

Summations, geometric and arithmetic progressions



Outline

Summations

Geometric progressions

Arithmetic progressions

Summations

Introduction: Summations

- ▶ In this section we introduce the **summation symbol** $\sum$, a compact notation to write sums of many terms
- ▶ We will see its **six main properties** (product by a constant, constant term, union of summations, decomposition, index translation and index reflection)
- ▶ Finally, we will compute some **important summations**: the sum of the first n natural numbers, of the first n odd numbers and of the first n even numbers

The summation symbol

Definition 1: summation

Let $a_1, a_2, \dots, a_n$ be n real numbers. Their sum

$$a_1 + a_2 + \cdots + a_n$$

can be written in compact form with the summation symbol:

$$\sum_{k=1}^n a_k$$

which is read: “sum for k from 1 to n of a_k ”. The symbol k is called the summation index.

- ▶ The summation symbol is therefore just a shorthand, which is nevertheless very useful when the terms a_k are defined explicitly as a function of the index k .

Example

$$\sum_{k=1}^{10} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$$

$$\sum_{k=3}^n k^2 = 3^2 + 4^2 + 5^2 + \cdots + n^2$$

Dummy index

- ▶ The summation index is a **dummy index**. This means that if we replace k with i , j or any other index (in all its occurrences), the value of the summation does not change.

Example

We have:

$$\sum_{k=1}^n k^2 = \sum_{i=1}^n i^2$$

since both symbols denote the sum of the squares of the first n natural numbers (without zero).

On the other hand, we have:

$$\sum_{k=1}^n k^2 \neq \sum_{k=1}^m k^2$$

since the two symbols denote the sum of, respectively, the first n or the first m squares (if $n \neq m$ the result will be different).

Main properties of summations

We now see six fundamental properties of summations, which will help us to manipulate and simplify expressions involving the symbol $\sum$.

Product by a constant

Observation 1: product by a constant

Given a summation $\sum_{k=1}^n a_k$ and a real number $c \in \mathbb{R}$, we have:

$$\sum_{k=1}^n (c \cdot a_k) = c \sum_{k=1}^n a_k$$

Proof

By the distributive property we have:

$$\underbrace{c a_1 + c a_2 + \cdots + c a_n}_{=\sum_{k=1}^n (c \cdot a_k)} = \underbrace{c (a_1 + a_2 + \cdots + a_n)}_{=c \sum_{k=1}^n a_k}$$



Summation with a constant term

Observation 2: summation with a constant term

For every natural number $n \geq 1$ and every real number $c \in \mathbb{R}$, we have:

$$\sum_{k=1}^n c = c \cdot n$$

Proof

We have:

$$\underbrace{c + c + \cdots + c}_{\text{that is, } c \text{ added } n \text{ times}} = c n$$

$= \sum_{k=1}^n c$



Union of summations

Observation 3: union of summations

Given two summations $\sum_{k=1}^n a_k$ and $\sum_{k=1}^n b_k$, we have:

$$\sum_{k=1}^n a_k + \sum_{k=1}^n b_k = \sum_{k=1}^n (a_k + b_k)$$

Proof

We have:

$$\begin{aligned} & \underbrace{a_1 + a_2 + \cdots + a_n + b_1 + b_2 + \cdots + b_n}_{=\sum_{k=1}^n a_k + \sum_{k=1}^n b_k} \\ &= \underbrace{a_1 + b_1 + a_2 + b_2 + \cdots + a_n + b_n}_{=\sum_{k=1}^n (a_k + b_k)} \end{aligned}$$



- ▶ The three properties that follow (decomposition, index translation and index reflection) are simply different ways of writing and/or ordering the terms of the summations
- ▶ To prove them it is enough to write out the terms of the two summations and observe that they are the same

