Numerical models (1/3): EX 1–5

Outline

The fifteen numerical models

EX 1 — The eight-seat van

EX 2 — Bus lines

EX 3 — Relay

EX 4 — Hub-and-spoke

EX 5 — Mutual funds bought in lots

Learning objectives (1/2)

At the end of this chapter you will be able to:

- write the variables and the model of an instance with explicit data, primal and dual;
- build by hand a feasible solution and a dual solution, and close the sandwich $LB \leq z(MILP) \leq UB$;

Learning objectives (2/2)

At the end of this chapter you will be able to:

- recognise in each statement the technique that the model needs;
- compare the hand-built bounds with the optimum given by the solver.

Prerequisites

- From this module, Chapter 2 «Relaxations, duality and bounds»: the dual of the LP relaxation and the dual solution built by hand
- From this module, Chapter 4 «Constructive heuristics»: constructive heuristics and the primal bound
- From this module, Chapter 6 «Links between variables»: the links between variables (activation, minimum lot, integer counts, min-max)

The fifteen numerical models

Introduction: The fifteen numerical models

- The course collects fifteen numerical models, from EX 1 to EX 15
- Explicit data, few variables, one technique per model
- They are the easiest ones: they come before the three families of problems and the mixed problems
- In these slides: the first five, from EX 1 to EX 5

The five parts of every numerical model

- the statement, with the data of the instance
- the **variables**, with their domain and their count
- the **model of the instance**, primal and dual
- a feasible solution built by hand, which gives the primal bound
- a dual solution built by hand, which gives the dual bound, and the comparison with the solver optimum

Every model has its own script and its own notebook.

Summary

- Small models with explicit data: they can be solved by hand too
- Always the same scheme: statement, variables, primal and dual model
- Two hand-built bounds place the optimum between two values *before* knowing it
- The solver gives the optimum, against which the two bounds are compared

EX 1 — The eight-seat van

Introduction: EX 1 — The eight-seat van

- Which groups of tourists to accept on an eight-seat van
- Three constraints on four binary variables: capacity, count, implication
- We build a primal bound and a dual bound by hand
- Removing one constraint at a time we see which constraint "bites"

EX 1 — which groups of tourists to accept

- A tour operator rents an eight-seat van
- Four groups show up, of 2, 3, 4 and 5 people, whose members do not want to be separated
- The groups offer 30, 50, 80 and 70 euros
- At most two groups can be accepted
- If group 2 is accepted then group 4 must be accepted as well
- The goal is to maximise the total earnings

The variables

There are four groups: one binary variable per group.

$$x_j = \begin{cases} 1 & \text{if group } j \text{ is accepted,} \\ 0 & \text{otherwise,} \end{cases} \qquad \forall j \in \{1, 2, 3, 4\}.$$

EX 1 on the instance — primal

EX 1 on the instance — primal

$$
\begin{aligned}
\max\ & 30x_1 + 50x_2 + 80x_3 + 70x_4 \\
\text{subject to}\ & 2x_1 + 3x_2 + 4x_3 + 5x_4 \leq 8 \\
& x_1 + x_2 + x_3 + x_4 \leq 2 \\
& x_2 - x_4 \leq 0 \\
& x_1, x_2, x_3, x_4 \in \{0, 1\}
\end{aligned}
$$

Description of the model

- The objective adds up the offers of the accepted groups
- First row: the capacity of the van, the people accepted do not exceed eight seats
- Second row: the count, at most two groups
- Third row: the logical precedence $x_2 \leq x_4$, i.e. $x_2 - x_4 \leq 0$
  - if $x_2 = 1$ then $x_4 = 1$; the other direction must not hold: group 4 can be accepted without group 2
- Last row: the four variables are binary

A feasible solution: the primal bound

Constructive heuristic on the decreasing offer $(3, 4, 2, 1)$, accepting a group only if all the constraints stay satisfied:

- Group 3 (offer 80): accepted, 4 seats, 1 group
- Group 4 (offer 70): rejected, 9 seats would be needed out of 8
- Group 2 (offer 50): rejected, it would force accepting group 4, and the groups would become three
- Group 1 (offer 30): accepted, 6 seats, 2 groups

$$\bar{x} = (1, 0, 1, 0), \qquad LB = 30 + 80 = 110.$$

The LP relaxation and its dual

- The LP relaxation replaces the binary domain with non-negativity only
- The dual attaches a non-negative variable to each of the three constraints:
  - α to the seats
  - β to the number of groups
  - γ to the implication
- The four columns of the primal give four dual constraints

EX 1 on the instance — dual of the LP relaxation

EX 1 on the instance — dual of the LP relaxation

$$
\begin{aligned}
\min\ & 8\alpha + 2\beta \\
\text{subject to}\ & 2\alpha + \beta \geq 30 \\
& 3\alpha + \beta + \gamma \geq 50 \\
& 4\alpha + \beta \geq 80 \\
& 5\alpha + \beta - \gamma \geq 70 \\
& \alpha \geq 0 \\
& \beta \geq 0 \\
& \gamma \geq 0
\end{aligned}
$$

A dual solution: the dual bound

The recipe: $\bar{\beta} = \bar{\gamma} = 0$ and only the seat is priced. The dual constraints become

$$2\alpha \geq 30, \qquad 3\alpha \geq 50, \qquad 4\alpha \geq 80, \qquad 5\alpha \geq 70,$$

and the seat gets the price no group can beat:

$$\bar{\alpha} = \max\left(\frac{30}{2}, \frac{50}{3}, \frac{80}{4}, \frac{70}{5}\right) = 20, \qquad UB = 8 \cdot 20 = 160.$$

Meaning: "a seat is worth as much as the group that pays best for it".

