

Invertible functions and inverse functions



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Invertible functions and inverse functions



Introduction: Invertible functions and inverse functions

- ▶ A function associates to each input x a unique output $y = f(x)$: we ask when it is possible to “go back”, that is, to recover from y the unique x that produces it
- ▶ In this section we define **invertible functions** and the **inverse function**, with a graphical criterion to recognize them
- ▶ We will prove that **strictly monotone** functions are invertible, and we will see that this is not a necessary condition

Invertible functions

- ▶ Given a real function of a real variable $f : D \rightarrow \mathbb{R}$, for every input x in the domain D there exists a unique output $y = f(x)$ in the image of the domain $f(D)$.
- ▶ If, in addition, for every output $y = f(x) \in f(D)$ there exists a unique input $x \in D$, then f is called **invertible**, and it establishes a **one-to-one correspondence** between the domain D of f and the image of the domain $f(D)$.

Definition 1: of invertible function

A function $f : D \rightarrow \mathbb{R}$ is **invertible** on the domain D if one of the following equivalent conditions holds:

$$\forall x_1, x_2 \in D, \quad x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

$$\forall x_1, x_2 \in D, \quad f(x_1) = f(x_2) \implies x_1 = x_2$$

$$\forall y \in f(D), \quad \exists! x \in D : y = f(x)$$

that is, if f is **injective**.

Warning

- ▶ **WARNING:** the definition of invertible function given here for real functions of a real variable is different from the usual one given for functions between arbitrary sets, which also requires surjectivity.
- ▶ In this case *injectivity is enough* because we define the inverse on the image of f and not on its whole codomain.

Inverse function

Definition 2: of inverse function

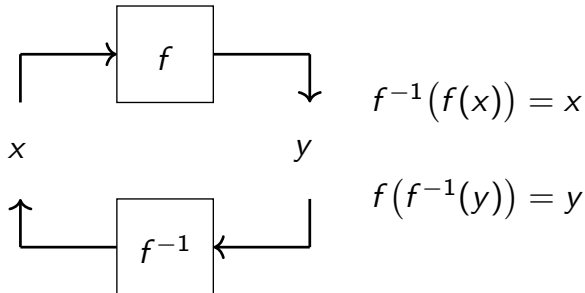
*Given an invertible function $f : D \rightarrow \mathbb{R}$, the function that associates to each output $y \in f(D)$ the unique input $x \in D$ such that $f(x) = y$ is called the **inverse function** of f and is denoted by the symbol f^{-1} .*

The pair f and f^{-1}

The pair f and f^{-1} is written as:

$$f : \begin{cases} y = f(x) \\ x \in D \end{cases} \quad f^{-1} : \begin{cases} x = f^{-1}(y) \\ y \in f(D) \end{cases}$$

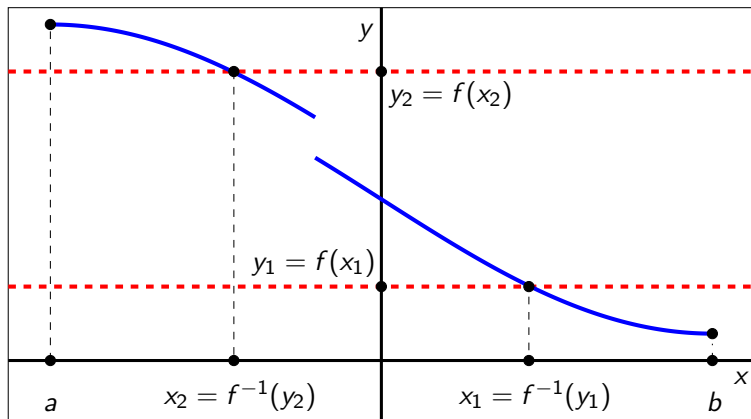
The black box of f^{-1} works backwards with respect to that of f , according to the following scheme:



Graphical criterion for invertibility

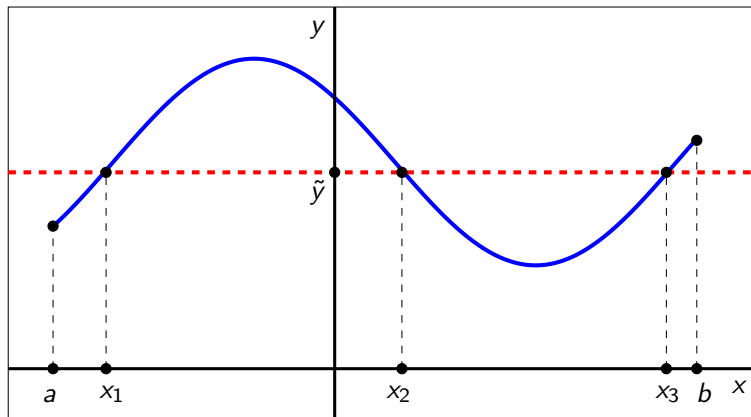
- ▶ The invertibility condition is equivalent to requiring that the graph of f be intersected in at most one point by every line parallel to the x -axis.
- ▶ We now see an example of the graph of an invertible function and one of a non-invertible function

Example



Graph of a function that is **invertible** on $[a, b]$, since every line parallel to the x -axis either does not intersect the graph of f or intersects it in exactly one point.