Observation 4: decomposition

Given two natural numbers $n \geq 1$ and $m \geq 1$, we have:

$$\sum_{k=1}^{n+m} a_k = \sum_{k=1}^n a_k + \sum_{k=n+1}^{n+m} a_k$$

Proof

The terms of the summation $\sum_{k=1}^{n+m} a_k$ can be grouped into the first n and the last m ones:

$$\begin{aligned} & \underbrace{a_1 + a_2 + \cdots + a_n}_{=\sum_{k=1}^n a_k} + \underbrace{a_{n+1} + a_{n+2} + \cdots + a_{n+m}}_{=\sum_{k=n+1}^{n+m} a_k} \\ &= \underbrace{a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots + a_{n+m}}_{=\sum_{k=1}^{n+m} a_k} \end{aligned}$$



Observation 5: index translation

Given a summation $\sum_{k=1}^n a_k$ and a natural number $m \geq 1$, we have:

$$\sum_{k=1}^n a_k = \sum_{k=1+m}^{n+m} a_{k-m} = \sum_{k=1-m}^{n-m} a_{k+m}$$

Proof (1/2)

In the summation $\sum_{k=1+m}^{n+m} a_{k-m}$ the index k ranges from $1+m$ to $n+m$, and therefore the index $k-m$ ranges from 1 to n :

$$\begin{aligned} & \underbrace{a_{(1+m)-m} + a_{(2+m)-m} + \cdots + a_{(n+m)-m}}_{=\sum_{k=1+m}^{n+m} a_{k-m}} \\ &= \underbrace{a_1 + a_2 + \cdots + a_n}_{=\sum_{k=1}^n a_k} \end{aligned}$$

Proof (2/2)

In the same way, in the summation $\sum_{k=1-m}^{n-m} a_{k+m}$ the index k ranges from $1 - m$ to $n - m$, and therefore the index $k + m$ ranges from 1 to n :

$$\begin{aligned} & \underbrace{a_{(1-m)+m} + a_{(2-m)+m} + \cdots + a_{(n-m)+m}}_{=\sum_{k=1-m}^{n-m} a_{k+m}} \\ &= \underbrace{a_1 + a_2 + \cdots + a_n}_{=\sum_{k=1}^n a_k} \end{aligned}$$



Observation 6: index reflection

Given a summation $\sum_{k=1}^n a_k$, we have:

$$\sum_{k=1}^n a_k = \sum_{k=1}^n a_{n-k+1} = \sum_{k=0}^{n-1} a_{n-k}$$

Proof (1/2)

In the summation $\sum_{k=1}^n a_{n-k+1}$ the index k ranges from 1 to n , and therefore the index $n - k + 1$ ranges from n to 1, that is, the terms are the same ones listed in reverse order:

$$\underbrace{a_n + a_{n-1} + \cdots + a_1}_{=\sum_{k=1}^n a_{n-k+1}} = \underbrace{a_1 + a_2 + \cdots + a_n}_{=\sum_{k=1}^n a_k}$$

Proof (2/2)

In the same way, in the summation $\sum_{k=0}^{n-1} a_{n-k}$ the index k ranges from 0 to $n-1$, and therefore the index $n-k$ ranges from n to 1:

$$\underbrace{a_n + a_{n-1} + \cdots + a_1}_{=\sum_{k=0}^{n-1} a_{n-k}} = \underbrace{a_1 + a_2 + \cdots + a_n}_{=\sum_{k=1}^n a_k}$$



Some important summations

Using the properties just seen, we now compute in closed form some important summations.

