

Complex numbers



Outline

Definition of $\mathbb{C}$ and field structure

Algebraic form of complex numbers

Trigonometric form of complex numbers

Quadratic equations

Definition of $\mathbb{C}$ and field structure

Introduction: Definition of $\mathbb{C}$ and field structure

- ▶ In this section we build the **complex numbers** as ordered pairs of real numbers, with suitable operations of sum and product
- ▶ We will see that with these operations $\mathbb{R}^2$ becomes a **field**, the field of complex numbers $\mathbb{C}$, which contains a copy of $\mathbb{R}$
- ▶ We will introduce the **imaginary unit** i , whose square is -1 : in $\mathbb{C}$ the equation $x^2 = -1$ has a solution

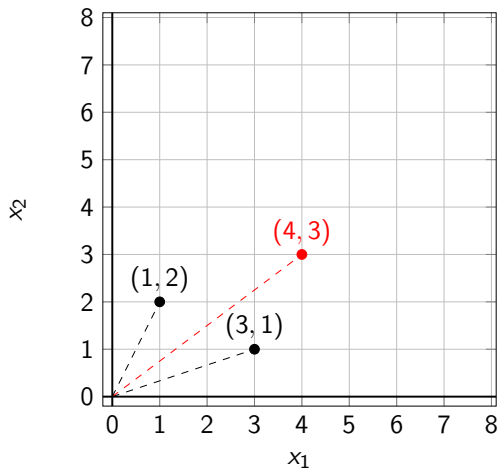
Sum and product in $\mathbb{R}^2$

- ▶ We have denoted by $\mathbb{R}^2$ (short for $\mathbb{R} \times \mathbb{R}$) the set of ordered pairs (a, b) of real numbers.
- ▶ On these pairs we directly define the operations of sum and product with the following rules:

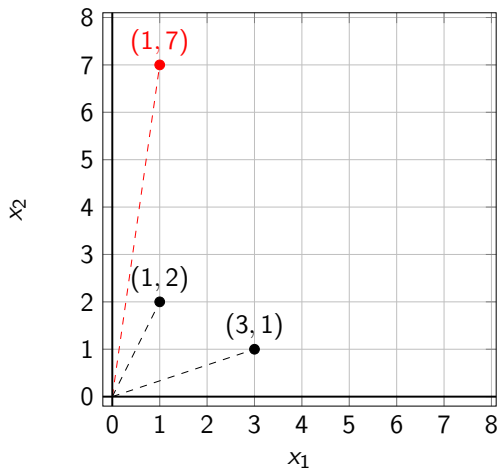
$$(a, b) + (c, d) = (a + c, b + d) \quad (1)$$

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc) \quad (2)$$

Example: sum



Example: product



Properties of sum and product (1/2)

- ▶ This “sum” and this “product” satisfy the **commutative**, **associative** and **distributive** properties
- ▶ We also observe that, $\forall (a, b) \in \mathbb{R}^2$:

$$(a, b) + (0, 0) = (0, 0) + (a, b) = (a, b)$$

hence the pair $(0, 0)$ is the **identity element for the sum**.
Moreover:

$$(a, b) \cdot (1, 0) = (1, 0) \cdot (a, b) = (a, b)$$

hence the pair $(1, 0)$ is the **identity element for the product**.

Properties of sum and product (2/2)

- We have:

$$(a, b) + (-a, -b) = (0, 0)$$

hence $(-a, -b)$ is the **opposite** of (a, b) .

- Moreover, if $(a, b) \neq (0, 0)$ then:

$$(a, b) \cdot \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right) = (1, 0)$$

hence the pair $(a/(a^2 + b^2), -b/(a^2 + b^2))$ is the **reciprocal** of (a, b) .

The field of complex numbers

Definition 1: field of complex numbers

*Properties R_1, R_2 are satisfied by the sum and the product defined above, and therefore the set $\mathbb{R}^2$ with this structure is a field, which we will call the **field of complex numbers** and denote by $\mathbb{C}$*

$\mathbb{C}$ is an extension of $\mathbb{R}$

- ▶ We now observe that $\mathbb{C}$ contains the subset $\mathbb{C}_0$ of pairs of the form $(a, 0)$; it is a subfield of $\mathbb{C}$, since the sum and product of pairs of this form are again pairs of the same form; indeed we have:

$$(a, 0) + (b, 0) = (a + b, 0) \quad \text{and} \quad (a, 0) \cdot (b, 0) = (a \cdot b, 0)$$

Moreover, $\mathbb{C}_0$ can be ordered by setting $(a, 0) < (b, 0)$ if $a < b$.

- ▶ If we then put the set of real numbers $\mathbb{R}$ in one-to-one correspondence with $\mathbb{C}_0$, by setting

$$(a, 0) \longleftrightarrow a$$

we can identify the real numbers $a \in \mathbb{R}$ with the complex numbers of the form $(a, 0) \in \mathbb{R}^2$. In this sense the field of complex numbers $\mathbb{C}$ is an **extension** of the field of real numbers $\mathbb{R}$.

The imaginary unit

Let us now consider the number $(0, 1) \in \mathbb{C}$. It has the remarkable property that:

$$(0, 1) \cdot (0, 1) = (-1, 0)$$

i.e., its square coincides with the real number -1 .