Example



Graph of a function that is **not invertible** on $[a, b]$, since for the indicated value $\tilde{y}$ there are several values x (precisely x_1 , x_2 and x_3) whose image is $\tilde{y}$.

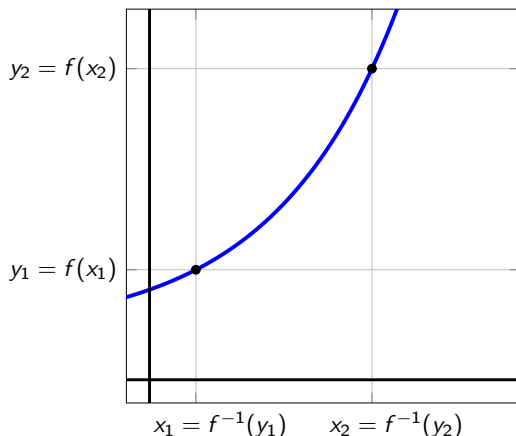
Strictly monotone functions

Theorem 1

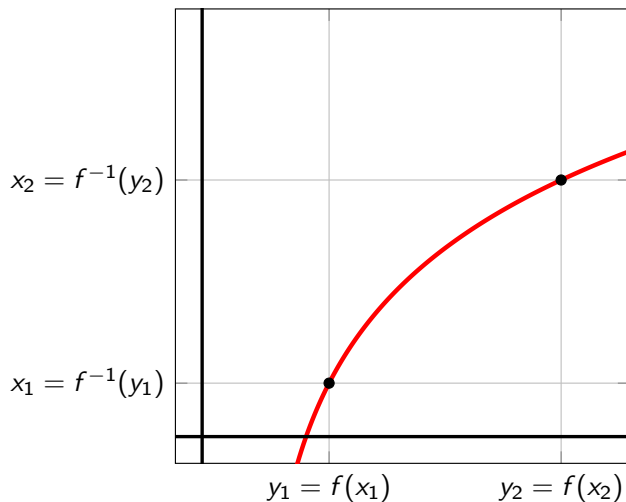
If a function $f : D \rightarrow \mathbb{R}$ is strictly increasing (decreasing) on D , then it is invertible on D . Moreover, its inverse function is strictly increasing (decreasing).

Graphical interpretation (1/2)

For the strictly increasing case, graphically we have:



Graphical interpretation (2/2)



Proof (1/2)

Consider the case of a function that is strictly increasing on D and two arbitrary values $x_1, x_2 \in D$ (the strictly decreasing case is analogous).

If $x_1 \neq x_2$, then either $x_1 < x_2$ or $x_1 > x_2$. Since f is strictly increasing, we have:

$$\text{if } x_1 < x_2, \text{ then } f(x_1) < f(x_2)$$

$$\text{if } x_1 > x_2, \text{ then } f(x_1) > f(x_2)$$

In both cases $f(x_1) \neq f(x_2)$, therefore f is invertible.

Proof (2/2)

Consider the inverse function f^{-1} , that is, $x = f^{-1}(y)$, and let us prove that it is strictly increasing. Consider two arbitrary values $y_1, y_2 \in f(D)$, with $y_1 < y_2$.

If we had $f^{-1}(y_1) = x_1 \geq x_2 = f^{-1}(y_2)$, since f is strictly increasing we would have $y_1 = f(x_1) \geq f(x_2) = y_2$, that is, $y_1 \geq y_2$. This cannot happen, since it contradicts $y_1 < y_2$ (**contradiction**).

Consequently, we have $f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2)$, that is, $f^{-1}(y_1) < f^{-1}(y_2)$, and hence f^{-1} is strictly increasing. □

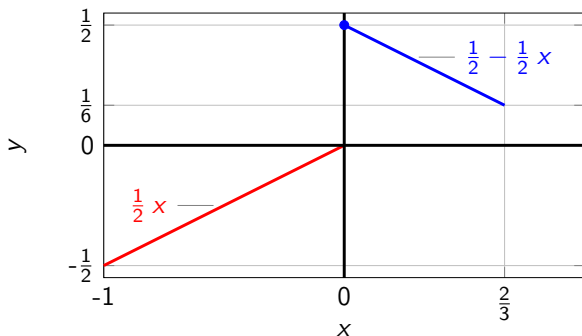
Strict monotonicity is not necessary

- ▶ However, a function can be invertible even without being strictly increasing or decreasing.

