The Riemann integral

Outline

Partitions and Riemann sums

The Riemann integral

Classes of integrable functions

Properties of the integral

Integral mean value theorem

Objectives

At the end of this chapter you will be able to:

- compute lower and upper sums with respect to a partition
- state the definition of the Riemann integral and interpret it as a signed area
- prove that monotone and continuous functions on $[a, b]$ are integrable
- use linearity, additivity, monotonicity and $\left|\int_a^b f\right| \leq \int_a^b |f|$
- state and prove the integral mean value theorem

Prerequisites

- From this module, «Ordered fields, supremum/infimum and the completeness axiom»: supremum and infimum of a set and their existence in $\mathbb{R}$
- From this module, «Summations, geometric and arithmetic progressions»: the summation symbol and the sum of the first n natural numbers
- From this module, «Continuous functions» and «Weierstrass theorem and intermediate value theorem»: continuity, maximum and minimum on a closed and bounded interval, intermediate values

Partitions and Riemann sums

Introduction: Partitions and Riemann sums

- We want to measure the area under the graph of a bounded function
- We divide $[a, b]$ into intervals and approximate the area with **inscribed** and **circumscribed** rectangles
- Refining the division, the two approximations get closer

Partitions

Definition 1: partition of an interval

*A **partition** of $[a, b]$ is a finite set $P = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$. The intervals are $I_k = [x_{k-1}, x_k]$, of length $\Delta x_k = x_k - x_{k-1}$.*

