

Sets



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Informal introduction to set theory



Introduction: Set theory

- ▶ In this section we informally introduce the concepts of **set**, **element** and **membership**
- ▶ We will see how a set is defined: **by listing**, **by a property** or **by construction**
- ▶ Sets are the basic language of all of mathematics and of this course

The three key concepts (1/2)

Set theory is based on the following three key concepts:

1. **Sets:** the notion of set is generally taken as **primitive** (that is, not reducible to more elementary concepts). The words *collection*, *class*, *aggregate*, *family* are used as synonyms of set.
2. **Elements:** a set is determined by its *elements*, in the sense that a set is defined when we have a **criterion** to establish whether a given element is or is not an element of this set. There are sets with a *finite number of elements* (for example the set of students of a university) and sets with an *infinite number of elements* (for example the set of points of the plane).

Example

Sets.

For example, we have the set of points of a plane, the set of students of a university, or the set of stars of a galaxy.

The three key concepts (2/2)

Sets are usually denoted by capital letters. The elements of a set are usually denoted by lowercase letters.

3. **Membership:** the concept of *membership* links elements to sets. When an object is an element of a set, we say that the element *belongs* to the set. To indicate that an element x belongs to a set A we write:

$$x \in A$$

Non-membership

- ▶ To indicate that an element x does *not* belong to a set A we write:

$$x \notin A$$

Definition by listing

- ▶ A first way to define sets is **definition by listing** (roster notation).

A set can be defined **by listing**, that is, by listing the elements that belong to it between curly brackets. This technique assumes that the set has a finite number of elements.

Example

Definition by listing.

- ▶ The notation

$$A = \{a, b, c\}$$

means that the set A has as elements the three letters a , b and c . For example, a belongs to A , that is, $a \in A$, while $d \notin A$.

- ▶ The set of vowels of the Latin alphabet can be defined by listing:

$$V = \{a, e, i, o, u\}$$

Two rules of listing

- ▶ When listing the elements of a set, each element is written **only once**: a set does not contain the same object more than once.
- ▶ The elements of a set are **not ordered**: $\{1, 2, 3\}$ and $\{3, 1, 2\}$ describe the same set, because the order in which we list the elements is irrelevant.

Definition by a property

- ▶ A second way to define sets is **definition by a property**.

A set can be defined **by a property** as follows:

$$A = \{ x \in U : p(x) \text{ is true} \}$$

where $p(x)$ is the property that the element x of the set U must have in order to belong to the set A . This technique can be used to define sets with a finite or even an infinite number of elements.

Example

Definition by a property. Consider for example the set of letters of the Latin alphabet:

$$U = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$$

Using for example the property $p(x)$ defined as “ x is a vowel” we can define the following set of vowels:

$$A = \underbrace{\{x \in U : x \text{ is a vowel}\}}_{\{a, e, i, o, u\}}$$

Universal set

Note that to define a set A by a property we need a set U to which all the elements of the set A we want to define belong. The set U plays the role of the **universal set**.

Definition 1: of universal set

*A **universal set** (or simply **universe**), denoted by U , is a set fixed in advance that contains all the objects under consideration in a given context: all the sets considered in that context have elements that belong to U .*

Universal set

- ▶ It is important that the property $p(x)$ we use makes sense for every x of the set U (universal set), and therefore is true or false (without ambiguity of meaning) for every particular $x \in U$.
- ▶ The set A will then consist of exactly those x belonging to U for which the property $p(x)$ is true.



Properties starting from any set

Fixing a single universal set for the whole discussion is convenient, but it is not necessary. We can start from **any set already known** B , which may change from one definition to the next:

$$A = \{ x \in B : p(x) \}$$

is the set made of exactly those elements of B for which the property $p(x)$ is true (“is true” is usually not written). The universal set is the special case in which $B = U$ is the same for all the definitions in a given context.

The form used everywhere

- ▶ This is the form used everywhere in mathematics. The property can be written in words or with a formula, and it can consist of several conditions, separated by commas or by “and”.
- ▶ When the starting set is clear from the context, it is often left implicit and we simply write $\{ x : p(x) \}$. For example, when discussing real numbers, $\{ x : x > 0 \}$ denotes the set of positive real numbers.
- ▶ The starting set is still fixed: it is only omitted from the notation.

Example

Properties with different starting sets.

- ▶ Starting from $B = \{a, b, c, d, e\}$ and from the same property “ x is a vowel” we obtain a set different from the one of the previous example:

$$\{x \in B : x \text{ is a vowel}\} = \{a, e\}$$

- ▶ Starting from $C = \{1, 2, 3, 4, 5, 6\}$ we can define:

$$\{x \in C : x \text{ is even}\} = \{2, 4, 6\}$$

- ▶ The conditions can be more than one:

$$\{x \in C : x \text{ is even, } x > 2\} = \{4, 6\}$$

Definition by construction

- ▶ A third way to define sets is **definition by construction**: instead of choosing the elements among those of a set that already contains them, we **build** them from the elements of a set already known.

Given a set B already known, a set can be defined **by construction** as follows:

$$A = \{ \text{expression built from } x : x \in B \}$$

that is, A is made of all the objects obtained by applying the same rule to each element x of B . There is no need to know in advance a set containing the elements of A .

Example

Definition by construction.

- ▶ The set of the squares of the elements of $\{1, 2, 3\}$:

$$\{ x^2 : x \in \{1, 2, 3\} \} = \{1, 4, 9\}$$

- ▶ The set of the doubles of the elements of $\{1, 2, 3, 4\}$:

$$\{ 2x : x \in \{1, 2, 3, 4\} \} = \{2, 4, 6, 8\}$$

- ▶ If the rule produces the same object from different elements, the object is counted only once:

$$\{ x^2 : x \in \{-1, 0, 1\} \} = \{0, 1\}$$

Beware of contradictions

One must be careful when defining sets, since contradictions may arise. There is a formal definition of the concept of set, developed to avoid contradictions, but it is beyond the scope of this course.

- ▶ In all the ways seen so far we always start from a set already known. The notation $\{ x : p(x) \}$ understood as “all the objects for which $p(x)$ is true”, with no starting set at all (not even an implicit one), does **not** in general define a set.
- ▶ For example, with the property $p(x)$ defined as “ $x \notin x$ ” we would obtain the set of all sets that do not contain themselves, which however is not a set in the formal definition of sets. Admitting this set would generate the contradiction: “the set of all sets that do not belong to themselves belongs to itself if and only if it does not belong to itself” (**Russell’s paradox**, discussed in the further topics at the end).

Summary

- ▶ Key concepts: **set** (primitive notion), **element**, **membership** ($x \in A$, $x \notin A$)
- ▶ Sets with capital letters, elements with lowercase letters
- ▶ Definition **by listing**: $A = \{a, b, c\}$ (finite sets); each element only once, the order does not matter
- ▶ Definition **by a property**: $A = \{x \in U : p(x) \text{ is true}\}$, with U the universal set, or more generally $\{x \in B : p(x)\}$ with B any set already known
- ▶ Definition **by construction**: $\{x^2 : x \in B\}$
- ▶ A naive definition may lead to contradictions (Russell's paradox)

Number sets

Introduction: Number sets

- ▶ In this section we introduce **positional numeral systems** and the decimal expansion
- ▶ We give an informal definition of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ by means of decimal expansions
- ▶ We will see why $0.\overline{9} = 1$ and examples of real numbers that are not rational

Definition 2: numeral system

*A **positional numeral system** is a way of encoding numbers using a **sequence of digits**, where each digit contributes differently to the number depending on its position. The number of distinct digits is the **base** of the system.*

The **decimal expansion** of a number is a numerical representation in base 10; it expresses a number as a sum of powers of 10, and it can be finite or infinite depending on the type of number.

