

The principle of mathematical induction



Outline

The principle of mathematical induction

Proofs based on the principle of mathematical induction

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Introduction: The principle of mathematical induction

- ▶ In this section we present **proof by induction**, a method to prove properties that hold for all natural numbers from some n_0 on
- ▶ We will see the two steps of the proof: the **base case of the induction** and the **inductive step**
- ▶ We will understand why the method works and what the delicate point is, often a source of misunderstandings

Proof by induction

- ▶ We now present a proof technique called proof by induction. This procedure can be applied to theorems with the following structure:

“ for every $n \in \mathbb{N}$, $n \geq n_0$, property $p(n)$ holds ”

- ▶ The number n_0 is the smallest integer for which we want the property to be true; if $n_0 = 0$ the theorem simply states that the property is true for every $n \in \mathbb{N}$.

The two steps of a proof by induction

A proof by induction consists of the following two steps:

1. We prove that $p(n)$ is true for $n = n_0$ (**base case of the induction**).
2. We prove that, if n is an arbitrary natural number $\geq n_0$, the fact that $p(n)$ is true implies that $p(n + 1)$ is true (**inductive step**).

We can then conclude that for every $n \geq n_0$, $p(n)$ is true.

Why the method works

- ▶ The validity of this proof technique, *intuitively*, is based on the following fact:
 1. By point 1, we know that $p(n_0)$ is true. Suppose for example $n_0 = 1$: we therefore know that $p(1)$ is true (this must be proved explicitly).
 2. By point 2, since $p(1)$ is true, $p(2)$ will be true: indeed, we have proved that for any n , if $p(n)$ is true then $p(n+1)$ is also true. But then, since $p(2)$ is true, $p(3)$ will be true; but then $p(4)$ is true, $\dots$ and so on, hence $p(n)$ is true for every $n \geq 1$.

The two phases of the proof

- ▶ In practice, the proof consists of two phases.
 1. Prove $p(n_0)$ directly;
 2. Assume $p(n)$ as a hypothesis (**inductive hypothesis**) and prove $p(n + 1)$.

The delicate point

- ▶ This is the delicate point, often subject to misunderstandings. “Assuming as a hypothesis” $p(n)$ does not mean assuming the thesis as a hypothesis. What must be proved is that:
“ for every $n \geq n_0$, if $p(n)$ is true then $p(n + 1)$ is also true ”
and not
“ if $p(n)$ is true for every n , then $p(n + 1)$ is also true”

Summary

Proof by induction:

- ▶ It applies to statements of the form “for every $n \geq n_0$, $p(n)$ holds”
- ▶ **Base case:** prove $p(n_0)$ directly
- ▶ **Inductive step:** assuming $p(n)$ (inductive hypothesis), prove $p(n+1)$, for an arbitrary $n \geq n_0$

Warning:

- ▶ We prove the implication “ $p(n) \Rightarrow p(n+1)$ ”, we do not assume the thesis

Proofs based on the principle of mathematical induction



Introduction: Proofs by induction

- ▶ In this section we apply the principle of induction to some important results
- ▶ We will prove **Bernoulli's inequality**, which we will use often in the study of limits
- ▶ We will then recover, this time by induction, the formulas for the **sum of the first n natural, odd and even numbers** and the one for the **sum of the terms of a geometric progression**
- ▶ Finally, we will prove the **inequality of arithmetic and geometric means**

Bernoulli's inequality

Observation 1: Bernoulli's inequality

For every integer $n \geq 0$, $x \in \mathbb{R}$, $x \geq -1$, we have:

$$(1 + x)^n \geq 1 + n x$$

Proof (1/3)

By induction on n .

► **Base case of the induction**

Let $n = 0$. Then the statement becomes:

$$(1 + x)^0 \geq 1 + 0x \quad \text{i.e.} \quad 1 \geq 1$$

which is clearly true.

