Improper integrals

Outline

Unbounded intervals and unbounded functions

Convergence tests

Integral test for series

Objectives

At the end of this chapter you will be able to:

- define and compute improper integrals on unbounded intervals and of unbounded functions
- use the standard integrals $\int_1^{+\infty} \frac{dx}{x^p}$ and $\int_0^1 \frac{dx}{x^p}$
- use the comparison test, the limit comparison test and absolute convergence
- use the integral test for series

Prerequisites

- From this module, «Fundamental theorem of calculus» and «Integration techniques and applications»: computing definite integrals with antiderivatives
- From this module, «Fundamental limits and asymptotic estimates» and «Monotone functions on an interval and invertibility»: asymptotic equivalences and limits of monotone functions
- From this module, «Series with non-negative terms»: comparison tests and the generalized harmonic series

Unbounded intervals and unbounded functions

Introduction: Unbounded intervals and unbounded functions

- The Riemann integral requires bounded functions on bounded intervals
- We extend it with a limit

Definitions

Definition 1: improper integral

$$\int_a^{+\infty} f = \lim_{\omega \to +\infty} \int_a^{\omega} f \ (f \text{ continuous on } [a, +\infty)); \quad \int_a^{b} f = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} f \ (f \text{ continuous on } (a, b]).$$

- Convergent, divergent or does not exist; several critical points: split
- $\int_0^{+\infty} e^{-x}\, dx = 1$, $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \pi$, $\int_0^{+\infty} \cos x\, dx$ does not exist

Standard integrals

Theorem 1: standard improper integrals

$\int_1^{+\infty} \frac{dx}{x^p}$ converges if and only if $p > 1$ (value $\frac{1}{p-1}$); $\int_0^1 \frac{dx}{x^p}$ converges if and only if $p < 1$ (value $\frac{1}{1-p}$).

- For $p \neq 1$: $\int_1^\omega x^{-p}\,dx = \frac{\omega^{1-p}-1}{1-p}$; for $p = 1$: $\log \omega$

Beware

- $\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -2$ is **wrong**
- The integrand is unbounded at 0: split, and $\int_0^1 \frac{dx}{x^2} = +\infty$
- $\int_0^{1/2} \frac{dx}{x \log^2 x} = \frac{1}{\log 2}$ (substitution $t = \log x$)

Summary

- Improper integral = limit of Riemann integrals
- At infinity we need $p > 1$, at zero $p < 1$

Convergence tests

Introduction: Convergence tests

- For $f \geq 0$ the integral converges or diverges to $+\infty$ (monotonicity)
- As for series: comparison and limit comparison

Comparison

Theorem 2: comparison test

If $0 \leq f \leq g$ *on* $[a, +\infty)$*:* $\int g$ *converges* $\Rightarrow \int f$ *converges;* $\int f$ *diverges* $\Rightarrow \int g$ *diverges.*

- $\int_a^\omega f \leq \int_a^\omega g \leq \int_a^{+\infty} g$: bounded and monotone
- $\int_0^{+\infty} e^{-x^2}\,dx$ converges: $e^{-x^2} \leq e^{-x}$ for $x \geq 1$



Limit comparison

Theorem 3: limit comparison test

$f, g > 0$, $\frac{f}{g} \to \ell \in (0, +\infty)$ (for example $f \sim g$): same behaviour.

- Eventually $\frac{\ell}{2} g \leq f \leq \frac{3\ell}{2} g$
- $\int_1^{+\infty} \frac{x+1}{x^3+2}\, dx$ converges ($\sim \frac{1}{x^2}$); $\int_0^1 \frac{dx}{e^x-1}$ diverges ($\sim \frac{1}{x}$)
- $\int_0^{+\infty} \frac{x^\alpha}{1+x^2}\, dx$ converges if and only if $-1 < \alpha < 1$

Absolute convergence

Theorem 4: absolute convergence

If $\int_a^{+\infty} |f|$ *converges, then* $\int_a^{+\infty} f$ *converges and* $\left|\int f\right| \leq \int |f|$.

- $0 \leq f + |f| \leq 2|f|$ and $f = (f + |f|) - |f|$
- $\int_1^{+\infty} \frac{\sin x}{x^2}\, dx$ converges

□

Summary

- Comparison and limit comparison for positive functions
- Absolute convergence for functions of variable sign

Integral test for series

Introduction: Integral test for series

- A series with positive decreasing terms is compared with an integral
- We recover the generalized harmonic series

Integral test

Theorem 5: integral test

f continuous, non-negative, non-increasing on $[1, +\infty)$: $\sum f(k)$ and $\int_1^{+\infty} f$ have the same behaviour, and $\sum_{k=2}^{n} f(k) \leq \int_1^n f \leq \sum_{k=1}^{n-1} f(k)$.

- $f(k+1) \leq \int_k^{k+1} f \leq f(k)$, then sum
- $\sum \frac{1}{k \log k}$ diverges, $\sum \frac{1}{k \log^2 k}$ converges
- $\log(n+1) \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \log n$

Summary

- Series and integral have the same behaviour for positive non-increasing f
- The partial sums of the harmonic series grow like $\log n$

Summary

- Improper integrals as limits; standard integrals at $+\infty$ and at 0
- Comparison, limit comparison, absolute convergence
- Integral test for series