

Integer part, fractional part and piecewise-defined functions

Outline

Integer part and fractional part functions

Piecewise-defined functions

The absolute-value function

Integer part and fractional part functions



Introduction: Integer part and fractional part functions

- ▶ Two functions that are typically encountered when writing *algorithms* are the integer part (floor) function and the fractional part (or decimal part) function
- ▶ In this section we define them, together with the **upper integer part (ceiling)** function
- ▶ We draw their graphs: they are “step” functions, piecewise constant, and the fractional part is periodic

Integer part and upper integer part

Definition 1: of the integer part function

The integer part (floor) function is:

$$f : \mathbb{R} \rightarrow \mathbb{Z}, x \mapsto [x] \quad (\text{or } x \mapsto \lfloor x \rfloor)$$

Definition 2: of the upper integer part function

The upper integer part (ceiling) function is:

$$f : \mathbb{R} \rightarrow \mathbb{Z}, x \mapsto \lceil x \rceil$$

Fractional part

Definition 3: of the fractional part function

The fractional part function is:

$$f : \mathbb{R} \rightarrow [0, 1), x \mapsto (x)$$

Monotonicity of the integer part and upper integer part

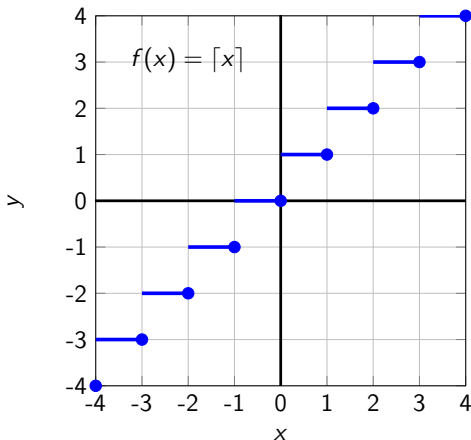
The functions $f(x) = \lfloor x \rfloor$ and $f(x) = \lceil x \rceil$ are **non-decreasing** in $\mathbb{R}$, but they are **not** increasing.

- ▶ **They are non-decreasing:** let $x_1 < x_2$. Then $\lfloor x_1 \rfloor \leq x_1 < x_2$ and, since $\lfloor x_2 \rfloor$ is the *largest* integer less than or equal to x_2 , we have $\lfloor x_1 \rfloor \leq \lfloor x_2 \rfloor$. In the same way $\lceil x_2 \rceil \geq x_2 > x_1$ and, since $\lceil x_1 \rceil$ is the *smallest* integer greater than or equal to x_1 , we have $\lceil x_1 \rceil \leq \lceil x_2 \rceil$
- ▶ **They are not increasing:** they are *constant* between two consecutive integers, for instance

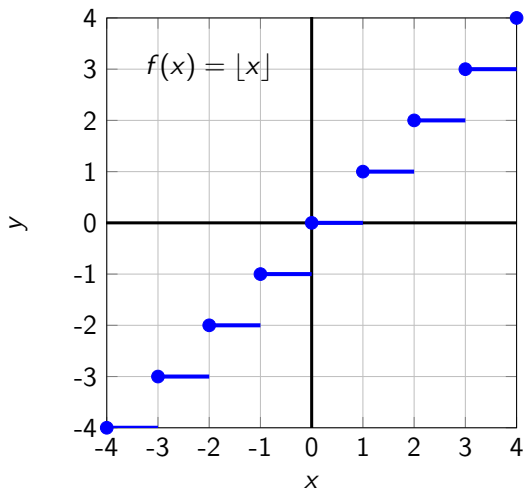
$$\left\lfloor \frac{1}{3} \right\rfloor = \left\lfloor \frac{2}{3} \right\rfloor = 0 \quad \text{and} \quad \left\lceil \frac{1}{3} \right\rceil = \left\lceil \frac{2}{3} \right\rceil = 1$$

Graph of $f(x) = \lceil x \rceil$

Graphs of the functions $f(x) = \lceil x \rceil$ and $f(x) = \lfloor x \rfloor$ on the interval $[-4, 4]$.