Informal definition of the main number sets (1/2)

1. We denote by $\mathbb{N}$ the set of **natural numbers**, that is, the set of numbers that can be written as **decimal expansions without decimal point and without sign**. We will use the informal notation:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$$

2. We denote by $\mathbb{Z}$ the set of **integers**, that is, the set of numbers that can be written as **decimal expansions without decimal point and with a sign**. We will use the informal notation:

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \dots\}$$

A convention and a notation

- ▶ We adopt the convention that **zero is a natural number**, that is $0 \in \mathbb{N}$
- ▶ Other textbooks exclude zero from $\mathbb{N}$: it is only a convention, but it must be stated once and for all
- ▶ We denote by $\mathbb{N}_{>0}$ the set of **positive natural numbers**:

$$\mathbb{N}_{>0} = \mathbb{N} \setminus \{0\} = \{1, 2, 3, 4, \dots\}$$

- ▶ Before going on with the other number sets, we introduce two definitions about natural numbers that we will use often in the course
- ▶ They are the definitions of **even** number and **odd** number

Even and odd numbers

Definition 3: of even number

*A natural number n is said to be **even** if there exists a natural number m such that:*

$$n = 2 m$$

Definition 4: of odd number

*A natural number n is said to be **odd** if there exists a natural number m such that:*

$$n = 2 m + 1$$

Example

Even and odd numbers.

- ▶ 0 is even, because $0 = 2 \cdot 0$, and 6 is even, because $6 = 2 \cdot 3$
- ▶ 1 is odd, because $1 = 2 \cdot 0 + 1$, and 7 is odd, because $7 = 2 \cdot 3 + 1$

The same two definitions are used for the integers, by requiring m to be an integer: for example $-4 = 2 \cdot (-2)$ is even and $-3 = 2 \cdot (-2) + 1$ is odd.

The well-ordering axiom

The **well-ordering axiom** states that every non-empty subset of $\mathbb{N}$ has a minimum.

- ▶ It is a property that is **assumed as an axiom** and that characterizes $\mathbb{N}$: it does not hold in the other number sets, since $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are themselves non-empty subsets without a minimum
- ▶ It is the axiom that justifies the **principle of mathematical induction**, which we will deal with later on

Every natural number is either even or odd

Observation 1

Every natural number is either even or odd, and no natural number is at the same time even and odd.

- The proof uses the well-ordering axiom just stated

Proof (1/2)

Every natural number is either even or odd. Consider the set

$$S = \{ n \in \mathbb{N} : n \text{ is neither even nor odd} \}$$

and suppose by contradiction that S is not empty: by the well-ordering axiom S has a minimum n_0 .

Since $0 = 2 \cdot 0$ is even and $1 = 2 \cdot 0 + 1$ is odd, we have $n_0 \geq 2$ and hence $n_0 - 2 \in \mathbb{N}$. As $n_0 - 2 < n_0$, the number $n_0 - 2$ does not belong to S :

- ▶ if $n_0 - 2 = 2m$, then $n_0 = 2(m + 1)$ is even;
- ▶ if $n_0 - 2 = 2m + 1$, then $n_0 = 2(m + 1) + 1$ is odd.

In both cases $n_0 \notin S$: a contradiction, hence S is empty.

Proof (2/2)

No natural number is at the same time even and odd. If a natural number n were even and odd, there would exist $m, k \in \mathbb{N}$ with

$$n = 2m \quad \text{and} \quad n = 2k + 1 \quad \implies \quad 2(m - k) = 1$$

But $m - k$ is an integer:

- ▶ if $m \leq k$ the left-hand side is less than or equal to 0;
- ▶ if $m \geq k + 1$ the left-hand side is greater than or equal to 2.

In no case can the left-hand side be equal to 1, and therefore such an n does not exist. □

Informal definition of the main number sets (2/2)

3. We denote by $\mathbb{Q}$ the set of **rational numbers**, that is, the set of numbers that can be written as **finite or infinite periodic decimal expansions**. In other words, it is the set of numbers that can be written as a fraction $\frac{p}{q}$ where p is an integer and q is a natural number different from zero.

Example

Rational numbers.

- ▶ For example, with $p = 2$ and $q = 5$ we have the fraction $\frac{2}{5}$, whose decimal expansion is 0.4.
- ▶ For example, with $p = 4$ and $q = 10$ we have the fraction $\frac{4}{10}$, whose decimal expansion is again 0.4. Note that a rational number can be written with more than one fraction.
- ▶ For example, with $p = 13$ and $q = 30$ we have the fraction $\frac{13}{30}$, whose decimal expansion is $0.4\overline{3} = 0.43333\dots$

However, we can represent every rational number different from 0 by a single fraction $\frac{p}{q}$ by choosing $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ coprime (that is, relatively prime: p and q are not both divisible by the same integer greater than 1).

The number $0.\overline{9}$

Observation 2

$$0.\overline{9} = 1$$

- ▶ There are different proofs of this observation, based on different mathematical techniques.

Proof (1/3)

1. A simple proof follows directly from the definition of 1 divided by 3; indeed, we have:

$$\frac{1}{3} = 0.\bar{3}$$

$$\frac{1}{3} \cdot 3 = 0.\bar{3} \cdot 3$$

$$1 = 0.\bar{9}$$

Proof (2/3)

2. Using algebraic arguments we can write:

$$x = 0.999 \dots$$

$$10x = 9.999 \dots \quad \text{multiplying by 10}$$

$$10x = 9 + 0.999 \dots \quad \text{separating the integer part from the fractional part}$$

$$10x = 9 + x \quad \text{by definition of } x$$

$$9x = 9 \quad \text{subtracting } x$$

$$x = 1 \quad \text{dividing by 9}$$

Proof (3/3)

3. A proof by contradiction is the following:

$$\begin{aligned}0.\bar{9} &\neq 1 \\0.\bar{9} \cdot 9 &\neq 1 \cdot 9 \\0.\bar{9} \cdot 9 + 0.\bar{9} &\neq 1 \cdot 9 + 0.\bar{9} \\0.\bar{9} \cdot 9 + 0.\bar{9} &\neq 9.\bar{9} \\0.\bar{9} \cdot (9 + 1) &\neq 9.\bar{9} \\0.\bar{9} \cdot (10) &\neq 9.\bar{9} \\9.\bar{9} &\neq 9.\bar{9} \quad \text{contradiction!}\end{aligned}$$



The real numbers

4. We denote by $\mathbb{R}$ the set of **real numbers**, that is, the set of numbers identified with finite or infinite decimal expansions, periodic or non-periodic.
- There are formal definitions of the sets of natural, integer and rational numbers that are beyond the scope of this course. Later we will give a formal definition of the real numbers.



Example

Real numbers.

- ▶ Consider for example the number

$$0.10110111011110\dots$$

obtained by putting after the decimal point one digit equal to 1, then 0, then two digits equal to 1, then 0, then three digits equal to 1 ... and so on. The string of digits after the decimal point is neither finite nor periodic: therefore this number is real but not rational.

- ▶ Other examples of real but not rational numbers are $\sqrt{2}$ and $\sqrt{3}$, or π and Euler's number e , which have infinite non-periodic decimal expansions and are therefore real but not rational numbers.

Example

Number sets defined by listing and by a property.

- ▶ The set of the first five prime numbers: $P = \{2, 3, 5, 7, 11\}$.
For example $2 \in P$, while $4 \notin P$.
- ▶ The set of **even numbers**, with universal set $\mathbb{Z}$:

$$E = \{x \in \mathbb{Z} : \text{there exists } k \in \mathbb{Z} \text{ such that } x = 2k\}$$

For example $-4 \in E$, because $-4 = 2 \cdot (-2)$, while $3 \notin E$, because $3 = 2k$ would give $k = \frac{3}{2} \notin \mathbb{Z}$.

- ▶ The set of **positive real numbers**, with universal set $\mathbb{R}$:

$$\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$$

It is an infinite set, which cannot be defined by listing.

Summary

- ▶ **Positional numeral system:** sequence of digits whose contribution depends on the position; base 10
- ▶ $\mathbb{N}$: expansions without decimal point and without sign; $\mathbb{Z}$: without decimal point and with a sign
- ▶ $\mathbb{Q}$: finite or infinite periodic expansions, fractions $\frac{p}{q}$
- ▶ $0.\overline{9} = 1$
- ▶ $\mathbb{R}$: arbitrary decimal expansions (e.g. $\sqrt{2}$, π , e are not rational)
- ▶ Convention $0 \in \mathbb{N}$ and notation $\mathbb{N}_{>0}$; **even** numbers ($n = 2m$) and **odd** numbers ($n = 2m + 1$)
- ▶ **Well-ordering axiom:** every non-empty subset of $\mathbb{N}$ has a minimum; from it, every natural number is either even or odd

Relations between sets



Introduction: Relations between sets

- ▶ In this section we define **equality**, **inclusion** and **strict inclusion** between sets
- ▶ We prove the reflexive, antisymmetric and transitive properties of inclusion: the antisymmetric property gives the **double inclusion** method to prove that two sets are equal
- ▶ We will learn not to confuse “belongs to” ($\in$) and “is contained in” ($\subseteq$)
- ▶ We will introduce the empty set, the cardinality and the power set

Definition 5: of equal sets

*Two sets A and B are **equal** when they have the same elements. We write*

$$A = B$$

and this means that every element that belongs to A also belongs to B and every element that belongs to B also belongs to A . If A and B are not equal we write $A \neq B$.

