

Trigonometric functions



Outline

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Trigonometric functions



Introduction: Trigonometric functions

- ▶ In this section we introduce the **sine** and **cosine** functions and, from them, the **tangent**, **cotangent**, **secant** and **cosecant**
- ▶ We study their symmetry, periodicity, sign and monotonicity, with the help of their graphs and of the unit circle
- ▶ Trigonometric functions are the natural tool to describe periodic phenomena and oscillations

Sine and cosine

Definition 1: sine function and cosine function

The trigonometric functions:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \sin x$$

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \cos x$$

*are called **sine** and **cosine**.*

Symmetry and periodicity

- ▶ The sine is an **odd** function, while the cosine is an **even** function. Hence we have:

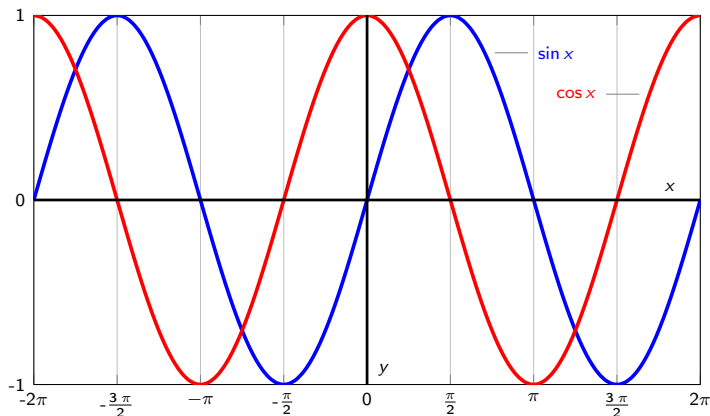
$$\sin(-x) = -\sin x, \quad \forall x \in \mathbb{R}$$

$$\cos(-x) = \cos x, \quad \forall x \in \mathbb{R}$$

- ▶ The sine and the cosine are periodic functions with period $T = 2\pi$ and we have the fundamental identity

$$\sin^2 x + \cos^2 x = 1, \quad \forall x \in \mathbb{R}$$

Graphs of sine and cosine



We have

$$\cos\left(x - \frac{\pi}{2}\right) = \sin x, \quad \forall x \in \mathbb{R}$$

Sign and monotonicity of the sine

- From the sign and the monotonicity of the sine function we have:

$$\sin x = 0 \iff x = k \pi, \forall k \in \mathbb{Z}$$

$$\begin{cases} \sin x > 0 & \text{if } x \in (2k\pi, 2k\pi + \pi), \forall k \in \mathbb{Z} \\ \sin x < 0 & \text{if } x \in (2k\pi - \pi, 2k\pi), \forall k \in \mathbb{Z} \end{cases}$$

$$\begin{cases} \text{if } x \in (2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}), \forall k \in \mathbb{Z} & f \text{ is increasing} \\ \text{if } x \in (2k\pi + \frac{\pi}{2}, 2k\pi + \frac{3\pi}{2}), \forall k \in \mathbb{Z} & f \text{ is decreasing} \end{cases}$$

Sign and monotonicity of the cosine

- From the sign and the monotonicity of the cosine function we have:

$$\cos x = 0 \iff x = k\pi - \frac{\pi}{2}, \forall k \in \mathbb{Z}$$

$$\begin{cases} \cos x > 0 & \text{if } x \in (2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}), \forall k \in \mathbb{Z} \\ \cos x < 0 & \text{if } x \in (2k\pi + \frac{\pi}{2}, 2k\pi + \frac{3\pi}{2}), \forall k \in \mathbb{Z} \end{cases}$$

$$\begin{cases} \text{if } x \in (2k\pi + \pi, 2k\pi + 2\pi), \forall k \in \mathbb{Z} & f \text{ is increasing} \\ \text{if } x \in (2k\pi, 2k\pi + \pi), \forall k \in \mathbb{Z} & f \text{ is decreasing} \end{cases}$$

Tangent and cotangent

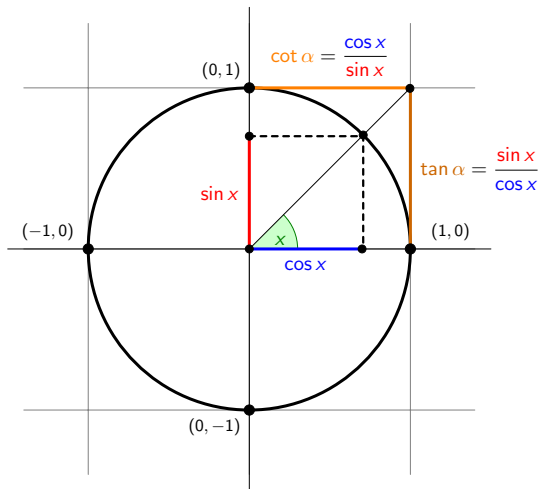
The trigonometric functions:

$$f : \mathbb{R} \setminus \left\{ k\pi - \frac{\pi}{2}, \forall k \in \mathbb{Z} \right\} \rightarrow \mathbb{R}, \quad f : x \mapsto \tan x = \frac{\sin x}{\cos x}$$