- **Uniform**: $\Delta x_k = \frac{b-a}{n}$; **finer**: $P \subseteq P'$

Lower and upper sums

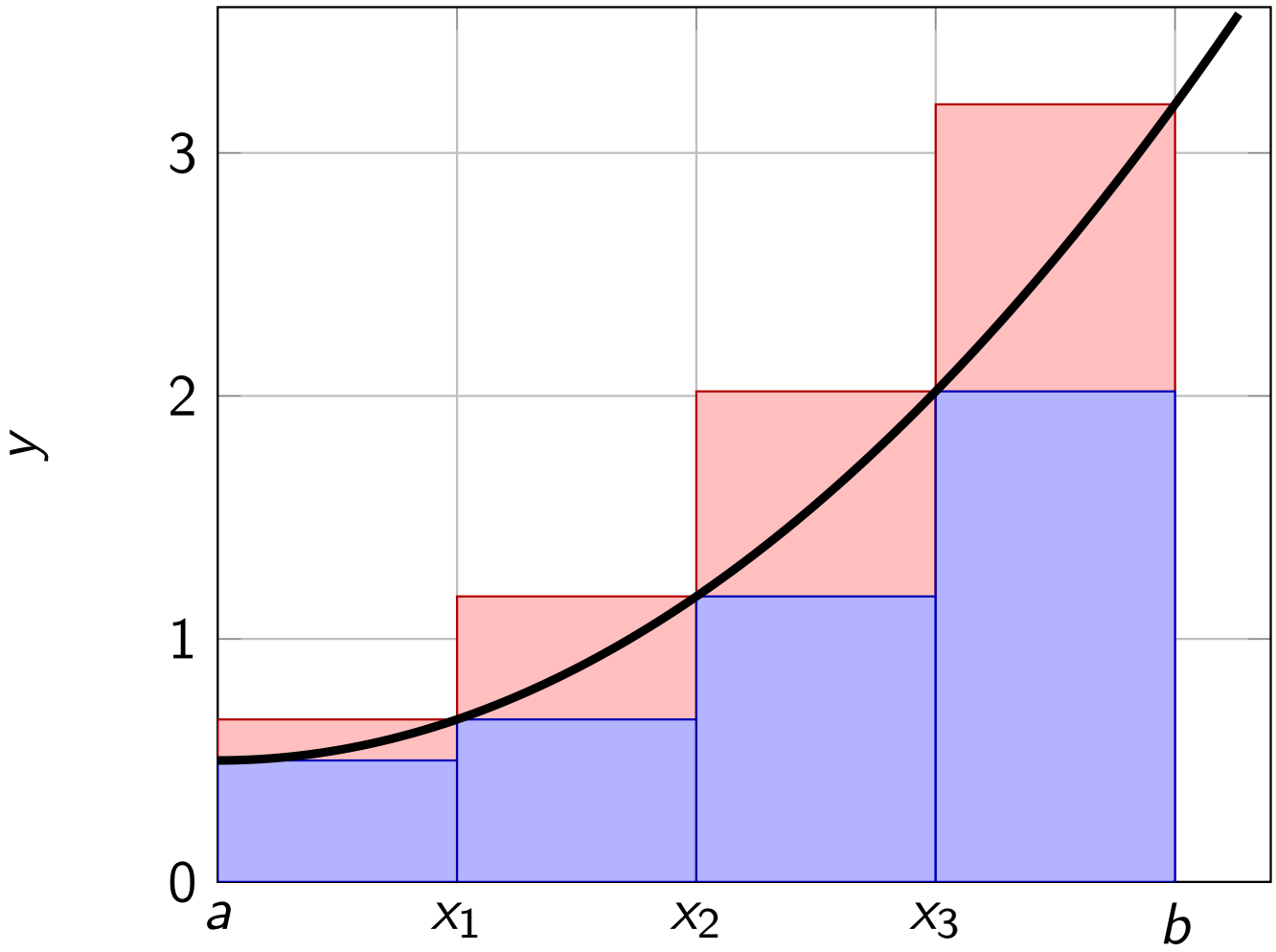
Definition 2: lower and upper sum

Let f be bounded on $[a, b]$, $m_k = \inf_{I_k} f$, $M_k = \sup_{I_k} f$:

$$s(f, P) = \sum_{k=1}^{n} m_k \, \Delta x_k, \qquad S(f, P) = \sum_{k=1}^{n} M_k \, \Delta x_k$$

- Always $s(f, P) \leq S(f, P)$

Graphical interpretation



Blue: lower sum; blue + red: upper sum.

Example

Example 1: sums of $f(x) = x$ on $[0, 1]$

With $x_k = \frac{k}{n}$: $m_k = \frac{k-1}{n}$, $M_k = \frac{k}{n}$, $\Delta x_k = \frac{1}{n}$. With $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$:

$$S(f, P_n) = \frac{n+1}{2n}, \qquad s(f, P_n) = \frac{n-1}{2n}$$

Both tend to $\frac{1}{2}$.

Refinements

Proposition 1: sums and refinements

If P' is finer than P: $s(f, P) \leq s(f, P') \leq S(f, P') \leq S(f, P)$.

Corollary 1

For every pair of partitions: $s(f, P_1) \leq S(f, P_2)$.

- Proof of the corollary: $P = P_1 \cup P_2$ refines both.

Proof of the proposition

- It is enough to add one point $y \in (x_{k-1}, x_k)$
- The infimum on a subinterval is $\geq m_k$:

$$m'(y - x_{k-1}) + m''(x_k - y) \geq m_k \, \Delta x_k$$

- Hence $s(f, P') \geq s(f, P)$; for the upper sums we argue with the suprema

□

Summary

- Lower sums use the infima, upper sums the suprema
- Refining, s increases and S decreases
- Every lower sum is $\leq$ every upper sum: $\sup_P s \leq \inf_P S$

The Riemann integral

Introduction: The Riemann integral

- A function is integrable when the approximations from below and from above meet
- The common value is the integral: a signed area
- A criterion with ε makes it checkable in practice

Definition

Definition 3: Riemann integrable function

f bounded on $[a,b]$ is ***integrable*** *if* $\sup_P s(f,P) = \inf_P S(f,P)$. *The common value is* $\int_a^b f(x)\,dx$.

- If $f \geq 0$: area of the region $\{a \leq x \leq b,\ 0 \leq y \leq f(x)\}$
- If f changes sign: area above the axis minus area below

Examples

- $\int_a^b c\,dx = c\,(b-a)$
- $\int_0^1 x\,dx = \frac{1}{2}$, $\int_0^1 x^2\,dx = \frac{1}{3}$ (with $\sum k^2 = \frac{n(n+1)(2n+1)}{6}$)
- **Dirichlet**: $D(x) = 1$ on $\mathbb{Q}$, 0 elsewhere. For every P: $s = 0$, $S = 1$: **not** integrable

Integrability criterion

Theorem 1: Riemann's integrability criterion

A bounded f is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there is P with $S(f, P) - s(f, P) < \varepsilon$.

- $(\Leftarrow)$ $0 \leq \inf S - \sup s \leq S(P) - s(P) < \varepsilon$
- $(\Rightarrow)$ P_1, P_2 within $\frac{\varepsilon}{2}$ of I, then $P = P_1 \cup P_2$

□

Riemann sums

- With $t_k \in I_k$: $\sigma(f, P) = \sum_{k=1}^{n} f(t_k)\,\Delta x_k$
- $s(f, P) \leq \sigma(f, P) \leq S(f, P)$: if $S - s \to 0$, then $\sigma \to \int_a^b f$
- Left endpoint, right endpoint or midpoint: the «Riemann sums» app of the lab

Summary

- Integrable: $\sup_P s = \inf_P S$; the integral is a signed area
- Criterion: $S(f, P) - s(f, P) < \varepsilon$ for some P
- The Dirichlet function is not integrable

Classes of integrable functions

Introduction: Classes of integrable functions

- Two large classes of integrable functions: **monotone** and **continuous** functions
- For continuous ones we need **uniform continuity** (Heine–Cantor theorem)

Monotone functions

Theorem 2: integrability of monotone functions

If $f : [a, b] \to \mathbb{R}$ is monotone, then it is integrable.