Definition 2: imaginary unit

*The pair $(0, 1) \in \mathbb{C}$ is denoted by the letter “ i ” and is called the **imaginary unit***

Summary

Il campo $\mathbb{C}$:

- ▶ $\mathbb{C}$ is $\mathbb{R}^2$ with sum $(a, b) + (c, d) = (a + c, b + d)$ and product $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$
- ▶ Identity elements $(0, 0)$ and $(1, 0)$; every $(a, b) \neq (0, 0)$ has a reciprocal
- ▶ Real numbers are identified with the pairs $(a, 0)$: $\mathbb{C}$ is an extension of $\mathbb{R}$

Imaginary unit:

- ▶ $i = (0, 1)$ and $i^2 = -1$

Algebraic form of complex numbers



Introduction: Algebraic form of complex numbers

- ▶ In this section we introduce the **algebraic form** $z = a + ib$, which allows us to compute with complex numbers as with algebraic expressions
- ▶ We will represent complex numbers in the **Gauss plane** and define **conjugate** and **modulus**
- ▶ We will see that $\mathbb{C}$ is not an ordered field, prove the triangle inequality and learn how to solve **equations in the complex field**

From pairs to the form $a + ib$

We observe that, if we write any complex number $(c, 0)$ simply as c , we have:

$$(a, b) = (a, 0) + \underbrace{(0, 1)}_{=i} \cdot (b, 0) = a + ib$$

With this notation, rules (1) and (2) are the ordinary rules of algebraic calculation, keeping in mind that $i^2 = -1$:

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

$$(a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc)$$

Definition 3: algebraic form, real part and imaginary part

The expression:

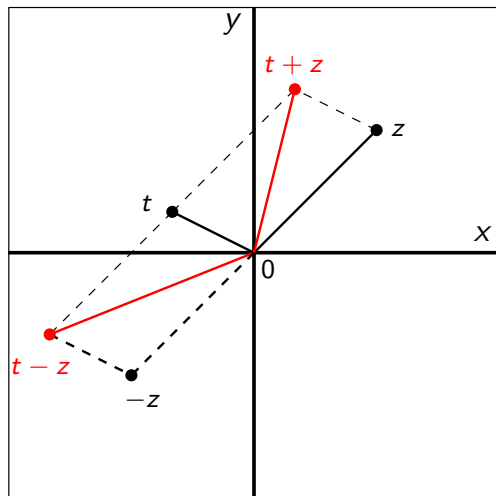
$$z = a + ib$$

*is called the **algebraic form of complex numbers**; a is called the **real part** of z and is denoted by $\Re(z)$ (or $\operatorname{Re}(z)$) while b is called the **imaginary part** and is denoted by $\Im(z)$ (or $\operatorname{Im}(z)$).*

Complex plane

- ▶ In a Cartesian plane, the complex numbers $a + ib$ can be represented as points with coordinates (a, b) . In this context:
 - ▶ the plane is called the **complex plane** or **Gauss plane**
 - ▶ the x, y axes are called the **real axis** and the **imaginary axis**
 - ▶ the points on the real axis are the **real numbers**
 - ▶ the points on the imaginary axis are the **purely imaginary numbers** (i.e., of the form ib)
- ▶ The **sum of two complex numbers** is the complex number whose coordinates are the sums of the coordinates: the geometric meaning of this fact is that the point $z + t$ is constructed from the points z, t according to the “**parallelogram rule**” (figure in the next slide).

Parallelogram rule



$\mathbb{C}$ is not ordered

Observation 1

The set of complex numbers $\mathbb{C}$ is not an ordered field

Proof

- ▶ We have seen that $\mathbb{C}$ satisfies the field axioms; however, it does not satisfy those of an ordered field, that is, it is not possible to define a relation $\leq$ between complex numbers in such a way that properties $R3$ hold
- ▶ It can be proved that properties $R3$ imply that the square of any number is never negative, and on the other hand, if a number is positive its opposite is negative. Now, in $\mathbb{C}$ we have:

$$1^2 = 1 \quad \text{and} \quad i^2 = -1$$

We therefore have two squares, each the opposite of the other. Neither of them, however, can be negative (because they are squares), and this is absurd (because between a and $-a$ one must be negative, if $a \neq 0$). We conclude that $\mathbb{C}$ is not an ordered field.



Complex conjugate

Definition 4: conjugate

*The complex number $a - ib$ is called the **complex conjugate** of $z = a + ib$ and is denoted by $\bar{z}$.*

We have:

$$z + \bar{z} = (a + ib) + (a - ib) = 2a = 2 \Re(z)$$

$$z - \bar{z} = (a + ib) - (a - ib) = 2ib = 2i \Im(z)$$

$$z \cdot \bar{z} = (a + ib) \cdot (a - ib) = a^2 - aib + iba - \underbrace{i^2}_{=-1} b^2 = a^2 + b^2 \geq 0$$

Properties of conjugation

- ▶ The conjugation operation has the following elementary properties with respect to sum and product:

$$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$$

$$\overline{(z_1 \cdot z_2)} = \bar{z}_1 \cdot \bar{z}_2$$

$$\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$$

Definition 5: modulus

The **modulus** of $z = a + ib$ is the non-negative real number $\sqrt{a^2 + b^2}$; it is denoted by $|z|$.

- ▶ If $z = a$ is real, its modulus is called the absolute value and is still denoted by $|a|$. The following properties hold:
 - a) $|z| = 0 \iff z = 0$, moreover $|z| \geq 0$
 - b) $|z| = |\bar{z}|$
 - c) $\Re(z) \leq |z|$ $\Im(z) \leq |z|$ $|z| \leq |\Re(z)| + |\Im(z)|$
 - d) $|z_1 + z_2| \leq |z_1| + |z_2|$ **triangle inequality**
 - e) $|z_1 + z_2| \geq ||z_1| - |z_2||$
- ▶ Properties a), b) , c) are immediately verified.

