no more than u_l (m constraints)
- Constraints (8.1c) ensure that every client's demand is satisfied exactly (n constraints)
- The last constraints define the variables of the model

The link between the variables (1/2)

- If a positive quantity ships from location l, the location must be installed:

$$\sum_{c=1}^{n} y_{lc} > 0 \quad \Longrightarrow \quad x_l = 1$$

- Contrapositive: $x_l = 0 \Longrightarrow \sum_{c=1}^{n} y_{lc} = 0$
- Imposed by constraints (8.1b): if $\sum_c y_{lc} > 0$ then $u_l x_l > 0$, hence $x_l = 1$; if $x_l = 0$ then $\sum_c y_{lc} \leq 0$ and $y_{lc} = 0$ for every c
- With $x_l = 1$ the constraint becomes $\sum_c y_{lc} \leq u_l$: the capacity

The link between the variables (2/2)

- The opposite direction

$$x_l = 1 \quad \Longrightarrow \quad \sum_{c=1}^{n} y_{lc} > 0$$

  is not imposed by the constraints, but follows from the objective in every optimal solution

- Since $i_l > 0$, if $\sum_c y_{lc} = 0$ and $x_l = 1$, setting $x_l = 0$ remains feasible and decreases the cost by i_l
- In every optimal solution no installed location goes unused
- A single family of constraints acts as both the **activation link** and the **capacity constraint**

The model in gurobipy

```
mod = gp.Model("capacitated_location")
x = mod.addVars(m, vtype=GRB.BINARY, name="x")
y = mod.addVars(m, n, name="y")
mod.setObjective(gp.quicksum(i[l] * x[l] for l in
    range(m))
    + gp.quicksum(t[l][c] * y[l, c] for l in range(m)
                  for c in range(n)), GRB.MINIMIZE)
mod.addConstrs((u[l] * x[l] - gp.quicksum(y[l, c]
    for c in range(n)) >= 0 for l in range(m)), name="
        capacity")
mod.addConstrs((gp.quicksum(y[l, c] for l in range(m))
    == d[c] for c in range(n)), name="demand")
```

The instance

$m = 2$ locations, $n = 3$ clients:

t_{lc}	$c = 1$	$c = 2$	$c = 3$
$l = 1$	4	5	6
$l = 2$	6	4	3

	$l = 1$	$l = 2$
u_l	50	50
i_l	60	90

	$c = 1$	$c = 2$	$c = 3$
d_c	8	25	27

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The model for the instance

The instance model

$$
\begin{array}{lrrrrrrrrl}
\min & 60x_1 & +90x_2 & +4y_{11} & +5y_{12} & +6y_{13} & +6y_{21} & +4y_{22} & +3y_{23} & \\
\text{subject to} & 50x_1 & & -y_{11} & -y_{12} & -y_{13} & & & & \geq 0 \\
& & 50x_2 & & & & -y_{21} & -y_{22} & -y_{23} & \geq 0 \\
& & & y_{11} & & & +y_{21} & & & = 8 \\
& & & & y_{12} & & & +y_{22} & & = 25 \\
& & & & & y_{13} & & & +y_{23} & = 27 \\
& x_1, & x_2 & & & & & & & \in \{0,1\} \\
& & & y_{11}, & y_{12}, & y_{13}, & y_{21}, & y_{22}, & y_{23} & \geq 0
\end{array}
$$

Constructive heuristic: the primal bound

- Locations are scanned in natural order; for each, the clients are scanned
- Each client is shipped the minimum of the location's residual capacity and the client's residual demand
- A location that shipped something is opened
- It is a constructive heuristic: a single solution is built, one element at a time, never backtracking

Execution on the instance (1/2)

- Location 1 ships 8 to client 1 (demand exhausted, residual capacity 42)
- then 25 to client 2 (demand exhausted, residual capacity 17)
- then 17 to client 3 (location 1's capacity exhausted, 10 still owed)
- Location 2 ships the remaining 10 to client 3

The feasible solution found is

$$\bar{x}_1 = \bar{x}_2 = 1, \quad \bar{y}_{11} = 8,\ \bar{y}_{12} = 25,\ \bar{y}_{13} = 17,\ \bar{y}_{23} = 10,$$

all other $\bar{y}_{lc} = 0$.

Execution on the instance (2/2)

The value of the solution is

$$\underbrace{60 + 90}_{\text{installation}} + \underbrace{4 \cdot 8 + 5 \cdot 25 + 6 \cdot 17 + 3 \cdot 10}_{\text{transport} = 289} = 150 + 289 = 439$$

Hence

$$z(MILP) \leq UB = 439.$$

LP relaxation and dual

- We associate a non-negative dual variable μ_l with each constraint (1) (a $\geq$ direction)
- and a free dual variable π_c with each constraint (2) (an equality)

Dual of the LP relaxation of model 8.1

$$\max \quad \sum_{c=1}^{n} d_c \, \pi_c \tag{8.2a}$$

$$\text{subject to} \quad u_l \, \mu_l \leq i_l, \qquad \forall l \in \{1, 2, \ldots, m\}, \tag{8.2b}$$

$$-\mu_l + \pi_c \leq t_{lc}, \quad \forall l \in \{1, 2, \ldots, m\}, \ \forall c \in \{1, 2, \ldots, n\}, \tag{8.2c}$$

$$\mu_l \geq 0, \qquad \forall l \in \{1, 2, \ldots, m\}, \tag{8.2d}$$

$$\pi_c \gtreqless 0, \qquad \forall c \in \{1, 2, \ldots, n\}. \tag{8.2e}$$

The dual for the instance

The instance model

$$
\begin{array}{lrrrrrl}
\max & & 8\pi_1 & +25\pi_2 & +27\pi_3 & & \\
\text{subject to} & 50\mu_1 & & & & \leq & 60 \\
& 50\mu_2 & & & & \leq & 90 \\
& -\mu_1 & +\pi_1 & & & \leq & 4 \\
& -\mu_1 & & +\pi_2 & & \leq & 5 \\
& -\mu_1 & & & +\pi_3 & \leq & 6 \\
& -\mu_2 & +\pi_1 & & & \leq & 6 \\
& -\mu_2 & & +\pi_2 & & \leq & 4 \\
& -\mu_2 & & & +\pi_3 & \leq & 3 \\
& \mu_1, \quad \mu_2 & & & & \geq & 0 \\
& & \pi_1, & \pi_2, & \pi_3 & \gtreqless & 0
\end{array}
$$

Description of the dual

- μ_l is the price of one unit of capacity of location l, π_c the value of one unit of demand of client c
- The objective prices all the demand to be met at those values
- Constraints (8.2b) are the columns of the x_l: the capacity u_l made available, priced at μ_l, cannot be worth more than the installation cost i_l
- Constraints (8.2c) are the columns of the y_{lc}: carrying one unit from l to c is worth π_c and consumes capacity at price μ_l; the balance cannot exceed t_{lc}

A hand-built dual solution (1/2)

- Set $\bar{\mu}_l = i_l / u_l$: constraint (8.2b) holds with equality, spreading the installation cost over the capacity

$$\bar{\mu}_1 = 60/50 = 6/5, \qquad \bar{\mu}_2 = 90/50 = 9/5$$

- Constraint (8.2c) becomes $\pi_c \leq t_{lc} + \bar{\mu}_l$ for every l: the largest feasible value is $\bar{\pi}_c = \min_l(t_{lc} + \bar{\mu}_l)$

$$\bar{\pi}_1 = \min(4 + 6/5,\ 6 + 9/5) = 26/5$$
$$\bar{\pi}_2 = \min(5 + 6/5,\ 4 + 9/5) = 29/5$$
$$\bar{\pi}_3 = \min(6 + 6/5,\ 3 + 9/5) = 24/5$$

A hand-built dual solution (2/2)

- The value of this feasible dual solution is

$$8 \cdot \tfrac{26}{5} + 25 \cdot \tfrac{29}{5} + 27 \cdot \tfrac{24}{5} = \tfrac{208+725+648}{5} = \tfrac{1581}{5}$$