Example

For example, the following piecewise-defined function is invertible:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } -1 \leq x < 0, \\ \frac{1}{2} - \frac{1}{2}x & \text{if } 0 \leq x \leq \frac{2}{3} \end{cases}$$



Non-invertible functions

- ▶ Two classes of functions that are certainly not invertible are:
 1. even (symmetric) functions, since:

$$f(-x) = f(x)$$

2. periodic functions, since:

$$f(x + T) = f(x)$$

Summary

Invertible and inverse functions:

- ▶ f is invertible on D if it is injective: equal outputs correspond to equal inputs
- ▶ The inverse f^{-1} is defined on the image $f(D)$ and associates to each y the unique x with $f(x) = y$
- ▶ Graphical criterion: every horizontal line intersects the graph in at most one point

Monotonicity:

- ▶ Strictly monotone $\Rightarrow$ invertible, with inverse monotone of the same type (the converse does not hold)
- ▶ Even functions and periodic functions are not invertible

Graph of the inverse function

Introduction: Graph of the inverse function

- ▶ In this section we see how to obtain the **graph** of f^{-1} from that of f : a symmetry with respect to the bisector $y = x$ is enough
- ▶ We then see how to find the **analytic expression** of the inverse by solving the equation $f(x) = y$ for x
- ▶ Examples: affine function, x^2 , exponential; and a case in which the inverse exists but cannot be written explicitly

Symmetry with respect to the bisector

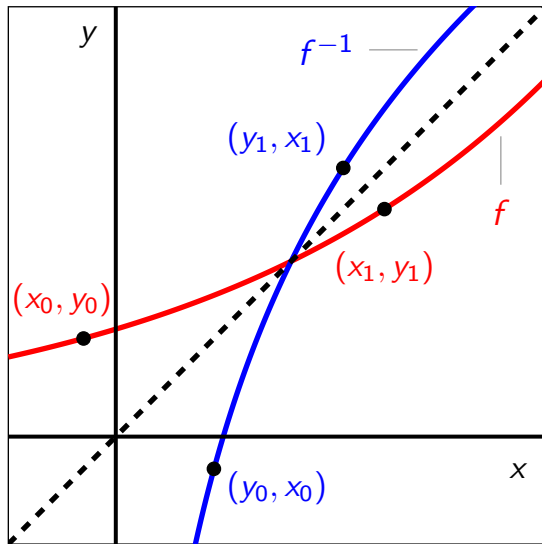
- ▶ The relations between an invertible function f and its inverse function f^{-1} :

$$f : \begin{cases} y = f(x) \\ x \in D \end{cases} \qquad f^{-1} : \begin{cases} x = f^{-1}(y) \\ y \in f(D) \end{cases}$$

indicate that if the point (x_0, y_0) lies on the graph of f , then the point (y_0, x_0) lies on the graph of f^{-1} .

- ▶ Since the points (x_0, y_0) and (y_0, x_0) are **symmetric with respect to the bisector** with equation $y = x$, the graph of f^{-1} is obtained from that of f by symmetry with respect to the bisector.

Symmetry with respect to the bisector



Analytic expression of the inverse

- ▶ If the analytic expression of f is known and f is invertible, the **analytic expression** of f^{-1} is found by trying to solve for x the equation:

$f(x) = y$ that is, find the input x that produces the output y

- ▶ Note that it is not always possible to find it, even in cases where the inverse function exists!

Inverse of an affine function

Given $a, b \in \mathbb{R}$, $a \neq 0$, we have:

$$f : \begin{cases} y = ax + b \\ x \in \mathbb{R} \end{cases}$$

$$f^{-1} : \begin{cases} x = \frac{y-b}{a} \\ y \in \mathbb{R} \end{cases}$$

Example (1/3)

The function $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto 2x + 3$ is strictly increasing, hence invertible on $\mathbb{R}$, and the equation

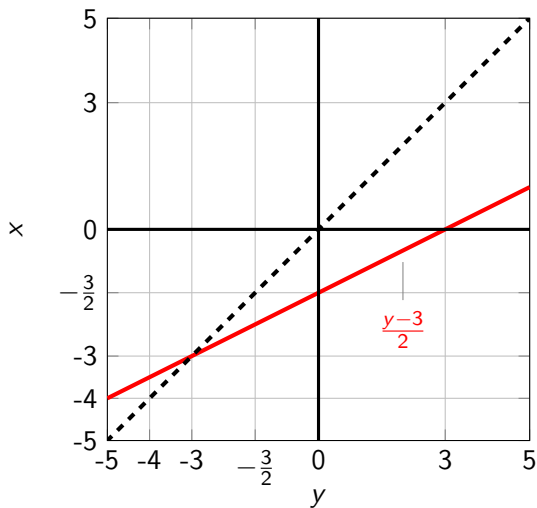
$$2x + 3 = y \text{ solved for } x, \quad \text{that is } x = \frac{y - 3}{2}$$

gives for f^{-1} the analytic expression:

$$f^{-1}(y) = \frac{y - 3}{2}$$

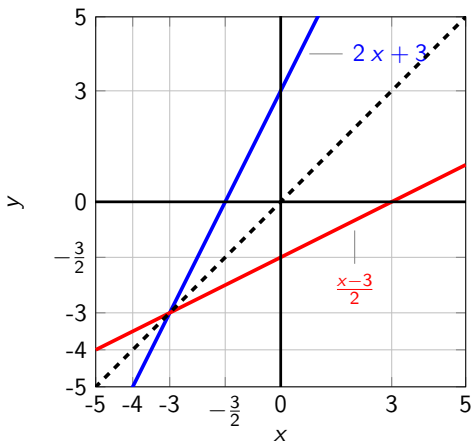
that is, the function: $f^{-1} : \mathbb{R} \rightarrow \mathbb{R}, y \mapsto \frac{y-3}{2}$.