Proof (1/3)

- ▶ Let us prove properties d) and e). They are equivalent to the following:

$$(|z_1| - |z_2|)^2 \leq |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$$

- ▶ Setting $z_1 = a + ib$, $z_2 = c + id$ we obtain:

$$\begin{aligned} |z_1 + z_2|^2 &= |(a + ib) + (c + id)|^2 = |(a + c) + i(b + d)|^2 \\ &= \left(\sqrt{(a + c)^2 + (b + d)^2} \right)^2 = (a + c)^2 + (b + d)^2 \end{aligned}$$

Proof (2/3)

and hence we have:

$$\begin{aligned}\left(\sqrt{a^2 + b^2} - \sqrt{c^2 + d^2}\right)^2 &\leq (a + c)^2 + (b + d)^2 \\ &\leq \left(\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2}\right)^2\end{aligned}$$

where, expanding the squares:

$$\begin{aligned}\left(\sqrt{a^2 + b^2} \mp \sqrt{c^2 + d^2}\right)^2 &= (a^2 + b^2) + (c^2 + d^2) \\ &\quad \mp 2\sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2} \\ (a + c)^2 + (b + d)^2 &= a^2 + 2ac + c^2 + b^2 + 2bd + d^2\end{aligned}$$

After simplification, this double inequality reduces to:

$$-\sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2} \leq ac + bd \leq \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2}$$

Proof (3/3)

which is equivalent to the following:

$$|ac + db| \leq \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2}$$

Squaring both sides we arrive at:

$$\underbrace{(ac + db)^2}_{=a^2c^2+d^2b^2+2acbd} \leq \underbrace{(a^2 + b^2) \cdot (c^2 + d^2)}_{=a^2c^2+a^2d^2+b^2c^2+b^2d^2}$$

that is, to

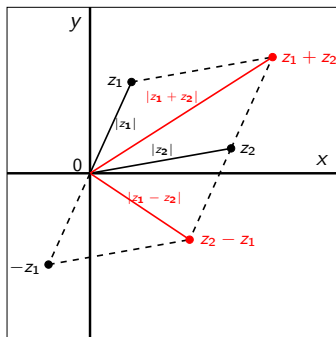
$$0 \leq -2acbd + a^2d^2 + b^2c^2 = (ad - bc)^2$$

which is true for every $a, b, c, d \in \mathbb{R}$.



Geometric interpretation of the modulus

- ▶ Geometrically, $|z|$ represents the distance of the point (or complex number) z from the origin; $|z_1 - z_2|$ represents the distance between the two points z_1 and z_2 ; inequalities d) and e) express the well-known theorem on the **lengths of the sides of a triangle** (in a non-degenerate triangle, the sum of the lengths of two sides is greater than the length of the third one):



Quotient of two complex numbers

- Using the concepts just introduced, we can write the quotient of two complex numbers in algebraic form:

$$\frac{a + ib}{c + id}$$

it suffices to multiply the numerator and the denominator by $c - id$, and we obtain:

$$\begin{aligned}\frac{a + ib}{c + id} &= \frac{(a + ib)(c - id)}{\underbrace{(c + id)(c - id)}_{=c^2+d^2}} = \frac{ac - aid + ibc - i^2bd}{c^2 + d^2} \\ &= \frac{(ac + bd)}{c^2 + d^2} + i \frac{(bc - ad)}{c^2 + d^2}\end{aligned}$$

Equations in the complex field

- ▶ Let us see how to solve an equation in the complex field, when it involves the unknown $z = x + iy$ also through $\Re(z)$, $\Im(z)$, z , $|z|$.
- ▶ We illustrate, with the following example, the procedure of transforming the equation in one complex unknown into a system of two equations in two real unknowns. The method consists in taking the real and imaginary parts of the equation.

Example (1/3)

- We want to solve:

$$z^2 + i \Im z + 2 \bar{z} = 0$$

We set $z = x + iy$, with x, y real unknowns, and rewrite the equation:

$$\begin{cases} z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \\ i \Im z = iy \\ 2\bar{z} = 2(x - iy) = 2x - 2iy \end{cases}$$

Substituting, we therefore obtain:

$$(x^2 - y^2 + 2ixy) + (iy) + (2x - 2iy) = 0$$

Example (2/3)

- Now, a complex number is zero if and only if its real part and imaginary part are zero. Therefore we separate the real part and the imaginary part of the left-hand side:

$$(x^2 - y^2 + 2x) + i(2xy + y - 2y) = 0$$

and set both equal to zero:

$$\begin{cases} x^2 - y^2 + 2x = 0 \\ 2xy - y = 0 \end{cases}$$

We have thus transformed the equation in one complex unknown into a system of two equations in two real unknowns.

Example (3/3)

- We solve the system. The second equation gives:

$$y = 0 \quad \text{or} \quad x = \frac{1}{2}$$

1. For $y = 0$ the first equation becomes $x^2 + 2x = 0$, which gives $x = 0$ or $x = -2$
2. For $x = \frac{1}{2}$ the first equation becomes $-y^2 + \frac{5}{4} = 0$, which gives $y = \pm \frac{\sqrt{5}}{2}$

- Hence the equation has the following 4 solutions:

$$z = 0, \quad z = -2, \quad z = \frac{1}{2} + i \frac{\sqrt{5}}{2}, \quad z = \frac{1}{2} - i \frac{\sqrt{5}}{2}$$

The method seen in this example can in principle be applied to any equation in $\mathbb{C}$, but a generic system of two equations in two unknowns is almost always impossible to solve algebraically.