Proof (2/3)

► Inductive step

Suppose it is true for n , and let us prove it for $(n + 1)$. By the inductive hypothesis, we have:

$$(1 + x)^n \geq 1 + nx$$

and moreover we have

$$(1 + x) \geq 0 \quad \text{since} \quad x \geq -1.$$

Proof (3/3)

Then we can write:

$$\begin{aligned}(1+x)^{n+1} &= (1+x) \cdot (1+x)^n \\ &\geq (1+x) \cdot (1+nx) \\ &= 1 + (n+1)x + nx^2 \\ &\geq 1 + (n+1)x\end{aligned}$$

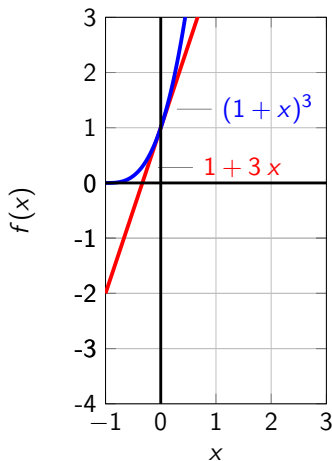
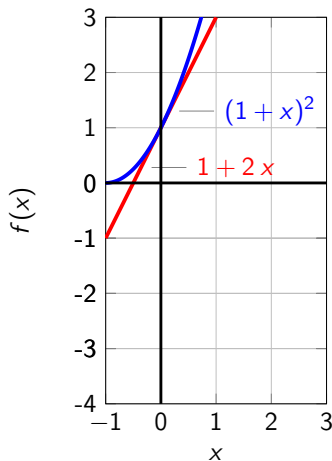
where in the last inequality we used the fact that $nx^2 \geq 0$. The chain of inequalities shows that, for $n+1$, we have

$$(1+x)^{n+1} \geq 1 + (n+1)x$$

which is exactly the desired statement, for $n+1$. □

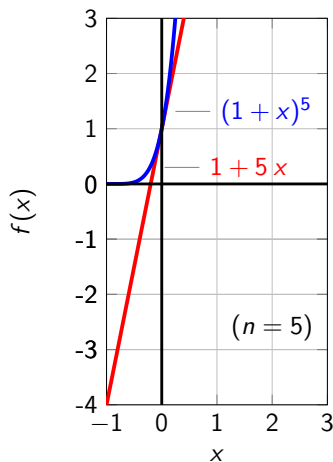
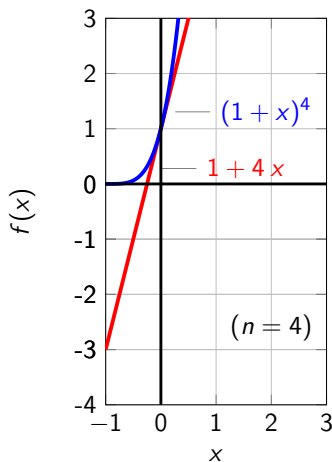
Example (1/2)

Bernoulli's inequality for $n = 2$ and $n = 3$:



Example (2/2)

Bernoulli's inequality for $n = 4$ and $n = 5$:



Some important summations

We now use the principle of induction to prove again the formula for the sum of the first n natural numbers.

Sum of the first n natural numbers

Observation 2: sum of the first n natural numbers

For every integer $n \geq 1$, we have:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

Proof (1/2)

By induction on n .

► **Base case of the induction**

Let $n = 1$. Then the statement becomes:

$$\sum_{k=1}^1 k = \frac{1(1+1)}{2} \quad \text{i.e.} \quad 1 = 1$$

which is clearly true.

► **Inductive step**

Suppose it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

Proof (2/2)

Hence we can write:

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \sum_{k=1}^n k + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}\end{aligned}$$

which is exactly the desired statement, for $n+1$. □

Sum of the first n odd numbers

Observation 3: sum of the first n odd numbers

For every integer $n \geq 1$, we have:

$$\sum_{k=0}^{n-1} (2k + 1) = n^2 \quad \text{or equivalently} \quad \sum_{k=1}^n (2k - 1) = n^2$$

Proof

By induction on n .

- ▶ **Base case of the induction:** for $n = 1$ the statement becomes $\sum_{k=0}^0 (2k+1) = 1^2$, i.e. $1 = 1$
- ▶ **Inductive step:** by the inductive hypothesis $\sum_{k=0}^{n-1} (2k+1) = n^2$, hence

$$\sum_{k=0}^{(n+1)-1} (2k+1) = \sum_{k=0}^{n-1} (2k+1) + 2n+1 = n^2 + 2n+1 = (n+1)^2$$

The second sum coincides with the first one ($k \rightarrow k+1$):

$$\sum_{k=1}^n (2k-1) = \sum_{k=0}^{n-1} (2(k+1)-1) = \sum_{k=0}^{n-1} (2k+1)$$



Sum of the first n even numbers

Observation 4: sum of the first n even numbers

For every integer $n \geq 1$, we have:

$$\sum_{k=1}^n 2k = n(n+1)$$

Proof

By induction on n .