Optimal solution

- The optimum is obtained by solving the model
- The two bounds are not passed to the solver: they place the optimum between two values before knowing it, and help interpret what the solver finds
- The optimum accepts groups 2 and 4, filling exactly the eight seats:

$$\tilde{x} = (0, 1, 0, 1), \qquad z(MILP) = 50 + 70 = 120.$$

The final comparison

	value	what it is
LB	110	heuristic solution
UB	160	dual bound
$z(LP)$	160	relaxation without the bounds
$z(LP^+)$	140	relaxation with the bounds
$z(MILP)$	120	MILP optimum

The gaps

- Gap of the heuristic with respect to the optimum:

$$\frac{|110 - 120|}{120} = 8.3\%.$$

- Certified gap, known *before* solving the MILP (it cannot use $z(MILP)$): in absolute terms $UB - LB = 50$; in percentage, in the solver convention,

$$\frac{|UB - LB|}{|UB|} = \frac{|160 - 110|}{160} = 31.3\%.$$

- The hand-built dual is *optimal* for the relaxation without the bounds; adding $x_j \leq 1$ it drops to 140

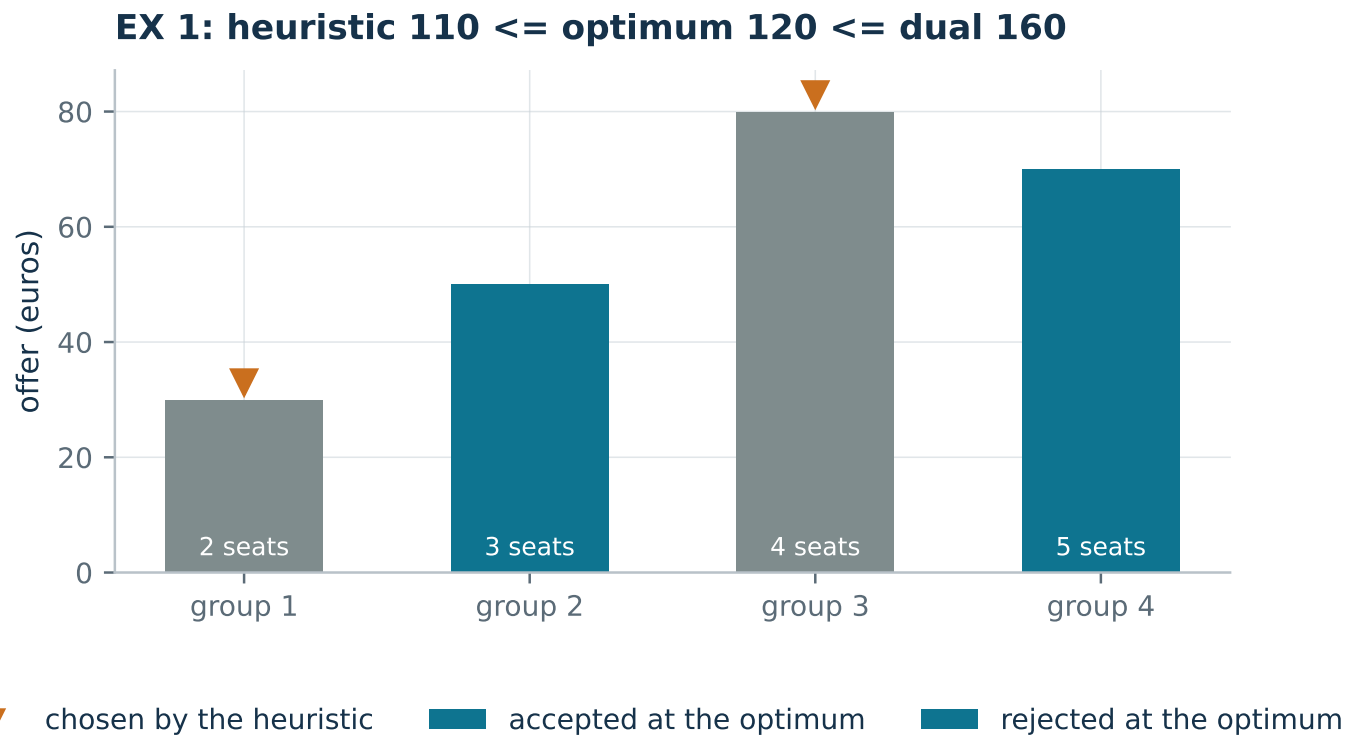
The contribution of each constraint

Removing one constraint at a time shows which one bites:

model	$z(MILP)$	groups
complete	120	2, 4
without the limit on the groups	120	2, 4
without the implication	130	2, 3
without both	130	2, 3

- The implication costs 10 euros
- The limit of two groups changes nothing: it is the seat capacity that bites

The three bounds side by side



Summary

- Capacity, count and implication $x_2 - x_4 \leq 0$ on four binaries
- Heuristic on the decreasing offer: $LB = 110$
- Dual "a seat is worth the group that pays best for it": $UB = 160$
- Optimum $z(MILP) = 120$, groups 2 and 4
- The constraint that bites is the capacity; the implication costs 10 euros

EX 2 — Bus lines

Introduction: EX 2 — Bus lines

- An assignment of lines to companies, with a capacity in number of lines
- Structure: *set partitioning* constraints plus capacity constraints
- We will see that the LP relaxation is exact, while the dual recipe does not reach it

EX 2 — assignment with a capacity in number of lines

A city council evaluates the offers of three bus companies for four bus lines. Cost at which each company would operate each line:

Company	Line 1	Line 2	Line 3	Line 4
1	10	4	9	7
2	1	2	3	10
3	8	9	10	1

Every line must be assigned to exactly one company and every company may operate at most two lines. The total cost is to be minimised.

The variables

Four lines and three companies: $3 \cdot 4 = 12$ binary variables, one per company–line pair.

$$x_{ij} = \begin{cases} 1 & \text{if company } i \text{ operates line } j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\forall i \in \{1, 2, 3\},\ \forall j \in \{1, 2, 3, 4\}.$$

The maximum number of lines per company is $p = 2$.

EX 2 on the instance — primal

EX 2 on the instance — primal

$$
\begin{aligned}
\min\ & 10x_{11}+4x_{12}+9x_{13}+7x_{14}+x_{21}+2x_{22}+3x_{23}+10x_{24}+8x_{31}+9x_{32}+10x_{33}+x_{34} \\
\text{subject to}\ & x_{11} + x_{21} + x_{31} = 1 \\
& x_{12} + x_{22} + x_{32} = 1 \\
& x_{13} + x_{23} + x_{33} = 1 \\
& x_{14} + x_{24} + x_{34} = 1 \\
& x_{11} + x_{12} + x_{13} + x_{14} \le 2 \\
& x_{21} + x_{22} + x_{23} + x_{24} \le 2 \\
& x_{31} + x_{32} + x_{33} + x_{34} \le 2 \\
& x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34} \in \{0,1\}
\end{aligned}
$$

Description of the model

- The objective adds up the costs of the chosen company–line pairs
- With $n = 3$ companies, $m = 4$ lines and $p = 2$:
  - four *set partitioning* constraints, one per line
  - three capacity constraints, one per company
- The last row declares the twelve variables binary
- It is problem 7.1 with the capacity counted in *number* of jobs instead of time

A feasible solution: the primal bound

Constructive heuristic on the lines, in order: every line goes to the cheapest company among those not yet full.

