

Factorials, binomial coefficients and absolute value



Outline

Factorials

Binomial coefficients

Absolute value



Factorials



Introduction: Factorials

- ▶ In this section we define the **factorial** $n!$ of a natural number n
- ▶ We will see that $n!$ grows very rapidly as n increases and we will study some of its elementary properties
- ▶ The factorial is the basic tool to define the **binomial coefficients** in the next section

Definition 1: factorial of n

The factorial of n is the product of the first n positive integers. It is denoted by $n!$ and is read “ n factorial”. In formulas:

$$n! = \prod_{k=1}^n k = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$$

By definition, we set $0! = 1$.

First values of the factorial

- The number $n!$ grows very rapidly as n increases. The first values are:

n	0	1	2	3	4	5	6	7	8	9	10
$n!$	1	1	2	6	24	120	720	5,040	40,320	362,880	3,628,800

Properties of the factorial

- Some properties of the factorial, which are immediate to verify, are:

$$n! = n \cdot (n - 1)!$$

$$\frac{n!}{(n - k)!} = n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot (n - k + 1), \quad \text{with } k \geq 1 \quad (1)$$

With $k \geq 1$, it becomes the product of k factors, starting from n and decreasing by one unit at a time.

Example

$$\frac{100!}{95!} = \frac{100!}{(100 - 5)!} = 100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 = 9,034,502,400$$

It is always convenient to simplify expressions containing the factorial as much as possible, before computing them!

Summary

Factorial:

- ▶ $n! = 1 \cdot 2 \cdot \dots \cdot n$, with $0! = 1$; it grows very rapidly
- ▶ $n! = n \cdot (n - 1)!$
- ▶ $\frac{n!}{(n-k)!}$ is the product of k decreasing factors starting from n
- ▶ It is convenient to simplify before computing

Binomial coefficients



Introduction: Binomial coefficients

- ▶ In this section we define the **binomial coefficients** $\binom{n}{k}$ and study their main properties
- ▶ We will prove **Newton's formula** for the n -th power of a binomial $(a + b)^n$
- ▶ Finally, we will see how to compute binomial coefficients recursively with **Pascal's triangle**

Definition 2: binomial coefficient

The **binomial coefficient** is defined as the number:

$$c_{n,k} = \frac{n!}{k! (n-k)!} \quad \text{with } 0 \leq k \leq n.$$

The binomial coefficient $c_{n,k}$ is usually denoted by the symbol: $\binom{n}{k}$ which is read “n choose k”.

- Given rule (1), we have:

$$c_{n,k} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)}{k!}$$

with $k \geq 1$, an expression that is more convenient for computing the binomial coefficient.

Symmetry of binomial coefficients

We have:

$$\binom{n}{n-k} = \binom{n}{k}$$

Since:

$$\binom{n}{n-k} = \frac{n!}{(n-k)! (n - (n-k))!} = \frac{n!}{k! (n-k)!} = \binom{n}{k}$$

Recursive relation (1/3)

We have:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k} \quad (2)$$

It also follows that:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Recursive relation (2/3)

Indeed, since $(n-1-(k-1))! = (n-k)! = (n-k)(n-k-1)!$
and $k! = k(k-1)!$:

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)! \underbrace{(n-1-(k-1))!}_{=(n-k)(n-k-1)!}} + \frac{(n-1)!}{k! (n-k-1)!} \\ &= \frac{(n-1)!}{(k-1)! (n-k)(n-k-1)!} \\ &\quad + \frac{(n-1)!}{k(k-1)! (n-k-1)!}\end{aligned}$$

Recursive relation (3/3)

Continuing:

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{k(n-1)! + (n-k)(n-1)!}{k(k-1)!(n-k)(n-k-1)!} \\ &= \frac{\overbrace{(n-1)!}^{=n!} n}{\underbrace{k(k-1)!}_{k!} \underbrace{(n-k)(n-k-1)!}_{(n-k)!}} \\ &= \frac{n!}{k!(n-k)!} = \binom{n}{k}\end{aligned}$$

Newton's formula

- ▶ The n -th power of a binomial $(a + b)$ can be computed with the following formula (from which the name binomial coefficient derives):

Observation 1: Newton's formula

For every integer $n \geq 0$, with $a, b \in \mathbb{R}$, we have:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Proof (1/4)

By induction on n .