- By weak duality

$$LB = 1581/5 \leq z(LP) \leq z(MILP) \leq UB = 439$$

- Managerial meaning: spreading each location's fixed cost over its capacity and then serving every client at the minimum marginal cost is a lower bound anyone would write down; the dual formalizes it

What the solver says (1/2)

- The optimum of the LP relaxation is exactly $z(LP) = 1581/5$: the hand-built dual solution is already optimal (strong duality)
- Strengthening the relaxation with $x_l \leq 1$ gives $z(LP^+) = 317$: a sizeable jump, the constraint $x_l \leq 1$ matters here
- The MILP has optimum $z(MILP) = 365$, with $\tilde{x}_1 = \tilde{x}_2 = 1$, $\tilde{y}_{11} = 8$, $\tilde{y}_{12} = 2$, $\tilde{y}_{22} = 23$, $\tilde{y}_{23} = 27$
- Both locations open, but location 1 serves only part of client 2 (the rest comes from location 2, cheaper for that client)

What the solver says (2/2)

UB (heuristic)	LB (hand dual)	$z(LP)$	$z(LP^+)$	$z(MILP)$	gap heuristic
439	1581/5	1581/5	317	365	20.3%

- The heuristic gap relative to the optimum is 20.3%

Optimal solution



For each open location, the liters shipped to each client.

Additional considerations

- If $u_l < d_c$, no single location can satisfy client c's entire demand alone: not the case in this instance, but worth checking before assuming one location suffices
- The constraints $y_{lc} \leq d_c \, x_l$ are valid inequalities implied jointly by constraints (1) and (2):
  - if $x_l = 0$ they force $y_{lc} = 0$
  - if $x_l = 1$ they are never tighter than $y_{lc} \leq d_c$

Question: minimum lot for every open location

- Every open location must ship at least 5 liters overall (it makes no sense to bear the installation cost for a negligible quantity)
- How does the model change?
- What is the new optimum for the instance?

The final comparison (1/2)

The three numbers that every problem of the course produces:

	value	what it is
UB	439	heuristic solution
LB	$\frac{1581}{5}$	certified bound
$z(MILP)$	365	MILP optimum

The final comparison (2/2)

- Gap of the heuristic with respect to the optimum:

$$\frac{|UB - z(MILP)|}{|z(MILP)|} = 20.3\%$$

- Certified gap: the one known *before* solving the MILP, which therefore cannot have it as its denominator
- In absolute terms $UB - LB = \frac{614}{5}$; as a percentage, against the heuristic solution,

$$\frac{UB - LB}{|UB|} = 28.0\%$$

A worked variant: conditional opening (1/2)

- Location 2 can only be installed if location 1 is also installed (say, a supervision requirement)
- It is a link between two variables of the same family, imposed by the single linear constraint

$$x_2 \leq x_1$$

  if $x_2 = 1$ then necessarily $x_1 = 1$

- On the instance the optimum already opens both locations: the constraint is not binding and the optimum stays 365

A worked variant: conditional opening (2/2)

- The constraint adds $\rho \leq 0$ to the dual, which moves fixed cost between the two locations:

$$\mu_1 = (i_1 + \rho)/u_1, \qquad \mu_2 = (i_2 - \rho)/u_2$$

- One tries ρ among the values that change the minimum defining π_c; the best is $\rho = 0$, because location 1 already has the lowest cost per liter

	value	what it is
UB	439	heuristic solution
LB	$\frac{1581}{5}$	dual certificate built by hand
$z(LP)$	325	relaxation without the bounds
$z(LP^+)$	325	relaxation with the bounds
$z(MILP)$	365	optimum of the MILP

Summary: Capacitated facility location

- Binary installation variables x_l and continuous transport variables y_{lc}
- $u_l x_l \geq \sum_c y_{lc}$: a single constraint acts as **aggregated activation** and as **capacity**
- The direction "location open $\Rightarrow$ used" follows from optimality, because $i_l > 0$
- Bounds: $LB = 1581/5$ (hand dual, already LP-optimal), $UB = 439$ (heuristic), $z(MILP) = 365$

p-median: at most k locations

Introduction: p-median

- At most k locations are chosen and every client is assigned to an open location
- We will see a **disaggregated activation** link: one constraint $x_l \geq y_{lc}$ for every location–client pair, derived from the CNF of an implication
- Here there is no opening cost, but a **budget** on the number of locations

Problem 8.2: p-median

- A company must choose at most $k \in \mathbb{Z}_{\geq 1}$ locations, among $m \in \mathbb{Z}_{\geq 1}$ candidates, where to open a service point, and assign each of the $n \in \mathbb{Z}_{\geq 1}$ clients to the most convenient open location
- For each location l and client c, $d_{lc} \in \mathbb{Q}_{>0}$ is the distance (or service cost) between l and c
- Every client must be assigned to exactly one open location
- Objective: minimize the sum of the distances between each client and the location it is assigned to

The problem in words

- **We decide** which locations to open (at most k) and which open location to assign each client to
- **The objective**: minimum sum of client–location distances
- **The constraints**: every client assigned to exactly one location; a client can only be assigned to an open location; at most k open locations
- This is the classic p-**median** problem

Data (model input)

Symbol	Type	Meaning
m	$\in \mathbb{Z}_{\geq 1}$	number of candidate locations, $l \in \{1, 2, \ldots, m\}$
n	$\in \mathbb{Z}_{\geq 1}$	number of clients, $c \in \{1, 2, \ldots, n\}$
d_{lc}	$\in \mathbb{Q}_{>0}$	distance between location l and client c
k	$\in \mathbb{Z}_{\geq 1}$	maximum number of open locations

Decision variables

We introduce the following m binary variables:

$$x_l = \begin{cases} 1 & \text{if location } l \text{ is open,} \\ 0 & \text{otherwise,} \end{cases} \qquad \forall l \in \{1, 2, \ldots, m\},$$

and the following $m\,n$ binary variables:

$$y_{lc} = \begin{cases} 1 & \text{if client } c \text{ is served by location } l, \\ 0 & \text{otherwise,} \end{cases}$$

$$\forall l \in \{1, 2, \ldots, m\},\ \forall c \in \{1, 2, \ldots, n\}.$$

The model

Model 8.2 — p-median

$$\min \quad \sum_{l=1}^{m} \sum_{c=1}^{n} d_{lc}\, y_{lc} \tag{8.3a}$$

$$\text{subject to} \quad \sum_{l=1}^{m} y_{lc} = 1, \qquad \forall c \in \{1, 2, \ldots, n\}, \tag{8.3b}$$

$$\sum_{l=1}^{m} x_l \leq k, \tag{8.3c}$$

$$x_l - y_{lc} \geq 0, \qquad \forall l \in \{1, 2, \ldots, m\}, \ \forall c \in \{1, 2, \ldots, n\}, \tag{8.3d}$$

$$x_l \in \{0, 1\}, \qquad \forall l \in \{1, 2, \ldots, m\}, \tag{8.3e}$$

$$y_{lc} \in \{0, 1\}, \quad \forall l \in \{1, 2, \ldots, m\}, \ \forall c \in \{1, 2, \ldots, n\}. \tag{8.3f}$$

Description of the objective function and constraints

- The objective function minimizes the sum of the distances between every client and the location it is assigned to
- Constraints (8.3b) ensure that every client is assigned to exactly one location (n constraints)
- Constraint (8.3c) limits to k the number of open locations (one constraint)
- Constraints (8.3d) link the assignment variables to the opening variables, in disaggregated form ($m\,n$ constraints)
- The last constraints define the variables of the model

The link between the variables (1/2)

- If client c is served by location l, then location l must be open:

$$y_{lc} = 1 \quad \Longrightarrow \quad x_l = 1$$

- In Boolean form, by its CNF,

$$y_{lc} \Longrightarrow x_l \qquad \equiv \qquad \texttt{NOT}\ y_{lc} \quad \texttt{OR} \quad x_l$$

  i.e. $x_l \geq y_{lc}$: exactly constraint (8.3d), imposed directly (the objective is not needed)
- Contrapositive: $x_l = 0 \Longrightarrow y_{lc} = 0$, imposed by the same constraint

