

Composite functions



Outline

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Introduction: Composite functions

- ▶ In this section we introduce the **composition** of two functions: we first apply one function and then another one to the result
- ▶ We will see when the composition is well defined and some examples with their graphs
- ▶ We will study the properties of composition: it is **not commutative**, but it is **associative**; finally we will introduce **iterate** functions

Definition 1: of composite function

Given two functions:

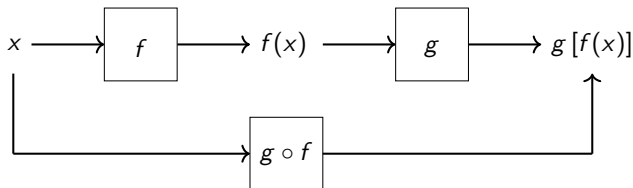
$$f : E \rightarrow \mathbb{R} \quad \text{and} \quad g : F \rightarrow \mathbb{R},$$

*if $f(E) \subseteq F$ (i.e., if for every $x \in E$ we have $f(x) \in F$), one can define the function $h : E \rightarrow \mathbb{R}$, the **composite** of f and g (in this order), denoted by the symbol $g \circ f$, through the formula*

$$h(x) = (g \circ f)(x) = g[f(x)]$$

Composition scheme

The composition scheme is the following:

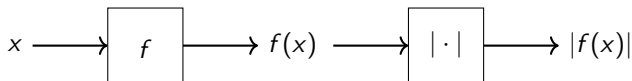


Example

The function $x \mapsto |f(x)|$ is actually “composed” of two functions:

1. given x , we compute $f(x)$
2. once $f(x)$ is computed, we compute $|f(x)|$

This amounts to operating in series with two black boxes, the first corresponding to f , the second to its absolute value, according to the following scheme:



Composition is not commutative

- ▶ It may happen that both compositions $(g \circ f)$ and $(f \circ g)$ are well defined, but in general

$$(f \circ g) \neq (g \circ f)$$

In other words, **the commutative property does not hold.**

- ▶ We now see two examples in which we compare $g \circ f$ and $f \circ g$

Example (1/3)

Consider the functions:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto x^2 \quad \text{and} \quad g : \mathbb{R} \rightarrow \mathbb{R}, \quad g : x \mapsto \cos x$$

that is, $f(x) = x^2$ and $g(x) = \cos x$.

1. Since g is defined on all of $\mathbb{R}$, $h = g \circ f$ is well defined on $\mathbb{R}$ and the following formula holds

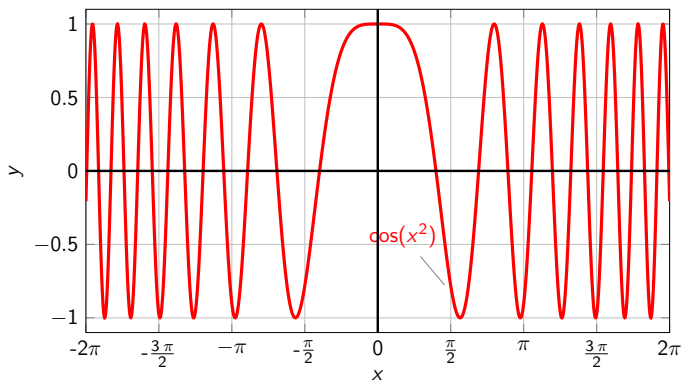
$$h(x) = (g \circ f)(x) = g[f(x)] = \cos(x^2)$$

2. Since f is defined on all of $\mathbb{R}$, $k = f \circ g$ is well defined on $\mathbb{R}$ and the following formula holds

$$k(x) = (f \circ g)(x) = f[g(x)] = \cos^2 x$$

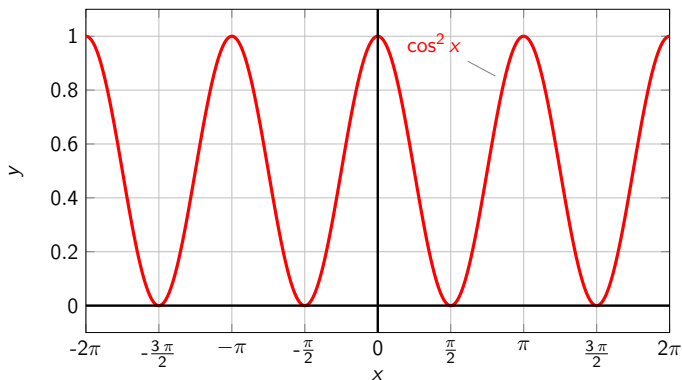
Example (2/3)

Graph of $h(x) = (g \circ f)(x) = \cos(x^2)$:



Example (3/3)

Graph of $k(x) = (f \circ g)(x) = \cos^2 x$:



Example (1/3)

Consider the functions:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto e^x \qquad g : [0, +\infty) \rightarrow \mathbb{R}, \quad g : x \mapsto \sqrt{x}$$

that is, $f(x) = e^x$ and $g(x) = \sqrt{x}$.