Example

Order and multiplicity of the elements.

- ▶ The concept of *order* among the elements is foreign to sets:

$$\{1, 2, 5\} = \{1, 5, 2\}$$

because the two sets have the same elements, the numbers 1, 2 and 5.

- ▶ The concept of *multiplicity of the elements* is also foreign to sets: the set of solutions of $x - 1 = 0$ is equal to the set of solutions of $(x - 1)^2 = 0$. The fact that the second equation has the solution $x = 1$ with algebraic multiplicity 2 does not change the set of its solutions: both sets are equal to $\{1\}$.

Subset

It may happen that only one of the two requirements expressed by the equality relation holds.

Definition 6: of subset

*Given two sets A and B , we say that A is a **subset** of B , and we write $A \subseteq B$, if every element of A is also an element of B , that is, if for every x the implication $x \in A \implies x \in B$ holds. We read “ A is contained in B ” or “ A is a subset of B ”.*

If we state that $A \subseteq B$, we do not exclude that $A = B$.

Definition 7: of proper subset

*Given two sets A and B , we say that A is a **proper subset** of B , or that A is strictly contained in B , and we write $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$. In this case we speak of **strict inclusion**.*

- In some texts the symbol $\subset$ denotes strict inclusion, in others it is a synonym of $\subseteq$. We will use $\subseteq$ for inclusion and $\subsetneq$ for strict inclusion; when it appears, the symbol $\subset$ has the same meaning as $\subseteq$.

Example

Inclusion and strict inclusion.

- ▶ $\{1, 4\} \subseteq \{1, 3, 4\}$, because 1 and 4 belong to $\{1, 3, 4\}$. The inclusion is strict, $\{1, 4\} \subsetneq \{1, 3, 4\}$, because $3 \in \{1, 3, 4\}$ but $3 \notin \{1, 4\}$.
- ▶ $\{1, 3, 4\} \subseteq \{1, 3, 4\}$, but the inclusion is not strict, because the two sets are equal.
- ▶ $\mathbb{N} \subsetneq \mathbb{Z}$: every natural number is an integer, and $-1 \in \mathbb{Z}$ but $-1 \notin \mathbb{N}$.

Membership and inclusion

Be careful not to confuse “belongs to” and “is contained in” (in symbols, $\in$ and $\subseteq$). In a sense, they are two ways of indicating that “something is inside something else”, but they have a fundamental logical difference:

- ▶ a set is contained in another set;
- ▶ an element belongs to a set.

Example

“Belongs to” vs “is contained in”. For example:

$$3 \in \{1, 3, 4\}; \quad \{3\} \subseteq \{1, 3, 4\}; \quad \{1, 4\} \subseteq \{1, 3, 4\}.$$

In particular, do not confuse 3 (which is a number) with $\{3\}$, which is the set that contains the number 3 as its only element.

Sometimes we consider sets whose elements are other sets. In this case too, the symbols $\in$ and $\subseteq$ are not interchangeable, but must be used correctly.

Example

“Belongs to” vs “is contained in”. For example, if we define the set

$$A = \left\{ \{1\}, \{2\}, \{1, 2\} \right\},$$

then it is correct to state that $\{2\} \in A$, because $\{2\}$ is a set that plays the role of an element of A , whereas it would be incorrect to say that $\{2\} \subseteq A$, because this would mean that $2 \in A$, while A has $\{2\}$ as an element but not 2.

Instead, the subset of A containing only the element $\{2\}$ is denoted by $\{\{2\}\}$ and we have $\{\{2\}\} \subseteq A$.

Properties of inclusion

We now see three properties of inclusion: reflexive, antisymmetric and transitive.

We prove them directly from the definition of subset.

Reflexive property

Proposition 1: reflexive property of inclusion

For every set A we have

$$A \subseteq A.$$

Proof

By the definition of subset we must check that for every x the implication

$$x \in A \implies x \in A$$

holds. The implication is true because the conclusion coincides with the hypothesis: if $x \in A$, then $x \in A$. Hence $A \subseteq A$. □

Antisymmetric property

Proposition 2: antisymmetric property of inclusion

Given two sets A and B , we have

$$A = B \iff A \subseteq B \text{ and } B \subseteq A.$$

Proof

- ▶ ($\implies$) Suppose $A = B$. By the definition of equal sets every element of A belongs to B , hence $A \subseteq B$. In the same way every element of B belongs to A , hence $B \subseteq A$.
- ▶ ($\impliedby$) Suppose $A \subseteq B$ and $B \subseteq A$. From $A \subseteq B$ it follows that every element of A belongs to B , and from $B \subseteq A$ it follows that every element of B belongs to A . Hence A and B have the same elements, that is, $A = B$.



The double inclusion method

- ▶ The antisymmetric property gives the most common method to prove that two sets A and B are equal, called **double inclusion**:
 1. we take an arbitrary element of A and show that it belongs to B (that is, $A \subseteq B$);
 2. we take an arbitrary element of B and show that it belongs to A (that is, $B \subseteq A$).

Transitive property

Proposition 3: transitive property of inclusion

Given three sets A , B and C , we have

$$A \subseteq B \text{ and } B \subseteq C \implies A \subseteq C.$$

Proof

Suppose $A \subseteq B$ and $B \subseteq C$, and let x be an arbitrary element of A .

- ▶ Since $A \subseteq B$, we have $x \in B$.
- ▶ Since $B \subseteq C$, from $x \in B$ it follows that $x \in C$.

Hence every element of A belongs to C , that is, $A \subseteq C$. □

Empty set

Definition 8: of empty set

The **empty set** is the set that contains no elements. It is denoted by $\emptyset$.

Observation 3

Given a set A , we have:

$$\emptyset \subseteq A$$

Proof

By the definition of subset we must prove that for every x the implication

$$x \in \emptyset \implies x \in A$$

holds.

- ▶ The hypothesis $x \in \emptyset$ is false for every x , because no element belongs to $\emptyset$.
- ▶ An implication with a false hypothesis is true whatever the conclusion is: we say that the implication is **vacuously true**.
- ▶ Reasoning by contradiction: if $\emptyset \subseteq A$ did not hold, there would exist an element of $\emptyset$ that does not belong to A ; but $\emptyset$ has no elements.



Definition 9: of cardinality of a set

*The number of elements of a set A is the **cardinality** of the set. It is denoted by $|A|$. If the cardinality of A is a natural number, the set A is called **finite**; otherwise it is called **infinite**.*

- ▶ The empty set is finite and has cardinality $|\emptyset| = 0$.
- ▶ A set with only one element is called a **singleton**: for example $|\{3\}| = 1$.
- ▶ A subset with k elements of a set A is called a **k -subset** of A .

Example

Cardinality and subsets with k elements. Given the set $A = \{1, 2, 3\}$ we have $|A| = 3$.

- ▶ The 2-subsets of A are $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$.
- ▶ The 1-subsets of A are the singletons $\{1\}$, $\{2\}$ and $\{3\}$.
- ▶ The only 0-subset of A is $\emptyset$ and the only 3-subset of A is A itself.

Definition 10: of power set

*Given a set A , the set whose elements are all the subsets of A is called the **power set** of A and is denoted by the symbol $\mathcal{P}(A)$. The notation 2^A is also used.*

- ▶ Every set A has two trivial subsets, namely A itself (by the reflexive property) and the empty set $\emptyset$; the two subsets coincide if A is empty.

Example

Power set. For example, given

$$A = \{1, 2, 3\},$$

then

$$\mathcal{P}(A) = \left\{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\} \right\}$$

Cardinality of the power set

Observation 4

Given a set A with n elements, the power set $\mathcal{P}(A)$ has 2^n elements:

$$|\mathcal{P}(A)| = 2^n = 2^{|A|}$$

Proof (1/2)

- ▶ If $n = 0$, the set A is empty and its only subset is $\emptyset$, hence $|\mathcal{P}(A)| = 1 = 2^0$.
- ▶ If $n \geq 1$, let $a_1, a_2, \dots, a_n$ be the elements of A . To build a subset S of A we must decide, for each of the elements $a_1, a_2, \dots, a_n$, whether or not S contains that element.
- ▶ Hence we must make a choice between two options n times, and each choice is independent of the others.