- **Line 1**: costs 10, 1, 8; the cheapest is company 2
- **Line 2**: costs 4, 2, 9; company 2 again, which is now full
- **Line 3**: companies 1 and 3 remain, costs 9 and 10: company 1
- **Line 4**: companies 1 and 3 remain, costs 7 and 1: company 3

$$\bar{x}_{21} = \bar{x}_{22} = \bar{x}_{13} = \bar{x}_{34} = 1, \qquad UB = 1 + 2 + 9 + 1 = 13.$$

The LP relaxation and its dual

- Relaxing $x_{ij} \in \{0, 1\}$ to $x_{ij} \geq 0$
- The dual has:
  - a free variable α_j for every line constraint
  - a variable $\pi_i \leq 0$ for every capacity constraint
- The twelve columns of the primal give twelve dual constraints

EX 2 on the instance — dual of the LP relaxation

EX 2 on the instance — dual of the LP relaxation

$$
\begin{array}{rrrrrrrrrrrrrr}
\max & \alpha_1 & + \alpha_2 & + \alpha_3 & + \alpha_4 & + 2\beta_1 & + 2\beta_2 & + 2\beta_3 & & \\
\text{subject to} & \alpha_1 & & & & + \beta_1 & & & \leq & 10 \\
& & \alpha_2 & & & + \beta_1 & & & \leq & 4 \\
& & & \alpha_3 & & + \beta_1 & & & \leq & 9 \\
& & & & \alpha_4 & + \beta_1 & & & \leq & 7 \\
& \alpha_1 & & & & & + \beta_2 & & \leq & 1 \\
& & \alpha_2 & & & & + \beta_2 & & \leq & 2 \\
& & & \alpha_3 & & & + \beta_2 & & \leq & 3 \\
& & & & \alpha_4 & & + \beta_2 & & \leq & 10 \\
& \alpha_1 & & & & & & + \beta_3 & \leq & 8 \\
& & \alpha_2 & & & & & + \beta_3 & \leq & 9 \\
& & & \alpha_3 & & & & + \beta_3 & \leq & 10 \\
& & & & \alpha_4 & & & + \beta_3 & \leq & 1 \\
& \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4 & & & & \gtreqless & 0 \\
& & & & & \beta_1, & \beta_2, & \beta_3 & \leq & 0
\end{array}
$$

Description of the dual

- α_j is the value of line j, which has to be assigned anyway
- π_i is the (non-positive) price of one of the p lines company i may run
- The objective prices the lines and the available slots
- The dual constraints are the columns of the x_{ij}: giving line j to company i satisfies its assignment constraint and takes one slot, and the balance cannot exceed the cost c_{ij} of that pairing

A dual solution: the dual bound

With the "zero out and saturate" recipe, the three capacity multipliers are set to zero,

$$\bar{\pi}_1 = 0, \qquad \bar{\pi}_2 = 0, \qquad \bar{\pi}_3 = 0,$$

and every line gets the lowest offer it received, $\bar{\alpha}_j = \min_i c_{ij}$, feasible by construction:

$$\bar{\alpha}_1 = 1, \qquad \bar{\alpha}_2 = 2, \qquad \bar{\alpha}_3 = 3, \qquad \bar{\alpha}_4 = 1,$$

$$LB = 1 + 2 + 3 + 1 = 7.$$

Meaning: "every line costs at least the lowest offer it received".

Optimal solution

- The optimum is obtained by solving the model; the two bounds place it and help interpret it
- The optimum assigns:
  - line 1 to company 2, line 2 to company 1
  - line 3 to company 2, line 4 to company 3

$$\tilde{x}_{21} = \tilde{x}_{12} = \tilde{x}_{23} = \tilde{x}_{34} = 1, \qquad z(MILP) = 1 + 4 + 3 + 1 = 9.$$

The final comparison

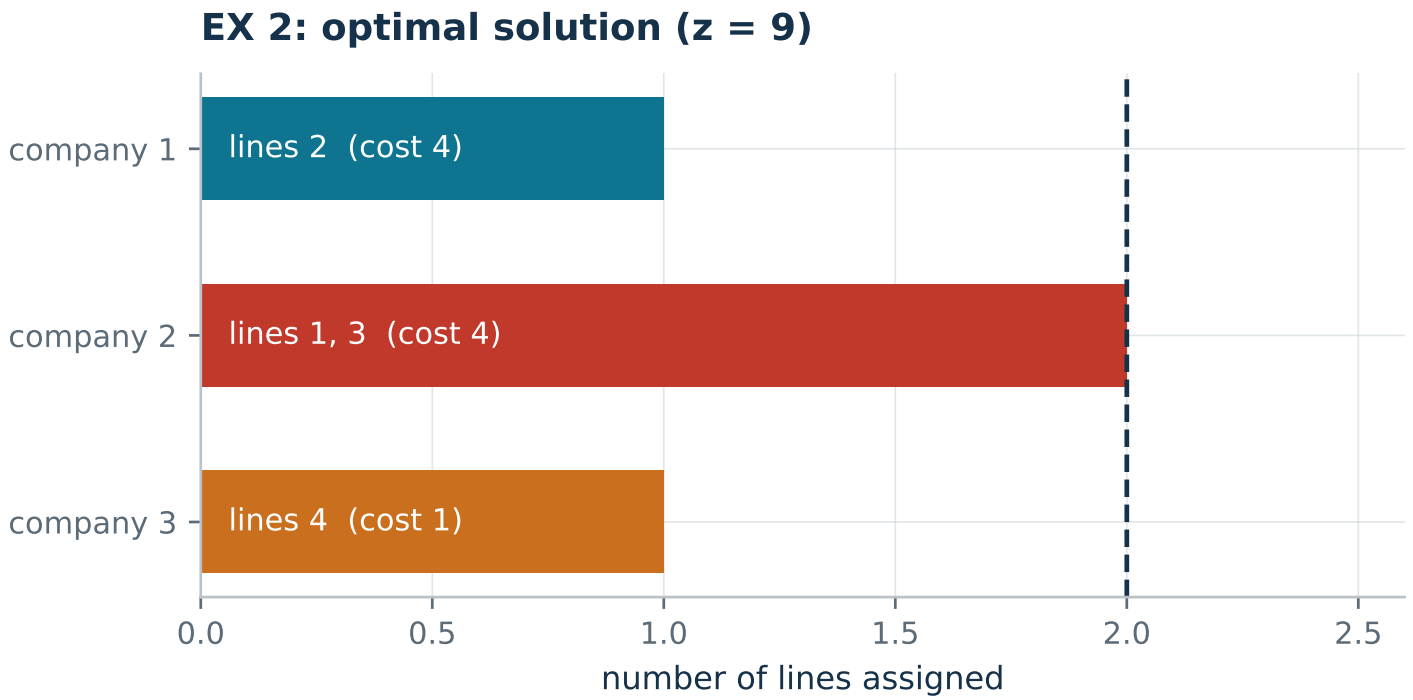
	value	what it is
UB	13	heuristic solution
LB	7	dual bound
$z(LP)$	9	LP relaxation
$z(MILP)$	9	MILP optimum