1. Since g is defined on $[0, +\infty)$ and $f(x) > 0$ for every $x \in \mathbb{R}$, $h = g \circ f$ is well defined on $\mathbb{R}$ and the following formula holds

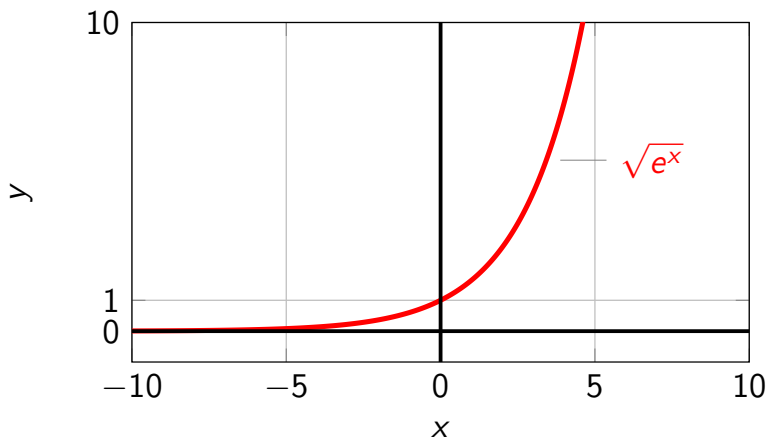
$$h(x) = (g \circ f)(x) = g[f(x)] = \sqrt{e^x}$$

2. Since g is defined only on $[0, +\infty)$, $k = f \circ g$ is well defined only on $\mathbb{R}_+$ and the following formula holds

$$k(x) = (f \circ g)(x) = f[g(x)] = e^{\sqrt{x}}$$

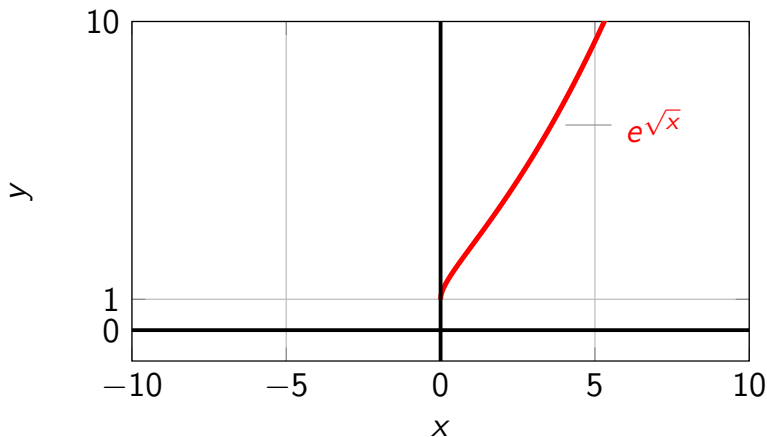
Example (2/3)

Graph of $h(x) = (g \circ f)(x) = \sqrt{e^x}$:



Example (3/3)

Graph of $k(x) = (f \circ g)(x) = e^{\sqrt{x}}$:



Associative property

- ▶ The composition operation can be extended to three or more factors. One can verify that if the composition $(f \circ g) \circ r$ exists, then $f \circ (g \circ r)$ also exists and they are equal

$$(f \circ g) \circ r = f \circ (g \circ r)$$

In other words, **the associative property holds.**

Iterate functions

- ▶ If a function $f : D \rightarrow \mathbb{R}$ is such that $f(D) \subseteq D$, then it can be composed with itself

$$f^2 = f \circ f \quad \text{that is} \quad f^2(x) = f[f(x)]$$

f^2 is called the **second iterate** of f . Similarly, f^n is called the **n -th iterate function** of f .

Summary

Composite functions:

- ▶ If $f(E) \subseteq F$, the composite $g \circ f$ is defined by $(g \circ f)(x) = g[f(x)]$: first apply f , then g
- ▶ The domain must be checked: e.g. $e^{\sqrt{x}}$ is defined only for $x \geq 0$, while $\sqrt{e^x}$ on all of $\mathbb{R}$

Properties:

- ▶ **Not commutative:** in general $f \circ g \neq g \circ f$ (e.g. $\cos(x^2) \neq \cos^2 x$)
- ▶ **Associative:** $(f \circ g) \circ r = f \circ (g \circ r)$
- ▶ If $f(D) \subseteq D$ we define the **iterates** $f^2 = f \circ f, \dots, f^n$