Sum of the first n natural numbers

Observation 7: sum of the first n natural numbers (without zero)

For every natural number $n \geq 1$, we have:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

Proof

Using the index reflection:

$$\begin{aligned}\sum_{k=1}^n k &= \frac{1}{2} \left(\sum_{k=1}^n k + \sum_{k=1}^n k \right) = \frac{1}{2} \left(\sum_{k=1}^n k + \sum_{k=1}^n (n - k + 1) \right) \\ &= \frac{1}{2} \sum_{k=1}^n (k + n - k + 1) = \frac{1}{2} \sum_{k=1}^n (n + 1) = \frac{n(n+1)}{2}\end{aligned}$$



Sum of the first n odd numbers

Observation 8: sum of the first n odd numbers

For every natural number $n \geq 1$, we have:

$$\sum_{k=1}^n (2k - 1) = n^2$$

or equivalently

$$\sum_{k=0}^{n-1} (2k + 1) = n^2$$

Proof (1/2)

Using the properties of summations we have:

$$\begin{aligned}\sum_{k=1}^n (2k - 1) &= 2 \sum_{k=1}^n k - \sum_{k=1}^n 1 \\ &= 2 \left(\frac{n(n+1)}{2} \right) - n = n^2 + n - n = n^2\end{aligned}$$

Proof (2/2)

For the second form:

$$\begin{aligned}\sum_{k=0}^{n-1} (2k + 1) &= 2 \sum_{k=0}^{n-1} k + \sum_{k=0}^{n-1} 1 \\&= 2 \sum_{k=1}^n (k - 1) + n = 2 \left(\sum_{k=1}^n k - \sum_{k=1}^n 1 \right) + n \\&= 2 \left(\frac{n(n+1)}{2} - n \right) + n = n^2 + n - 2n + n = n^2\end{aligned}$$



Sum of the first n even numbers

Observation 9: sum of the first n even numbers (without zero)

For every natural number $n \geq 1$, we have:

$$\sum_{k=1}^n 2k = n(n+1)$$

Proof

$$\sum_{k=1}^n 2k = 2 \sum_{k=1}^n k = 2 \left(\frac{n(n+1)}{2} \right) = n(n+1)$$



- ▶ The three important summations we have just seen can also be proved by induction, in the chapter “[Principle of induction](#)” of Part 1
- ▶ Here we have obtained them using only the properties of summations

Summary

Summations:

- ▶ $\sum_{k=1}^n a_k$ is a compact notation for $a_1 + a_2 + \cdots + a_n$; the index k is a **dummy index**
- ▶ The six properties: product by a constant, constant term, union of summations, decomposition, index translation and index reflection

Important summations ($n \geq 1$):

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^n (2k-1) = n^2, \quad \sum_{k=1}^n 2k = n(n+1)$$

Geometric progressions



Introduction: Geometric progressions

- ▶ In this section we introduce **geometric progressions**: sequences in which each term is obtained from the previous one by multiplying it by a constant, the **common ratio**
- ▶ We will derive the **closed formula of the k -th term** and the formula for the **sum of the first n terms** of a geometric progression
- ▶ This formula appears often in the course (for example in the study of geometric series)

Geometric progression

Definition 2: geometric progression

*A sequence of real numbers is a **geometric progression** if the ratio between each term (starting from the second one) and the previous one is constant. This constant is called the common ratio of the progression.*

Terms of a geometric progression

Given the first term $a \in \mathbb{R}$ and the common ratio $q \in \mathbb{R}$, the associated geometric progression is:

$$a, \quad a q, \quad a q^2, \quad a q^3, \quad a q^4, \quad \dots$$

Each term (starting from the second one) is obtained from the previous one by multiplying it by q . The k -th term ($k \in \mathbb{N}$, $k \geq 1$) can be written as $a q^{k-1}$ and we have:

$$a q^{1-1} = a q^0 = a \quad \text{first term, in position } k = 1$$

$$a q^{2-1} = a q^1 = a q \quad \text{second term, in position } k = 2$$

$$a q^{3-1} = a q^2 \quad \text{third term, in position } k = 3$$

$$a q^{4-1} = a q^3 \quad \text{fourth term, in position } k = 4$$

$\dots$

$\dots$

Observation 10: k -th term of a geometric progression

Given the first term $a \in \mathbb{R}$ and the common ratio $q \in \mathbb{R}$, let t_k denote the term in position k . The recursive relation that defines the progression is:

$$t_1 = a \quad \text{and} \quad t_k = t_{k-1} q \quad \text{for every } k \geq 2$$

and the closed formula of the k -th term is:

$$t_k = a q^{k-1} \quad \text{for every } k \geq 1$$

Proof

We apply repeatedly the recursive relation:

$$t_1 = a = a q^0$$

$$t_2 = t_1 q = a q = a q^1$$

$$t_3 = t_2 q = a q q = a q^2$$

$$t_4 = t_3 q = a q^2 q = a q^3$$

$$\vdots$$

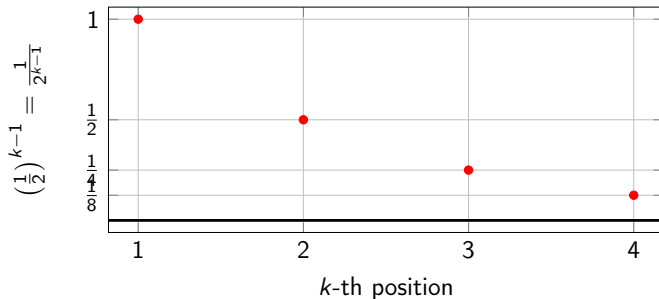
$$t_k = t_{k-1} q = a q^{k-2} q = a q^{k-1}$$



Example (1/3)

- With $a = 1$ and $q = \frac{1}{2}$, the first 4 terms are:

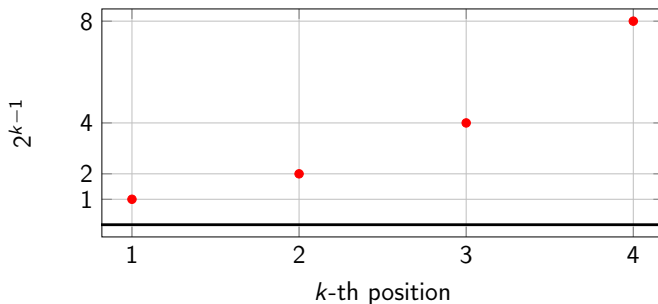
$$1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}$$



Example (2/3)

- With $a = 1$ and $q = 2$, the first 4 terms are:

1, 2, 4, 8

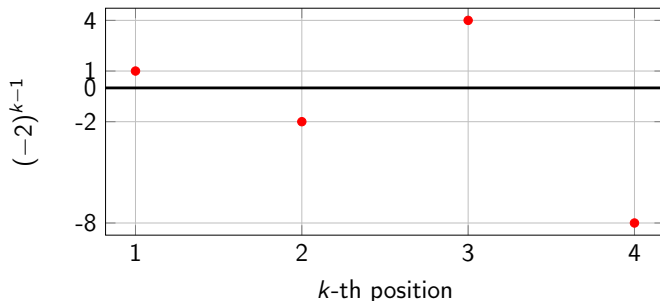


Example (3/3)

- ▶ With $a = 1$ and $q = -2$, the first 4 terms are:

$$1, \quad -2, \quad 4, \quad -8$$

- ▶ With a negative common ratio the terms change sign at every step



Sums of the terms of geometric progressions

We now see how to compute in closed form the sum of the first n terms of a geometric progression.

Sum of the first n terms

Observation 11: sum of the first n terms of the geometric progression ($a = 1$)

Given $q \in \mathbb{R}$, for every natural number $n \geq 1$ we have:

$$\sum_{k=1}^n q^{k-1} = \begin{cases} \frac{q^n - 1}{q - 1} & \text{if } q \neq 1 \\ n & \text{otherwise} \end{cases}$$

Proof (1/2)