Summary

Algebraic form:

- ▶ $z = a + ib$, with $\Re(z) = a$ and $\Im(z) = b$; we compute as in algebraic calculation with $i^2 = -1$
- ▶ Gauss plane: the sum follows the parallelogram rule
- ▶ $\mathbb{C}$ is not an ordered field

Conjugate and modulus:

- ▶ $\bar{z} = a - ib$, $z \bar{z} = a^2 + b^2 = |z|^2$
- ▶ $|z_1 + z_2| \leq |z_1| + |z_2|$ and $|z_1 + z_2| \geq ||z_1| - |z_2||$
- ▶ Equations in $\mathbb{C}$: separate the real part and the imaginary part

Trigonometric form of complex numbers



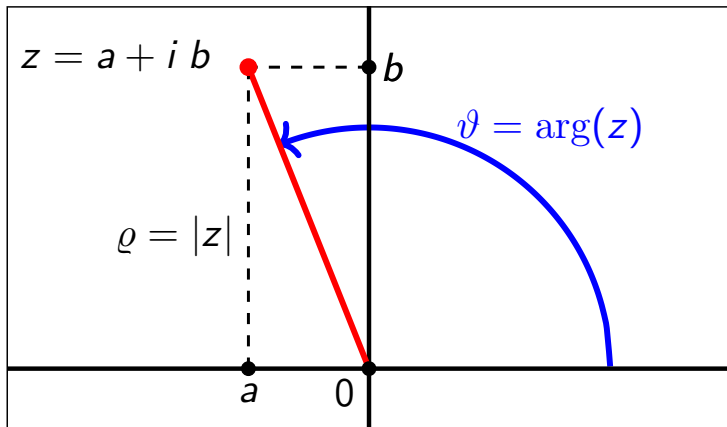
Introduction: Trigonometric form of complex numbers

- ▶ In this section we represent complex numbers with **polar coordinates**: modulus and argument
- ▶ With the trigonometric form, products, quotients and powers become simple (**De Moivre's formulas**) and acquire a geometric meaning
- ▶ We will compute the **n -th roots** of complex numbers and introduce the **exponential form** and Euler's formula

Polar coordinates

- ▶ As is known from Geometry, the points of the plane can be identified not only by their Cartesian coordinates, but also by their **polar coordinates**:
 1. $\varrho \rightarrow$ **polar radius**, i.e., the distance of the point from the origin
 2. $\vartheta \rightarrow$ **polar angle**, i.e., the angle that the line joining the point to the origin forms with the positive x-axis, measured counterclockwise.
- ▶ It is clear that a pair ϱ, ϑ , with $\varrho > 0$, identifies a well-defined point of the plane; conversely, a point of the plane uniquely determines the coordinate ϱ , but the angle ϑ , measured in radians, is determined only up to multiples of 2π .

Polar coordinates: figure



Argument of a complex number

- ▶ Given a complex number z , its **modulus** $|z|$ **coincides with the polar radius** ϱ of the point representing it in the complex plane.
- ▶ We call **argument** of z , and denote by $\arg(z)$, any of the angles ϑ associated with the point z . In this way the argument of z is not uniquely determined. Often this indeterminacy does not cause any problem. At other times, however, it is preferable to assign a well-defined argument to a complex number. This can be done in infinitely many ways, by fixing any interval of length 2π within which the angle ϑ is allowed to vary
- ▶ The intervals most commonly used for this purpose are $[0, 2\pi)$ and $(-\pi, \pi]$; the argument of z is then called the **principal argument**.

Example

- ▶ The number $-i$ has argument $-\pi/2$ or $3\pi/2$ or any other value of the form $-\pi/2 + 2k\pi$ with $k \in \mathbb{Z}$. Its principal argument will be $3\pi/2$ if we adopt the convention $\vartheta \in [0, 2\pi)$, and $-\pi/2$ with the convention $\vartheta \in (-\pi, \pi]$.
- ▶ Positive real numbers have principal argument 0 and negative ones π , with both conventions.

Relations between Cartesian and polar coordinates

Given the number $z = a + ib$, from trigonometry we immediately obtain the relations between the Cartesian coordinates a , b and the polar ones ϱ , ϑ :

$$a = \varrho \cos \vartheta \quad \text{and} \quad b = \varrho \sin \vartheta$$

The inverse relations are:

$$\varrho = \sqrt{a^2 + b^2}, \quad \cos \vartheta = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \sin \vartheta = \frac{b}{\sqrt{a^2 + b^2}}$$

Trigonometric form

Definition 6: trigonometric form

A complex number $z = a + ib$ can also be written in the form

$$z = \varrho (\cos \vartheta + i \sin \vartheta)$$

*which is called the **trigonometric form** of complex numbers.*

Example

Let us write the following complex number in trigonometric form:

$$z = \sqrt{3} + i$$

We have $\varrho = \sqrt{a^2 + b^2} = \sqrt{3 + 1} = 2$, hence:

$$\sqrt{3} + i = 2 \left(\frac{\sqrt{3}}{2} + i \frac{1}{2} \right) = 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

De Moivre's formulas: product

- The trigonometric form is convenient for expressing products and quotients of complex numbers. Indeed, if we have:

$$z_1 = \varrho_1 (\cos \vartheta_1 + i \sin \vartheta_1) \quad z_2 = \varrho_2 (\cos \vartheta_2 + i \sin \vartheta_2)$$

for the **product** we obtain:

$$\begin{aligned} z_1 z_2 &= \varrho_1 \varrho_2 \cdot \left\{ \cos \vartheta_1 \cos \vartheta_2 - \sin \vartheta_1 \sin \vartheta_2 \right. \\ &\quad \left. + i (\sin \vartheta_1 \cos \vartheta_2 + \cos \vartheta_1 \sin \vartheta_2) \right\} \quad (3) \\ &= \varrho_1 \varrho_2 \cdot \left\{ \cos (\vartheta_1 + \vartheta_2) + i \sin (\vartheta_1 + \vartheta_2) \right\} \end{aligned}$$