Proof (2/2)

- ▶ Every sequence of n choices determines a subset of A and, conversely, every subset S is obtained by including a_i if and only if $a_i \in S$.
- ▶ Different sequences give different subsets: if they differ in the choice about a_i , then a_i belongs to one of the two subsets but not to the other.

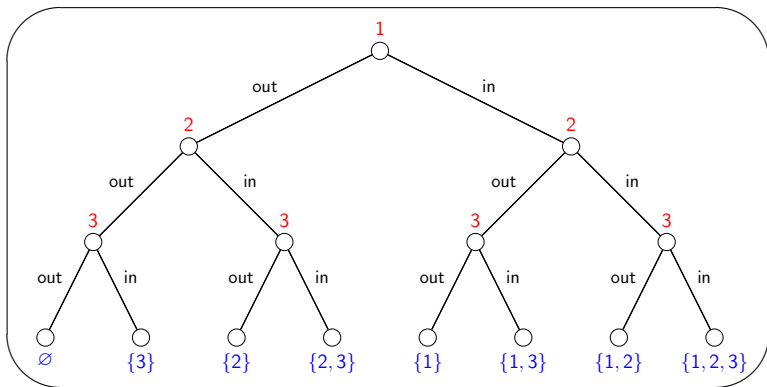
The number of subsets is therefore the number of sequences of n choices between two options:

$$\underbrace{2 \cdot 2 \cdots 2}_{n \text{ times}} = 2^n.$$



Example

Construction of the power set with a binary tree. Given the set $A = \{1, 2, 3\}$ with $n = 3$ elements, the cardinality of its power set is $2^3 = 8$. The construction of $\mathcal{P}(A)$ can be visualized through the following **binary tree** (an undirected, connected and acyclic graph) in which at each level we decide whether or not to include the object in the subset:



Summary (1/2)

- ▶ $A = B$: same elements; $A \subseteq B$: every element of A is an element of B
- ▶ $A \subsetneq B$: $A \subseteq B$ and $A \neq B$ (strict inclusion)
- ▶ Inclusion is reflexive, antisymmetric and transitive
- ▶ $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$: **double inclusion** method
- ▶ $\in$ links elements and sets, $\subseteq$ links sets and sets

Summary (2/2)

- ▶ $\emptyset$ is contained in every set (vacuously true implication)
- ▶ $|A|$ is the cardinality; finite and infinite sets; singletons and k -subsets
- ▶ $\mathcal{P}(A) = 2^A$: power set, $|\mathcal{P}(A)| = 2^n$ if $|A| = n$

Operations on sets

Introduction: Operations on sets

- ▶ In this section we define **intersection**, **union**, **difference**, **complement** and **Cartesian product**, and the **partitions** of a set
- ▶ We will represent them graphically with **Venn diagrams**
- ▶ We will prove their properties (commutative, associative, distributive, absorption, De Morgan's laws) and the properties of the cardinalities

Intersection and union

Definition 11: of intersection of sets

The intersection of two sets $A, B \subseteq U$ is the set defined by:

$$A \cap B = \{x \in U : x \in A \text{ and } x \in B\}$$

It is the set of elements that belong both to the first and to the second set.

Definition 12: of union of sets

*The **union** of two sets $A, B \subseteq U$ is the set defined by:*

$$A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$$

It is the set of elements that belong to the first or to the second set, where “or” is meant in the non-exclusive sense.

Definition 13: of difference of sets

The *difference* of two sets $A, B \subseteq U$ is the set defined by:

$$A \setminus B = \{x \in A : x \notin B\}$$

It is the set of elements that belong to the first but not to the second set. The symbol “ $\setminus$ ” can also be written “ $-$ ” by analogy with arithmetic subtraction.

Example

Intersection, union and difference. Given the sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$, we have

$$A \cap B = \{3, 4\}, \quad A \cup B = \{1, 2, 3, 4, 5\},$$

$$A \setminus B = \{1, 2\}, \quad B \setminus A = \{5\}.$$

In particular $A \setminus B \neq B \setminus A$: in the difference the order of the two sets matters.

Example

Universal sets. For example, in questions of arithmetic we could have $U = \mathbb{N}$; if we consider sets made up only of integers it is natural to choose $U = \mathbb{Z}$, while in questions of analysis we could have $U = \mathbb{R}$.

Definition 14: of set complementation and complementary sets

The **complementation** of a set $A \subseteq U$ is the set defined by:

$$\overline{A} = \{x \in U : x \notin A\}$$

This set is called the **complement** of A with respect to U .

- By the definition of difference, $U \setminus A = \{x \in U : x \notin A\}$, hence

$$\overline{A} = U \setminus A.$$

Properties of the complement

Proposition 4: properties of the complement

For every set $A \subseteq U$, where U is the universal set, we have:

$$\overline{\overline{A}} = A, \quad A \cap \overline{A} = \emptyset, \quad A \cup \overline{A} = U,$$

$$\overline{U} = \emptyset, \quad \overline{\emptyset} = U.$$

Proof (1/2)

All the sets involved are subsets of U : it is enough to consider the elements $x \in U$.

- ▶ $x \in \overline{\overline{A}}$ if and only if $x \notin \overline{A}$, that is, if and only if it is not true that $x \notin A$, that is, if and only if $x \in A$. Hence $\overline{\overline{A}} = A$.
- ▶ If there existed $x \in A \cap \overline{A}$, we would have $x \in A$ and $x \notin A$: this is impossible. Hence $A \cap \overline{A} = \emptyset$.
- ▶ $A \cup \overline{A} \subseteq U$ by the definition of union. Conversely, let $x \in U$: if $x \in A$, then $x \in A \cup \overline{A}$; if $x \notin A$, then $x \in \overline{A} \subseteq A \cup \overline{A}$. By double inclusion $A \cup \overline{A} = U$.

Proof (2/2)

- ▶ $\overline{U} = \{x \in U : x \notin U\}$ has no elements, because no element of U can fail to belong to U . Hence $\overline{U} = \emptyset$.
- ▶ $\overline{\emptyset} = \{x \in U : x \notin \emptyset\}$, and every $x \in U$ satisfies $x \notin \emptyset$. Hence $\overline{\emptyset} = U$.



Identity laws

Proposition 5: identity laws

For every set $A \subseteq U$, where U is the universal set, we have:

$$A \cap \emptyset = \emptyset, \quad A \cap U = A, \quad A \cup \emptyset = A, \quad A \cup U = U.$$

Proof

- ▶ No x can belong both to A and to $\emptyset$, because $\emptyset$ has no elements. Hence $A \cap \emptyset = \emptyset$.
- ▶ If $x \in A \cap U$, then $x \in A$. Conversely, if $x \in A$, then $x \in U$ because $A \subseteq U$, and hence $x \in A \cap U$.
- ▶ $x \in A \cup \emptyset$ if and only if $x \in A$ or $x \in \emptyset$; the second possibility never occurs, hence $A \cup \emptyset = A$.
- ▶ $A \cup U \subseteq U$ by the definition of union; conversely, if $x \in U$, then $x \in A \cup U$. Hence $A \cup U = U$.



Disjoint sets

Definition 15: of disjoint sets

Two sets A and B are **disjoint** if they have no elements in common:

$$A \cap B = \emptyset$$

Example

Disjoint sets.

- ▶ The sets $\{1, 2\}$ and $\{3, 4\}$ are disjoint.
- ▶ The sets $\{1, 2\}$ and $\{2, 3\}$ are not disjoint, because $\{1, 2\} \cap \{2, 3\} = \{2\}$.
- ▶ By the properties of the complement, every set $A \subseteq U$ and its complement $\bar{A}$ are disjoint.

Union of a family of sets

- ▶ Disjoint sets allow us to “split” a set into pieces that do not overlap.
- ▶ Given a family $\mathcal{F}$ of sets, the **union of the family** is the set of elements that belong to at least one of the sets of the family:

$$\bigcup_{B \in \mathcal{F}} B.$$

- ▶ For a family made up of two sets B and C we recover the union $B \cup C$.