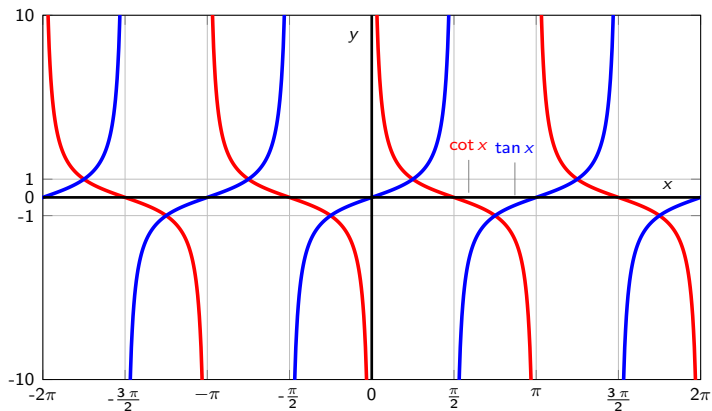
$$f : \mathbb{R} \setminus \{ k\pi, \forall k \in \mathbb{Z} \} \rightarrow \mathbb{R}, \quad f : x \mapsto \cot x = \frac{\cos x}{\sin x}$$

are called **tangent** and **cotangent**; they are both odd and periodic with period $T = \pi$.

The unit circle



Graphs of tangent and cotangent



Secant and cosecant

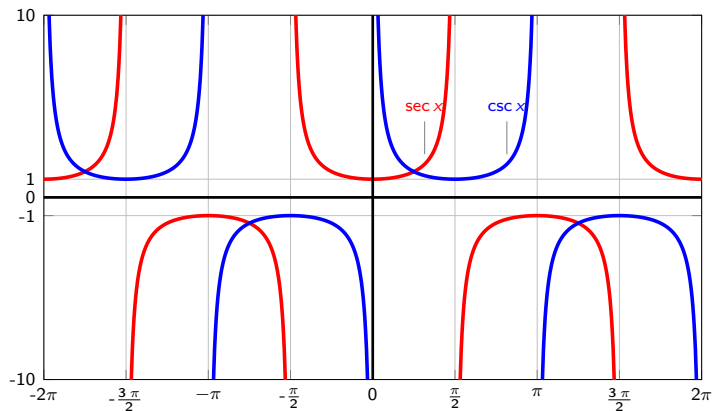
The trigonometric functions:

$$f : \mathbb{R} \setminus \left\{k\pi - \frac{\pi}{2}, \forall k \in \mathbb{Z}\right\} \rightarrow \mathbb{R}, \quad f : x \mapsto \sec x = \frac{1}{\cos x}$$

$$f : \mathbb{R} \setminus \{k\pi, \forall k \in \mathbb{Z}\} \rightarrow \mathbb{R}, \quad f : x \mapsto \csc x = \frac{1}{\sin x}$$

are called **secant** and **cosecant** and they are periodic with period $T = 2\pi$. The secant function is even, while the cosecant function is odd.

Graphs of secant and cosecant



Summary

Trigonometric functions:

- ▶ $\sin$ (odd) and $\cos$ (even) are defined on $\mathbb{R}$, periodic with period 2π , with $\sin^2 x + \cos^2 x = 1$
- ▶ $\tan = \frac{\sin}{\cos}$ and $\cot = \frac{\cos}{\sin}$: odd, periodic with period π
- ▶ $\sec = \frac{1}{\cos}$ (even) and $\csc = \frac{1}{\sin}$ (odd): periodic with period 2π
- ▶ Sign and monotonicity can be read on the unit circle

Values of the trigonometric functions



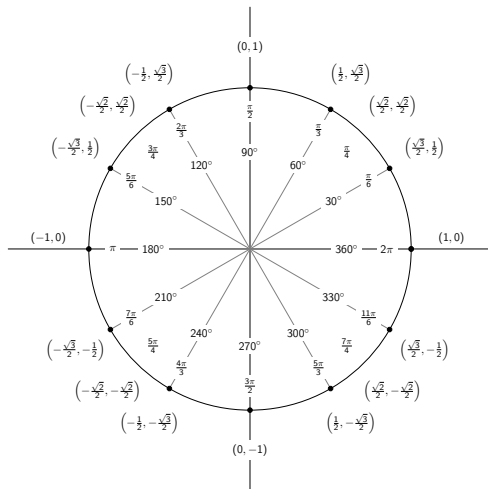
Introduction: Values of the trigonometric functions