- Heuristic gap: $\frac{|13-9|}{9} = 44.4\%$
- Certified gap: $UB - LB = 6$, i.e. $\frac{|13-7|}{13} = 46.2\%$

Exact relaxation, inexact dual recipe

- The relaxation is *exact*: $z(LP) = z(MILP) = 9$
- Yet the hand-built bound stops at 7 and the heuristic is off by 44%
- Zeroing the capacity multipliers throws away exactly the information that makes the problem hard: the cheapest company cannot take all the lines
- The relaxation is exact, the dual recipe is not: they are two different things

The optimal solution



Summary

- Set partitioning on the lines plus capacity on the companies
- Line-by-line heuristic: $UB = 13$
- Dual "every line costs at least its lowest offer": $LB = 7$
- Optimum $z(MILP) = 9 = z(LP)$: the relaxation is exact
- The dual recipe zeroes out exactly the capacity information

EX 3 — Relay

Introduction: EX 3 — Relay

- An assignment with more resources (swimmers) than tasks (legs)
- The same structure as EX 2 with unit capacity
- We will see why the relaxation is exact here: the matrix is totally unimodular

EX 3 — assignment with more resources than tasks

A coach must form a relay team by choosing three swimmers out of four. Three 50-metre legs; recorded times (seconds):

Stroke	Carlo	Paolo	Fabio	Marco
Backstroke	37	32	33	37
Breaststroke	43	33	42	34
Butterfly	33	28	38	30

The total relay time is to be minimised.

The variables

Three strokes and four swimmers: twelve binary variables, one per stroke–swimmer pair.

$$x_{sn} = \begin{cases} 1 & \text{if stroke } s \text{ is swum by swimmer } n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\forall s \in \{1, 2, 3\}, \ \forall n \in \{1, 2, 3, 4\}.$$

One swimmer stays out: it is an assignment with more resources than tasks.

EX 3 on the instance — primal

EX 3 on the instance — primal

$$
\begin{aligned}
\min\ & 37x_{11}+32x_{12}+33x_{13}+37x_{14}+43x_{21}+33x_{22}+42x_{23}+34x_{24}+33x_{31}+28x_{32}+38x_{33}+30x_{34} \\
\text{subject to}\ & x_{11}+x_{12}+x_{13}+x_{14} = 1 \\
& x_{21}+x_{22}+x_{23}+x_{24} = 1 \\
& x_{31}+x_{32}+x_{33}+x_{34} = 1 \\
& x_{11}+x_{21}+x_{31} \le 1 \\
& x_{12}+x_{22}+x_{32} \le 1 \\
& x_{13}+x_{23}+x_{33} \le 1 \\
& x_{14}+x_{24}+x_{34} \le 1 \\
& x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34} \in \{0,1\}
\end{aligned}
$$

Description of the model

- The objective sums the times of the assignments chosen
- The first three rows: every stroke is swum by exactly one swimmer
- The four that follow: every swimmer swims at most one stroke
- It is the structure of the *assignment* problem: equalities on the rows of the matrix, inequalities on the columns
- The last row declares the twelve variables binary

A feasible solution: the primal bound

Constructive heuristic on the strokes in the given order, each to the fastest swimmer still free:

- backstroke: the fastest of the four is Paolo, with 32
- breaststroke: the fastest of the three left is Marco, with 34
- butterfly: the fastest of the two left is Carlo, with 33

$\bar{x}_{12} = \bar{x}_{24} = \bar{x}_{31} = 1, \qquad UB = 32 + 34 + 33 = 99$ seconds.

SAPIENZA UNIVERSITÀ DI ROMA

EX 3 on the instance — dual of the LP relaxation

EX 3 on the instance — dual of the LP relaxation

$$
\begin{array}{rrrrrrrrrrr}
\text{max} & \alpha_1 & +\alpha_2 & +\alpha_3 & +\beta_1 & +\beta_2 & +\beta_3 & +\beta_4 & & \\
\text{subject to} & \alpha_1 & & & +\beta_1 & & & & \leq & 37 \\
& \alpha_1 & & & & +\beta_2 & & & \leq & 32 \\
& \alpha_1 & & & & & +\beta_3 & & \leq & 33 \\
& \alpha_1 & & & & & & +\beta_4 & \leq & 37 \\
& & \alpha_2 & & +\beta_1 & & & & \leq & 43 \\
& & \alpha_2 & & & +\beta_2 & & & \leq & 33 \\
& & \alpha_2 & & & & +\beta_3 & & \leq & 42 \\
& & \alpha_2 & & & & & +\beta_4 & \leq & 34 \\
& & & \alpha_3 & +\beta_1 & & & & \leq & 33 \\
& & & \alpha_3 & & +\beta_2 & & & \leq & 28 \\
& & & \alpha_3 & & & +\beta_3 & & \leq & 38 \\
& & & \alpha_3 & & & & +\beta_4 & \leq & 30 \\
& \alpha_1, & \alpha_2, & \alpha_3 & & & & & \gtreqless & 0 \\
& & & & \beta_1, & \beta_2, & \beta_3, & \beta_4 & \leq & 0
\end{array}
$$