Definition 16: of partition

Given a set A , a family $\mathcal{F}$ of subsets of A is a **partition** of A if:

- ▶ every set of the family is nonempty: $B \neq \emptyset$ for every $B \in \mathcal{F}$;
- ▶ the sets of the family are pairwise disjoint: if $B, C \in \mathcal{F}$ and $B \neq C$, then $B \cap C = \emptyset$;
- ▶ their union is A : $A = \bigcup_{B \in \mathcal{F}} B$, that is, every element of A belongs to at least one set of the family.

The sets of the family are called the **blocks** of the partition.

Every element in exactly one block

Observation 5

*A family $\mathcal{F}$ of nonempty subsets of a set A is a partition of A if and only if every element of A belongs to **exactly one** set of the family.*

Proof

- ▶ ($\implies$) Let $x \in A$. Since the union of the family is A , the element x belongs to at least one set of the family. If it belonged to two different sets $B, C \in \mathcal{F}$, we would have $x \in B \cap C = \emptyset$: this is impossible.
- ▶ ($\impliedby$) The blocks are nonempty by hypothesis. Every element of A lies in at least one block and every element of the blocks lies in A : by double inclusion the union is A . If two different sets $B, C \in \mathcal{F}$ had a common element, this element would belong to two blocks, against the hypothesis: hence $B \cap C = \emptyset$.



Example

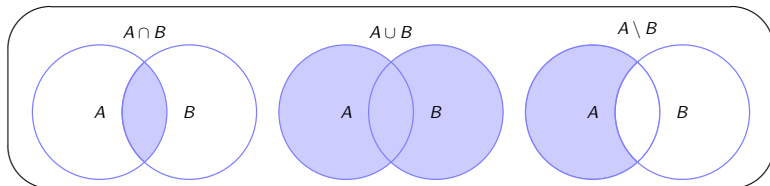
Partitions.

- ▶ $\{\{1, 2\}, \{3\}, \{4, 5, 6\}\}$ is a partition of $\{1, 2, 3, 4, 5, 6\}$: every element belongs to exactly one block.
- ▶ $\{\{1, 2\}, \{2, 3\}\}$ is not a partition of $\{1, 2, 3\}$: the element 2 belongs to two sets.
- ▶ $\{\{1\}, \{2\}\}$ is not a partition of $\{1, 2, 3\}$: the element 3 does not belong to any set.
- ▶ With universal set $\mathbb{Z}$, the set E of even numbers and its complement $\overline{E}$ (the odd numbers) form a partition of $\mathbb{Z}$: they are nonempty ($0 \in E$, $1 \in \overline{E}$), $E \cap \overline{E} = \emptyset$ and $E \cup \overline{E} = \mathbb{Z}$.

Venn diagrams

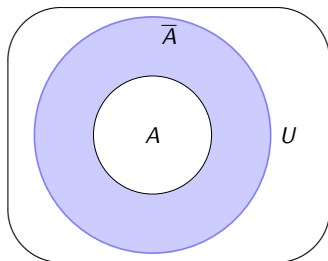
Venn diagrams are graphical representations in which sets are represented as regions of the plane.

- Venn diagrams of intersection, union and difference:



Venn diagrams

- Venn diagram of the complement of a set:



Properties of intersection

Observation 6: properties of intersection

Given three sets A , B and C , intersection has the following properties:

- ▶ **Commutative:** $A \cap B = B \cap A$
- ▶ **Associative:** $A \cap (B \cap C) = (A \cap B) \cap C$
- ▶ **Idempotence:** $A \cap A = A$

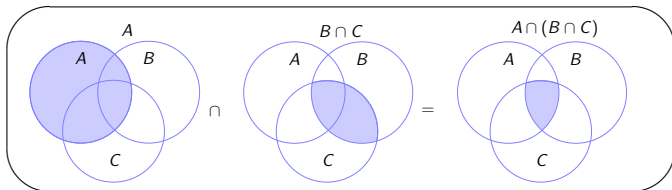
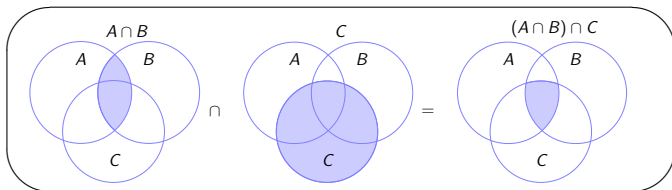
Proof (1/2)

We compare the elements of the two sides.

- ▶ **Commutative:** $x \in A \cap B$ if and only if $x \in A$ and $x \in B$, that is, if and only if $x \in B$ and $x \in A$, that is, if and only if $x \in B \cap A$.
- ▶ **Associative:** $x \in A \cap (B \cap C)$ if and only if x belongs to all three sets A , B and C , that is, if and only if $x \in A \cap B$ and $x \in C$, that is, if and only if $x \in (A \cap B) \cap C$.
- ▶ **Idempotence:** $x \in A \cap A$ if and only if $x \in A$ and $x \in A$, that is, if and only if $x \in A$.

Proof (2/2)

Associative property: first row $(A \cap B) \cap C$, second row $A \cap (B \cap C)$.



Properties of union

Observation 7: properties of union

Given three sets A , B and C , union has the following properties:

- ▶ **Commutative:** $A \cup B = B \cup A$
- ▶ **Associative:** $A \cup (B \cup C) = (A \cup B) \cup C$
- ▶ **Idempotence:** $A \cup A = A$

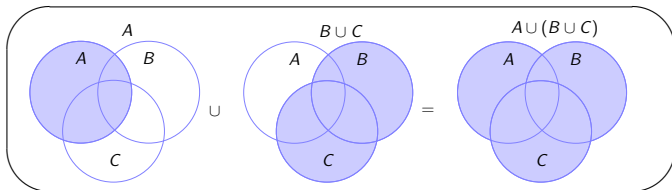
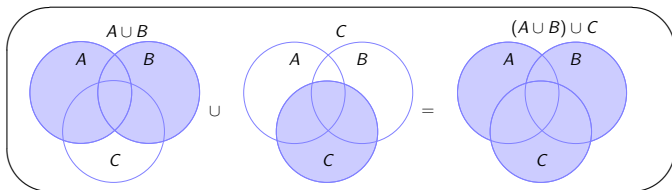
Proof (1/2)

We compare the elements of the two sides.

- ▶ **Commutative:** $x \in A \cup B$ if and only if $x \in A$ or $x \in B$, that is, if and only if $x \in B$ or $x \in A$, that is, if and only if $x \in B \cup A$.
- ▶ **Associative:** $x \in A \cup (B \cup C)$ if and only if x belongs to at least one of the three sets A , B and C , that is, if and only if $x \in A \cup B$ or $x \in C$, that is, if and only if $x \in (A \cup B) \cup C$.
- ▶ **Idempotence:** $x \in A \cup A$ if and only if $x \in A$ or $x \in A$, that is, if and only if $x \in A$.

Proof (2/2)

Associative property: first row $(A \cup B) \cup C$, second row $A \cup (B \cup C)$.



Distributive properties

Observation 8: distributive properties (linking union and intersection)

Given three sets A , B and C we have:

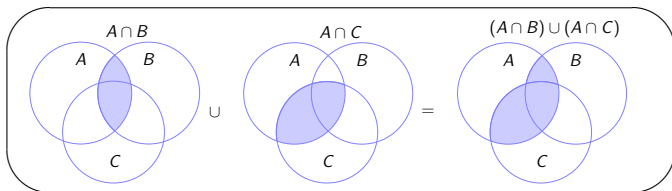
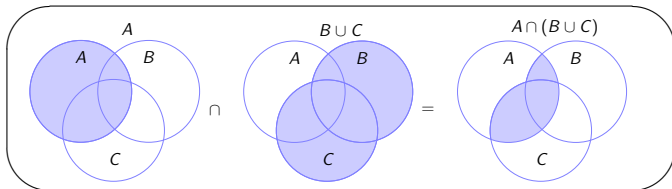
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

The distributive properties link union and intersection to each other.