The link between the variables (2/2)

- Unlike problem 8.1, here there is no opening cost that would discourage open-but-unused locations
- Constraint (8.3c) only caps the number: the direction "$x_l = 1 \Rightarrow$ at least one client served by l" is neither imposed nor guaranteed by optimality
- The link is **disaggregated**, one row per pair (l, c) as in problem 7.5, because it is binary against binary for each pair, not a total against a single activation

The model in gurobipy

```
mod = gp.Model("p_median")
x = mod.addVars(m, vtype=GRB.BINARY, name="x")
y = mod.addVars(m, n, vtype=GRB.BINARY, name="y")
mod.setObjective(gp.quicksum(dist[l][c] * y[l, c]
    for l in range(m) for c in range(n)), GRB.MINIMIZE
        )
mod.addConstrs((y.sum("*", c) == 1 for c in range(n)),
    name="assign")
mod.addConstr(x.sum() <= k, name="number_of_locations"
    )
mod.addConstrs((x[l] - y[l, c] >= 0 for l in range(m)
    for c in range(n)), name="link")
```

The instance

$m = 3$ locations, $n = 3$ clients, $k = 2$:

d_{lc}	$c = 1$	$c = 2$	$c = 3$
$l = 1$	5	6	10
$l = 2$	3	12	9
$l = 3$	10	9	4

The model for the instance

The instance model

$$
\begin{array}{lrrr

Constructive heuristic: the primal bound

- The first k locations are opened in natural order; then every client is served by the nearest open location
- On the instance, locations 1 and 2 are opened ($x_1 = x_2 = 1$)
- Client 1: the nearest open location is 2 (distance 3)
- Client 2: the nearest open location is 1 (distance 6)
- Client 3: the nearest open location is 2 (distance 9)

$$\bar{y}_{21} = \bar{y}_{12} = \bar{y}_{23} = 1, \quad \text{all other } \bar{y}_{lc} = 0$$

of value $3 + 6 + 9 = 18$. Hence $z(MILP) \leq UB = 18$.

LP relaxation and dual

A free dual variable μ_c for constraints (1), a non-positive ϱ for constraint (2), a non-negative π_{lc} for constraints (3):

Dual of the LP relaxation of model 8.2

$$\max \quad \sum_{c=1}^{n} \mu_c + k\,\varrho \tag{8.4a}$$

$$\text{subject to} \quad \varrho + \sum_{c=1}^{n} \pi_{lc} \leq 0, \qquad \forall l \in \{1,2,\ldots,m\}, \tag{8.4b}$$

$$\mu_c - \pi_{lc} \leq d_{lc}, \quad \forall l \in \{1,2,\ldots,m\}, \ \forall c \in \{1,2,\ldots,n\}, \tag{8.4c}$$

$$\mu_c \gtreqless 0, \qquad \forall c \in \{1,2,\ldots,n\}, \tag{8.4d}$$

$$\varrho \leq 0, \tag{8.4e}$$

$$\pi_{lc} \geq 0, \quad \forall l \in \{1,2,\ldots,m\}, \ \forall c \in \{1,2,\ldots,n\}. \tag{8.4f}$$

The dual for the instance

The instance model

$$
\begin{aligned}
\max\ & \mu_1+\mu_2+\mu_3+2\varrho \\
\text{subject to}\ & \varrho+\pi_{11}+\pi_{12}+\pi_{13} \le 0 \\
& \varrho +\pi_{21}+\pi_{22}+\pi_{23} \le 0 \\
& \varrho +\pi_{31}+\pi_{32}+\pi_{33} \le 0 \\
& \mu_1 -\pi_{11} \le 5 \\
& \mu_2 -\pi_{12} \le 6 \\
& \mu_3 -\pi_{13} \le 10 \\
& \mu_1 -\pi_{21} \le 3 \\
& \mu_2 -\pi_{22} \le 12 \\
& \mu_3 -\pi_{23} \le 9 \\
& \mu_1 -\pi_{31} \le 10 \\
& \mu_2 -\pi_{32} \le 9 \\
& \mu_3 -\pi_{33} \le 4 \\
& \mu_1,\ \mu_2,\ \mu_3 \ge 0 \\
& \varrho \le 0 \\
& \pi_{11},\ \pi_{12},\ \pi_{13},\ \pi_{21},\ \pi_{22},\ \pi_{23},\ \pi_{31},\ \pi_{32},\ \pi_{33} \ge 0
\end{aligned}
$$

Description of the dual

- μ_c is the value of the assignment of client c, ϱ the (non-positive) price of the budget of k locations, π_{lc} the price of the link "to serve c from l one must open l"
- Constraints (8.4b) are the columns of the x_l: opening l consumes budget and releases the links; the balance cannot be positive, because opening a location costs nothing in the objective
- Constraints (8.4c) are the columns of the y_{lc}: serving c from l is worth μ_c and consumes the link π_{lc}; the balance cannot exceed the distance d_{lc}

A hand-built dual solution

- With $\bar{\varrho} = 0$ and $\bar{\pi}_{lc} = 0$, constraint (8.4b) holds and constraint (8.4c) becomes $\mu_c \leq d_{lc}$ for every l
- The largest feasible value is $\bar{\mu}_c = \min_l d_{lc}$, the distance to the nearest location in absolute terms (open or not):

$$\bar{\mu}_1 = 3, \qquad \bar{\mu}_2 = 6, \qquad \bar{\mu}_3 = 4$$

- Value $3 + 6 + 4 + 2 \cdot 0 = 13$; by weak duality

$$LB = 13 \leq z(LP) \leq z(MILP) \leq UB = 18$$

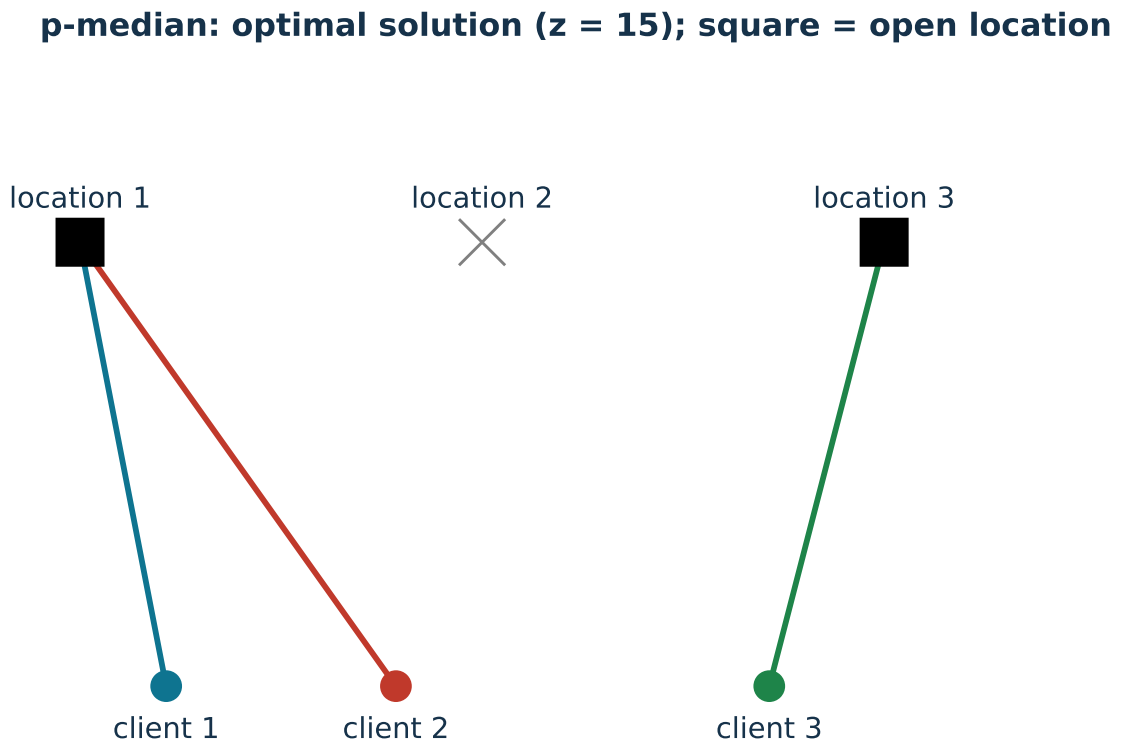
- "No client can cost less than its nearest location" is a trivial bound that ignores the budget k

What the solver says

- $z(LP) = 15 = z(LP^+)$: the constraint $y_{lc} \leq 1$ is never relevant here
- The MILP has optimum $z(MILP) = 15$, with $\tilde{x}_1 = \tilde{x}_3 = 1$ (locations 1 and 3, not 1 and 2) and $\tilde{y}_{11} = \tilde{y}_{12} = \tilde{y}_{33} = 1$
- The LP relaxation is already integral on this instance; the heuristic, which opens the first k locations regardless of cost, loses 20.0%

UB (heuristic)	LB (hand dual)	$z(LP)$	$z(LP^+)$	$z(MILP)$	gap heuristic
18	13	15	15	15	20.0%

Optimal solution



Square = open location, cross = closed location, lines = client–location assignment.