De Moivre's formulas: quotient

- For the **quotient**, if $z_2 \neq 0$, we have:

$$\frac{z_1}{z_2} = \frac{\varrho_1}{\varrho_2} \cdot \frac{\cos \vartheta_1 + i \sin \vartheta_1}{\cos \vartheta_2 + i \sin \vartheta_2},$$

multiplying numerator and denominator by $(\cos \vartheta_2 - i \sin \vartheta_2)$ and, taking into account that $(\cos \vartheta_2)^2 + (\sin \vartheta_2)^2 = 1$, we obtain:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{\varrho_1}{\varrho_2} \cdot \left\{ (\cos \vartheta_1 + i \sin \vartheta_1) \cdot (\cos \vartheta_2 - i \sin \vartheta_2) \right\} \\ &= \frac{\varrho_1}{\varrho_2} \cdot \left\{ \cos (\vartheta_1 - \vartheta_2) + i \sin (\vartheta_1 - \vartheta_2) \right\} \end{aligned}$$

Modulus and argument of product and quotient

- Therefore the modulus of the product and of the quotient of two complex numbers is, respectively, the product and the quotient of the moduli; the argument is, respectively, the sum and the difference of the arguments:

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2| \quad \text{and} \quad \arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

De Moivre's formulas: powers

- ▶ Formula (3) generalizes to the case of any number of factors $z_1, z_2, \dots, z_n$:

$$z_1 z_2 \dots z_n = \varrho_1 \varrho_2 \dots \varrho_n \cdot \left\{ \cos(\vartheta_1 + \dots + \vartheta_n) + i \sin(\vartheta_1 + \dots + \vartheta_n) \right\}$$

If, moreover, all the factors are equal, we obtain:

$$z^n = \varrho^n \cdot \left\{ \cos(n \vartheta) + i \sin(n \vartheta) \right\}$$

- ▶ These relations on products and quotients of complex numbers are known as De Moivre's formulas.

Example

Write in algebraic form: $z = (1 + i)^7$. We determine the modulus and argument of $(1 + i)$ and then apply De Moivre's formula.

$$|1+i| = \sqrt{2} \quad \text{and} \quad \arg(1+i) = \frac{\pi}{4} \quad \left(\cos \vartheta = \frac{1}{\sqrt{2}}, \quad \sin \vartheta = \frac{1}{\sqrt{2}} \right)$$

Hence we have:

$$|(1+i)^7| = \left(\sqrt{2}\right)^7 = 2^{\frac{7}{2}} = 2^{3+\frac{1}{2}} = 8\sqrt{2} \quad \text{and} \quad \arg(1+i)^7 = \frac{7}{4} \pi$$

Consequently:

$$\begin{aligned} (1+i)^7 &= 8\sqrt{2} \left(\underbrace{\cos \frac{7}{4} \pi}_{=\frac{1}{\sqrt{2}}} + i \underbrace{\sin \frac{7}{4} \pi}_{=-\frac{1}{\sqrt{2}}} \right) \\ &= 8\sqrt{2} \left(\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) = 8 - 8i \end{aligned}$$

Geometric interpretation of the product

- ▶ De Moivre's formulas allow us to give a geometric interpretation of the product of complex numbers.
- ▶ Let z , to begin with, be a complex number with modulus 1, hence of the form $(\cos \vartheta + i \sin \vartheta)$. Then, multiplying a number by z means adding ϑ to its argument, i.e., performing a **rotation by the angle ϑ** .
- ▶ If z has modulus ϱ instead of 1, in addition to a rotation we perform a **dilation by the factor ϱ** .

Example

- ▶ multiplying by i means performing a rotation by $\frac{\pi}{2}$;
- ▶ multiplying by -1 means performing a rotation by π ;
- ▶ multiplying by $(1 + i)$ means performing a dilation by the factor $\sqrt{2}$ and a rotation by $\frac{\pi}{4}$

Example (1/2)

- We want to solve: $z^3 - |z| = 0$. We rewrite the equation in the form $z^3 = |z|$ and we set $z = \varrho (\cos \vartheta + i \sin \vartheta)$ (trigonometric form) and rewrite the equation:

$$\begin{cases} z^3 = \varrho^3 (\cos 3\vartheta + i \sin 3\vartheta) \\ |z| = \varrho \end{cases}$$

Substituting, we therefore obtain:

$$\varrho^3 (\cos 3\vartheta + i \sin 3\vartheta) = \varrho$$

Example (2/2)

The equation is satisfied if and only if the two sides have equal moduli and arguments that differ by multiples of 2π (the right-hand side has argument 0), that is:

$$\begin{cases} \varrho^3 = \varrho \\ 3\vartheta = 2k\pi \quad \text{with } k \in \mathbb{Z} \end{cases}$$

The first equation gives $\varrho = 0$ and $\varrho = 1$ (careful: ϱ must be ≥ 0 because it is the modulus of the complex number; therefore $\varrho = -1$ is not acceptable); the second one gives $\vartheta = \frac{2k\pi}{3}, k \in \mathbb{Z}$. Hence:

$$z = 0, \quad z = \cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3} \quad \text{with } k \in \mathbb{Z}$$

Explicitly:

$$z = 0, \quad z = 1, \quad z = -\frac{1}{2} + i \frac{\sqrt{3}}{2}, \quad z = -\frac{1}{2} - i \frac{\sqrt{3}}{2}$$

n -th roots of complex numbers

Definition 7: n -th root of a complex number

Given a complex number w , we say that z is an n -th (complex) root of w if $z^n = w$.