Example (2/3)



Example (3/3)

The graphs $y = 2x + 3$ and $y = \frac{x-3}{2}$ are symmetric with respect to the bisector $y = x$



Inverse of x^2

We have:

$$f : \begin{cases} y = x^2 \\ x \geq 0 \end{cases}$$

$$f^{-1} : \begin{cases} x = \sqrt{y} \\ y \geq 0 \end{cases}$$

Example (1/2)

The function $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$ is strictly increasing on the interval $[0, +\infty)$, hence invertible there; the equation

$$x^2 = y \text{ solved for } x, \text{ that is } x = \sqrt{y}$$

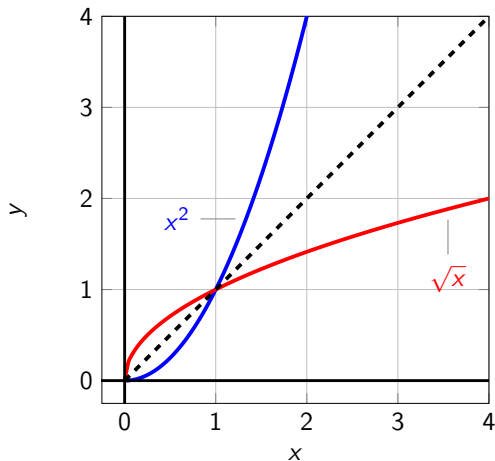
gives for f^{-1} the analytic expression

$$f^{-1}(y) = \sqrt{y}$$

that is, the function: $f^{-1} : [0, +\infty) \rightarrow \mathbb{R}, y \mapsto \sqrt{y}$.

Example (2/2)

The graphs $y = x^2$ and $y = \sqrt{x}$ are symmetric with respect to the bisector $y = x$



Inverse of the exponential

Given $a \in \mathbb{R}_+$, $a \neq 1$, we have:

$$f : \begin{cases} y = a^x \\ x \in \mathbb{R} \end{cases}$$

$$f^{-1} : \begin{cases} x = \log_a y \\ y > 0 \end{cases}$$

Example (1/2)

The function $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto e^x$ is strictly increasing on all of $\mathbb{R}$ and hence invertible; the equation

$$e^x = y \text{ solved for } x, \text{ that is } x = \log y$$

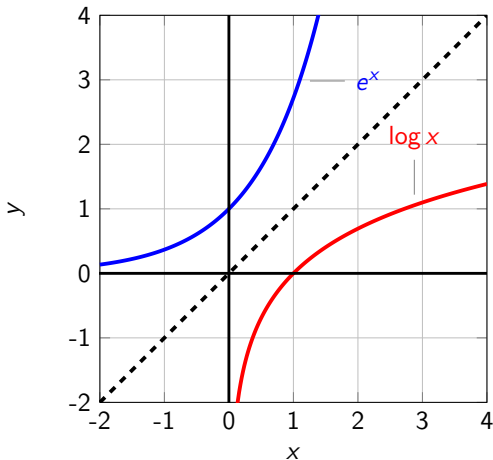
gives for f^{-1} the analytic expression

$$f^{-1}(y) = \log y$$

that is, the function: $f^{-1} : (0, +\infty) \rightarrow \mathbb{R}, y \mapsto \log y$.

Example (2/2)

The graphs $y = e^x$ and $y = \log x$ are symmetric with respect to the bisector $y = x$



An inverse that cannot be written explicitly

- ▶ Now consider the function:

$$f(x) = x + e^x$$

Being the sum of two functions that are strictly increasing on all of $\mathbb{R}$, $f(x)$ is strictly increasing and hence invertible on all of $\mathbb{R}$ (see the Theorem on strictly monotone functions).

- ▶ However, we would try in vain to solve the equation $x + e^x = y$ for x . In other words, f^{-1} exists, but we cannot write it explicitly.

Summary

Graph and expression of the inverse:

- ▶ (x_0, y_0) lies on the graph of f if and only if (y_0, x_0) lies on the graph of f^{-1} : the graphs are symmetric with respect to the bisector $y = x$
- ▶ The expression of f^{-1} is found by solving $f(x) = y$ for x

Examples:

- ▶ $y = ax + b \Leftrightarrow x = \frac{y-b}{a}$; $y = x^2 (x \geq 0) \Leftrightarrow x = \sqrt{y}$;
 $y = a^x \Leftrightarrow x = \log_a y$
- ▶ $f(x) = x + e^x$ is invertible, but f^{-1} cannot be written explicitly

Inverse power functions



Introduction: Inverse power functions

- ▶ In this section we study the inverse of the **power** functions $y = x^\alpha$
- ▶ The inverse of a power is again a power, with exponent $\frac{1}{\alpha}$ (the **roots**)
- ▶ The domain on which the power is invertible depends on the exponent: even powers must be restricted to $x \geq 0$, odd powers are invertible on all of $\mathbb{R}$

Powers and their inverses (1/2)