Additional considerations

- Constraint (2) is "at most k", not "exactly k": opening fewer than k locations is never advantageous, because one more location cannot worsen the assignment (no client is forced to use it)
- The model, however, does not impose equality; the worked variant below checks that on the instance the optimum does not change
- Constraints (3), summed over c for a fixed l, give

$$n\, x_l \geq \sum_{c=1}^{n} y_{lc}$$

an aggregated valid inequality, weaker than the disaggregated one

Question: proximity coverage for one client

- Client 1 must be served within distance 4 (a service-level requirement)
- How is this modelled?
- What is the new optimum?

The final comparison (1/2)

The three numbers that every problem of the course produces:

	value	what it is
UB	18	heuristic solution
LB	13	certified bound
$z(MILP)$	15	MILP optimum

The final comparison (2/2)

- Gap of the heuristic with respect to the optimum:

$$\frac{|UB - z(MILP)|}{|z(MILP)|} = 20.0\%$$

- Certified gap, known *before* solving the MILP: in absolute terms $UB - LB = 5$; as a percentage, against the heuristic solution,

$$\frac{UB - LB}{|UB|} = 27.8\%$$

A worked variant: exactly k open locations (1/2)

- For organisational reasons, exactly k locations must be open
- It suffices to add the linear constraint

$$\sum_{l=1}^{m} x_l \geq k$$

  which together with constraint (2) imposes equality
- On the instance the optimum already opens exactly $2 = k$ locations: the constraint is not binding and the optimum stays 15

A worked variant: exactly k open locations (2/2)

- The dual gains $\sigma \geq 0$ with right-hand side k; the column of the x_l imposes $\varrho + \sigma \leq 0$, and the objective contains $k(\varrho + \sigma) \leq 0$
- The maximum is at $\varrho + \sigma = 0$: the relaxation does not move, and neither does the integer optimum
- With no opening cost the equality constraint is redundant; it bites only when opening costs something

	value	what it is
UB	18	heuristic solution
LB	13	dual certificate built by hand
$z(LP)$, $z(LP^+)$	15	relaxations without/with the bounds
$z(MILP)$	15	optimum of the MILP

Summary: p-median

- Binary opening variables x_l and assignment variables y_{lc}, budget $\sum_l x_l \leq k$
- **Disaggregated** link $x_l \geq y_{lc}$, from the CNF of $y_{lc} \Rightarrow x_l$
- With no opening cost, "at most k" and "exactly k" give the same optimum
- Bounds: $LB = 13$, $UB = 18$, $z(MILP) = z(LP) = 15$ (relaxation already integral)

Signal coverage with interference

Introduction: Signal coverage with interference

- Antennas are installed to cover clients: a client is covered if it receives enough signal and not too much interference
- We will see an **if and only if** link: one direction imposed by two families of constraints, the other following from the objective
- It is a **maximisation** problem: the heuristic gives the lower bound, the dual the upper bound

Problem 8.3: coverage with interference (1/2)

- An operator must choose at most $k \in \mathbb{Z}_{\geq 1}$ locations, among $m \in \mathbb{Z}_{\geq 1}$ candidates, where to install an antenna, to serve $n \in \mathbb{Z}_{\geq 1}$ clients
- $s_{lc} \in \mathbb{Q}_{\geq 0}$ is the signal strength received by client c if location l is installed
- For each client c, $p_c \in \mathbb{Q}_{>0}$ is the profit if the client is covered
- The operator wants to decide which locations to install to maximize the total profit

Problem 8.3: coverage with interference (2/2)

A client c is **covered** if and only if:

- the total signal strength received from all installed locations is at least the threshold $t \in \mathbb{Q}_{>0}$

and

- at most one installed location generates for that client a signal greater than or equal to the interference limit $b \in \mathbb{Q}_{>0}$

The problem in words

- **We decide** which locations to install (at most k)
- **The objective**: maximum total profit, the sum of the profits of the covered clients
- **The constraints**: a client is covered if and only if it receives enough signal and not too much interference; the number of installed locations does not exceed k

Data (model input)

Symbol	Type	Meaning
m	$\in \mathbb{Z}_{\geq 1}$	number of candidate locations, $l \in \{1, 2, \ldots, m\}$
n	$\in \mathbb{Z}_{\geq 1}$	number of clients, $c \in \{1, 2, \ldots, n\}$
s_{lc}	$\in \mathbb{Q}_{\geq 0}$	signal received by client c if l is installed
p_c	$\in \mathbb{Q}_{>0}$	profit if client c is covered
t	$\in \mathbb{Q}_{>0}$	total signal threshold
b	$\in \mathbb{Q}_{>0}$	interference limit
k	$\in \mathbb{Z}_{\geq 1}$	maximum number of installed locations

For every client c: $\mathcal{L}_c = \{l : s_{lc} \geq b\}$, the "strong" locations.

Decision variables

We introduce the following m binary variables:

$$x_l = \begin{cases} 1 & \text{if location } l \text{ is installed,} \\ 0 & \text{otherwise,} \end{cases} \qquad \forall l \in \{1, 2, \ldots, m\},$$

and the following n binary variables:

$$y_c = \begin{cases} 1 & \text{if client } c \text{ is covered,} \\ 0 & \text{otherwise,} \end{cases} \qquad \forall c \in \{1, 2, \ldots, n\}.$$

The model

Model 8.3 — coverage with interference

$$\max \quad \sum_{c=1}^{n} p_c \, y_c \tag{8.5a}$$

$$\text{subject to} \quad -\sum_{l=1}^{m} s_{lc} \, x_l + t \, y_c \leq 0, \qquad \forall c \in \{1, 2, \ldots, n\}, \tag{8.5b}$$

$$\sum_{l \in \mathcal{L}_c} x_l + (m-1) \, y_c \leq m, \qquad \forall c \in \{1, 2, \ldots, n\}, \tag{8.5c}$$

$$\sum_{l=1}^{m} x_l \leq k, \tag{8.5d}$$

$$x_l \in \{0, 1\}, \quad \forall l \in \{1, 2, \ldots, m\}, \tag{8.5e}$$

$$y_c \in \{0, 1\}, \quad \forall c \in \{1, 2, \ldots, n\}. \tag{8.5f}$$

Description of the objective function and constraints

- The objective function maximizes the total profit of the covered clients
- Constraints (1) link coverage to the received signal: if a client is covered, the total signal received is at least t (n constraints)
- Constraints (2) link coverage to interference: if a client is covered, at most one location in $\mathcal{L}_c$ is installed (n constraints)
- Constraint (3) limits to k the number of installed locations
- The last constraints define the variables of the model

The link between the variables: an "if and only if" (1/2)

The client is covered if and only if the threshold and interference conditions hold:

$$y_c = 1 \iff \left(\sum_{l=1}^{m} s_{lc}\, x_l \geq t \right) \quad \texttt{AND} \quad \left(\sum_{l \in \mathcal{L}_c} x_l \leq 1 \right)$$

One direction, imposed by the constraints. If $y_c = 1$:

- constraint (1) forces $\sum_l s_{lc} x_l \geq t$: the threshold is met
- constraint (2) forces $\sum_{l \in \mathcal{L}_c} x_l \leq m - (m-1) = 1$: at most one strong location

The link between the variables: an "if and only if" (2/2)

The other direction, from optimality.