If $q \neq 1$, we prove the equation in the following equivalent form:

$$(q - 1) \sum_{k=1}^n q^{k-1} = q^n - 1$$

Applying the properties of summations, in particular the product by a constant and the index translation, we get:

$$\begin{aligned}(q - 1) \sum_{k=1}^n q^{k-1} &= q \sum_{k=1}^n q^{k-1} - \sum_{k=1}^n q^{k-1} = \\&= \sum_{k=1}^n q^k - \sum_{k=1}^n q^{k-1} = \sum_{k=1}^n q^k - \sum_{k=0}^{n-1} q^k = \\&= \sum_{k=1}^{n-1} q^k + q^n - \left(1 + \sum_{k=1}^{n-1} q^k \right) = q^n - 1\end{aligned}$$

Proof (2/2)

Dividing by $q - 1 \neq 0$ we obtain:

$$\sum_{k=1}^n q^{k-1} = \frac{q^n - 1}{q - 1}$$

If $q = 1$, we have instead:

$$\sum_{k=1}^n q^{k-1} = \sum_{k=1}^n 1 = n$$



The common ratio can be negative or zero

- ▶ The proof never uses the sign of the common ratio: the formula holds for every $q \in \mathbb{R}$
- ▶ The only case to be made precise is $q = 0$: the first term is $q^{1-1} = 0^0$, which here is understood to be equal to 1 (convention $0^0 = 1$)
- ▶ With this convention, for every $n \geq 1$:

$$\sum_{k=1}^n 0^{k-1} = 1 + 0 + \cdots + 0 = 1 \quad \text{and} \quad \frac{0^n - 1}{0 - 1} = \frac{-1}{-1} = 1$$

Extension to the case $a \in \mathbb{R}$

- Given $q \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 1$ and $a \in \mathbb{R}$, the formula extends as follows:

$$\sum_{k=1}^n a q^{k-1} = \begin{cases} a \left(\frac{q^n - 1}{q - 1} \right) & \text{if } q \neq 1 \\ a n & \text{otherwise} \end{cases}$$

since

$$\sum_{k=1}^n a q^{k-1} = a \sum_{k=1}^n q^{k-1}$$

Equivalent form

► Clearly:

$$\frac{q^n - 1}{q - 1} = \frac{1 - q^n}{1 - q}$$

and hence we also have:

$$\sum_{k=1}^n a q^{k-1} = \begin{cases} a \left(\frac{1-q^n}{1-q} \right) & \text{if } q \neq 1 \\ a n & \text{otherwise} \end{cases}$$

- ▶ The formula for the sum of the first n terms can also be proved by induction, in the chapter “[Principle of induction](#)” of Part 1
- ▶ We now see three numerical examples, which check the formula with a direct computation

Example (1/4)

- With $a = 2$ and $q = \frac{1}{2}$, the first 4 terms are:

$$2, \quad 1, \quad \frac{1}{2}, \quad \frac{1}{4} \quad \text{and} \quad 2 + 1 + \frac{1}{2} + \frac{1}{4} = \frac{8 + 4 + 2 + 1}{4} = \frac{15}{4}$$

the sum of the first $n = 4$ terms is given by the formula:

$$\begin{aligned} \sum_{k=1}^4 2 \left(\frac{1}{2} \right)^{k-1} &= \sum_{k=1}^4 \frac{2}{2^{k-1}} = 2 \frac{1 - \frac{1}{2^4}}{1 - \frac{1}{2}} \\ &= 2 \frac{1 - \frac{1}{16}}{\frac{1}{2}} = \frac{15}{16} \cdot 4 = \frac{15}{4} \end{aligned}$$

Example (2/4)

- With $a = 1$ and $q = \frac{1}{2}$, the first 4 terms are:

$$1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8} \quad \text{and} \quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{8 + 4 + 2 + 1}{8} = \frac{15}{8}$$

the sum of the first $n = 4$ terms is given by the formula:

$$\sum_{k=1}^4 \left(\frac{1}{2}\right)^{k-1} = \sum_{k=1}^4 \frac{1}{2^{k-1}} = \frac{1 - \frac{1}{2^4}}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{16}}{\frac{1}{2}} = \frac{15}{16} \cdot 2 = \frac{15}{8}$$