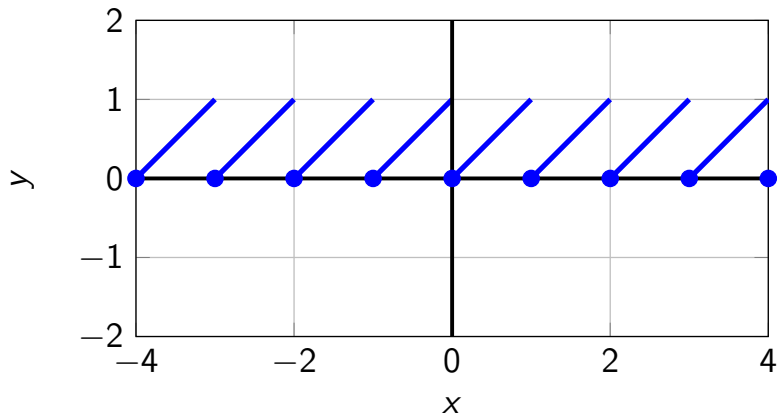


Graph of $f(x) = \lfloor x \rfloor$



Graph of the fractional part

Note that the fractional part is a periodic function with period 1.



Summary

Integer part and fractional part:

- ▶ Integer part $\lfloor x \rfloor$ and upper integer part $\lceil x \rceil$: functions from $\mathbb{R}$ to $\mathbb{Z}$, **non-decreasing** (but not increasing), piecewise constant (“step” graphs)
- ▶ Fractional part $\{x\}$: function from $\mathbb{R}$ to $[0, 1)$, periodic with period 1
- ▶ They are typical functions in the writing of algorithms

Piecewise-defined functions

Introduction: Piecewise-defined functions

- ▶ Starting from the elementary functions, new ones can be built by using **different analytic definitions on different intervals**
- ▶ In this section we give the definition of piecewise-defined function and we see an example with its graph
- ▶ Piecewise functions are widely used to model situations in which the “rule” changes depending on the values of the variable

Piecewise-defined functions

Definition 4: of piecewise-defined function

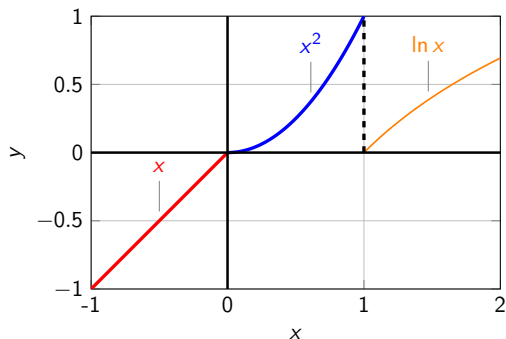
*A function f whose value $f(x)$ is computed through different “instructions” depending on the interval in which x lies is called a **piecewise-defined function**.*

Example

Consider for example

$$f(x) = \begin{cases} \ln x & \text{if } x > 1, \\ x^2 & \text{if } 0 < x \leq 1, \\ x & \text{if } x \leq 0 \end{cases}$$

its graph is:



Piecewise functions and algorithms

- ▶ From the *logical / algorithmic* point of view, one can say that a function of this kind has the peculiarity of being built using not only mathematical functions but also the **logical function** “if ... then”.