A dual solution: the dual bound

The same structure as EX 2 with $p = 1$: $\bar{\beta}_n = 0$ for every swimmer and the largest feasible α_s,

$$\bar{\alpha}_s = \min_n t_{sn} : \qquad \bar{\alpha}_1 = 32, \quad \bar{\alpha}_2 = 33, \quad \bar{\alpha}_3 = 28, \qquad LB = 93.$$

- Meaning: "every stroke costs at least the time of its best specialist"
- The bound is weak exactly when two strokes share the same best specialist: here Paolo is the fastest in both backstroke and butterfly, and the bound is not attainable

Optimal solution

- The optimum is obtained by solving the model; the two bounds place it and help interpret it
- Backstroke to Fabio (33), breaststroke to Marco (34), butterfly to Paolo (28); Carlo stays out

$$\tilde{x}_{13} = \tilde{x}_{24} = \tilde{x}_{32} = 1, \qquad z(MILP) = 33 + 34 + 28 = 95 \text{ seconds.}$$

The final comparison

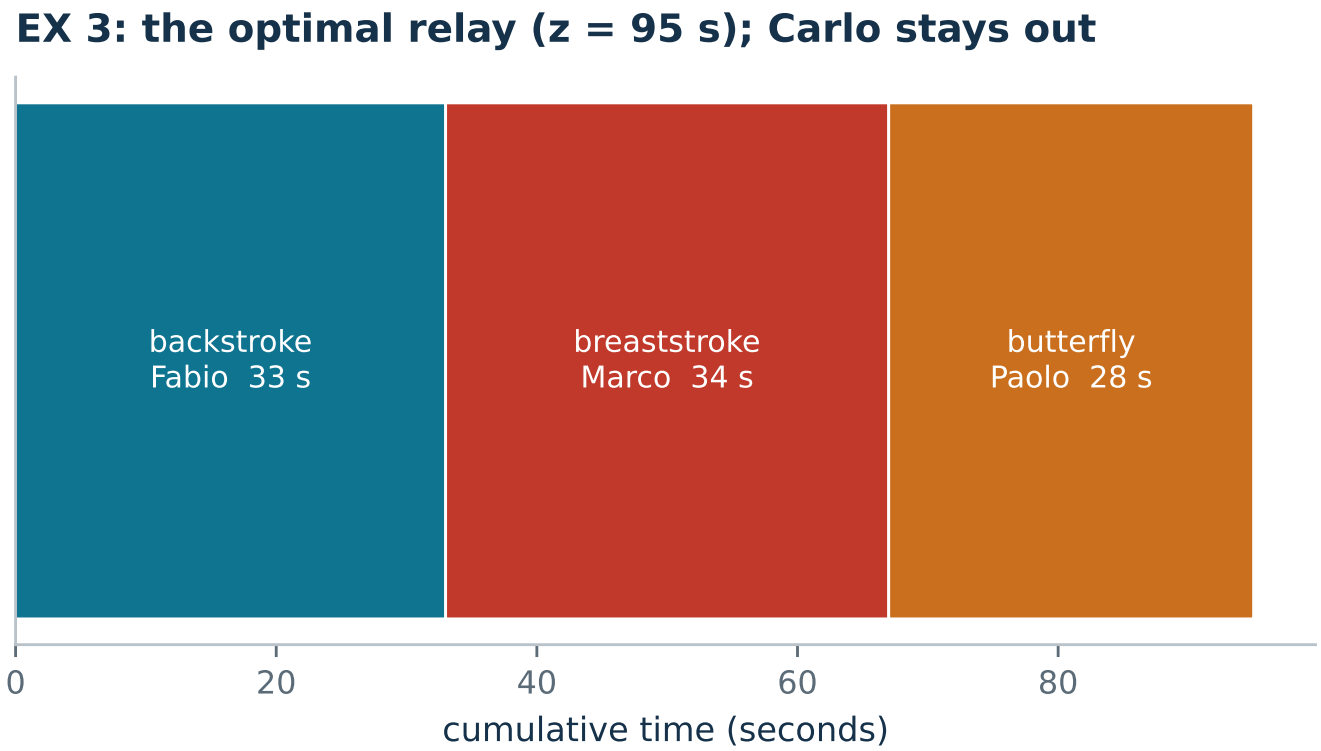
	value	what it is
UB	99	heuristic solution
LB	93	dual bound
$z(LP)$	95	LP relaxation
$z(MILP)$	95	MILP optimum

- Heuristic gap: $\frac{|99-95|}{95} = 4.2\%$
- Certified gap: $UB - LB = 6$, i.e. $\frac{|99-93|}{99} = 6.1\%$

Why the relaxation is exact here

- The matrix of the model is that of the assignment problem: totally unimodular
- All the vertices of the relaxation are integer, so $z(LP) = z(MILP)$ and integrality costs nothing
- The same phenomenon as alldiff (Section "Alldiff and binary expansion")
- This stops being true as soon as a capacity constraint with coefficients other than 0 and 1 is added

The optimal relay



Summary

- Assignment with more resources than tasks: equalities on the strokes, inequalities on the swimmers
- Stroke-by-stroke heuristic: $UB = 99$
- Dual "the best specialist": $LB = 93$
- Optimum $z(MILP) = 95 = z(LP)$: totally unimodular matrix

EX 4 — Hub-and-spoke

Introduction: EX 4 — Hub-and-spoke

- The minimum number of hubs covering eight cities: a pure *set covering*
- Constructive covering heuristic for the primal bound
- Dual constructive heuristic for the dual bound
- The two hand-built bounds coincide: optimality is proved without the solver

EX 4 — the minimum number of hubs

An airline uses a *hub-and-spoke* system: a hub serves directly every city within 1000 miles. The cities within 1000 miles are:

City	Cities within 1000 miles
Atlanta	Atlanta, Chicago, Houston, New York
Chicago	Atlanta, Chicago, New York
Denver	Denver, Los Angeles
Houston	Atlanta, Houston
Los Angeles	Denver, Los Angeles, San Francisco
New York	Atlanta, Chicago, New York
San Francisco	Los Angeles, San Francisco, Seattle
Seattle	San Francisco, Seattle

The minimum number of hubs covering every city is wanted.