Given $\alpha \in \mathbb{R}, \alpha \neq 0$, we have:

$$f : \begin{cases} y = x^\alpha \\ x > 0 \end{cases} \qquad f^{-1} : \begin{cases} x = y^{\frac{1}{\alpha}} \\ y > 0 \end{cases}$$

If $\alpha > 0$ we have:

$$f : \begin{cases} y = x^\alpha \\ x \geq 0 \end{cases} \qquad f^{-1} : \begin{cases} x = y^{\frac{1}{\alpha}} \\ y \geq 0 \end{cases}$$

Powers and their inverses (2/2)

If $\alpha = \frac{m}{n} \in \mathbb{Q}$, with $m \in \mathbb{Z}$, $n \in \mathbb{N}_+$ odd and coprime, we have:

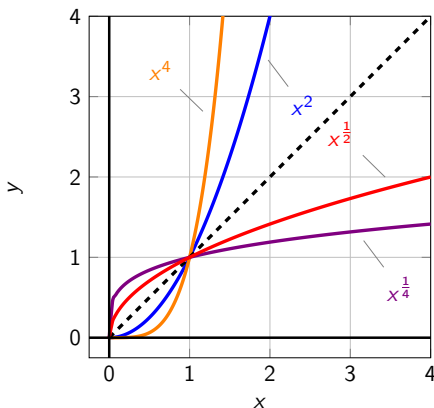
$$f : \begin{cases} y = x^{\frac{m}{n}} \\ x \in \mathbb{R} \end{cases} \qquad f^{-1} : \begin{cases} x = y^{\frac{n}{m}} \\ y \in \mathbb{R} \end{cases}$$

Even powers

The **even powers**:

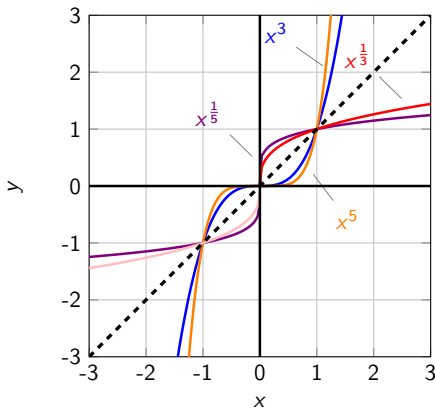
$$x^{2n} \quad \text{with } n = 1, 2, \dots$$

are not invertible on the whole real line, but only on the half-line $x \geq 0$.



Odd powers

The **odd powers** x^{2n+1} (with $n = 1, 2, \dots$) and the **powers with rational exponent** $x^{\frac{m}{n}}$ (with n, m positive odd integers), being strictly increasing monotone functions, are invertible from $-\infty$ to $+\infty$.



Summary

Inverse powers:

- ▶ The inverse of $y = x^\alpha$ ($x > 0$) is $x = y^{\frac{1}{\alpha}}$; if $\alpha > 0$ this also holds on $[0, +\infty)$
- ▶ Even powers x^{2n} : invertible only on $x \geq 0$ (inverses: $x^{\frac{1}{2}}, x^{\frac{1}{4}}, \dots$)
- ▶ Odd powers x^{2n+1} and rational exponents with m, n odd: strictly increasing, invertible on all of $\mathbb{R}$ (inverses: $x^{\frac{1}{3}}, x^{\frac{1}{5}}, \dots$)

Inverse trigonometric functions

Introduction: Inverse trigonometric functions

- ▶ The trigonometric functions are periodic and hence **not invertible**: to invert them we restrict them to intervals on which they are strictly monotone
- ▶ In this way we obtain the **arcsine**, the **arccosine** and the **arctangent**, with their graphs
- ▶ These functions are used to write the solutions of trigonometric equations and inequalities with non-standard angles

Trigonometric functions are not invertible

- ▶ Being periodic, the trigonometric functions cannot be invertible. Indeed, for example, the equation

$$y = \sin x$$

has infinitely many solutions if $-1 \leq y \leq 1$ (the output y corresponds to infinitely many inputs), or has no real solutions if $|y| > 1$.

- ▶ To speak of inverse functions of sine, cosine and tangent, we will need to restrict ourselves to intervals on which these functions are strictly monotone and therefore invertible.

Arcsine

- An interval on which the sine function is invertible is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Definition 3: of arcsine

*The inverse function of the sine on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is the **arcsine**:*

$$f : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad f : y \mapsto \arcsin y$$

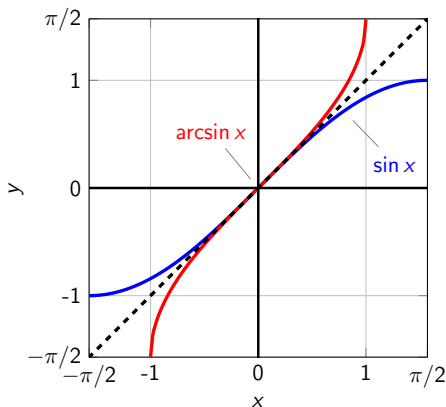
We have:

$$f : \begin{cases} y = \sin x \\ x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases}$$

$$f^{-1} : \begin{cases} x = \arcsin y \\ y \in [-1, 1] \end{cases}$$

Graph of the arcsine

The graph of the arcsine is obtained from the arc of the sine curve restricted to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (strictly monotone), by symmetry with respect to the bisector $y = x$:



Arccosine

- ▶ An interval on which the cosine function is invertible is $[0, \pi]$.

