Integration techniques and applications

Outline

Substitution and by parts

Rational functions

Areas and mean value

Objectives

At the end of this chapter you will be able to:

- use the formulas of integration by substitution and by parts
- integrate rational functions with partial fractions
- compute the area between two graphs and the mean value of a function

Prerequisites

- From this module, «Fundamental theorem of calculus»: antiderivatives, fundamental formula of integral calculus, basic antiderivatives
- From this module, «Rules for computing derivatives»: derivative of a product and of a composite function
- From this module, «Limits of polynomials and rational functions»: polynomials, roots and rational functions
- From this module, «The Riemann integral»: area under a graph and integral mean

Substitution and by parts

Introduction: Substitution and by parts

- The chain rule read backwards gives substitution
- The product rule read backwards gives integration by parts

Substitution

Theorem 1: integration by substitution

f continuous, g with continuous derivative:

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(t)\, dt$$

- $F(g(x))$ is an antiderivative of the integrand (chain rule)
- $t = g(x)$, $dt = g'(x)\, dx$; the limits change too



Examples

Example 1: substitution

$$\int_0^1 x\sqrt{1+x^2}\,dx = \tfrac{1}{2}\int_1^2 \sqrt{t}\,dt = \frac{2\sqrt{2}-1}{3}$$

$$\int_0^1 \sqrt{1-x^2}\,dx \overset{x=\sin t}{=} \int_0^{\pi/2} \cos^2 t\,dt = \frac{\pi}{4}$$

By parts

Theorem 2: integration by parts

$$\int_a^b f\, g'\, dx = \big[f\, g\big]_a^b - \int_a^b f'\, g\, dx$$

- $(fg)' = f'g + fg'$ and the fundamental formula
- $\int x\, e^x\, dx = (x-1)e^x + c$, $\int \log x\, dx = x \log x - x + c$
- $\int e^x \sin x\, dx = \frac{e^x(\sin x - \cos x)}{2} + c$ (the integral comes back)

Summary

- Substitution: integrand, differential and limits change together
- By parts: differentiate the factor that becomes simpler

Rational functions

Introduction: Rational functions

- Division if $\deg P \geq \deg Q$
- Then partial fractions according to the factors of the denominator

Second-degree denominator

- $\Delta > 0$: $\frac{A}{x-x_1} + \frac{B}{x-x_2}$
- $\Delta = 0$: $\frac{A}{x-x_1} + \frac{B}{(x-x_1)^2}$
- $\Delta < 0$: numerator = multiple of $2x + p$ + constant; logarithm and arctangent

Example 2: partial fractions

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c, \qquad \int \frac{x+3}{x^2+2x+5}\,dx = \frac{1}{2}\log(x^2+2x+5) + \arctan\frac{x+1}{2} + c$$

Summary

- Division, then partial fractions
- Repeated factors: one term for each power

Areas and mean value

Introduction: Areas and mean value

- The area between two graphs is an integral of the difference
- The mean value extends the arithmetic mean

Area between two graphs

Proposition 1: area between two graphs

If $f \leq g$ on $[a, b]$, the area between the graphs is $\int_a^b (g - f)\, dx$.

- Between $y = x^2$ and $y = x + 2$: intersections -1, 2; area $\int_{-1}^{2} (x + 2 - x^2)\, dx = \frac{9}{2}$
- If the graphs cross: $\int_a^b |g - f|$; between $\sin x$ and $\cos x$ on $[0, \frac{\pi}{2}]$ it is $2\sqrt{2} - 2$

Mean value

- $\frac{1}{b-a}\int_a^b f(x)\,dx$
- Mean of $\sin x$ on $[0,\pi]$: $\frac{2}{\pi}$
- Velocity $v(t)=3t^2$ on $[0,2]$: distance 8, mean velocity 4

Summary

- Area: $\int_a^b |g - f|$ after the intersections
- Mean value: $\frac{1}{b-a} \int_a^b f$

Summary

- Substitution and change of variable
- By parts, also with the integral that comes back
- Rational functions: division and partial fractions
- Areas between graphs and mean value