The n n -th roots

Theorem 1

Let $w \in \mathbb{C}$, $w \neq 0$, and n an integer ≥ 1 . There exist exactly n complex n -th roots $z_0, z_1, \dots, z_{n-1}$ of w ; setting

$$w = r (\cos \varphi + i \sin \varphi) \quad \text{and} \quad z_k = \varrho_k (\cos \vartheta_k + i \sin \vartheta_k)$$

we have

$$\begin{cases} \varrho_k = r^{1/n} \\ \vartheta_k = \frac{\varphi + 2k\pi}{n} \end{cases} \quad k = 0, 1, 2, \dots, n-1$$

Proof (1/2)

- ▶ The numbers z_k are clearly roots of w , as can be seen by computing z_k^n with De Moivre's formula. Let us show that there are no others.
- ▶ If a number $R(\cos \psi + i \sin \psi)$ is an n -th root of w , we must have

$$R^n = r \quad \text{and} \quad n\psi = \varphi + 2h\pi \quad \text{with} \quad h \in \mathbb{Z}$$

or equivalently:

$$R = r^{1/n} \quad \text{and} \quad \psi = \varphi/n + 2h\pi/n \quad \text{with} \quad h \in \mathbb{Z}$$

Proof (2/2)

- ▶ Giving h the values $0, 1, \dots, n-1$ we find exactly the numbers z_k .
- ▶ Giving h any other value $\bar{h}$ different from the previous ones, it can be written in the form $\bar{h} = k + mn$ ($m \in \mathbb{Z}$ is the quotient and k is the remainder of the division of $\bar{h}$ by n), so that we would have $\psi = \frac{\varphi}{n} + \frac{2k\pi}{n} + 2m\pi = \vartheta_k + 2m\pi$ and we would find again the same z_k as before.



Example

- ▶ Let us compute $\sqrt[5]{1+i}$: the complex number $(1+i)$ has modulus $\sqrt{2}$ and argument $\pi/4$. The fifth roots will therefore be:

$$\begin{aligned}\sqrt[5]{1+i} &= \sqrt[5]{\sqrt{2}} \left[\cos \left(\frac{\frac{\pi}{4} + 2k\pi}{5} \right) + i \sin \left(\frac{\frac{\pi}{4} + 2k\pi}{5} \right) \right] \\ &= \sqrt[10]{2} \left[\cos \left(\frac{\pi}{20} + \frac{2}{5}k\pi \right) + i \sin \left(\frac{\pi}{20} + \frac{2}{5}k\pi \right) \right]\end{aligned}$$

with $k = 0, 1, 2, 3, 4$.

- ▶ The angles found are not standard angles, but if desired, their sines and cosines can be computed approximately using a calculator.

Notation for complex roots

- ▶ Unfortunately, a somewhat ambiguous notation is used for complex roots, the same one used to denote the arithmetic root; that is, $\sqrt[n]{z}$ or $z^{1/n}$ denotes the set of the n complex roots of z .
- ▶ This can create confusion when z is real. Indeed, the symbol $\sqrt{4}$, understood as the arithmetic root of 4, is 2; understood as the complex root of 4, it is the set of the two numbers $+2$ and -2 .

Example (1/2)

- Let us compute $\sqrt[3]{-1}$: the complex number (-1) has modulus 1 and argument π . The cube roots will therefore be:

$$\sqrt[3]{-1} = \underbrace{\sqrt[3]{1}}_{=1} \left[\cos \left(\frac{\pi}{3} + \frac{2}{3}k\pi \right) + i \sin \left(\frac{\pi}{3} + \frac{2}{3}k\pi \right) \right]$$

with $k = 0, 1, 2$, that is, the numbers z_k of the form:

$$\cos \vartheta_k + i \sin \vartheta_k \quad \text{with} \quad \vartheta_k = \frac{\pi}{3} + \frac{2}{3}k\pi \quad \text{and} \quad k = 0, 1, 2$$

Example (2/2)

► Explicitly we have:

$$z_0 = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i \frac{\sqrt{3}}{2} = \frac{1}{2}(1 + i\sqrt{3})$$

$$z_1 = \cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) + i \sin\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) = \cos(\pi) + i \sin(\pi) = -1$$

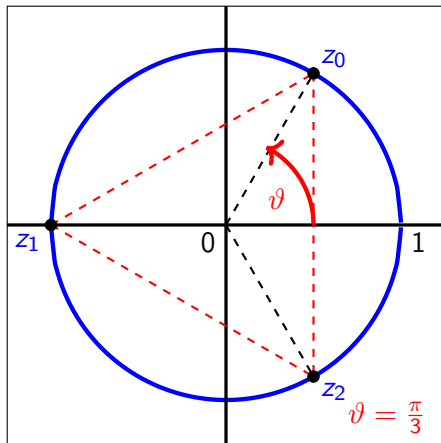
$$\begin{aligned} z_2 &= \cos\left(\frac{\pi}{3} + \frac{4\pi}{3}\right) + i \sin\left(\frac{\pi}{3} + \frac{4\pi}{3}\right) \\ &= \cos\left(\frac{5\pi}{3}\right) + i \sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - i \frac{\sqrt{3}}{2} = \frac{1}{2}(1 - i\sqrt{3}) \end{aligned}$$

Arrangement of the roots in the Gauss plane

- ▶ The arrangement of the roots of complex numbers in the Gauss plane is not random.
- ▶ Indeed, if $w = r(\cos \varphi + i \sin \varphi)$, the n -th roots $z_0, z_1, \dots, z_{n-1}$ of w lie at the vertices of the regular polygon with n sides inscribed in the circle with center 0 and radius $r^{1/n}$, with the vertex z_0 located at the point with argument $\vartheta = \varphi/n$.