Definition 4: of arccosine

*The inverse function of the cosine on the interval $[0, \pi]$ is the **arccosine**:*

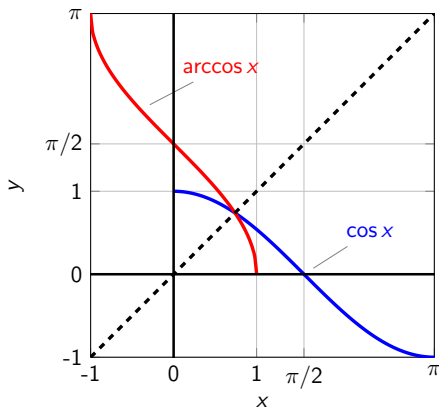
$$f : [-1, 1] \rightarrow [0, \pi], \quad f : y \mapsto \arccos y$$

We have:

$$f : \begin{cases} y = \cos x \\ x \in [0, \pi] \end{cases} \qquad f^{-1} : \begin{cases} x = \arccos y \\ y \in [-1, 1] \end{cases}$$

Graph of the arccosine

The graph of the arccosine is obtained from the arc of the cosine curve restricted to the interval $[0, \pi]$ (strictly monotone), by symmetry with respect to the bisector $y = x$:



Arctangent

- Note that on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ the tangent is strictly monotone and hence invertible. Its inverse function is called the arctangent ($\arctan$), and it is defined on $\mathbb{R}$.

Definition 5: of arctangent

*The inverse function of the tangent on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ is the **arctangent**:*

$$f : \mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad f : y \mapsto \arctan y$$

Arctangent

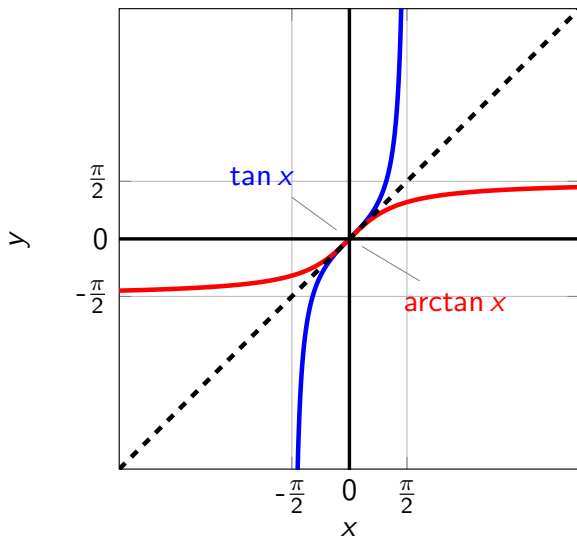
We have:

$$f : \begin{cases} y = \tan x \\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases}$$

$$f^{-1} : \begin{cases} x = \arctan y \\ y \in \mathbb{R} \end{cases}$$

- The graph of the arctangent is obtained from that of the tangent restricted to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (strictly monotone), by symmetry with respect to the bisector $y = x$.

Graph of the arctangent



Trigonometric equations and inequalities

- ▶ By means of the inverse trigonometric functions, we can express the solutions of a **trigonometric equation or inequality** when it involves angles that are not standard angles.

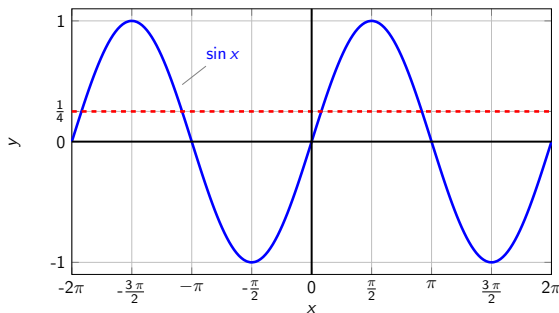
Example (1/3)

The solutions of:

$$\sin x = \frac{1}{4}$$

are:

$$(i) \ x = \arcsin \frac{1}{4} + 2k\pi; \quad (ii) \ x = \pi - \arcsin \frac{1}{4} + 2k\pi \quad (k \in \mathbb{Z})$$



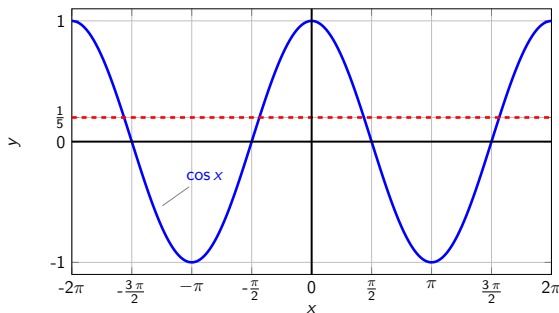
Example (2/3)

The solutions of:

$$\cos x < \frac{1}{5}$$

are:

$$\arccos \frac{1}{5} + 2k\pi < x < 2\pi - \arccos \frac{1}{5} + 2k\pi \quad (k \in \mathbb{Z})$$



