

The Fibonacci sequence



Outline

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Introduction: The Fibonacci sequence

- ▶ In this section we define the **Fibonacci sequence**, in which each term is the sum of the two previous ones
- ▶ We will introduce the **golden ratio** ϕ and its conjugate $\hat{\phi}$, roots of the equation $x^2 = x + 1$
- ▶ We will prove by induction an **explicit formula** for the Fibonacci numbers and we will see that F_i is the integer closest to $\phi^i / \sqrt{5}$

The Fibonacci sequence

Definition 1: Fibonacci sequence

$$F_0 = 0,$$

$$F_1 = 1,$$

$$F_i = F_{i-1} + F_{i-2}, \quad \text{with } i \geq 2, i \in \mathbb{N}$$

- Therefore, each Fibonacci number is the sum of the two previous ones, which gives the sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

Golden ratio

- ▶ The Fibonacci sequence is related to the **golden ratio** ϕ and to the **conjugate of the golden ratio** $\hat{\phi}$.

Definition 2: golden ratio

*The **golden ratio** ϕ and the **conjugate of the golden ratio** $\hat{\phi}$ are the two roots of the equation:*

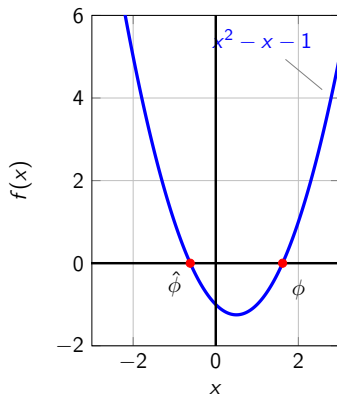
$$x^2 = x + 1$$

and are given by the formulas:

$$\begin{aligned}\phi &= \frac{1 + \sqrt{5}}{2} & \hat{\phi} &= \frac{1 - \sqrt{5}}{2} \\ &\approx 1.61803 & &\approx -0.61803\end{aligned}$$

Graph of $f(x) = x^2 - x - 1$

The following figure shows the graph of the function $f(x) = x^2 - x - 1$ on the interval $[-3, 3]$; the red dots correspond to the roots of the equation $x^2 = x + 1$.



ϕ and $\hat{\phi}$ satisfy $x^2 = x + 1$

The golden ratio ϕ and the conjugate of the golden ratio $\hat{\phi}$ clearly satisfy the equation $x^2 = x + 1$ since:

$$\phi^2 = \left(\frac{1 + \sqrt{5}}{2} \right)^2 = \frac{1 + 2\sqrt{5} + 5}{4} = \frac{3 + \sqrt{5}}{2} = \frac{1 + \sqrt{5}}{2} + 1 = \phi + 1$$

$$\hat{\phi}^2 = \left(\frac{1 - \sqrt{5}}{2} \right)^2 = \frac{1 - 2\sqrt{5} + 5}{4} = \frac{3 - \sqrt{5}}{2} = \frac{1 - \sqrt{5}}{2} + 1 = \hat{\phi} + 1$$

Explicit formula for the Fibonacci numbers

Using the properties of ϕ and $\hat{\phi}$, we now derive an explicit formula for the Fibonacci numbers.

Explicit formula

Observation 1: closed formula of the Fibonacci sequence

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}, \quad i = 0, 1, 2, \dots$$

Proof (1/2)

By induction on i .

► Base case of the induction

We prove that the formula holds for $i = 0$ and $i = 1$:

$$F_0 = \frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = \frac{1 - 1}{\sqrt{5}} = 0, \quad F_1 = \frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1.$$

► Inductive step

Suppose it is true for $i = k$ and $i = k - 1$ with $k \geq 1$, and let us prove it for $i = k + 1$. By the definition of the Fibonacci sequence, we have $F_{k+1} = F_k + F_{k-1}$, while by the inductive hypothesis we have:

$$F_k = \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}} \quad \text{and} \quad F_{k-1} = \frac{\phi^{k-1} - \hat{\phi}^{k-1}}{\sqrt{5}}$$

Proof (2/2)

Hence we can write (using $\phi + 1 = \phi^2$ and $\hat{\phi} + 1 = \hat{\phi}^2$):

$$\begin{aligned} F_{k+1} &= F_k + F_{k-1} = \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}} + \frac{\phi^{k-1} - \hat{\phi}^{k-1}}{\sqrt{5}} \\ &= \frac{(\phi^k - \hat{\phi}^k) + (\phi^{k-1} - \hat{\phi}^{k-1})}{\sqrt{5}} = \frac{(\phi^k + \phi^{k-1}) - (\hat{\phi}^k + \hat{\phi}^{k-1})}{\sqrt{5}} \\ &= \frac{\phi^{k-1}(\phi + 1) - \hat{\phi}^{k-1}(\hat{\phi} + 1)}{\sqrt{5}} = \frac{\phi^{k-1}(\phi^2) - \hat{\phi}^{k-1}(\hat{\phi}^2)}{\sqrt{5}} \\ &= \frac{\phi^{k+1} - \hat{\phi}^{k+1}}{\sqrt{5}} \end{aligned}$$

which is exactly the desired statement, for $i = k + 1$. □

F_i as the integer closest to $\phi^i/\sqrt{5}$

Observation 2: computing F_i by rounding

$$F_i = \left\lfloor \frac{\phi^i}{\sqrt{5}} + \frac{1}{2} \right\rfloor, \quad i = 0, 1, 2, \dots$$