- If both conditions are satisfied, constraints (1) and (2) still allow both $y_c = 0$ and $y_c = 1$: nothing forces $y_c = 1$
- But $p_c > 0$ and y_c appears only in these two constraints: raising y_c from 0 to 1 remains feasible and raises the objective by p_c
- Hence in every optimal solution $y_c = 1$ whenever both conditions are satisfied
- The same pattern as problem 7.6: one direction from the constraints, the other from optimality in a maximisation problem

The model in gurobipy

```
mod = gp.Model("coverage_interference")
x = mod.addVars(m, vtype=GRB.BINARY, name="x")
y = mod.addVars(n, vtype=GRB.BINARY, name="y")
mod.setObjective(gp.quicksum(p[c] * y[c]
    for c in range(n)), GRB.MAXIMIZE)
mod.addConstrs((-gp.quicksum(s[l][c] * x[l] for l in
    range(m))
    + t * y[c] <= 0 for c in range(n)), name="
        threshold")
mod.addConstrs((gp.quicksum(x[l] for l in L[c])
    + (m - 1) * y[c] <= m for c in range(n)), name="
        interference")
mod.addConstr(x.sum() <= k, name="budget")
```

The instance

$m = 3$ locations, $n = 5$ clients, $t = 5$, $b = 4$, $k = 2$:

s_{lc}	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$
$l = 1$	6	0	5	3	1
$l = 2$	4	5	2	0	0
$l = 3$	0	7	5	4	2
p_c	10	20	5	15	25

With $b = 4$: $\mathcal{L}_1 = \{1, 2\}$, $\mathcal{L}_2 = \{2, 3\}$, $\mathcal{L}_3 = \{1, 3\}$, $\mathcal{L}_4 = \{3\}$, $\mathcal{L}_5 = \emptyset$.

The model for the instance

The instance model

$$
\begin{array}{rrrrrrrrrrrrrrr}
\max & & & & & 10y_1 & +20y_2 & +5y_3 & +15y_4 & +25y_5 & & \\
\text{subject to} & -6x_1 & -4x_2 & & & +5y_1 & & & & & \leq & 0 \\
& & -5x_2 & -7x_3 & & & +5y_2 & & & & \leq & 0 \\
& -5x_1 & -2x_2 & -5x_3 & & & & +5y_3 & & & \leq & 0 \\
& -3x_1 & & -4x_3 & & & & & +5y_4 & & \leq & 0 \\
& -x_1 & & -2x_3 & & & & & & +5y_5 & \leq & 0 \\
& x_1 & +x_2 & & & +2y_1 & & & & & \leq & 3 \\
& & x_2 & +x_3 & & & +2y_2 & & & & \leq & 3 \\
& x_1 & & +x_3 & & & & +2y_3 & & & \leq & 3 \\
& & & x_3 & & & & & +2y_4 & & \leq & 3 \\
& & & & & & & & & 2y_5 & \leq & 3 \\
& x_1 & +x_2 & +x_3 & & & & & & & \leq & 2 \\
& x_1, & x_2, & x_3 & & & & & & & \in & \{0,1\} \\
& & & & & y_1, & y_2, & y_3, & y_4, & y_5 & \in & \{0,1\}
\end{array}
$$

Constructive heuristic: the primal bound (1/2)

- The first k locations are opened in natural order
- A client is covered if the total signal of the open locations reaches the threshold t and at most one location of $\mathcal{L}_c$ is open
- On the instance, locations 1 and 2 are opened
- Client 1: signal $6 + 4 = 10 \geq 5$, but both strong locations of $\mathcal{L}_1 = \{1, 2\}$ are open: **not covered**
- Client 2: signal $0 + 5 = 5 \geq 5$, only strong location 2 is open: **covered**

Constructive heuristic: the primal bound (2/2)

- Client 3: signal $5 + 2 = 7 \geq 5$, only strong location 1 is open: **covered**
- Client 4: signal $3 + 0 = 3 < 5$: **not covered**
- Client 5: signal $1 + 0 = 1 < 5$: **not covered**

- The solution found covers clients 2 and 3, of value $20 + 5 = 25$

Hence

$$z(MILP) \geq LB = 25.$$

LP relaxation and dual

Non-negative dual variables: π_c for constraints (1), λ_c for constraints (2), μ for constraint (3):

Dual of the LP relaxation of model 8.3

$$\min \quad m \sum_{c=1}^{n} \lambda_c + k\,\mu \tag{8.6a}$$

$$\text{subject to} \quad -\sum_{c=1}^{n} s_{lc}\,\pi_c + \sum_{c\,:\,l \in \mathcal{L}_c} \lambda_c + \mu \geq 0, \quad \forall l \in \{1, 2, \ldots, m\}, \tag{8.6b}$$

$$t\,\pi_c + (m-1)\,\lambda_c \geq p_c, \quad \forall c \in \{1, 2, \ldots, n\}, \tag{8.6c}$$

$$\pi_c \geq 0, \quad \forall c \in \{1, 2, \ldots, n\}, \tag{8.6d}$$

$$\lambda_c \geq 0, \quad \forall c \in \{1, 2, \ldots, n\}, \tag{8.6e}$$

$$\mu \geq 0. \tag{8.6f}$$

The dual for the instance

The instance model

$$
\begin{array}{rrrr

Description of the dual

- π_c is the price of the threshold constraint of client c, λ_c that of its interference constraint, μ the price of one unit of budget
- Constraints (8.6b) are the columns of the x_l: antenna l brings signal, enters the interference constraints and consumes budget; the balance stays non-negative, because the x_l do not appear in the objective
- Constraints (8.6c) are the columns of the y_c: serving c requires t units of signal at price π_c and $m-1$ units of interference at price λ_c, which together must cover the reward p_c

A hand-built dual solution

- With $\bar{\pi}_c = 0$ and $\bar{\mu} = 0$, constraint (8.6b) holds and constraint (8.6c) becomes $(m-1)\bar{\lambda}_c \geq p_c$, i.e. $\bar{\lambda}_c = p_c/2$:

$$\bar{\lambda}_1 = 5, \quad \bar{\lambda}_2 = 10, \quad \bar{\lambda}_3 = 5/2, \quad \bar{\lambda}_4 = 15/2, \quad \bar{\lambda}_5 = 25/2$$

- Value: $3 \cdot (5 + 10 + 5/2 + 15/2 + 25/2) = 3 \cdot 75/2 = 225/2$
- By weak duality