The variables

There are $n = 8$ cities and $m = 8$ candidate hubs: one binary variable per city.

$$y_j = \begin{cases} 1 & \text{if city } j \text{ becomes a hub,} \\ 0 & \text{otherwise,} \end{cases} \qquad \forall j \in \{1, 2, \ldots, 8\}.$$

Numbering the cities in alphabetical order, the data is the set $S_i \subseteq \{1, 2, \ldots, 8\}$ of the cities within 1000 miles of city i, i.e. those that can cover it.

The sets S_i

city i	S_i
1 Atlanta	$\{1, 2, 4, 6\}$
2 Chicago	$\{1, 2, 6\}$
3 Denver	$\{3, 5\}$
4 Houston	$\{1, 4\}$
5 Los Angeles	$\{3, 5, 7\}$
6 New York	$\{1, 2, 6\}$
7 San Francisco	$\{5, 7, 8\}$
8 Seattle	$\{7, 8\}$

EX 4 on the instance — primal

EX 4 on the instance — primal

$$
\begin{aligned}
\min \quad & y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 \\
\text{subject to} \quad & y_1 + y_2 + y_4 + y_6 \geq 1 \\
& y_1 + y_2 + y_6 \geq 1 \\
& y_3 + y_5 \geq 1 \\
& y_1 + y_4 \geq 1 \\
& y_3 + y_5 + y_7 \geq 1 \\
& y_1 + y_2 + y_6 \geq 1 \\
& y_5 + y_7 + y_8 \geq 1 \\
& y_7 + y_8 \geq 1 \\
& y_1, \; y_2, \; y_3, \; y_4, \; y_5, \; y_6, \; y_7, \; y_8 \in \{0, 1\}
\end{aligned}
$$

Description of the model

- A pure *set covering*: eight covering constraints, all costs equal to 1
- "Within 1000 miles" is a symmetric relation: $j \in S_i \iff i \in S_j$
- Hence the matrix of the model is symmetric
- The dual of the LP relaxation attaches a variable $\pi_i \geq 0$ to each covering constraint
- The column of y_j collects the cities that j can cover

EX 4 on the instance — dual of the LP relaxation

EX 4 on the instance — dual of the LP relaxation

$$
\begin{array}{rl}
\max & \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 + \pi_7 + \pi_8 \\
\text{subject to} & \pi_1 + \pi_2 + \pi_4 + \pi_6 \le 1 \\
& \pi_1 + \pi_2 + \pi_6 \le 1 \\
& \pi_3 + \pi_5 \le 1 \\
& \pi_1 + \pi_4 \le 1 \\
& \pi_3 + \pi_5 + \pi_7 \le 1 \\
& \pi_1 + \pi_2 + \pi_6 \le 1 \\
& \pi_5 + \pi_7 + \pi_8 \le 1 \\
& \pi_7 + \pi_8 \le 1 \\
& \pi_1, \ \pi_2, \ \pi_3, \ \pi_4, \ \pi_5, \ \pi_6, \ \pi_7, \ \pi_8 \ge 0
\end{array}
$$

A feasible solution: the primal bound

Constructive covering heuristic, criterion "cost per newly covered city":

- **Step 1.** Ratios 1/4 for Atlanta, 1/3 for Chicago, New York and San Francisco, 1/2 for the others: **Atlanta** is chosen (covers Atlanta, Chicago, Houston, New York)
- **Step 2.** Denver, Los Angeles, San Francisco, Seattle remain; ratios 1/2, 1/3, 1/3, 1/2: **Los Angeles** is chosen (covers Denver, Los Angeles, San Francisco)
- **Step 3.** Seattle remains; ratio 1/1 for San Francisco and Seattle: **San Francisco** is chosen

Three hubs, so $UB = 3$.

A dual solution: the meaning of the dual

Relaxing to $y_j \geq 0$, with $\pi_i \geq 0$ for every covering constraint ($\geq$ direction in a minimisation):

- π_i is the price attached to city i
- The objective adds up the prices of the eight cities
- The constraints are the columns of the y_j, one per hub: the cities hub j covers, priced at those values, cannot be worth more than what it costs to open it, namely 1

A dual solution: the dual bound

Dual constructive heuristic on the cities, in order: π_i is raised until the first opposing dual constraint becomes tight, and the residuals are updated.

- **Atlanta**: four hubs with residual 1; $\bar{\pi}_1 = 1$, the residuals of Atlanta, Chicago, Houston, New York drop to zero
- **Chicago**, **Houston**, **New York**: hubs at residual zero, $\bar{\pi} = 0$
- **Denver**: Denver and Los Angeles with residual 1; $\bar{\pi}_3 = 1$
- **Los Angeles**, **San Francisco**: one hub at residual zero, $\bar{\pi} = 0$
- **Seattle**: San Francisco and Seattle with residual 1; $\bar{\pi}_8 = 1$

Value $LB = 1 + 1 + 1 = 3$.

What this bound proves, in words

- The three cities with $\pi_i = 1$ — Atlanta, Denver, Seattle — are pairwise "far apart"
- No city can serve as a hub for two of them
- Hence at least three hubs are needed
- The dual has formalised an argument one could tell without writing it: that is the point of the constructive heuristic recipe