- ▶ In this section we collect the **notable values** of the trigonometric functions at the most common angles
- ▶ We also see how the functions change for **related angles** ($-x$, $\frac{\pi}{2} \pm x$, $\pi \pm x$)
- ▶ These tables are used constantly in exercises

Notable values

	cos	sin	tan	cot	sec	csc
0	1	0	0	$\pm\infty$	1	$\pm\infty$
$x = \frac{\pi}{6} (30^\circ)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
$x = \frac{\pi}{4} (45^\circ)$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$x = \frac{\pi}{3} (60^\circ)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2}{3}\sqrt{3}$
$x = \frac{\pi}{2} (90^\circ)$	0	1	$\pm\infty$	0	$\pm\infty$	1
$x = \pi (180^\circ)$	-1	0	0	$\pm\infty$	-1	$\pm\infty$

Notable values on the unit circle



Related angles

	cos	sin	tan	cot
x	$\cos x$	$\sin x$	$\tan x$	$\cot x$
$-x$	$\cos x$	$-\sin x$	$-\tan x$	$-\cot x$
$\frac{\pi}{2} + x$	$-\sin x$	$\cos x$	$-\cot x$	$-\tan x$
$\frac{\pi}{2} - x$	$\sin x$	$\cos x$	$\cot x$	$\tan x$
$\pi + x$	$-\cos x$	$-\sin x$	$\tan x$	$\cot x$
$\pi - x$	$-\cos x$	$\sin x$	$-\tan x$	$-\cot x$

Summary

Values of the trigonometric functions:

- ▶ Notable values at $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ of $\cos, \sin, \tan, \cot, \sec, \csc$
- ▶ Related angles: the functions at $-x, \frac{\pi}{2} \pm x, \pi \pm x$ are expressed through the functions at x

Main trigonometric formulas



Introduction: Main trigonometric formulas

- ▶ In this section we collect the main **trigonometric identities**
- ▶ Relations among the functions, addition, double-angle, triple-angle, half-angle, parametric, sum-to-product and product-to-sum formulas
- ▶ They are essential to simplify expressions, solve equations and compute limits and integrals

Relations among the trigonometric functions

	$\sin x$	$\cos x$	$\tan x$	$\cot x$
$\sin x$	-	$\pm\sqrt{1 - \cos^2 x}$	$\pm\sqrt{\frac{\tan^2 x}{1 + \tan^2 x}}$	$\pm\sqrt{\frac{1}{1 + \cot^2 x}}$
$\cos x$	$\pm\sqrt{1 - \sin^2 x}$	-	$\pm\sqrt{\frac{1}{1 + \tan^2 x}}$	$\pm\sqrt{\frac{\cot^2 x}{1 + \cot^2 x}}$
$\tan x$	$\pm\sqrt{\frac{\sin^2 x}{1 - \sin^2 x}}$	$\pm\sqrt{\frac{1 - \cos^2 x}{\cos^2 x}}$	-	$\frac{1}{\cot x}$
$\cot x$	$\pm\sqrt{\frac{1 - \sin^2 x}{\sin^2 x}}$	$\pm\sqrt{\frac{\cos^2 x}{1 - \cos^2 x}}$	$\frac{1}{\tan x}$	-

where the symbol $\pm$ means that the sign depends on the quadrant in which x lies.

Addition and subtraction

$$\sin(x_1 \pm x_2) = \sin x_1 \cos x_2 \pm \sin x_2 \cos x_1$$

$$\cos(x_1 \pm x_2) = \cos x_1 \cos x_2 \mp \sin x_1 \sin x_2$$

$$\tan(x_1 \pm x_2) = \frac{\tan x_1 \pm \tan x_2}{1 \mp \tan x_1 \tan x_2}$$

$$\cot(x_1 \pm x_2) = \frac{\cot x_1 \cot x_2 \mp 1}{\cot x_2 \pm \cot x_1}$$

Double-angle formulas

$$\sin(2x) = 2 \sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$= 2 \cos^2 x - 1$$

$$= 1 - 2 \sin^2 x$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\cot(2x) = \frac{\cot^2 x - 1}{2 \cot x}$$

Triple-angle formulas

$$\sin(3x) = 3 \sin x - 4 \sin^3 x$$

$$\cos(3x) = 4 \cos^3 x - 3 \cos x$$

Half-angle formulas (1/2)

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$$

Half-angle formulas (2/2)

$$\tan \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$$

$$\cot \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{1 - \cos x}} = \frac{\sin x}{1 - \cos x} = \frac{1 + \cos x}{\sin x}$$

Parametric formulas (tangent half-angle substitution)