Example (1/3)

- The figure shows the cube roots of -1 : z_0, z_1, z_2 from the previous exercise:



Example (2/3)

- Sixth roots of i : $z_0, z_1, z_2, z_3, z_4, z_5$,

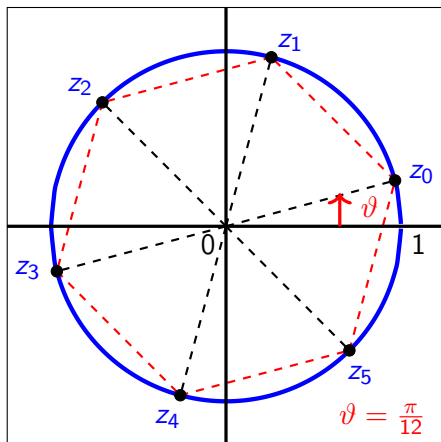
$$\sqrt[6]{i} = \underbrace{\sqrt[6]{1}}_{=1} \left[\cos \left(\frac{\pi}{2 \cdot 6} + \frac{2}{6} k \pi \right) + i \sin \left(\frac{\pi}{2 \cdot 6} + \frac{2}{6} k \pi \right) \right]$$

with $k = 0, 1, 2, 3, 4, 5$, that is, the numbers z_k of the form:

$$\cos \vartheta_k + i \sin \vartheta_k \quad \text{with} \quad \vartheta_k = \frac{\pi}{12} + \frac{1}{3} k \pi, \quad k = 0, 1, 2, 3, 4, 5$$

Example (3/3)

- The figure shows the sixth roots of i : $z_0, z_1, z_2, z_3, z_4, z_5$:

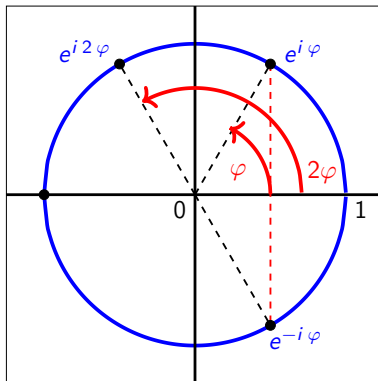


Exponential form of complex numbers

- The following notation is useful:

$$e^{i\varphi} = \cos \varphi + i \sin \varphi \quad \text{with } \varphi \in \mathbb{R}$$

where e is Napier's constant.



Exponential form

With this notation, the *trigonometric form* can be equivalently rewritten as:

$$z = |z| (\cos \varphi + i \sin \varphi) = |z| e^{i \varphi}, \quad \varphi = \arg(z) \quad \text{with } z \neq 0$$

This notation is called the **exponential form** of complex numbers.

- This form is useful since we have:

$$e^{i \varphi_1} \cdot e^{i \varphi_2} = e^{i (\varphi_1 + \varphi_2)}$$

that is, it is very easy to compute products of complex numbers. **De Moivre's formulas** become:

$$(e^{i \varphi})^n = e^{i n \varphi}$$

Properties of the exponential form and Euler's formula

- The exponential notation is widely used in applications because it greatly simplifies manipulations involving trigonometric quantities. We have:

$$e^{2k\pi i} = 1, \quad e^{i(\varphi+2k\pi)} = e^{i\varphi} \quad \forall k \in \mathbb{Z}, \quad |e^{i\varphi}| = 1$$

We also have:

$$e^{i\pi} = -1, \quad e^{i\frac{\pi}{2}} = i, \quad e^{i\frac{3\pi}{2}} = -i, \quad e^{i\frac{3\pi}{4}} = \frac{-1+i}{\sqrt{2}}$$

Rewriting, we obtain **Euler's formula**:

$$e^{i\pi} + 1 = 0$$

which links in a simple way five of the most important constants: 0, 1, i , π , e .

Complex exponential

The exponential notation can be generalized consistently with the properties of powers, by defining the **complex exponential**:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y), \quad \forall z = x + iy \in \mathbb{C}$$

- From the definition of exponential form it follows that

$$e^{-i\varphi} = \cos(-\varphi) + i \sin(-\varphi) = \cos \varphi - i \sin \varphi$$

and we easily obtain:

$$\cos \varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}, \quad \sin \varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i} \quad \text{with } \varphi \in \mathbb{R}$$

These formulas allow us to express sine and cosine as combinations of complex exponentials.

Summary

Trigonometric form:

- ▶ $z = \varrho(\cos \vartheta + i \sin \vartheta)$, with $\varrho = |z|$ and $\vartheta = \arg(z)$ (defined up to $2k\pi$)
- ▶ De Moivre: moduli multiply and arguments add;
 $z^n = \varrho^n(\cos n\vartheta + i \sin n\vartheta)$

Roots and exponential form:

- ▶ Every $w \neq 0$ has n n -th roots, vertices of a regular polygon with n sides
- ▶ $e^{i\varphi} = \cos \varphi + i \sin \varphi$, $z = |z|e^{i\varphi}$, Euler's formula $e^{i\pi} + 1 = 0$

Quadratic equations

Introduction: Quadratic equations

- ▶ In this section we derive the **quadratic formula** with the method of completing the square
- ▶ We will see that, when the discriminant is negative, the solutions exist in the complex field and are two complex conjugate numbers
- ▶ We will conclude with the **fundamental theorem of algebra**: in $\mathbb{C}$ every polynomial equation of degree n has exactly n roots

Quadratic equations

- ▶ A **second-degree equation** or **quadratic equation** in one **real** unknown x is an algebraic equation in which the highest power of the unknown is 2, and it can always be written in the form:

$$a x^2 + b x + c = 0 \quad (a \neq 0)$$

where a , b and c are real numbers.

- ▶ The solutions are also called **roots** or **zeros** of the equation.

Quadratic formula

Observation 2: quadratic formula

Given a quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$), its zeros or roots are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- ▶ A quadratic equation is called a “complete quadratic equation” when all its coefficients are different from 0. It is solved with the so-called **method of completing the square**, so called because the equation is modified until its left-hand side becomes the square of a binomial.

Proof

Isolating the constant term, we obtain:

$$a x^2 + b x = -c$$

$$\underbrace{4 a^2 x^2}_{=(2 a x)^2} + \underbrace{4 a b x}_{=2 (2 a x) b} = -4 a c$$

$$(2 a x)^2 + 2 (2 a x) b + b^2 = b^2 - 4 a c$$

$$(2 a x + b)^2 = \underbrace{b^2 - 4 a c}_{:=\Delta \text{ (discriminant)}}$$

$$2 a x + b = \pm \sqrt{b^2 - 4 a c}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4 a c}}{2 a}$$



The role of the discriminant

- ▶ If the discriminant Δ is negative there are no real solutions.
- ▶ If $\Delta = 0$, the quadratic formula becomes:

$$x = -\frac{b}{2a}$$

therefore there is only one root, with *multiplicity two*.