$$LB = 25 \leq z(MILP) \leq z(LP) \leq UB = 225/2$$

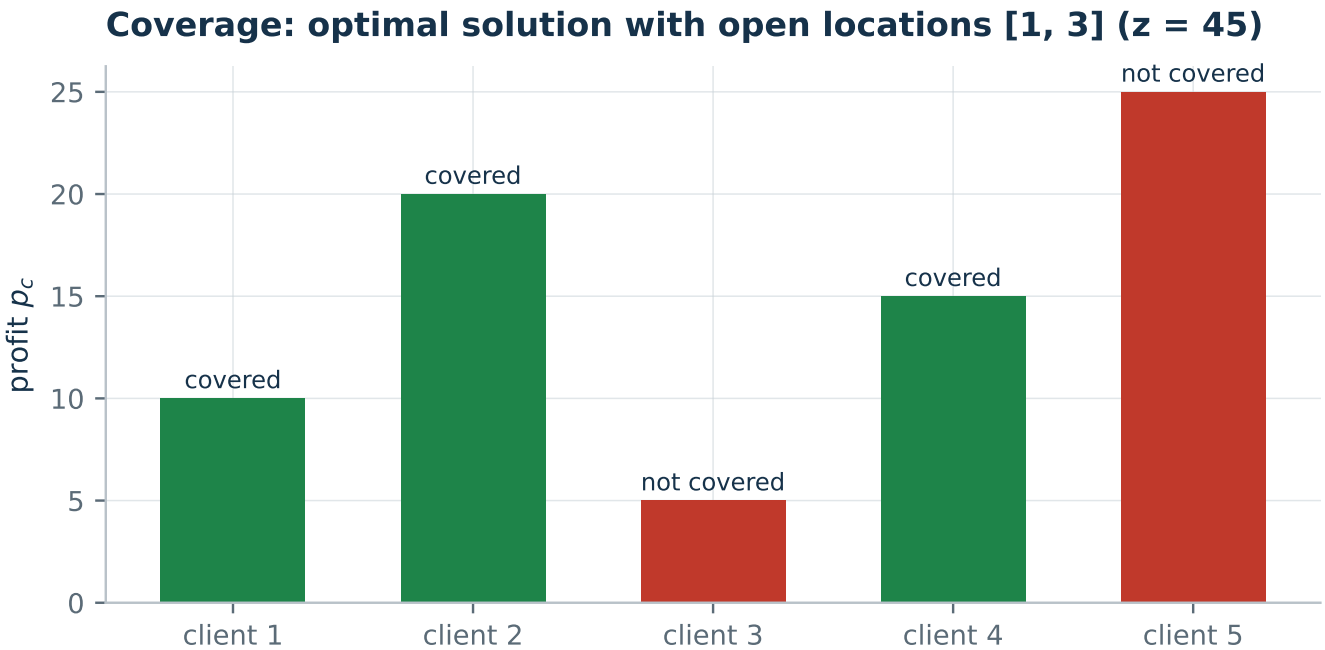
- A maximisation problem: the heuristic gives the lower bound, the dual the upper bound

What the solver says

- $z(LP) = 41925/646 \approx 64.9$; with $x_l \leq 1$ one gets $z(LP^+) = 125/2 = 62.5$
- The MILP has optimum $z(MILP) = 45$, with $\tilde{x}_1 = \tilde{x}_3 = 1$ and $\tilde{y}_1 = \tilde{y}_2 = \tilde{y}_4 = 1$ (clients 1, 2, 4 covered)
- The heuristic gap is 44.4%: opening the first k locations is much less effective than on the minimum-cost problems

LB (heuristic)	UB (hand dual)	$z(LP)$	$z(LP^+)$	$z(MILP)$	gap heuristic
25	225/2	41925/646	125/2	45	44.4%

Optimal solution



Profit of each client, green if covered, red if not covered.

Additional considerations

- Client 5 can never be covered: $\mathcal{L}_5 = \emptyset$, but the maximum signal is $1 + 0 + 2 = 3 < t = 5$ even opening all locations; y_5 can be fixed to 0 upfront
- Constraints (2), for a client with $|\mathcal{L}_c| \leq 1$, are redundant (never binding): the case of clients 4 and 5
- Client 2 is not among them: $\mathcal{L}_2 = \{2, 3\}$ has two elements

Question: conditional installation

- For maintenance reasons, location 1 can only be installed if location 3 is also installed
- How is this modelled?
- What is the new optimum?

The final comparison (1/2)

The three numbers that every problem of the course produces:

	value	what it is
LB	25	heuristic solution
UB	$\frac{225}{2}$	certified bound
$z(MILP)$	45	MILP optimum

The final comparison (2/2)

- Gap of the heuristic with respect to the optimum:

$$\frac{|LB - z(MILP)|}{|z(MILP)|} = 44.4\%$$

- Certified gap, known *before* solving the MILP: in absolute terms $UB - LB = \frac{175}{2}$; as a percentage, against the heuristic solution,

$$\frac{UB - LB}{|LB|} = 350.0\%$$

A worked variant: guaranteed minimum coverage (1/2)

- By contract, at least 3 clients must be covered
- Add the linear constraint

$$\sum_{c=1}^{n} y_c \geq 3$$

- On the instance the optimum already covers 3 clients: the constraint is not binding and the optimum stays 45

A worked variant: guaranteed minimum coverage (2/2)

- The constraint adds $\omega \leq 0$ to the dual; it is best left at zero: lowering it forces every λ_c up by $-\omega/(m-1)$, and those λ weigh more than the right-hand side saves
- The heuristic, with few locations, tries every choice of k locations and keeps the best feasible one

	value	what it is
UB	$\frac{225}{2}$	dual certificate built by hand
LB	45	heuristic solution
$z(LP)$	$\frac{41925}{646}$	relaxation without the bounds
$z(LP^+)$	$\frac{125}{2}$	relaxation with the bounds
$z(MILP)$	45	optimum of the MILP

Summary: Coverage with interference

- Binary installation variables x_l and coverage variables y_c
- **If and only if** link: threshold (1) and interference (2) impose $y_c = 1 \Rightarrow$ conditions; the opposite direction follows from optimality ($p_c > 0$)
- The interference constraint uses the coefficient $m - 1$ to switch itself off when $y_c = 0$
- Bounds (maximisation): $LB = 25$, $UB = 225/2$, $z(MILP) = 45$

Hub location with maximum cost

Introduction: Hub location with maximum cost

- Terminals are connected to hubs with a capacity; one pays the activation of the hubs and, for each hub, the **maximum** connection cost
- We find again the **aggregated activation** of problem 8.1 together with the **maximum variable** of problem 7.4
- We will also see a common trap in the construction of a dual solution

Problem 8.4: hub with maximum connection cost (1/2)

- In a computer network there are $n \in \mathbb{Z}_{\geq 1}$ terminals, each to be connected to exactly one hub
- There are $m \in \mathbb{Z}_{\geq 1}$ hubs available, and each hub can serve at most $k \in \mathbb{Z}_{\geq 1}$ terminals
- A hub must be activated before it can serve any terminals: $f_j \in \mathbb{Q}_{\geq 0}$ is the activation cost of hub j
- $c_{ij} \in \mathbb{Q}_{\geq 0}$ is the cost of connecting terminal i to hub j

Problem 8.4: hub with maximum connection cost (2/2)

- Objective: decide how to assign terminals to hubs to minimize the sum
  - of the activation costs of the hubs used
  - of the maximum connection cost of each hub
- subject to the constraints described

The problem in words

- **We decide** which hubs to activate and which hub to connect each terminal to
- **The objective**: activation costs plus, for each hub, the highest connection cost among the terminals assigned to it (not the sum: the bottleneck)
- **The constraints**: every terminal connected to exactly one hub; an inactive hub can serve no one, and an active one serves at most k terminals

Data (model input)

Symbol	Type	Meaning
n	$\in \mathbb{Z}_{\geq 1}$	number of terminals, $i \in \{1, 2, \ldots, n\}$
m	$\in \mathbb{Z}_{\geq 1}$	number of hubs, $j \in \{1, 2, \ldots, m\}$
k	$\in \mathbb{Z}_{\geq 1}$	capacity (terminals) of each hub
c_{ij}	$\in \mathbb{Q}_{\geq 0}$	cost of connecting terminal i to hub j
f_j	$\in \mathbb{Q}_{\geq 0}$	activation cost of hub j

Decision variables

- $n\,m$ binary variables: $x_{ij} = 1$ if terminal i is connected to hub j, 0 otherwise
- m binary variables: $y_j = 1$ if hub j is activated, 0 otherwise
- m non-negative continuous variables:

  z_j = maximum connection cost among the terminals assigned to hub j

with $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$.