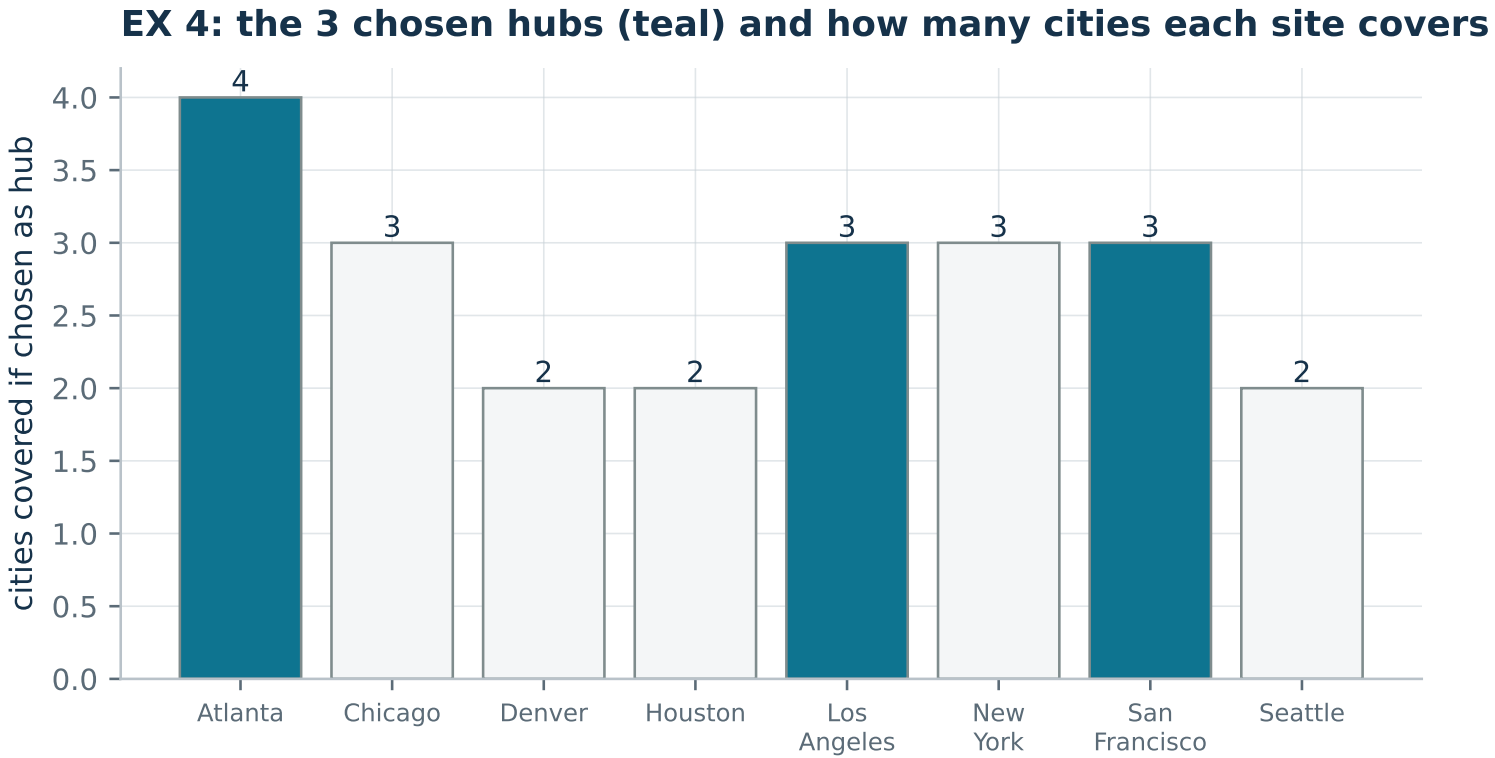
The final comparison

$z(MILP) = 3$, with hubs in Atlanta, Los Angeles and San Francisco: the same solution as the constructive heuristic.

UB (constructive heuristic)	3
LB (dual by hand)	3
$z(LP)$	3
$z(MILP)$	3
heuristic gap	0.0%

The two hand-built bounds coincide: optimality is proved without the solver.

How many cities every hub would cover



For every city, how many it would cover as a hub; in teal the three chosen ones.

Summary

- Pure set covering, symmetric matrix
- Heuristic "cost per newly covered city": $UB = 3$
- Dual constructive heuristic: $LB = 3$
- Atlanta, Denver, Seattle are pairwise far apart: at least three hubs are needed
- $LB = UB$: optimality proved by hand

EX 5 — Mutual funds bought in lots

Introduction: EX 5 — Mutual funds bought in lots

- An integer, non-binary knapsack: the variables count lots
- A share constraint written in linear form
- The heuristic trips over the best fund
- A dual attempt that looks natural but is not feasible

EX 5 — four mutual funds

- An investor must allocate 100 million lire among four mutual funds
- Shares can only be bought in indivisible lots, costing 12, 15, 18 and 25 million
- Estimated annual return: 1/6 of the capital for the first fund, 1/3 for the second, 1/9 for the third, 16% for the fourth
- The number of lots of the second fund cannot exceed 50% of the total number of lots
- The goal is to maximise the expected annual return

The variables

x_j = number of lots of fund j. The return of one lot is

$$12 \cdot \tfrac{1}{6} = 2, \qquad 15 \cdot \tfrac{1}{3} = 5, \qquad 18 \cdot \tfrac{1}{9} = 2, \qquad 25 \cdot 0.16 = 4.$$

The share constraint

$$x_2 \leq \tfrac{1}{2}(x_1 + x_2 + x_3 + x_4),$$

multiplied by 2 and moved to the left, becomes

$$-x_1 + x_2 - x_3 - x_4 \leq 0.$$

EX 5 — the model written out in full

EX 5 — the model written out in full

$$
\begin{aligned}
\max \quad & 2x_1 & +5x_2 & +2x_3 & +4x_4 & \\
\text{subject to} \quad & 12x_1 & +15x_2 & +18x_3 & +25x_4 & \leq 100 \\
& -x_1 & +x_2 & -x_3 & -x_4 & \leq 0 \\
& x_1, & x_2, & x_3, & x_4 & \in \mathbb{Z}_{\geq 0}
\end{aligned}
$$

A feasible solution: the primal bound

Return per million invested: 1/3 (second), 1/6 (first), 4/25 (fourth), 1/9 (third).