Piecewise-defined functions:

- ▶ The value $f(x)$ is computed with different “instructions” depending on the interval in which x lies
- ▶ They are built starting from the elementary functions
- ▶ Besides mathematical functions, they use the logical function “if . . . then”

The absolute-value function



Introduction: The absolute-value function

- ▶ The absolute value of a real number was defined in Part 2; here we look at it as a **function**
- ▶ It is the most important example of a piecewise-defined function, and we will use it constantly in the study of limits
- ▶ We study its sign, monotonicity, boundedness, image and symmetry

The absolute-value function

In the chapter “[Factorials, binomial coefficients and triangle inequality](#)” of Part 2 the **absolute value** of $a \in \mathbb{R}$ was defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Definition 5: of the absolute-value function

*The **absolute-value function** is:*

$$f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto |x|$$

Sign and symmetry

- ▶ The absolute-value function is *nonnegative* in $\mathbb{R}$ and *positive* in $\mathbb{R} \setminus \{0\}$:

$$|x| \geq 0, \quad \forall x \in \mathbb{R} \quad \text{and} \quad |x| = 0 \iff x = 0$$

Indeed, if $x \geq 0$ then $|x| = x \geq 0$, and if $x < 0$ then $|x| = -x > 0$

- ▶ It is an *even* function, since $|-x| = |x|$ for every $x \in \mathbb{R}$: its graph is symmetric with respect to the y -axis

Monotonicity

The absolute-value function is *decreasing* in $(-\infty, 0]$ and *increasing* in $[0, +\infty)$, but it is **not** monotonic in $\mathbb{R}$.

- ▶ If $x_1 < x_2 \leq 0$, then $|x_1| = -x_1 > -x_2 = |x_2|$
- ▶ If $0 \leq x_1 < x_2$, then $|x_1| = x_1 < x_2 = |x_2|$
- ▶ It is not monotonic in $\mathbb{R}$ because $f(-1) = 1 > f(0) = 0$ and $f(0) = 0 < f(1) = 1$



Boundedness and image

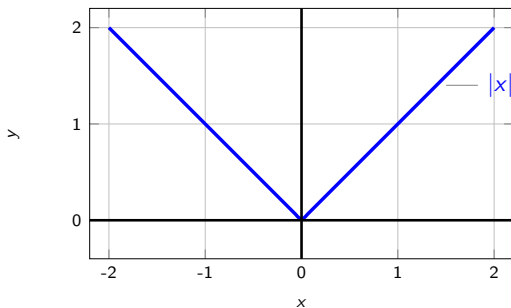
- ▶ The function is *bounded below* in $\mathbb{R}$: the number 0 is a lower bound for its values
- ▶ It is **not** bounded above: for every $M \in \mathbb{R}$ with $M > 0$ we have $f(M + 1) = M + 1 > M$
- ▶ Its image is

$$f(\mathbb{R}) = [0, +\infty),$$

because $|x| \geq 0$ for every x and, conversely, every $y \geq 0$ is the image of y itself: $|y| = y$

Graph of $f(x) = |x|$

The graph consists of the two half-lines $y = -x$ for $x \leq 0$ and $y = x$ for $x \geq 0$.



The absolute-value function:

- ▶ $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto |x|$: a piecewise-defined function
- ▶ **Sign**: nonnegative in $\mathbb{R}$, positive in $\mathbb{R} \setminus \{0\}$
- ▶ **Monotonicity**: decreasing in $(-\infty, 0]$, increasing in $[0, +\infty)$, not monotonic in $\mathbb{R}$
- ▶ **Boundedness and image**: bounded below, not bounded above, $f(\mathbb{R}) = [0, +\infty)$
- ▶ **Symmetry**: it is an even function

Summary

- ▶ **Integer part** $\lfloor x \rfloor$ and **upper integer part** $\lceil x \rceil$: step functions, **non-decreasing**, with values in $\mathbb{Z}$
- ▶ **Fractional part** $\{x\}$: values in $[0, 1)$, periodic with period 1
- ▶ **Piecewise-defined functions**: different analytic definitions on different intervals, built with the “if ... then” logic
- ▶ **Absolute-value function** $x \mapsto |x|$: the most important example of a piecewise-defined function, decreasing in $(-\infty, 0]$ and increasing in $[0, +\infty)$