The model

Model 8.4 — hub with maximum cost

$$\min \quad \sum_{j=1}^{m} f_j\, y_j + \sum_{j=1}^{m} z_j \tag{8.7a}$$

$$\text{subject to} \quad \sum_{j=1}^{m} x_{ij} = 1, \qquad \forall i \in \{1, 2, \ldots, n\}, \tag{8.7b}$$

$$-\sum_{i=1}^{n} x_{ij} + k\, y_j \geq 0, \qquad \forall j \in \{1, 2, \ldots, m\}, \tag{8.7c}$$

$$-c_{ij}\, x_{ij} + z_j \geq 0, \qquad \forall i \in \{1, 2, \ldots, n\},\ \forall j \in \{1, 2, \ldots, m\}, \tag{8.7d}$$

$$x_{ij} \in \{0, 1\}, \quad \forall i \in \{1, 2, \ldots, n\},\ \forall j \in \{1, 2, \ldots, m\}, \tag{8.7e}$$

$$y_j \in \{0, 1\}, \qquad \forall j \in \{1, 2, \ldots, m\}, \tag{8.7f}$$

$$z_j \geq 0, \qquad \forall j \in \{1, 2, \ldots, m\}. \tag{8.7g}$$

Description of the objective function and constraints

- The objective function minimizes the activation costs of the hubs used plus the maximum connection cost of each hub
- Constraints (1): every terminal connected to exactly one hub (n constraints)
- Constraints (2) link assignment and activation and impose the capacity: an inactive hub serves no one, an active one serves at most k terminals (m constraints)
- Constraints (3) link assignment and the maximum variable: if i is connected to j, then $z_j \geq c_{ij}$ ($n\,m$ constraints)
- The last constraints define the variables of the model

First link: aggregated activation (1/2)

- If at least one terminal is connected to j, then j must be activated:

$$\sum_{i=1}^{n} x_{ij} \geq 1 \implies y_j = 1,$$

$$\text{contrapositive:} \quad y_j = 0 \implies \sum_{i=1}^{n} x_{ij} = 0$$

- Imposed directly, in aggregated form, by constraint (2): if $y_j = 0$ it becomes $\sum_i x_{ij} \leq 0$; if $y_j = 1$ it becomes $\sum_i x_{ij} \leq k$, the capacity
- The opposite direction, $y_j = 1 \Rightarrow \sum_i x_{ij} \geq 1$, is not imposed: $y_j = 1$ with all $x_{ij} = 0$ satisfies (2)

First link: aggregated activation (2/2)

The opposite direction follows from optimality, with a strength that depends on the sign of f_j:

- if $f_j > 0$: setting $y_j = 0$ on an empty hub stays feasible and reduces the cost; hence **in every optimum** no activated hub is left empty
- if $f_j = 0$ (allowed, $f_j \in \mathbb{Q}_{\geq 0}$): the exchange neither worsens nor improves, and only **there exists an optimum** with the empty hubs switched off

On the instance $f = (5, 6, 7) > 0$ and the strong version holds; but a proof must be written with the assumption it uses, not with the convenient one.

Second link: maximum variable

- If terminal i is connected to hub j, then $z_j \geq c_{ij}$:

$$x_{ij} = 1 \quad \Longrightarrow \quad z_j \geq c_{ij}$$

  imposed by constraint (3), equivalent to $z_j \geq c_{ij} x_{ij}$
- Over all terminals connected to j: $z_j \geq \max_{i\,:\,x_{ij}=1} c_{ij}$ (maximum over the empty set $= 0$, from $z_j \geq 0$)
- The objective minimizes $\sum_j z_j$ and z_j appears nowhere else: in every optimum

$$z_j = \max_{i\,:\,x_{ij}=1} c_{ij}$$

  the same pattern as problem 7.4

The model in gurobipy

```
mod = gp.Model("hub_max")
x = mod.addVars(n, m, vtype=GRB.BINARY, name="x")
y = mod.addVars(m, vtype=GRB.BINARY, name="y")
z = mod.addVars(m, name="z")
mod.setObjective(gp.quicksum(f[j] * y[j] for j in
    range(m))
    + z.sum(), GRB.MINIMIZE)
mod.addConstrs((gp.quicksum(x[i, j] for j in range(m))
     == 1
    for i in range(n)), name="assignment")
mod.addConstrs((-gp.quicksum(x[i, j] for i in range(n)
    )
    + k * y[j] >= 0 for j in range(m)), name="
        activation")
mod.addConstrs((-c[i][j] * x[i, j] + z[j] >= 0
    for i in range(n) for j in range(m)), name="
        maximum")
```

The instance

$n = 3$ terminals, $m = 3$ hubs, $k = 2$:

c_{ij}	$j = 1$	$j = 2$	$j = 3$
$i = 1$	5	10	2
$i = 2$	5	4	6
$i = 3$	5	4	6
f_j	5	6	7

The model for the instance

The instance model

$$
\begin{array}{lrrr

Constructive heuristic: the primal bound (1/2)

- A **next-fit** heuristic, inspired by bin packing: one hub at a time is filled up to k terminals, then the algorithm moves to the next one
- If all hubs have been used before every terminal is assigned, the algorithm fails
- It is the same generic heuristic used for the scheduling problems of the previous family: hubs are the "machines" with capacity k, terminals the "jobs" with unit time

Constructive heuristic: the primal bound (2/2)

- Terminal 1 goes to hub 1 (residual capacity $2 \to 1$), terminal 2 goes to hub 1 ($1 \to 0$)
- Terminal 3 no longer fits on hub 1, it moves to hub 2
- Hubs 1 and 2 are activated: $\max(c_{11}, c_{21}) = 5$ and $c_{32} = 4$

$$\bar{x}_{11} = \bar{x}_{21} = \bar{x}_{32} = 1, \quad \bar{y}_1 = \bar{y}_2 = 1, \quad \bar{z}_1 = 5, \ \bar{z}_2 = 4$$

of value $5 + 6 + 5 + 4 = 20$. Hence

$$z(MILP) \leq UB = 20.$$

LP relaxation and dual

A free dual variable α_i for (1), non-negative β_j for (2) and γ_{ij} for (3):

Dual of the LP relaxation of model 8.4

$$\max \quad \sum_{i=1}^{n} \alpha_i \tag{8.8a}$$

$$\text{subject to} \quad \alpha_i - \beta_j - c_{ij}\,\gamma_{ij} \leq 0, \quad \forall i \in \{1,2,\ldots,n\}, \ \forall j \in \{1,2,\ldots,m\}, \tag{8.8b}$$

$$k\,\beta_j \leq f_j, \qquad \forall j \in \{1,2,\ldots,m\}, \tag{8.8c}$$

$$\sum_{i=1}^{n} \gamma_{ij} \leq 1, \qquad \forall j \in \{1,2,\ldots,m\}, \tag{8.8d}$$

$$\alpha_i \gtreqless 0, \qquad \forall i \in \{1,2,\ldots,n\}, \tag{8.8e}$$

$$\beta_j \geq 0, \qquad \forall j \in \{1,2,\ldots,m\}, \tag{8.8f}$$

$$\gamma_{ij} \geq 0, \quad \forall i \in \{1,2,\ldots,n\}, \ \forall j \in \{1,2,\ldots,m\}. \tag{8.8g}$$

The dual for the instance

The instance model

$$
\begin{array}{rrrr

Description of the dual

- α_i is the value of terminal i, β_j the price of one of the k slots of hub j, γ_{ij} the price of the link that pushes the maximum of hub j above c_{ij}
- Constraints (8.8b) are the columns of the x_{ij}: assigning i to j satisfies the assignment, takes one slot and raises the maximum up to c_{ij}; the balance cannot be positive
- Constraints (8.8c) are the columns of the y_j: the k slots cannot be worth more than the fixed cost f_j
- Constraints (8.8d) are the columns of the z_j: z_j weighs 1 in the primal objective, so the prices of the links sum to at most one

A hand-built dual solution

- Set $\bar{\gamma}_{ij} = 0$ (constraint (8.8d) is satisfied) and $\bar{\beta}_j = f_j/k$, the largest value allowed by (8.8c):

$$\bar{\beta} = (5/2,\ 3,\ 7/2)$$

- Constraint (8.8b) then becomes $\alpha_i \leq \bar{\beta}_j$ for *every* hub j: the largest feasible value is $\bar{\alpha}_i = \min_j \bar{\beta}_j = 5/2$
- Value $3 \cdot 5/2 = 15/2$; by weak duality

$$LB = 15/2 \leq z(LP) \leq z(MILP) \leq UB = 20$$

A common trap

Observation 1: A common trap

Setting $\bar{\gamma}_{ij} = 0$ only for the "inconvenient" hubs and hoping that constraint (8.8b) bounds α_i only against the cheapest hub is a mistake: the constraint holds for every j, so α_i is bounded by the minimum over all hubs.