For f non-decreasing and P_n uniform, the sum telescopes:

$$S(f, P_n) - s(f, P_n) = \frac{(b-a)(f(b) - f(a))}{n} \to 0$$

□

Uniform continuity

Definition 4: uniformly continuous function

f is uniformly continuous on I if $\forall \varepsilon > 0 \ \exists \delta > 0$: $x, y \in I$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

- x^2 on $[0, 1]$: $|x^2 - y^2| \leq 2|x - y|$, $\delta = \frac{\varepsilon}{2}$
- $\frac{1}{x}$ on $(0, 1]$ is not: $x_n = \frac{1}{n}$, $y_n = \frac{1}{n+1}$

Heine–Cantor theorem

Theorem 3: Heine–Cantor theorem

If f is continuous on $[a, b]$, then it is uniformly continuous on $[a, b]$.

- By contradiction: $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon$
- Bolzano–Weierstrass: $x_{n_j} \to c \in [a, b]$, hence also $y_{n_j} \to c$
- By continuity $f(x_{n_j}) - f(y_{n_j}) \to 0$: contradiction

□

Continuous functions

Theorem 4: integrability of continuous functions

If $f : [a, b] \to \mathbb{R}$ is continuous, then it is integrable.

- Heine–Cantor: $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$
- Partition with $\Delta x_k < \delta$: $M_k - m_k < \frac{\varepsilon}{b-a}$ (Weierstrass)
- $S - s < \varepsilon$: conclude with the criterion

□

Summary

- Monotone $\Rightarrow$ integrable (even with discontinuities, like $[x]$)
- Continuous on $[a, b] \Rightarrow$ uniformly continuous $\Rightarrow$ integrable
- Changing f at finitely many points does not change the integral

Properties of the integral

Introduction: Properties of the integral

- The properties we will always use in computations: linearity, additivity, monotonicity
- Conventions: $\int_a^a f = 0$, $\int_b^a f = -\int_a^b f$

Linearity and additivity

Theorem 5: linearity of the integral

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$$

Theorem 6: additivity with respect to the interval

For $a < c < b$: $\int_a^b f = \int_a^c f + \int_c^b f$

- Idea: $s(f) + s(g) \leq s(f+g) \leq S(f+g) \leq S(f) + S(g)$; partitions containing c

Monotonicity and estimates

Theorem 7: monotonicity of the integral

If $f \leq g$ on $[a, b]$, then $\int_a^b f \leq \int_a^b g$.

- If $m \leq f \leq M$: $m(b-a) \leq \int_a^b f \leq M(b-a)$
- Example: $\frac{1}{e} \leq \int_0^1 e^{-x^2}\, dx \leq 1$

Absolute value

Theorem 8: integral of the absolute value

If f is integrable, so is $|f|$ and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

- $\big||f(x)| - |f(y)|\big| \leq |f(x) - f(y)|$: $S(|f|) - s(|f|) \leq S(f) - s(f)$
- $-|f| \leq f \leq |f|$ and monotonicity
- $\int_{-1}^{1} x\,dx = 0 < 1 = \int_{-1}^{1} |x|\,dx$

Summary

- Linearity, additivity with respect to the interval, monotonicity
- Estimates $m(b-a) \leq \int_a^b f \leq M(b-a)$
- $\left|\int_a^b f\right| \leq \int_a^b |f|$, sometimes with strict inequality

Integral mean value theorem

Introduction: Integral mean value theorem

- The integral mean is the height of the rectangle with the same area as the region under the graph
- For continuous functions it is a value taken by the function

Integral mean value theorem

Theorem 9: integral mean value theorem

If f is continuous on $[a,b]$, there is $c \in [a,b]$ with $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

- Weierstrass: $m \leq \frac{1}{b-a}\int_a^b f \leq M$
- Intermediate values: the mean is taken at some point c
- Example: the mean of x^2 on $[0,1]$ is $\frac{1}{3}$, $c = \frac{1}{\sqrt{3}}$

Summary

- Integral mean: $\frac{1}{b-a}\int_a^b f$
- For continuous f it is taken at least at one point
- Without continuity it may not be (step function)

Summary

- Lower and upper sums, the integral as $\sup_P s = \inf_P S$
- Integrability criterion with ε
- Monotone and continuous functions are integrable (Heine–Cantor)
- Linearity, additivity, monotonicity, $\left|\int f\right| \leq \int |f|$
- Integral mean value theorem