► **Base case of the induction**

Let $n = 0$. Then the statement becomes: $(a + b)^0 = \binom{0}{0} a^0 b^0$
i.e. $1 = 1$ which is clearly true.

► **Inductive step**

Suppose it is true for n , and let us prove it for $(n + 1)$. By the inductive hypothesis, we have:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Proof (2/4)

Then:

$$\begin{aligned}(a+b)^{n+1} &= (a+b) \cdot (a+b)^n = (a+b) \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \\&= \sum_{k=0}^n \binom{n}{k} a^{n-k+1} b^k + \underbrace{\sum_{k=0}^n \binom{n}{k} a^{n-k} b^{k+1}}_{= \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n-k+1} b^k}\end{aligned}$$

Proof (3/4)

Isolating the term with $k = 0$ in the first sum and the one with $k = n + 1$ in the second:

$$\begin{aligned}(a + b)^{n+1} &= a^{n+1} + \sum_{k=1}^n \binom{n}{k} a^{n-k+1} b^k \\ &\quad + b^{n+1} + \sum_{k=1}^n \binom{n}{k-1} a^{n-k+1} b^k\end{aligned}$$

Proof (4/4)

Collecting terms and using the recursive relation:

$$\begin{aligned}(a + b)^{n+1} &= a^{n+1} + b^{n+1} + \sum_{k=1}^n \underbrace{\left(\binom{n}{k} + \binom{n}{k-1} \right)}_{= \binom{n+1}{k}} a^{n-k+1} b^k \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k\end{aligned}$$

which is exactly the desired statement, for $n + 1$. □

Sum of the binomial coefficients

Observation 2

For every integer $n \geq 0$ and integer k such that $0 \leq k \leq n$, we have:

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

Proof

We write:

$$2^n = (1 + 1)^n$$

We now apply Newton's formula:

$$(1 + 1)^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^n \binom{n}{k}$$



Recursive computation: Pascal's triangle

Relation (2) allows us to compute the binomial coefficients $\binom{n}{k}$ by means of the so-called **Pascal's triangle** (or Tartaglia's triangle). The rules for building the triangle are:

1. At the top of the triangle we place the number $\binom{0}{0} = 1$ (by definition).
2. On the sides we place the numbers $\binom{n}{0} = \binom{n}{n} = 1$ for every $n \geq 1$.
3. For $0 < k < n$, the number $\binom{n}{k}$ is written at the intersection of the n -th row and the k -th column.
4. The number $\binom{n}{k}$ is the sum of the two numbers located in the previous row, the one in the same column and the one in the previous column.

Example (1/2)

Pascal's triangle with $n \leq 10$ and $k \leq 10$ is:

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$
$n = 0$	1										
$n = 1$	1	1									
$n = 2$	1	2	1								
$n = 3$	1	3	3	1							
$n = 4$	1	4	6	4	1						
$n = 5$	1	5	10	10	5	1					
$n = 6$	1	6	15	20	15	6	1				
$n = 7$	1	7	21	35	35	21	7	1			
$n = 8$	1	8	28	56	70	56	28	8	1		
$n = 9$	1	9	36	84	126	126	84	36	9	1	
$n = 10$	1	10	45	120	210	252	210	120	45	10	1

Example (2/2)

To compute the binomial coefficient $\binom{5}{3}$, corresponding to the blue cell, we can use relation (2) and add the two binomial coefficients $\binom{4}{2}$ and $\binom{4}{3}$, corresponding to the red cells:

$$\binom{5}{3} = \binom{4}{2} + \binom{4}{3} = 6 + 4 = 10.$$

Example

Thanks to the previous Pascal's triangle and to Newton's formula we can compute:

$$(a + b)^5 = a^5 + 5 a^4 b + 10 a^3 b^2 + 10 a^2 b^3 + 5 a b^4 + b^5$$



Summary

Binomial coefficients:

- ▶ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, with $0 \leq k \leq n$
- ▶ Symmetry: $\binom{n}{n-k} = \binom{n}{k}$; recursive relation:
 $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$

Newton's formula and Pascal's triangle:

- ▶ $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$ and hence $\sum_{k=0}^n \binom{n}{k} = 2^n$
- ▶ Pascal's triangle computes the binomial coefficients recursively