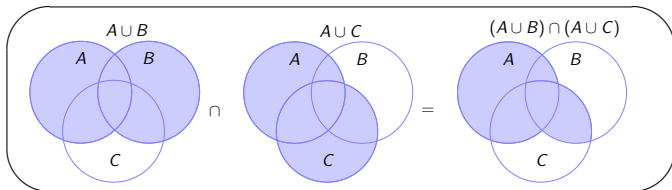
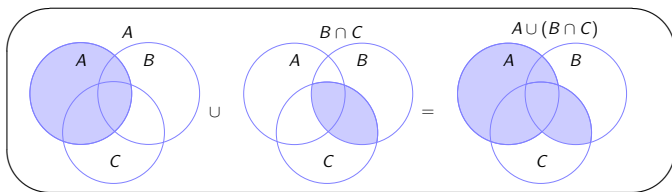
Proof (1/2)

First property: first row the left-hand side, second row the right-hand side.



Proof (2/2)

Second property: first row the left-hand side, second row the right-hand side.



Absorption laws

Proposition 6: absorption laws

Given two sets A and B we have:

$$A \cap (A \cup B) = A, \quad A \cup (A \cap B) = A.$$

Proof

We prove the two laws by double inclusion.

- ▶ If $x \in A \cap (A \cup B)$, then $x \in A$. Conversely, if $x \in A$, then $x \in A \cup B$, and hence $x \in A \cap (A \cup B)$.
- ▶ If $x \in A$, then $x \in A \cup (A \cap B)$. Conversely, if $x \in A \cup (A \cap B)$, then $x \in A$ or $x \in A \cap B$; in the second case, again, $x \in A$.



Inclusion, union and intersection

Proposition 7: inclusion, union and intersection

Given two sets A and B , the following statements are equivalent:

$$A \subseteq B, \quad A \cap B = A, \quad A \cup B = B.$$

Proof

- ▶ $(A \subseteq B \Rightarrow A \cap B = A)$ Every element of $A \cap B$ belongs to A . Conversely, if $x \in A$, then $x \in B$, and hence $x \in A \cap B$.
- ▶ $(A \cap B = A \Rightarrow A \subseteq B)$ If $x \in A$, then $x \in A \cap B$, and hence $x \in B$.
- ▶ $(A \subseteq B \Rightarrow A \cup B = B)$ Every element of B belongs to $A \cup B$. Conversely, if $x \in A \cup B$, then $x \in A$ (and hence $x \in B$) or $x \in B$.
- ▶ $(A \cup B = B \Rightarrow A \subseteq B)$ If $x \in A$, then $x \in A \cup B = B$.



De Morgan's laws

Proposition 8: De Morgan's laws (first version)

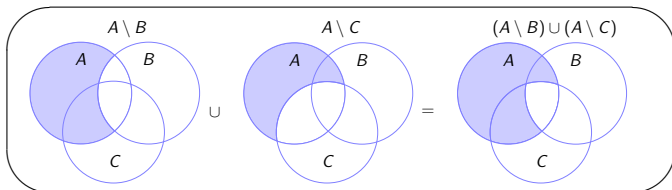
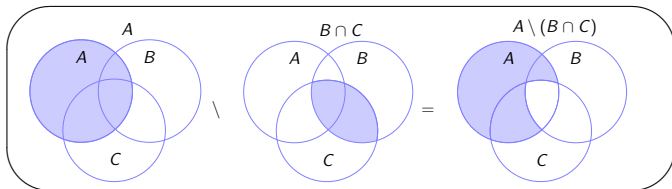
Given three sets A , B and C we have:

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$

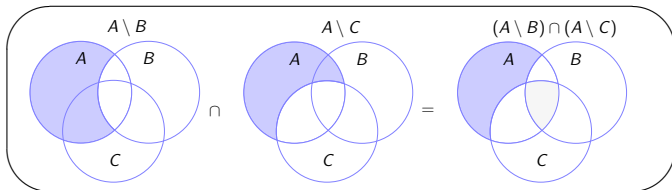
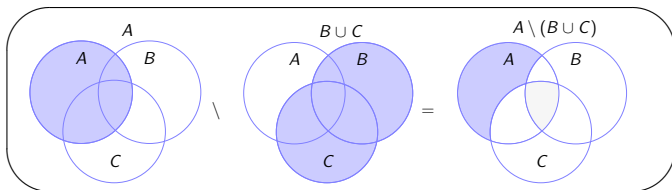
Proof (1/2)

First law: first row the left-hand side, second row the right-hand side.



Proof (2/2)

Second law: first row the left-hand side, second row the right-hand side.



De Morgan's laws

Proposition 9: De Morgan's laws (second version)

Given the sets $B, C \subseteq U$, we have

$$\overline{B \cap C} = \overline{B} \cup \overline{C}$$

$$\overline{B \cup C} = \overline{B} \cap \overline{C}$$

Proof

These laws can be derived by setting A equal to U in the previous De Morgan's laws and recalling that $\overline{B} = U \setminus B$ and $\overline{C} = U \setminus C$:

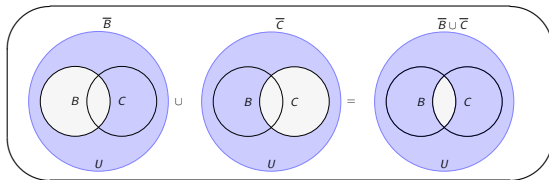
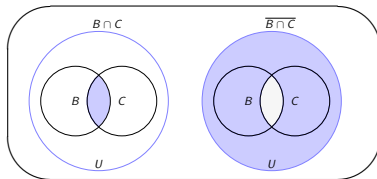
$$\underbrace{U \setminus (B \cap C)}_{=\overline{B \cap C}} = \underbrace{(U \setminus B)}_{=\overline{B}} \cup \underbrace{(U \setminus C)}_{=\overline{C}}$$

$$\underbrace{U \setminus (B \cup C)}_{=\overline{B \cup C}} = \underbrace{(U \setminus B)}_{=\overline{B}} \cap \underbrace{(U \setminus C)}_{=\overline{C}}$$



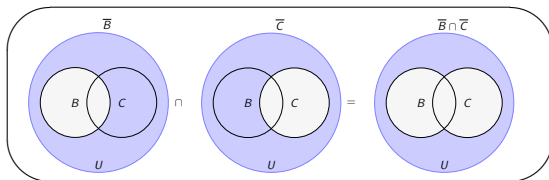
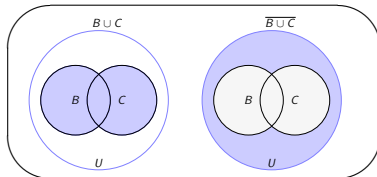
Proof (graphical) (1/2)

First law: first row the left-hand side, second row the right-hand side.



Proof (graphical) (2/2)

Second law: first row the left-hand side, second row the right-hand side.



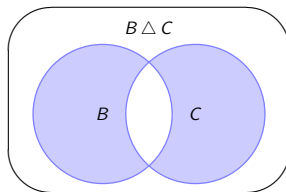
Symmetric difference

- ▶ The set of elements that belong to a set B or to a set C but not to both (meaning “or” in the exclusive sense) is called the **symmetric difference** of B and C :

$$B \triangle C = (B \setminus C) \cup (C \setminus B).$$

- ▶ It is also obtained by setting $U = B \cup C$ and taking the complement of the intersection:

$$B \triangle C = (B \cup C) \setminus (B \cap C).$$



Proof

We prove that the two expressions of the symmetric difference coincide.

- ▶ $x \in (B \setminus C) \cup (C \setminus B)$ if and only if $(x \in B \text{ and } x \notin C)$ or $(x \in C \text{ and } x \notin B)$, that is, if and only if x belongs to only one of the two sets B and C .
- ▶ $x \in (B \cup C) \setminus (B \cap C)$ if and only if x belongs to at least one of the two sets but not to both, that is, again, if and only if x belongs to only one of the two sets.



Properties of cardinality

We now see how to count the number of elements of the union of two finite sets.

We start from the case of two disjoint sets.

Union of disjoint sets

Proposition 10: cardinality of the union of disjoint sets

If A and B are two disjoint finite sets, that is, $|A \cap B| = 0$, then

$$|A \cup B| = |A| + |B|.$$

More generally, if $A_1, A_2, \dots, A_m$ are pairwise disjoint finite sets, then

$$|A_1 \cup A_2 \cup \dots \cup A_m| = |A_1| + |A_2| + \dots + |A_m|.$$

Proof

- ▶ Every element of $A \cup B$ belongs to A or to B , and no element belongs to both, because $A \cap B = \emptyset$.
- ▶ Counting first the $|A|$ elements of A and then the $|B|$ elements of B we count every element of $A \cup B$ exactly once:
 $|A \cup B| = |A| + |B|$.
- ▶ In the same way, every element of $A_1 \cup A_2 \cup \dots \cup A_m$ belongs to only one of the sets $A_1, A_2, \dots, A_m$: counting their elements one after the other, we count every element of the union exactly once.