Example (3/3)

The solutions of:

$$\tan x \geq 3$$

are:

$$\arctan 3 + k \pi \leq x < \frac{\pi}{2} + k \pi \quad (k \in \mathbb{Z})$$

Summary

Inverse trigonometric functions:

- ▶ $\arcsin : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$, inverse of the sine restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$
- ▶ $\arccos : [-1, 1] \rightarrow [0, \pi]$, inverse of the cosine restricted to $[0, \pi]$
- ▶ $\arctan : \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$, inverse of the tangent restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$

Application:

- ▶ They allow us to write the solutions of trigonometric equations and inequalities with non-standard angles

Inverse hyperbolic functions



Introduction: Inverse hyperbolic functions

- ▶ In this section we invert the **hyperbolic sine** and the **hyperbolic cosine** (the latter restricted to $x \geq 0$)
- ▶ Unlike the trigonometric functions, here we can find the **analytic expression** of the inverse, by solving a quadratic equation in e^x
- ▶ We obtain the **inverse hyperbolic sine** and the **inverse hyperbolic cosine**, expressed through the logarithm

Inverse hyperbolic sine (1/3)

- Consider the **hyperbolic sine** function:

$$y = \operatorname{Sh} x = \frac{e^x - e^{-x}}{2}$$

It is defined and strictly increasing on all of $\mathbb{R}$, therefore it is invertible.

- To solve the equation for x , multiplying both sides by e^x we obtain:

$$\begin{aligned} 0 &= e^x y - e^x \frac{e^x - e^{-x}}{2} \\ &= 2 e^x y - (e^x e^x - \underbrace{e^x e^{-x}}_{=e^{x-x}=e^0}) \\ &= 2 y e^x - e^{2x} + 1 \\ &= e^{2x} - 2 y e^x - 1 \end{aligned}$$

Inverse hyperbolic sine (2/3)

- ▶ This is a quadratic equation in the unknown e^x . Setting $t = e^x$ we obtain

$$0 = t^2 - 2 y t - 1$$

We obtain:

$$t = y \pm \sqrt{y^2 + 1} \quad \text{and substituting back} \quad e^x = y \pm \sqrt{y^2 + 1}$$

since $e^x > 0$, the solution with the minus sign must be discarded. Hence we are left with:

$$e^x = \underbrace{y + \sqrt{y^2 + 1}}_{>0, \forall y \in \mathbb{R}} \quad \text{and hence} \quad x = \log \left(y + \sqrt{y^2 + 1} \right)$$

Inverse hyperbolic sine (3/3)

- ▶ This is the analytic expression of the inverse function of $\text{Sh } x$, which is called the **inverse hyperbolic sine** (area hyperbolic sine), and is also denoted by SettSh . It is defined for every real y .

Definition 6: of inverse hyperbolic sine

*The inverse function of the hyperbolic sine is the **inverse hyperbolic sine**:*

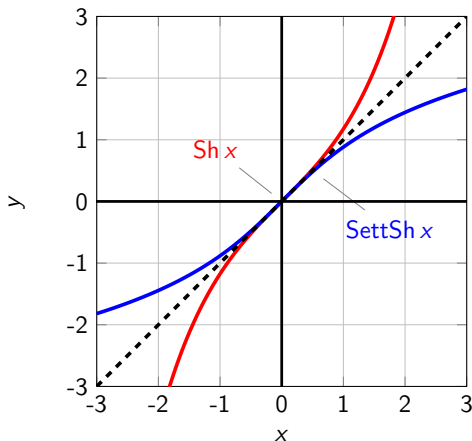
$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : y \mapsto \text{SettSh } y$$

We have:

$$f : \begin{cases} y = \text{Sh } x \\ x \in \mathbb{R} \end{cases} \qquad f^{-1} : \begin{cases} x = \text{SettSh } y \\ y \in \mathbb{R} \end{cases}$$

Graph of the inverse hyperbolic sine

The graph of the inverse hyperbolic sine is obtained from that of the hyperbolic sine on $\mathbb{R}$ (strictly monotone), by symmetry with respect to the bisector $y = x$:



Inverse hyperbolic cosine (1/3)

- Consider the **hyperbolic cosine** function:

$$y = \operatorname{Ch} x = \frac{e^x + e^{-x}}{2}$$

It is defined on all of $\mathbb{R}$, strictly increasing for $x \geq 0$, decreasing for $x \leq 0$. Therefore it is not invertible on all of $\mathbb{R}$.

- Its restriction to $x \geq 0$, however, is invertible. We want to determine the analytic expression of the inverse function of this restriction.
- Proceeding as before, we obtain:

$$e^x = y \pm \sqrt{y^2 - 1}$$

Inverse hyperbolic cosine (2/3)

- ▶ This time both numbers $y \pm \sqrt{y^2 - 1}$ are positive; recall, however, that we are reasoning only for $x \geq 0$, which is equivalent to choosing the plus sign. Therefore:

$$x = \log \left(y + \sqrt{y^2 - 1} \right)$$

This is the analytic expression of the inverse function of $\text{Ch } x$, which is called the inverse hyperbolic cosine (area hyperbolic cosine), and is also denoted by SettCh . Note that it is defined for $y \geq 1$.