Absolute value

Introduction: Absolute value

- ▶ In this section we define the **absolute value** (or modulus) of a real number
- ▶ We will prove the **triangle inequality** and the **reverse triangle inequality** in $\mathbb{R}$
- ▶ These inequalities are fundamental tools to estimate distances and will be used constantly in the study of limits

Definition 3: absolute value

The **absolute value** of a real number $a \in \mathbb{R}$ (or **modulus** of a) is the non-negative number defined as follows:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

A consequence of the definition

From the definition of absolute value it immediately follows that:

$$\forall \varepsilon \geq 0, a \in \mathbb{R}, \quad |a| \leq \varepsilon \iff -\varepsilon \leq a \leq \varepsilon \quad (3)$$

Triangle inequality in $\mathbb{R}$

Observation 3: triangle inequality in $\mathbb{R}$

$$|b + c| \leq |b| + |c| \quad \forall b, c \in \mathbb{R} \quad (4)$$

Proof

We write the two relations:

$$-|b| \leq b \leq |b|, \quad -|c| \leq c \leq |c|$$

and add them side by side:

$$-(|b| + |c|) \leq b + c \leq |b| + |c|$$

Hence, by (3), with $\varepsilon = |b| + |c| \geq 0$ and $a = b + c$, (4) follows.



Other forms of the triangle inequality (1/2)

- The triangle inequality is also used in the following form:

$$|d - e| \leq |d - f| + |e - f| \quad \forall d, e, f \in \mathbb{R}$$

To obtain it, it suffices to set in (4):

$$b = d - f, \quad c = f - e$$

we obtain:

$$|d - f + f - e| = |d - e| \leq |d - f| + |f - e| = |d - f| + |e - f|$$

since:

$$|f - e| = |e - f| \quad \forall e, f \in \mathbb{R}$$

Other forms of the triangle inequality (2/2)

- ▶ Moreover, the triangle inequality can also be written in the following form:

$$|g| \leq |g - h| + |h| \quad \text{i.e.} \quad |g| - |h| \leq |g - h| \quad \forall g, h \in \mathbb{R} \quad (5)$$

To obtain it, it suffices to set in (4):

$$b = g - h, \quad c = h$$

Reverse triangle inequality in $\mathbb{R}$

Observation 4: reverse triangle inequality in $\mathbb{R}$

$$||g| - |h|| \leq |g - h|, \quad \forall g, h \in \mathbb{R}.$$

Proof

From (5) we have

$$|g| - |h| \leq |g - h| \quad \forall g, h \in \mathbb{R}$$

Similarly, swapping g and h in (5) we obtain:

$$|h| - |g| \leq |h - g| = |g - h| \quad \text{that is} \quad |g| - |h| \geq -|g - h|$$

hence we have

$$-(|g - h|) \leq |g| - |h| \leq |g - h| \quad \forall g, h \in \mathbb{R}$$

Hence, by (3), with $\varepsilon = |g - h| \geq 0$ and $a = |g| - |h|$, the reverse triangle inequality follows. □

Further properties of the absolute value

- ▶ Inequality (4) can easily be extended to the case of k terms:

$$\left| \sum_{i=1}^k b_i \right| \leq \sum_{i=1}^k |b_i|.$$

- ▶ The following immediate properties also hold:

$$|b c| = |b| |c|, \quad |-b| = |b| \quad \forall b, c \in \mathbb{R},$$

$$\left| \frac{b}{c} \right| = \frac{|b|}{|c|} \quad \forall b, c \in \mathbb{R}, \quad c \neq 0.$$

Summary

Absolute value:

- ▶ $|a| = a$ if $a \geq 0$, $|a| = -a$ if $a < 0$
- ▶ $|a| \leq \varepsilon \iff -\varepsilon \leq a \leq \varepsilon$

Inequalities:

- ▶ Triangle: $|b + c| \leq |b| + |c|$, extendable to k terms
- ▶ Reverse triangle: $||g| - |h|| \leq |g - h|$
- ▶ $|bc| = |b| |c|$, $|b/c| = |b|/|c|$, $|-b| = |b|$

Final summary

- ▶ The factorial $n!$ is the product of the first n positive integers ($0! = 1$) and grows very rapidly
- ▶ The binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ are symmetric and satisfy the recursive relation of Pascal's triangle
- ▶ Newton's formula: $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$
- ▶ The absolute value satisfies the triangle inequality and the reverse triangle inequality