Cardinality of the union

Proposition 11: cardinality of the union

For any two finite sets A and B , we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof (1/2)

Every element of $A \cup B$ belongs to exactly one of the three sets

$$A \setminus B, \quad A \cap B, \quad B \setminus A,$$

which are therefore pairwise disjoint with union $A \cup B$. By the cardinality of the union of disjoint sets

$$|A \cup B| = |A \setminus B| + |A \cap B| + |B \setminus A|.$$

Proof (2/2)

In the same way A is the union of the two disjoint sets $A \setminus B$ and $A \cap B$, and B is the union of the two disjoint sets $B \setminus A$ and $A \cap B$:

$$|A| = |A \setminus B| + |A \cap B|, \quad |B| = |B \setminus A| + |A \cap B|.$$

Adding:

$$\begin{aligned} |A| + |B| &= |A \setminus B| + |A \cap B| + |B \setminus A| + |A \cap B| \\ &= |A \cup B| + |A \cap B|. \end{aligned}$$



Inequality for the cardinality of the union

Proposition 12: inequality for the cardinality of the union

For any two finite sets A and B , we have

$$|A \cup B| \leq |A| + |B|.$$

Proof. By the cardinality of the union, and since $|A \cap B| \geq 0$:

$$|A \cup B| = |A| + |B| - |A \cap B| \leq |A| + |B|.$$



Cardinality of a subset

Proposition 13: cardinality of a subset

Given two finite sets A and B with $A \subseteq B$, we have

$$|A| \leq |B|.$$

If moreover $A \subsetneq B$, then $|A| < |B|$.

Proof

- ▶ Since $A \subseteq B$, the set B is the union of the two disjoint sets A and $B \setminus A$, hence

$$|B| = |A| + |B \setminus A| \geq |A|.$$

- ▶ If $A \subsetneq B$, then $A \neq B$ and, by the antisymmetric property, $B \subseteq A$ cannot hold: there exists $b \in B$ with $b \notin A$, that is, $b \in B \setminus A$. Then $|B \setminus A| \geq 1$ and

$$|B| = |A| + |B \setminus A| \geq |A| + 1 > |A|.$$



Ordered pairs

There is another operation on sets, which can be performed on any two sets. To introduce it we need the concept of *ordered pair*: unlike the set $\{a, b\}$, in an ordered pair it matters which element comes first.

Definition 17: of ordered pair

Given two elements a and b , the **ordered pair** (a, b) is the set

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

The element a is called the **first component** and the element b the **second component** of the pair.

Characteristic property

- ▶ This definition, due to Kuratowski, expresses the ordered pair using only sets. What matters is the following property.

Proposition 14: characteristic property of ordered pairs

Given the elements a , b , c and d , we have

$$(a, b) = (c, d) \iff a = c \text{ and } b = d.$$

Proof (1/2)

- ▶ ($\Leftarrow$) If $a = c$ and $b = d$, the sets $\{\{a\}, \{a, b\}\}$ and $\{\{c\}, \{c, d\}\}$ are written with the same elements, hence they are equal.
- ▶ ($\Rightarrow$) Suppose $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$.

Case $a = b$. Then $(a, b) = \{\{a\}\}$ has only one element, hence $\{\{c\}, \{c, d\}\}$ also has $\{a\}$ as its only element: from $\{c\} = \{a\}$ it follows that $c = a$, and from $\{c, d\} = \{a\}$ it follows that $d = a$. Hence $a = c$ and $b = a = d$.

Proof (2/2)

Case $a \neq b$. The pair (a, b) has two distinct elements: $\{a\}$, which has one element, and $\{a, b\}$, which has two.

- ▶ Also $c \neq d$, otherwise (c, d) would have only one element.
- ▶ $\{a\}$ has only one element and $\{c, d\}$ has two, hence $\{a\} = \{c\}$, that is, $a = c$.
- ▶ $\{a, b\}$ has two elements, hence it is not equal to $\{c\}$: we must have $\{a, b\} = \{c, d\} = \{a, d\}$.
- ▶ Since $b \in \{a, d\}$ and $b \neq a$, we obtain $b = d$.



In particular, if $a \neq b$ then $(a, b) \neq (b, a)$, while $\{a, b\} = \{b, a\}$.

Cartesian product

Definition 18: of Cartesian product

*Given two (not necessarily distinct) sets A and B , the set consisting of all ordered pairs (a, b) , with $a \in A$ and $b \in B$, is called the **Cartesian product** of A and B and is denoted by the symbol $A \times B$:*

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Proposition 15: cardinality of the Cartesian product

When A and B are two finite sets: $|A \times B| = |A| \cdot |B|$.

Proof

- ▶ If $A = \emptyset$, then $A \times B = \emptyset$ and $|A \times B| = 0 = 0 \cdot |B|$.
- ▶ Otherwise let $a_1, a_2, \dots, a_m$ be the elements of A and, for $i \in \{1, 2, \dots, m\}$, let $R_i = \{(a_i, b) : b \in B\}$.
- ▶ By the characteristic property, every pair of $A \times B$ belongs to only one of the sets R_i (the one of its first component), and every R_i has $|B|$ elements, because $(a_i, b) = (a_i, b')$ if and only if $b = b'$.

By the cardinality of the union of disjoint sets

$$|A \times B| = |R_1| + |R_2| + \dots + |R_m| = \underbrace{|B| + \dots + |B|}_{m \text{ times}} = |A| \cdot |B|.$$



Example

Cartesian product.

$$\{a, b\} \times \{a, b, c\} = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c)\}$$

$$|\{a, b\}| = 2, \quad |\{a, b, c\}| = 3, \quad |\{a, b\} \times \{a, b, c\}| = 2 \cdot 3 = 6$$

The Cartesian product is not commutative: the pair (c, a) belongs to $\{a, b, c\} \times \{a, b\}$, but it does not belong to $\{a, b\} \times \{a, b, c\}$, because $c \notin \{a, b\}$.

Cartesian product of n sets

Definition 19: of ordered n -tuple and of Cartesian product of n sets

Given $n \geq 3$ elements $a_1, a_2, \dots, a_n$, the **ordered n -tuple** is the ordered pair

$$(a_1, a_2, \dots, a_n) = ((a_1, a_2, \dots, a_{n-1}), a_n).$$

Given $n \geq 2$ sets $A_1, A_2, \dots, A_n$, their **Cartesian product** is

$$A_1 \times \dots \times A_n = \{(a_1, \dots, a_n) : a_i \in A_i, i \in \{1, 2, \dots, n\}\}.$$

Cardinality of the product of n sets

- ▶ Applying repeatedly the characteristic property:
 $(a_1, \dots, a_n) = (b_1, \dots, b_n)$ if and only if $a_i = b_i$ for every $i \in \{1, 2, \dots, n\}$.

Proposition 16: cardinality of the Cartesian product of n sets

If $A_1, A_2, \dots, A_n$ are finite sets, then

$$|A_1 \times A_2 \times \dots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|.$$

Proof

For $n = 2$ it is the previous proposition. For $n \geq 3$, by the definition of n -tuple

$$A_1 \times A_2 \times \cdots \times A_n = (A_1 \times A_2 \times \cdots \times A_{n-1}) \times A_n,$$

hence

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1 \times A_2 \times \cdots \times A_{n-1}| \cdot |A_n|.$$

Repeating the argument on $A_1 \times \cdots \times A_{n-1}$, then on $A_1 \times \cdots \times A_{n-2}$, and so on down to $A_1 \times A_2$, we obtain $|A_1| \cdot |A_2| \cdots |A_n|$. □

The powers A^n and the spaces $\mathbb{R}^n$

- ▶ The Cartesian product of n sets equal to A is denoted by

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}, \quad |A^n| = |A|^n \text{ if } A \text{ is finite.}$$

For example $\{0, 1\}^3$ has $2^3 = 8$ elements: the triples of 0 and 1 correspond to the “in” or “out” choices of the binary tree of the power set.