Example (3/4)

- With $a = 1$ and $q = 2$, the first 4 terms are:

$$1, \quad 2, \quad 4, \quad 8 \quad \text{and} \quad 1 + 2 + 4 + 8 = 15$$

the sum of the first $n = 4$ terms is given by the formula:

$$\sum_{k=1}^4 2^{k-1} = \frac{2^4 - 1}{2 - 1} = 16 - 1 = 15$$

Example (4/4)

- With $a = 1$ and $q = -2$, the first 4 terms are:

$$1, \quad -2, \quad 4, \quad -8 \quad \text{and} \quad 1 - 2 + 4 - 8 = -5$$

the sum of the first $n = 4$ terms is given by the formula:

$$\sum_{k=1}^4 (-2)^{k-1} = \frac{(-2)^4 - 1}{-2 - 1} = \frac{15}{-3} = -5$$

Summary

Geometric progressions:

- ▶ First term a , common ratio q : terms $a, aq, aq^2, \dots$
- ▶ Recursive relation $t_1 = a, t_k = t_{k-1} q$; closed formula $t_k = a q^{k-1}$

Sum of the first n terms:

- ▶ If $q \neq 1$: $\sum_{k=1}^n a q^{k-1} = a \frac{q^n - 1}{q - 1} = a \frac{1 - q^n}{1 - q}$
- ▶ If $q = 1$: $\sum_{k=1}^n a q^{k-1} = a n$

Arithmetic progressions



Introduction: Arithmetic progressions

- ▶ In this section we introduce **arithmetic progressions**: sequences in which each term is obtained from the previous one by adding a constant, the **common difference**
- ▶ They are the additive analogue of geometric progressions: there one multiplies by q , here one adds d
- ▶ Here too we will derive the **closed formula of the k -th term** and the formula for the **sum of the first n terms**, which we will use in the study of series

Arithmetic progression

Definition 3: arithmetic progression

*A sequence of real numbers is an **arithmetic progression** if the difference between each term (starting from the second one) and the previous one is constant. This constant is called the common difference of the progression.*

Terms of an arithmetic progression

Given the first term $a \in \mathbb{R}$ and the common difference $d \in \mathbb{R}$, the associated arithmetic progression is:

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad a + 4d, \quad \dots$$

Each term (starting from the second one) is obtained from the previous one by adding d to it. The k -th term ($k \in \mathbb{N}$, $k \geq 1$) can be written as $a + (k - 1)d$ and we have:

$$a + (1 - 1)d = a + 0d = a \quad \text{first term } (k = 1)$$

$$a + (2 - 1)d = a + 1d = a + d \quad \text{second term } (k = 2)$$

$$a + (3 - 1)d = a + 2d \quad \text{third term } (k = 3)$$

$$a + (4 - 1)d = a + 3d \quad \text{fourth term } (k = 4)$$

$\dots$

$\dots$

Observation 12: k -th term of an arithmetic progression

Given the first term $a \in \mathbb{R}$ and the common difference $d \in \mathbb{R}$, let t_k denote the term in position k . The recursive relation that defines the progression is:

$$t_1 = a \quad \text{and} \quad t_k = t_{k-1} + d \quad \text{for every } k \geq 2$$

and the closed formula of the k -th term is:

$$t_k = a + (k - 1) d \quad \text{for every } k \geq 1$$

Proof

We apply repeatedly the recursive relation:

$$t_1 = a = a + 0 d$$

$$t_2 = t_1 + d = a + d = a + 1 d$$

$$t_3 = t_2 + d = a + d + d = a + 2 d$$

$$t_4 = t_3 + d = a + 2 d + d = a + 3 d$$

$$\vdots$$

$$t_k = t_{k-1} + d = a + (k-2) d + d = a + (k-1) d$$



The same formula is obtained by a second argument, which does not use the dots: we add all the consecutive differences and the sum turns out to be **telescopic**.