- Of the second, the best, no lot is bought: on its own it would break the share
- Of the first, 8 lots are bought (96 million); the 4 million left are not enough for any other lot
- The heuristic has tripped over the best fund: buying a lot of it pays, but only *together* with a lot of another fund

$$z(MILP) \geq LB = 16.$$

EX 5 on the instance — dual

EX 5 on the instance — dual

$$
\begin{aligned}
\min\ & 100\alpha & \\
\text{subject to}\ & 12\alpha - \beta \geq 2 \\
& 15\alpha + \beta \geq 5 \\
& 18\alpha - \beta \geq 2 \\
& 25\alpha - \beta \geq 4 \\
& \alpha \geq 0 \\
& \beta \geq 0
\end{aligned}
$$

An attempt that is not feasible

Observation 1: An attempt that is not feasible

A choice that comes naturally is $\bar{\alpha} = 5/32$, $\bar{\beta} = 1/8$. *First constraint:*

$$12 \cdot \frac{5}{32} - \frac{1}{8} = \frac{60}{32} - \frac{4}{32} = \frac{56}{32} = \frac{7}{4} < 2.$$

It is violated by $1/4$*: the pair is not dual feasible, and* $100 \cdot 5/32$ *is not a bound. A dual value is a bound only after checking all the constraints.*

A dual solution: the dual bound

The correct recipe is $\bar{\beta} = 0$ and

$$\bar{\alpha} = \max_j \frac{p_j}{c_j} = \max\left(\frac{2}{12}, \frac{5}{15}, \frac{2}{18}, \frac{4}{25}\right) = \frac{1}{3},$$

that is, a million is worth what it returns in the best fund. All the constraints become $c_j\,\bar{\alpha} \geq p_j$ and are satisfied:

$$z(MILP) \leq UB = 100 \cdot \frac{1}{3} = \frac{100}{3} \approx 33.33.$$

The final comparison

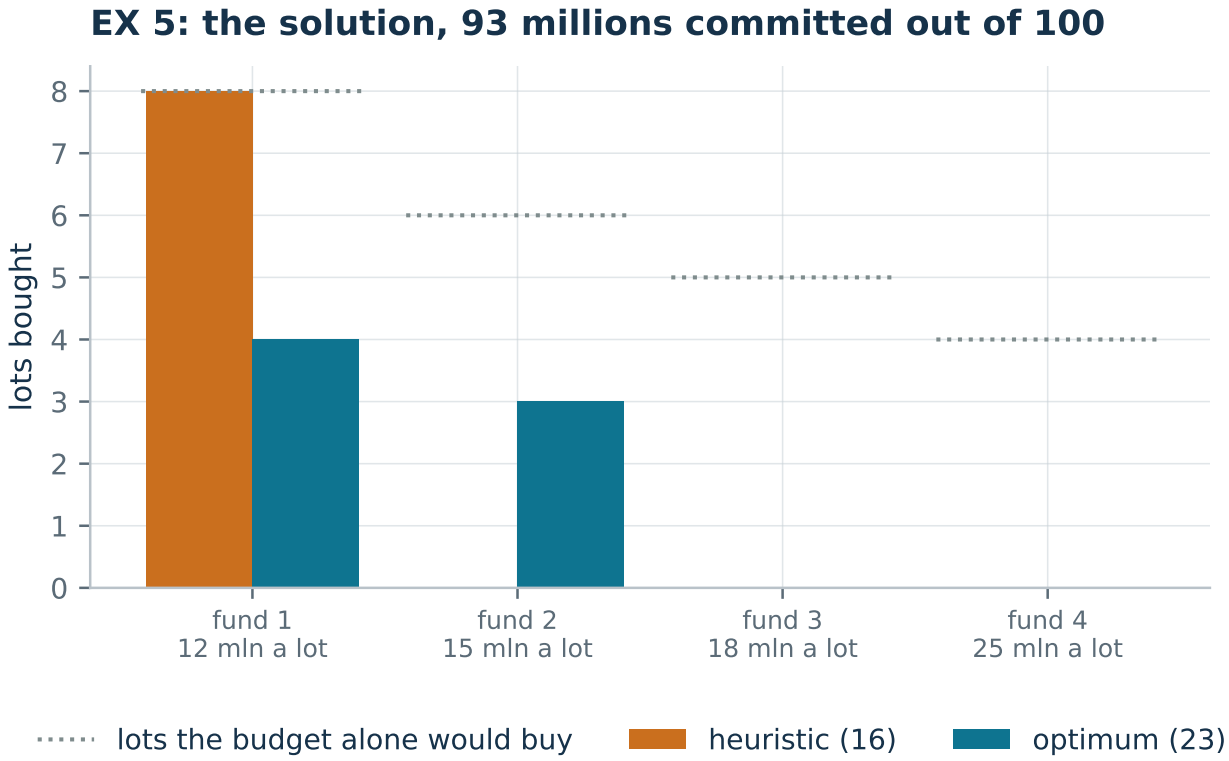
LB	UB	$z(LP)$	$z(LP^+)$	$z(MILP)$
16	$100/3$	$700/27$	$700/27$	23

- The optimum buys 4 lots of the first fund and 3 of the second: 93 million for a return of 23
- The heuristic stops at 16 and is 30% off
- $700/27 - 23 = 79/27$ is the price of integrality: the relaxation would buy lots in pieces

What the bounds and the constraints say

- The hand-built dual bound, 100/3, stays far away: "a million is worth what it returns in the best fund" ignores that the best fund cannot be bought on its own
- It is a valid recipe, not a good one
- Dropping the share constraint the optimum would rise from 23 to 30: six lots of the second fund and nothing else
- The share costs 7 million of return a year
- This is how one measures what a constraint does: remove it and see how much the optimum moves

Heuristic and optimum compared



The lots bought by the heuristic and at the optimum, with the number of lots the budget alone would allow for each fund.

Summary

- Integer knapsack with a linear share constraint
- Heuristic on the return per million: $LB = 16$, it trips over the best fund
- A dual value is a bound only if *all* the dual constraints hold
- Dual "a million is worth what it returns in the best fund": $UB = 100/3$
- Optimum $z(MILP) = 23$; the share costs 7

Final summary (1/2)

The scheme of the numerical models:

- Variables, primal and dual model on the instance
- A primal bound from a constructive heuristic, a dual bound from a dual recipe
- The solver optimum always lies between the two

What to measure:

- The heuristic gap and the certified gap
- The cost of a constraint: remove it and see how much the optimum moves

Final summary (2/2)

- **EX 1** van: $110 \leq 120 \leq 160$
- **EX 2** bus lines: $7 \leq 9 \leq 13$, exact relaxation
- **EX 3** relay: $93 \leq 95 \leq 99$, totally unimodular matrix
- **EX 4** hub-and-spoke: $3 = 3 = 3$, optimality proved by hand
- **EX 5** funds: $16 \leq 23 \leq 100/3$, beware of dual feasibility