- ▶ A typical use of the Cartesian product is $\mathbb{R} \times \mathbb{R}$, which is abbreviated by the symbol $\mathbb{R}^2$ and denotes the set of ordered pairs of real numbers.
- ▶ Similarly, $\mathbb{R}^n$ is the set of *ordered n -tuples of real numbers*

$$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R}, i \in \{1, 2, \dots, n\} \}$$

Example

Subsets of the plane. When we study subsets of the plane we choose $U = \mathbb{R}^2$ as universal set. For example, the set of points of the plane with both coordinates positive (the *first quadrant*) is

$$Q_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 > 0\}.$$

Since $(x_1, x_2) \in Q_1$ if and only if $x_1 \in \mathbb{R}_{>0}$ and $x_2 \in \mathbb{R}_{>0}$, we have

$$Q_1 = \mathbb{R}_{>0} \times \mathbb{R}_{>0}.$$

Summary (1/3)

- ▶ $A \cap B$: common elements; $A \cup B$: elements of at least one of the two; $A \setminus B$: elements of A not in B
- ▶ $\bar{A} = U \setminus A$: complement; properties of the complement and identity laws
- ▶ Disjoint if $A \cap B = \emptyset$; **partition**: nonempty blocks, every element in exactly one block
- ▶ Venn diagrams: graphical representation of the operations

Summary (2/3)

- ▶ $\cap$ and $\cup$: commutative, associative, idempotent; distributive properties
- ▶ Absorption: $A \cap (A \cup B) = A$, $A \cup (A \cap B) = A$
- ▶ $A \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B$
- ▶ De Morgan's laws: $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$,
 $\overline{B \cap C} = \overline{B} \cup \overline{C}$, ...
- ▶ Symmetric difference $B \triangle C = (B \setminus C) \cup (C \setminus B)$

Summary (3/3)

- ▶ $|A \cup B| = |A| + |B|$ if A and B are disjoint; in general $|A \cup B| = |A| + |B| - |A \cap B| \leq |A| + |B|$
- ▶ $A \subseteq B \Rightarrow |A| \leq |B|$; $A \subsetneq B \Rightarrow |A| < |B|$ (finite sets)
- ▶ Ordered pair $(a, b) = \{\{a\}, \{a, b\}\}$: $(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$
- ▶ $A \times B$: ordered pairs, $|A \times B| = |A| \cdot |B|$;
 $|A_1 \times \cdots \times A_n| = |A_1| \cdots |A_n|$

Further topics

Introduction: Further topics

- ▶ In this section we discuss **Russell's paradox**, which shows the limits of a naive definition of set
- ▶ We mention the Zermelo-Fraenkel axioms and we prove that no set is an element of itself
- ▶ Finally we see a further proof of $0.\overline{9} = 1$ based on the Archimedean property

Russell's paradox

“can a set be an element of itself or not?”

- ▶ For example, the set of all books in a library is not an element of itself (a set of books is not a book).
- ▶ On the other hand, reasoning naively, the collection of all sets with more than 20 elements would seem to be an element of itself. We will see at the end of this section that in the formal definition this collection is not a set.
- ▶ Two categories of sets can thus be defined:
 1. sets that are not elements of themselves
 2. sets that are elements of themselves

Russell's paradox

If we consider the set of all sets that are not elements of themselves, is it an element of itself or not?

Let us call this set S ; two hypotheses can be made:

- 1 If we suppose $S \in S$, then S contains itself as an element and therefore does not belong to S . Hence $S \notin S$, and we have a contradiction.
- 2 If we suppose $S \notin S$, then S does not contain itself as an element and therefore belongs to S . Hence $S \in S$ and we have another contradiction.

Russell's paradox: The set of all sets that do not belong to themselves belongs to itself if and only if it does not belong to itself.

Zermelo-Fraenkel axioms

- ▶ The formal definition of the concept of set is based on the **Zermelo-Fraenkel system of axioms**, abbreviated as **ZF**. This system of axioms includes the standard axioms of axiomatic set theory on which, together with the *axiom of choice*, all of ordinary mathematics is based.
- ▶ The **axiom of pairing** states that “Given two objects, there exists a set whose elements are the two objects”.
- ▶ The **axiom of regularity** states that “Every non-empty set A contains an element disjoint from A ”.

Observation 9

No set is an element of itself.

Proof

Let A be a set.

- ▶ By the axiom of pairing, applied to the two objects A and A , there exists the set $\{A, A\}$, which coincides with $\{A\}$: hence $\{A\}$ is a non-empty set.
- ▶ By the axiom of regularity applied to $\{A\}$, there exists an element of $\{A\}$ disjoint from $\{A\}$. The only element of $\{A\}$ is A , hence $A \cap \{A\} = \emptyset$.
- ▶ If $A \in A$ held, since also $A \in \{A\}$, we would have $A \in A \cap \{A\} = \emptyset$: this is impossible. Hence $A \notin A$.



- ▶ In particular, the collection of all sets with more than 20 elements is not a set in ZF: if it were, it would have more than 20 elements and would be an element of itself.
- ▶ The other ZF axioms are not part of the basic course in Mathematical Analysis.

A further proof of $0.\overline{9} = 1$

- ▶ A further proof of

$$0.\overline{9} = 1$$

starts from the assumption that two numbers are equal if and only if their difference is equal to zero, and it is based on computing the value of $1 - 0.\overline{9}$.

- ▶ This proof is based on the fact that 0 is the only non-negative number less than all the reciprocals of the positive integers, or equivalently that there is no number greater than every natural number. This is the **Archimedean property**, which holds for the rational and the real numbers.



The Archimedean property

The **Archimedean property** states that for every real number x there exists a natural number n such that $n > x$.

- ▶ Equivalently: for every real number $\varepsilon > 0$ there exists $n \in \mathbb{N}_{>0}$ with $\frac{1}{n} < \varepsilon$ (it is enough to take $n > \frac{1}{\varepsilon}$)
- ▶ Hence the only non-negative number that is less than or equal to $\frac{1}{n}$ for every n is zero: this is the form in which we use it
- ▶ It is not an additional axiom: in the chapter on [ordered fields](#) we will prove it starting from the supremum property

Proof (1/2)

We write the number $0.999\dots$ with n digits after the decimal point as $0.(9)_n$, hence $0.(9)_1 = 0.9$, $0.(9)_2 = 0.99$, $0.(9)_3 = 0.999$, and so on.

Given $\frac{1}{10^n} = 0.0\dots 01$, with n digits after the decimal point, the addition rules for decimal numbers imply

$$0.(9)_n + \frac{1}{10^n} = 1 \quad \text{moreover} \quad 0.(9)_n < 1, \forall n \in \mathbb{N}.$$

We must prove that 1 is the smallest number that is not less than all the $0.(9)_n$. For this it is enough to prove that, if a number x is not greater than 1 and not less than all the $0.(9)_n$, then $x = 1$.

Proof (2/2)

So let x be such that

$$0.(9)_n \leq x \leq 1$$

for every positive integer n . Hence

$$1 - 1 \leq 1 - x \leq 1 - 0.(9)_n$$

which, using basic arithmetic and the first equality established above, simplifies to

$$0 \leq 1 - x \leq \frac{1}{10^n}$$

This implies that the difference between 1 and x is less than the reciprocal of any positive integer. Hence this difference must be zero, and therefore $x = 1$; which in turn implies

$$0.999 \dots = 1$$



Summary

- ▶ **Russell's paradox:** the set of the sets that do not belong to themselves leads to a contradiction
- ▶ The axiomatic theory (ZF) avoids the paradoxes: axiom of pairing, axiom of regularity
- ▶ No set is an element of itself: the collection of the sets with more than 20 elements is not a set
- ▶ $0.\overline{9} = 1$ also via the **Archimedean property**



Summary (1/2)

Sets and elements:

- ▶ Set, element and membership; definition by listing or by a property; universal set
- ▶ Number sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ by means of decimal expansions

Relations between sets:

- ▶ Equality, inclusion and strict inclusion; difference between $\in$ and $\subseteq$; double inclusion
- ▶ Empty set, cardinality, power set (2^n elements)

Summary (2/2)

Operations on sets:

- ▶ Intersection, union, difference, complement, partitions
- ▶ Commutative, associative, distributive properties, absorption, De Morgan's laws
- ▶ Cardinality of the union; ordered pairs and Cartesian product

Further topics:

- ▶ Russell's paradox and Zermelo-Fraenkel axioms