Proof

Since $|\hat{\phi}| < 1$, we have

$$\frac{|\hat{\phi}^i|}{\sqrt{5}} \leq \frac{1}{\sqrt{5}} < \frac{1}{2} \quad \text{and} \quad -\frac{1}{2} < \frac{\hat{\phi}^i}{\sqrt{5}} < \frac{1}{2}, \quad i = 0, 1, 2, \dots$$

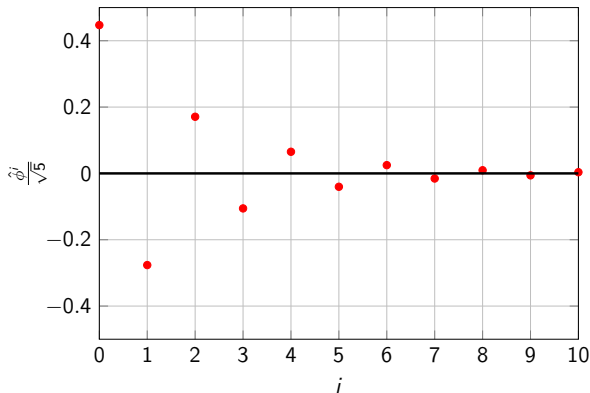
Since $F_i \in \mathbb{N}$ and $F_i = \frac{\phi^i}{\sqrt{5}} - \frac{\hat{\phi}^i}{\sqrt{5}}$, the i -th Fibonacci number F_i is equal to $\frac{\phi^i}{\sqrt{5}}$ rounded to the nearest integer:

$$F_i = \left\lfloor \frac{\phi^i}{\sqrt{5}} + \frac{1}{2} \right\rfloor, \quad i = 0, 1, 2, \dots$$



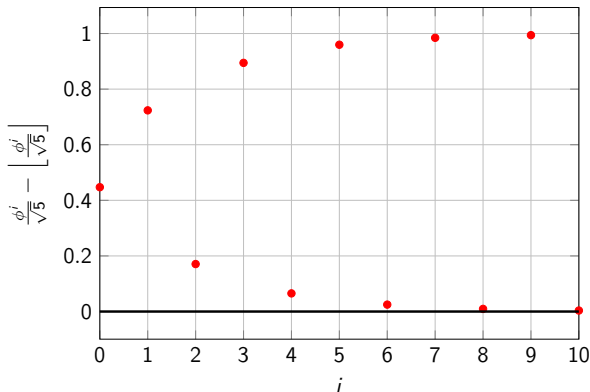
Values of $\hat{\phi}^i/\sqrt{5}$

- The values of $\frac{\hat{\phi}^i}{\sqrt{5}}$ with $i = 0, 1, \dots, 10$ are:



Values of the fractional part of $\phi^i/\sqrt{5}$

- The values of $\frac{\phi^i}{\sqrt{5}} - \left\lfloor \frac{\phi^i}{\sqrt{5}} \right\rfloor$ with $i = 0, 1, \dots, 10$ are:

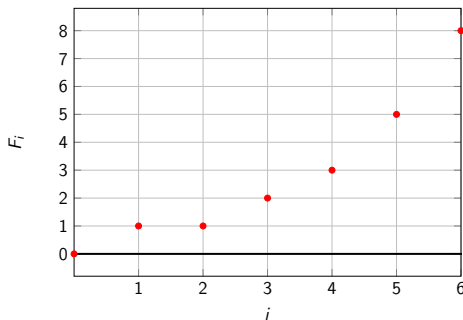


Values of F_i

- Since $F_i \in \mathbb{N}$, we moreover have:

$$\left(\frac{\phi^i}{\sqrt{5}} - \left\lfloor \frac{\phi^i}{\sqrt{5}} \right\rfloor \right) - \frac{\hat{\phi}^i}{\sqrt{5}} \in \{0, 1\}, \quad i = 0, 1, 2, \dots$$

- The values of F_i , with $i = 0, 1, \dots, 10$, are:



Summary

Fibonacci sequence:

- ▶ $F_0 = 0, F_1 = 1, F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$:
 $0, 1, 1, 2, 3, 5, 8, 13, \dots$

Golden ratio:

- ▶ $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$ are the roots of $x^2 = x + 1$

Formulas for F_i :

- ▶ $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$ (proved by induction)
- ▶ $F_i = \left\lfloor \frac{\phi^i}{\sqrt{5}} + \frac{1}{2} \right\rfloor$, since $\left| \frac{\hat{\phi}^i}{\sqrt{5}} \right| < \frac{1}{2}$

Final summary

- ▶ The Fibonacci sequence is defined recursively: each term is the sum of the two previous ones
- ▶ It is related to the golden ratio ϕ and to its conjugate $\hat{\phi}$, roots of $x^2 = x + 1$
- ▶ The explicit formula $F_i = (\phi^i - \hat{\phi}^i)/\sqrt{5}$ holds, proved by induction
- ▶ Since $|\hat{\phi}| < 1$, F_i is the integer closest to $\phi^i/\sqrt{5}$