Alternative proof (1/2)

We write the recursive relation in the form $t_j - t_{j-1} = d$ for every position j from 2 to k :

$$t_2 - t_1 = d$$

$$t_3 - t_2 = d$$

$$t_4 - t_3 = d$$

$$\vdots$$

$$t_k - t_{k-1} = d$$

We now add all these $k - 1$ equalities: on the right-hand side we add d to itself $k - 1$ times and we obtain $d(k - 1)$.

Alternative proof (2/2)

On the left-hand side the sum is **telescopic**: every term appears twice, once with the sign $+$ and once with the sign $-$, except t_k and t_1 :

$$\underbrace{(t_2 - t_1) + (t_3 - t_2) + \cdots + (t_k - t_{k-1})}_{=\sum_{j=2}^k (t_j - t_{j-1})} = t_k - t_1 = t_k - a$$

Equating the two sides we get $t_k - a = d(k-1)$, that is:

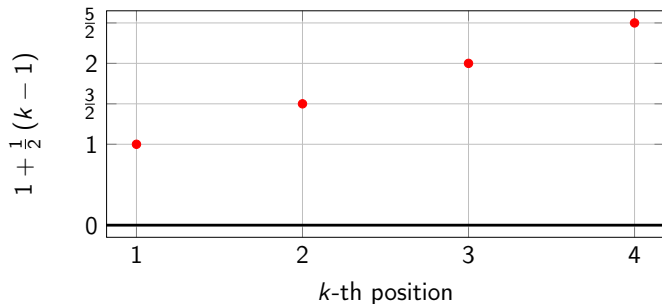
$$t_k = a + (k-1)d \quad \text{for every } k \geq 1$$



Example (1/2)

- With $a = 1$ and $d = \frac{1}{2}$, the first 4 terms are:

$$1, \quad \frac{3}{2}, \quad 2, \quad \frac{5}{2}$$

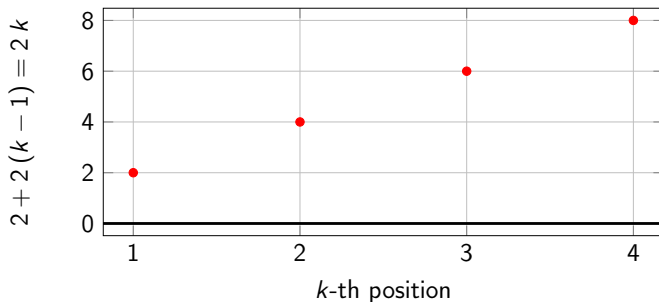


Example (2/2)

- With $a = 2$ and $d = 2$, the first 4 terms are:

2, 4, 6, 8

that is, the first 4 even numbers (without zero)



Sums of the terms of arithmetic progressions

We now see how to compute in closed form the sum of the first n terms of an arithmetic progression: we use the index reflection, the same technique as for the sum of the first n natural numbers.

Sum of the first n terms

Observation 13: sum of the first n terms of the arithmetic progression ($a = 0$)

Given $d \in \mathbb{R}$, for every natural number $n \geq 1$ we have:

$$\sum_{k=1}^n d(k-1) = d \left(\frac{n(n-1)}{2} \right)$$

Proof

Using the product by a constant and the index reflection:

$$\begin{aligned}\sum_{k=1}^n d(k-1) &= \frac{d}{2} \left(\sum_{k=1}^n (k-1) + \sum_{k=1}^n (k-1) \right) \\&= \frac{d}{2} \left(\sum_{k=1}^n (k-1) + \sum_{k=1}^n (n-k) \right) \\&= \frac{d}{2} \sum_{k=1}^n (k-1 + n-k) = \frac{d}{2} \sum_{k=1}^n (n-1) \\&= d \left(\frac{n(n-1)}{2} \right)\end{aligned}$$