- ▶ Finally, if $\Delta < 0$, the equation has no real solutions. In particular, there are always two solutions, but they belong to the field of complex numbers: they are **two complex conjugate numbers** and are computed with the two formulas:

$$x_+ = \frac{-b}{2a} + i \left(\frac{\sqrt{4ac - b^2}}{2a} \right), \quad x_- = \frac{-b}{2a} - i \left(\frac{\sqrt{4ac - b^2}}{2a} \right)$$

where i is the **imaginary unit** ($i^2 = -1$).

Quadratic equations in $\mathbb{C}$

A quadratic equation in one **complex** unknown z has the following form

$$a z^2 + b z + c = 0 \quad (a \neq 0)$$

where a , b and c are complex numbers. It is solved with the same formula as in the case of a real unknown:

$$z = \frac{-b \pm \sqrt{b^2 - 4 a c}}{2 a}$$

where the square root is understood in the complex sense (the $\pm$ sign is superfluous, because in the complex field the root denotes two numbers, each the opposite of the other).

Example (1/2)

- We want to solve:

$$z^2 + 2 i z - \sqrt{3}i = 0 \quad a = 1, b = 2 i, c = -\sqrt{3}i$$

we have (switching to the trigonometric form to compute the root):

$$\begin{aligned} z &= \frac{-2 i \pm \sqrt{2^2 i^2 + 2^2 \sqrt{3}i}}{2} = \frac{-2 i \pm 2\sqrt{-1 + \sqrt{3}i}}{2} \\ &= -i \pm \sqrt{-1 + \sqrt{3}i} \end{aligned}$$

Example (2/2)

Continuing:

$$\begin{aligned} z &= -i \pm \sqrt{2 \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2} \right)} = -i \pm \sqrt{2 \left(\cos \frac{2}{3}\pi + i \sin \frac{2}{3}\pi \right)} \\ &= -i \pm \sqrt{2} \left(\underbrace{\cos \frac{\pi}{3}}_{=\frac{1}{2}} + i \underbrace{\sin \frac{\pi}{3}}_{=\frac{\sqrt{3}}{2}} \right) \\ &= \pm \frac{\sqrt{2}}{2} + i \left(-1 \pm \frac{\sqrt{6}}{2} \right) \end{aligned}$$

Roots of polynomials in $\mathbb{C}$

- ▶ The previous theorem tells us that a polynomial of the form $z^n + a$ (with a complex) has exactly n roots in $\mathbb{C}$; in the real field, instead, the equation $x^n + a = 0$ may have two, one, or no roots (examples: $x^2 - 1 = 0$, $x^3 - 1 = 0$, $x^2 + 1 = 0$); we now know that such an equation always has n roots in $\mathbb{C}$, but only occasionally one or two of them lie in $\mathbb{R}$.
- ▶ The result is much more general, as stated by the following theorem, whose proof we do not report.

Fundamental theorem of algebra

Theorem 2: fundamental theorem of algebra

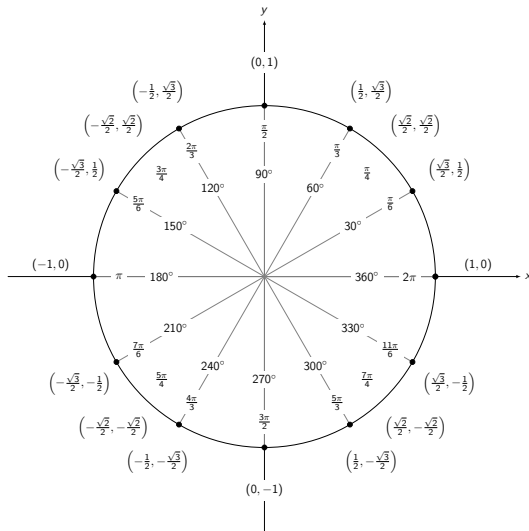
A polynomial equation of the form

$$a_0 + a_1 z + \cdots + a_n z^n = 0 \quad (a_n \neq 0)$$

with arbitrary complex coefficients has exactly n roots in $\mathbb{C}$, if each of them is counted with its multiplicity.

If $P(z)$ is a polynomial in z of degree n and z_0 is one of its roots, we say that z_0 has multiplicity k (k integer, ≥ 1) if the formula $P(z) = (z - z_0)^k Q(z)$ holds, where Q is a polynomial such that $Q(z_0) \neq 0$.

Cosines and sines of the main angles



Summary

Quadratic equations:

- ▶ $ax^2 + bx + c = 0$ has solutions $x = \frac{-b \pm \sqrt{\Delta}}{2a}$, with $\Delta = b^2 - 4ac$ (completing the square)
- ▶ If $\Delta < 0$ the solutions are two complex conjugate numbers
- ▶ In $\mathbb{C}$ the same formula also holds with complex coefficients

Fundamental theorem of algebra:

- ▶ A polynomial of degree n has exactly n roots in $\mathbb{C}$, counted with multiplicity

Final summary

- ▶ $\mathbb{C}$ is the field of pairs of real numbers with suitable sum and product; it contains $\mathbb{R}$ and the imaginary unit i , with $i^2 = -1$
- ▶ Algebraic form $z = a + ib$: conjugate, modulus, triangle inequality; $\mathbb{C}$ is not ordered
- ▶ Trigonometric and exponential forms: De Moivre's formulas and n -th roots
- ▶ In $\mathbb{C}$ quadratic equations always have solutions and every polynomial of degree n has n roots