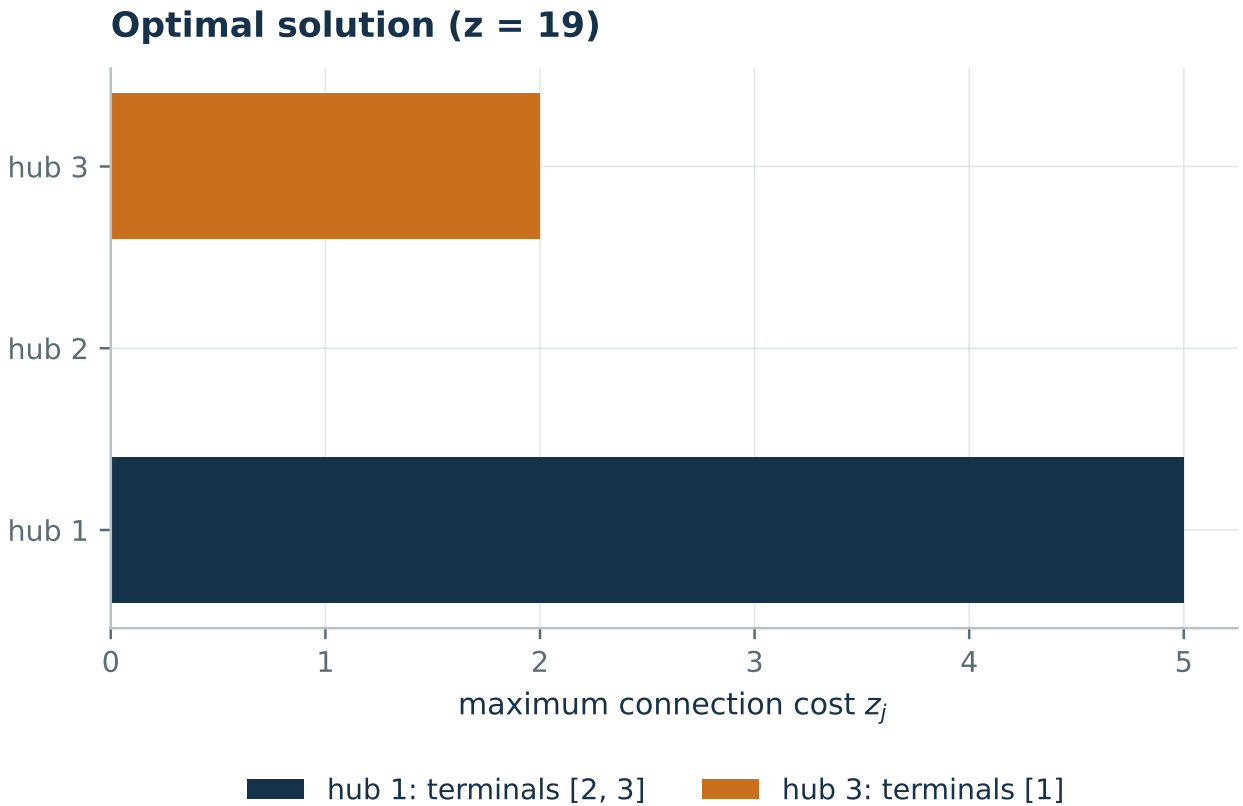
For instance, setting $\bar{\gamma}_{i1} = 1/n$ only for the cheapest hub and leaving $\bar{\beta}_j = 0$ elsewhere is not feasible: it violates constraint (8.8b) for the other hubs.

What the solver says

- $z(LP) = 25/2$; with $x_{ij} \leq 1$ one gets $z(LP^+) = 1015/78 \approx 13.0$
- The MILP has optimum $z(MILP) = 19$: $\tilde{x}_{13} = \tilde{x}_{21} = \tilde{x}_{31} = 1$, $\tilde{y}_1 = \tilde{y}_3 = 1$, $\tilde{z}_1 = 5$, $\tilde{z}_3 = 2$
- Terminal 1 alone on hub 3 ($c_{13} = 2$), terminals 2 and 3 on hub 1; next-fit still comes close (5.3%)

UB (heuristic)	LB (hand dual)	$z(LP)$	$z(LP^+)$	$z(MILP)$	gap heuristic
20	15/2	25/2	1015/78	19	5.3%

Optimal solution



Maximum connection cost of each activated hub, with the terminals assigned.

Additional considerations

- The disaggregated constraints $x_{ij} \leq y_j$ are valid inequalities implied by the aggregated constraint (2): the same activation logic in a different form
- Setting $M_j = \max_i c_{ij}$, the constraints $z_j \leq M_j\, y_j$ are not valid inequalities (the model allows $z_j > 0$ with $y_j = 0$), but they are **optimality-preserving constraints**: in every optimum $z_j = 0$ when $y_j = 0$

Question: disaggregated activation link

- *Add* the disaggregated links $x_{ij} \leq y_j$ to model 8.4
- Does the optimum for the instance change? Does the relaxation change?
- What happens if, instead of adding them, one *replaces* the aggregated constraint (2) by them?

The final comparison (1/2)

The three numbers that every problem of the course produces:

	value	what it is
UB	20	heuristic solution
LB	$\frac{15}{2}$	certified bound
$z(MILP)$	19	MILP optimum

The final comparison (2/2)

- Gap of the heuristic with respect to the optimum:

$$\frac{|UB - z(MILP)|}{|z(MILP)|} = 5.3\%$$

- Certified gap, known *before* solving the MILP: in absolute terms $UB - LB = \frac{25}{2}$; as a percentage, against the heuristic solution,

$$\frac{UB - LB}{|UB|} = 62.5\%$$

A worked variant: forbidden connection (1/2)

- Terminal 1 cannot be connected to hub 2 (a security requirement)
- The variable is fixed to zero with the linear constraint

$$x_{12} = 0$$

- On the instance the optimum does not already use x_{12} (terminal 1 is on hub 3): the constraint is not binding and the optimum stays 19

A worked variant: forbidden connection (2/2)

- Forbidding a connection removes a column from the primal, hence *removes* a constraint from the dual: α_1 can rise
- It is given one γ on each of the hubs left to it and the bound grows: one more constraint in the primal *improves* the certificate

	value	what it is
UB	20	heuristic solution
LB	$\frac{21}{2}$	dual certificate built by hand
$z(LP)$	$\frac{25}{2}$	relaxation without the bounds
$z(LP^+)$	$\frac{79}{6}$	relaxation with the bounds
$z(MILP)$	19	optimum of the MILP

Summary: Hub with maximum cost

- Binary assignment variables x_{ij} and activation variables y_j, continuous maximum variables z_j
- **Aggregated activation** $\sum_i x_{ij} \leq k\, y_j$, as in problem 8.1
- **Maximum variable** $z_j \geq c_{ij} x_{ij}$: in every optimum z_j is exactly the maximum
- In the dual, the constraint holds for *every* hub: α_i is bounded by the minimum
- Bounds: $LB = 15/2$, $UB = 20$, $z(MILP) = 19$

Summary (1/2)

The four links:

- aggregated activation: $\sum_c y_{lc} \leq u_l x_l$ (capacitated facility location)
- disaggregated activation: $x_l \geq y_{lc}$ (p-median)
- if and only if: one direction from the constraints, the other from the objective (coverage)
- aggregated activation and maximum variable $z_j \geq c_{ij} x_{ij}$ (hub)

Directions of the links:

- the direction the constraints do not impose must be proved from optimality, with the right assumptions (for instance $i_l > 0$, $f_j > 0$, $p_c > 0$)

Summary (2/2)

The two bounds:

- the constructive heuristic gives a primal bound, a hand-built dual solution a dual bound
- together they form the "sandwich" that the solver tries to close

Problem	heuristic	hand dual	$z(MILP)$
8.1 capacitated	439	1581/5	365
8.2 p-median	18	13	15
8.3 coverage (max)	25	225/2	45
8.4 hub	20	15/2	19