(the term in position k is $a_k = k-1$, hence $a_{n-k+1} = n-k$) □

Extension to the case $a \in \mathbb{R}$

- Given $d \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 1$ and $a \in \mathbb{R}$, the formula extends as follows:

$$\sum_{k=1}^n (a + d(k-1)) = a n + d \left(\frac{n(n-1)}{2} \right)$$

since

$$\sum_{k=1}^n (a + d(k-1)) = \underbrace{\sum_{k=1}^n a}_{=a n} + \sum_{k=1}^n d(k-1)$$

Equivalent form

- ▶ Reflecting the index, the term in position k and the one in position $n - k + 1$ have constant sum $t_k + t_{n-k+1} = t_1 + t_n$, and therefore:

$$\sum_{k=1}^n (a + d(k-1)) = \frac{n}{2} \left(\underbrace{a}_{=t_1} + \underbrace{a + (n-1)d}_{=t_n} \right) = n \left(\frac{t_1 + t_n}{2} \right)$$

- ▶ The sum of the first n terms is therefore n times the average of the first and the last term

Special cases already known

- ▶ With $a = 1$ and $d = 1$ the progression is $1, 2, 3, \dots$ and the formula gives

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

that is, the sum of the first n natural numbers (without zero)

- ▶ With $a = 1$ and $d = 2$ the progression is $1, 3, 5, \dots$ and the formula gives

$$n + 2 \frac{n(n-1)}{2} = n^2$$

that is, the sum of the first n odd numbers

Example (1/2)

- With $a = 1$ and $d = \frac{1}{2}$, the first 4 terms are:

$$1, \quad \frac{3}{2}, \quad 2, \quad \frac{5}{2} \quad \text{and} \quad 1 + \frac{3}{2} + 2 + \frac{5}{2} = \frac{14}{2} = 7$$

the sum of the first $n = 4$ terms is given by the formula:

$$\sum_{k=1}^4 \left(1 + \frac{1}{2} (k - 1) \right) = 1 \cdot 4 + \frac{1}{2} \left(\frac{4 \cdot 3}{2} \right) = 4 + 3 = 7$$

Example (2/2)

- With $a = 2$ and $d = 2$, the first 4 terms are:

$$2, 4, 6, 8 \quad \text{and} \quad 2 + 4 + 6 + 8 = 20$$

the sum of the first $n = 4$ terms is given by the formula:

$$\sum_{k=1}^4 (2 + 2(k-1)) = 2 \cdot 4 + 2 \left(\frac{4 \cdot 3}{2} \right) = 8 + 12 = 20$$

and with the equivalent form: $4 \left(\frac{2+8}{2} \right) = 20$

Summary

Arithmetic progressions:

- ▶ First term a , common difference d : terms $a, a + d, a + 2d, \dots$
- ▶ Recursive relation $t_1 = a, t_k = t_{k-1} + d$; closed formula $t_k = a + (k - 1) d$

Sum of the first n terms:

- ▶
$$\sum_{k=1}^n (a + d(k - 1)) = a n + d \frac{n(n - 1)}{2} = n \frac{t_1 + t_n}{2}$$
- ▶ Special cases: sum of the first n natural numbers ($a = d = 1$) and of the first n odd numbers ($a = 1, d = 2$)

Final summary

- ▶ The symbol $\sum$ allows us to write sums of many terms in compact form; its index is a dummy index
- ▶ The six properties of summations (product by a constant, constant term, union, decomposition, index translation and index reflection) allow us to compute sums in closed form
- ▶ Important summations:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^n (2k-1) = n^2, \quad \sum_{k=1}^n 2k = n(n+1)$$

- ▶ In a geometric progression with common ratio $q \neq 1$:

$$\sum_{k=1}^n a q^{k-1} = a \frac{q^n - 1}{q - 1}$$

- ▶ In an arithmetic progression:

$$\sum_{k=1}^n (a + d(k-1)) = a n + d \frac{n(n-1)}{2}$$