Fundamental theorem of calculus

Sapienza Università di Roma

Outline

Antiderivatives

Integral function and fundamental theorem

Basic antiderivatives

Derivative of an integral function

Objectives

At the end of this chapter you will be able to:

- describe all the antiderivatives of a function on an interval
- state and prove the two parts of the fundamental theorem
- compute definite integrals with basic antiderivatives
- differentiate integral functions with variable limits

Prerequisites

- From this module, «The Riemann integral»: definition and properties of the integral, integral mean value theorem
- From this module, «The derivative function», «Derivatives of elementary functions» and «Rules for computing derivatives»: derivatives and the chain rule
- From this module, «Mean value theorem, maxima and minima»: a function with zero derivative on an interval is constant
- From this module, «L'Hôpital's rule and differentiability»: L'Hôpital's rule

Antiderivatives

Introduction: Antiderivatives

- Looking for a function knowing its derivative
- On an interval, antiderivatives differ by a constant

Antiderivatives

Definition 1: antiderivative

*$F : I \to \mathbb{R}$ is an **antiderivative** of f on the interval I if F is differentiable and $F' = f$ on I.*

- $\frac{x^3}{3}$ and $\frac{x^3}{3} + 5$ are antiderivatives of x^2
- Indefinite integral: $\int f(x)\,dx = F(x) + c$

Characterization of antiderivatives

Theorem 1: characterization of antiderivatives

If F is an antiderivative of f on the interval I, then G is an antiderivative if and only if $G = F + c$.

- $(G - F)' = 0$ on an interval: $G - F$ is constant (Lagrange)
- Without an interval it is false: $\log|x|$ on $\mathbb{R} \setminus \{0\}$



Summary

- Antiderivative: $F' = f$
- On an interval, all antiderivatives are $F + c$

Integral function and fundamental theorem

Introduction: Integral function and fundamental theorem

- The integral function $F(x) = \int_{x_0}^{x} f(t)\,dt$ is an antiderivative of a continuous f
- Hence a definite integral is computed with an antiderivative

Integral function

Definition 2: integral function

For f integrable on $[a, b]$ and $x_0 \in [a, b]$: $F(x) = \int_{x_0}^{x} f(t)\, dt$.

Proposition 1: continuity of the integral function

If $|f| \leq L$: $|F(x) - F(y)| \leq L\,|x - y|$.

First part

Theorem 2: fundamental theorem of calculus, first part

If f is continuous on $[a, b]$, then $F(x) = \int_{x_0}^{x} f(t)\,dt$ is differentiable and $F'(x) = f(x)$.

- $\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f = f(c_h)$ (mean value theorem)
- $c_h \to x$ and f is continuous: $f(c_h) \to f(x)$

□

Second part

Theorem 3: fundamental theorem of calculus, second part

If f is continuous on $[a, b]$ and G is an antiderivative: $\int_a^b f(x)\,dx = G(b) - G(a)$.

- $G = F + c$ with $F(x) = \int_a^x f$: $G(b) - G(a) = F(b) - F(a)$
- $\int_0^\pi \sin x\,dx = \big[-\cos x\big]_0^\pi = 2$

Summary

- $F' = f$ for continuous f: every continuous function has an antiderivative
- $\int_a^b f = \left[G\right]_a^b$ for any antiderivative G

Basic antiderivatives

Introduction: Basic antiderivatives

- The table of derivatives read backwards
- It also holds in composite form, with $g'(x)$

Table

$x^\alpha\ (\alpha \neq -1)$	$\frac{x^{\alpha+1}}{\alpha+1}$	$\sin x$	$-\cos x$
$\frac{1}{x}$	$\log \lvert x \rvert$	$\cos x$	$\sin x$
e^x	e^x	$\frac{1}{\cos^2 x}$	$\tan x$
a^x	$\frac{a^x}{\log a}$	$\frac{1}{1+x^2}$	$\arctan x$
$\operatorname{Sh} x$	$\operatorname{Ch} x$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$

- $\int \frac{g'(x)}{g(x)}\,dx = \log|g(x)| + c$: $\int \tan x\,dx = -\log|\cos x| + c$

Example

Example 1: basic antiderivatives

$$\int \left(3x^2 - \tfrac{4}{x} + 2e^x\right) dx = x^3 - 4\log|x| + 2e^x + c$$

$$\int \frac{x}{1+x^2}\, dx = \tfrac{1}{2}\log(1+x^2) + c, \qquad \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$$

Summary

- Each row of the table is checked by differentiating
- The formulas hold on intervals where the function is defined

Derivative of an integral function

Introduction: Derivative of an integral function

- Functions defined by integrals, even without an elementary antiderivative
- They are used in limits and in the study of functions

Variable limits

Theorem 4: differentiation of integral functions

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(t)\,dt = f(\beta(x))\,\beta'(x) - f(\alpha(x))\,\alpha'(x)$$

- $G = F(\beta) - F(\alpha)$ and the chain rule
- $\frac{d}{dx}\int_1^{x^2} e^{t^2}\,dt = 2x\,e^{x^4}$

Examples

Example 2: limit and study

$$\lim_{x\to 0} \frac{1}{x^3} \int_0^x \sin(t^2)\, dt = \lim_{x\to 0} \frac{\sin(x^2)}{3x^2} = \frac{1}{3} \text{ (L'Hôpital).}$$

$F(x) = \int_0^x e^{-t^2}\, dt$: increasing ($F' > 0$), inflection at 0, odd.

Summary

- Derivative with variable limits: $f(\beta)\beta' - f(\alpha)\alpha'$
- With L'Hôpital's rule for limits of integral functions

Summary

- Antiderivatives and their characterization on an interval
- The integral function is continuous; differentiable with $F' = f$ if f is continuous
- $\int_a^b f = G(b) - G(a)$
- Basic antiderivatives and composite form
- Derivative of $\int_{\alpha(x)}^{\beta(x)} f$