Inverse hyperbolic cosine (3/3)

Definition 7: of inverse hyperbolic cosine

*The inverse function of the hyperbolic cosine on the interval $[0, +\infty)$ is the **inverse hyperbolic cosine**:*

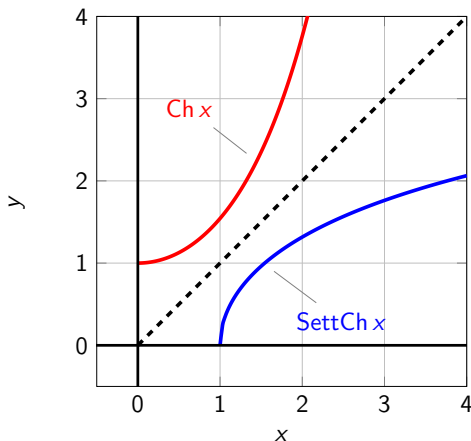
$$f : [1, +\infty) \rightarrow [0, +\infty), \quad f : y \mapsto \text{SettCh } y$$

We have:

$$f : \begin{cases} y = \text{Ch } x \\ x \in [0, +\infty) \end{cases} \qquad f^{-1} : \begin{cases} x = \text{SettCh } y \\ y \in [1, +\infty) \end{cases}$$

Graph of the inverse hyperbolic cosine

The graph of the inverse hyperbolic cosine is obtained from that of the hyperbolic cosine on $[0, +\infty)$ (strictly monotone), by symmetry with respect to the bisector $y = x$.



Example (1/2)

- ▶ The equation:

$$\operatorname{Sh} x = 2$$

has the unique solution:

$$x = \operatorname{SettSh} 2 = \log(2 + \sqrt{5})$$

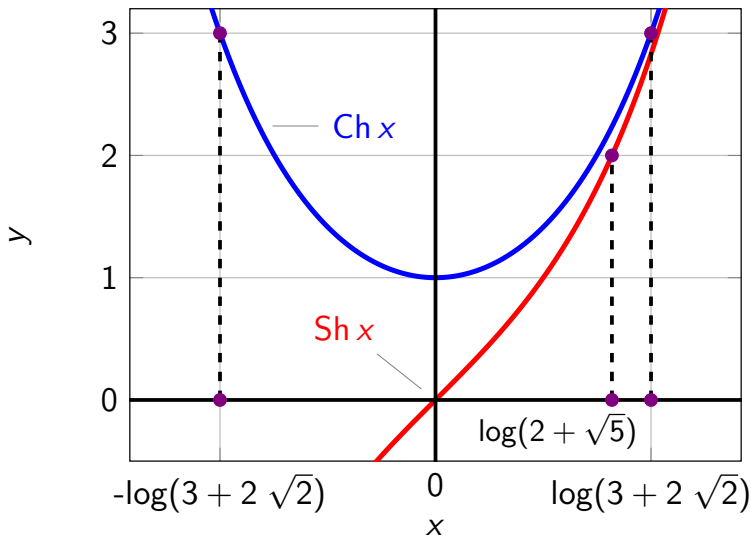
- ▶ The equation:

$$\operatorname{Ch} x = 3$$

has two solutions:

$$x = \pm \operatorname{SettCh} 3 = \pm \log(3 + 2\sqrt{2})$$

Example (2/2)



Inverse hyperbolic functions:

- ▶ Sh is strictly increasing on $\mathbb{R}$: its inverse is the inverse hyperbolic sine $\text{SettSh } y = \log \left(y + \sqrt{y^2 + 1} \right)$, defined for every $y \in \mathbb{R}$
- ▶ Ch is invertible only if restricted to $x \geq 0$: its inverse is the inverse hyperbolic cosine $\text{SettCh } y = \log \left(y + \sqrt{y^2 - 1} \right)$, defined for $y \geq 1$
- ▶ The expressions are obtained by solving a quadratic equation in $t = e^x$

Summary (1/2)

Invertible functions:

- ▶ f is invertible if it is injective; the inverse f^{-1} is defined on $f(D)$
- ▶ Strictly monotone $\Rightarrow$ invertible; even or periodic $\Rightarrow$ not invertible
- ▶ The graph of f^{-1} is symmetric to that of f with respect to the bisector $y = x$
- ▶ The expression of f^{-1} is obtained by solving $f(x) = y$ for x (not always possible)

Summary (2/2)

Inverses of the elementary functions:

- ▶ Powers: the inverse of x^α is $y^{\frac{1}{\alpha}}$ (even powers only for $x \geq 0$)
- ▶ Exponential and logarithm: $y = a^x \leftrightarrow x = \log_a y$
- ▶ Trigonometric (on intervals of monotonicity): $\arcsin$, $\arccos$, $\arctan$
- ▶ Hyperbolic: $\text{SettSh } y = \log \left(y + \sqrt{y^2 + 1} \right)$,
 $\text{SettCh } y = \log \left(y + \sqrt{y^2 - 1} \right)$