- ▶ **Base case of the induction:** for $n = 1$ the statement becomes $\sum_{k=1}^1 2k = 1 \cdot 2$, i.e. $2 = 2$
- ▶ **Inductive step:** by the inductive hypothesis $\sum_{k=1}^n 2k = n(n+1)$, hence

$$\begin{aligned}\sum_{k=1}^{n+1} 2k &= \sum_{k=1}^n 2k + 2(n+1) = n^2 + n + 2(n+1) \\ &= (n+1)^2 + (n+1) = (n+1)((n+1) + 1)\end{aligned}$$

which is exactly the desired statement, for $n+1$. □

Sum of the terms of the geometric progression

With the same method we recover the formula for the sum of the first n terms of a geometric progression.

Sum of the first n terms of the geometric progression

Observation 5: sum of the first n terms of the geometric progression ($a = 1$)

Given $q \in \mathbb{R}_+$, for every integer $n \geq 1$ we have:

$$\sum_{k=1}^n q^{k-1} = \begin{cases} \frac{q^n - 1}{q - 1} & \text{if } q \neq 1 \\ n & \text{otherwise} \end{cases}$$

Proof (1/2)

By induction on n .

► Base case of the induction

Let $n = 1$. Then the statement becomes:

$$\sum_{k=1}^1 q^{k-1} = \frac{q^1 - 1}{q - 1} \quad \text{i.e.} \quad 1 = 1$$

which is clearly true.

► Inductive step

Suppose it is true for n , and let us prove it for $(n + 1)$. By the inductive hypothesis, we have:

$$\sum_{k=1}^n q^{k-1} = \frac{q^n - 1}{q - 1}$$

Proof (2/2)

Hence we can write

$$\begin{aligned}\sum_{k=1}^{n+1} q^{k-1} &= \sum_{k=1}^n q^{k-1} + q^n = \frac{q^n - 1}{q - 1} + q^n \\ &= \frac{q^n - 1 + q^{n+1} - q^n}{q - 1} = \frac{q^{n+1} - 1}{q - 1}\end{aligned}$$

which is exactly the desired statement, for $n + 1$. □

Arithmetic and geometric means

Finally, by induction we prove that the arithmetic mean of non-negative numbers is always greater than or equal to their geometric mean.



Inequality of arithmetic and geometric means

Observation 6: inequality of arithmetic and geometric means

For every integer $n \geq 2$ and every vector $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}_{\geq 0}^n$, we have:

$$\frac{\sum_{i=1}^n a_i}{n} \geq \sqrt[n]{\prod_{i=1}^n a_i}$$

- ▶ The *arithmetic mean* is greater than or equal to the *geometric mean*
- ▶ If $a_i = a$ for every $i \in \{1, 2, \dots, n\}$, both means equal a : this is the only case of equality

Proof (1/4)

By induction on n .

► **Base case of the induction**

Let $n = 2$. Since both sides are non-negative, we can square them:

$$\begin{aligned}\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} &\iff \frac{a_1^2 + 2 a_1 a_2 + a_2^2}{4} \geq a_1 a_2 \\ &\iff (a_1 - a_2)^2 \geq 0\end{aligned}$$

which is clearly true.

Proof (2/4)

► Inductive step

Suppose it is true for n , and let us prove it for $(n + 1)$. Let

$$\alpha = \frac{\sum_{i=1}^{n+1} a_i}{n + 1}$$

If $a_i = \alpha$ for every i , equality holds. Otherwise, reordering the values, we may assume

$$a_n > \alpha \quad \text{and} \quad a_{n+1} < \alpha \implies (a_n - \alpha)(\alpha - a_{n+1}) > 0$$

Proof (3/4)

Let $y = a_n + a_{n+1} - \alpha \geq a_n - \alpha > 0$ (since $a_{n+1} \geq 0$). Then:

$$n \alpha = \sum_{i=1}^{n-1} a_i + y \iff \alpha = \frac{\sum_{i=1}^{n-1} a_i + y}{n}$$

that is, α is the arithmetic mean of the n non-negative numbers $a_1, \dots, a_{n-1}$ and y .

By the inductive hypothesis:

$$\alpha^{n+1} = \alpha^n \alpha \geq \left(\prod_{i=1}^{n-1} a_i \right) y \alpha$$

Proof (4/4)

Moreover:

$$(a_n - \alpha)(\alpha - a_{n+1}) = y \alpha - a_n a_{n+1} > 0 \implies y \alpha > a_n a_{n+1}$$

Hence, since $\prod_{i=1}^{n-1} a_i \geq 0$:

$$\alpha^{n+1} \geq \prod_{i=1}^{n+1} a_i \iff \frac{\sum_{i=1}^{n+1} a_i}{n+1} \geq \sqrt[n+1]{\prod_{i=1}^{n+1} a_i}$$

which is exactly the desired statement, for $n+1$. □

Fibonacci sequence

We close with an application of induction in which the base case checks **two** values.

The Fibonacci sequence

- ▶ The **Fibonacci sequence** is defined by:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2$$

- ▶ The first terms are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...
- ▶ The solutions of $x^2 - x - 1 = 0$ are the **golden ratio** and its **conjugate**:

$$\phi = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \hat{\phi} = \frac{1 - \sqrt{5}}{2}$$

- ▶ Since they are solutions of $x^2 = x + 1$: $\phi^2 = \phi + 1$ and $\hat{\phi}^2 = \hat{\phi} + 1$

Observation 7: closed formula of the Fibonacci sequence

For every integer $n \geq 0$, we have:

$$F_n = \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}}$$

- ▶ The n -th term is computed directly, without going through all the preceding ones

Proof (1/2)

By induction on n . The recurrence links F_{n+1} to the **two** preceding terms: the base case checks $n = 0$ and $n = 1$, the inductive step assumes the statement true for n and $n - 1$.

► Base case of the induction

Let $n = 0$ and $n = 1$. Then the statement becomes:

$$F_0 = \frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = 0 \quad \text{and} \quad F_1 = \frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

which is clearly true.

Proof (2/2)

► Inductive step

By definition $F_{n+1} = F_n + F_{n-1}$, and by the inductive hypothesis the closed formulas for F_n and F_{n-1} hold. Hence:

$$\begin{aligned} F_{n+1} &= \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}} + \frac{\phi^{n-1} - \hat{\phi}^{n-1}}{\sqrt{5}} = \frac{(\phi^n + \phi^{n-1}) - (\hat{\phi}^n + \hat{\phi}^{n-1})}{\sqrt{5}} \\ &= \frac{\phi^{n-1}(\phi + 1) - \hat{\phi}^{n-1}(\hat{\phi} + 1)}{\sqrt{5}} = \frac{\phi^{n-1} \phi^2 - \hat{\phi}^{n-1} \hat{\phi}^2}{\sqrt{5}} = \frac{\phi^{n+1} - \hat{\phi}^{n+1}}{\sqrt{5}} \end{aligned}$$

which is exactly the desired statement, for $n + 1$. □

Summary

Results proved by induction:

- ▶ **Bernoulli's inequality:** $(1 + x)^n \geq 1 + nx$ for every $n \geq 0$ and $x \geq -1$
- ▶ $\sum_{k=1}^n k = \frac{n(n+1)}{2}$ for every $n \geq 1$
- ▶ $\sum_{k=0}^{n-1} (2k+1) = n^2$ and $\sum_{k=1}^n 2k = n(n+1)$ for every $n \geq 1$
- ▶ $\sum_{k=1}^n q^{k-1} = \frac{q^n - 1}{q - 1}$ for every $n \geq 1$ and $q \neq 1$
- ▶ **Arithmetic mean $\geq$ geometric mean** for $n \geq 2$ non-negative real numbers
- ▶ **Closed formula of Fibonacci:** $F_n = (\phi^n - \hat{\phi}^n)/\sqrt{5}$, with two base cases

Common scheme:

- ▶ Check the base case, then the inductive step from n to $n+1$ using the inductive hypothesis

Final summary

- ▶ Proof by induction proves statements of the form “for every $n \geq n_0$, $p(n)$ holds”
- ▶ Two steps are needed: the **base case** ($p(n_0)$ true) and the **inductive step** ($p(n) \Rightarrow p(n+1)$)
- ▶ The inductive hypothesis is not the thesis: we prove an implication valid for every $n \geq n_0$
- ▶ By induction we proved Bernoulli's inequality, the sum of the first n natural numbers, the sum of the terms of a geometric progression and the inequality of arithmetic and geometric means