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

$$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$$

Sum-to-product formulas (prosthaphaeresis)

$$\sin x_1 + \sin x_2 = 2 \sin \frac{x_1 + x_2}{2} \cos \frac{x_1 - x_2}{2}$$

$$\sin x_1 - \sin x_2 = 2 \sin \frac{x_1 - x_2}{2} \cos \frac{x_1 + x_2}{2}$$

$$\cos x_1 + \cos x_2 = 2 \cos \frac{x_1 + x_2}{2} \cos \frac{x_1 - x_2}{2}$$

$$\cos x_1 - \cos x_2 = -2 \sin \frac{x_1 + x_2}{2} \sin \frac{x_1 - x_2}{2}$$

Product-to-sum formulas (Werner's formulas)

$$\sin x_1 \sin x_2 = \frac{1}{2} \left(\cos(x_1 - x_2) - \cos(x_1 + x_2) \right)$$

$$\sin x_1 \cos x_2 = \frac{1}{2} \left(\sin(x_1 - x_2) + \sin(x_1 + x_2) \right)$$

$$\cos x_1 \cos x_2 = \frac{1}{2} \left(\cos(x_1 - x_2) + \cos(x_1 + x_2) \right)$$

Additional formulas

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin x \cos x = \frac{\sin(2x)}{2}$$

Summary

Main trigonometric formulas:

- ▶ Each trigonometric function can be expressed through the others (up to the sign, which depends on the quadrant)
- ▶ **Addition/subtraction, double-angle, triple-angle, half-angle** formulas
- ▶ **Parametric** formulas: $\sin x$ and $\cos x$ in terms of $\tan \frac{x}{2}$
- ▶ **Sum-to-product** (prosthaphaeresis) and **product-to-sum** (Werner) formulas
- ▶ $\sin^2 x$, $\cos^2 x$ and $\sin x \cos x$ in terms of the double angle

An application: oscillations



Vibration phenomena

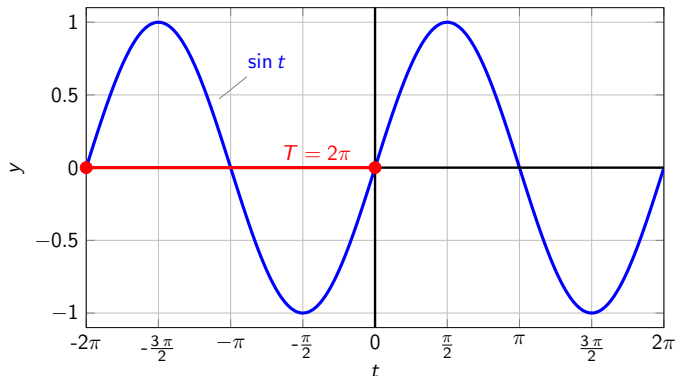


Introduction: Vibration phenomena

- ▶ In this section we use sine and cosine to describe phenomena that **oscillate in time**
- ▶ We introduce **elementary vibrations** and the quantities that characterize them: **period**, **amplitude**, **angular frequency**, **frequency** and **phase**
- ▶ We will see that the superposition of two vibrations with the same angular frequency is again an elementary vibration

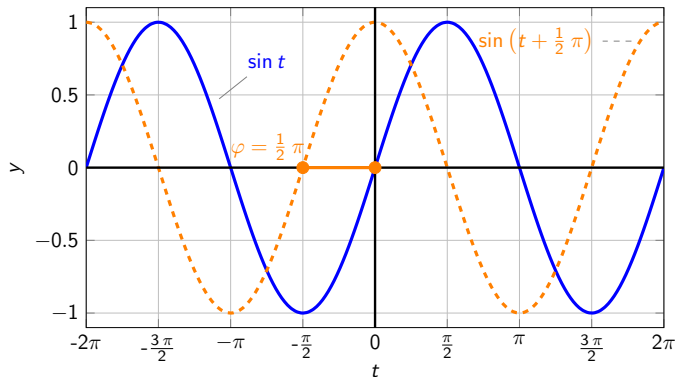
Sine and cosine in time

- We have seen that the sine and cosine functions are periodic with **period** 2π . We will use the variable t to denote time.



Phase shift

- Recall that $\sin(t + \varphi)$ corresponds to a **phase shift** φ (horizontal translation).



Elementary vibrations

The functions:

$$t \mapsto a \sin(\omega t), \quad t \mapsto b \cos(\omega t)$$

where a , b and ω are positive real numbers, are called **elementary vibrations**.

They are periodic functions with

$$\text{period} \quad T = \frac{2\pi}{\omega}$$

Example: period

For example, considering $t' = t + \frac{2\pi}{\omega}$:

$$\sin \left[\omega \underbrace{\left(t + \frac{2\pi}{\omega} \right)}_{t'} \right] = \sin(\omega t + 2\pi) = \sin(\omega t)$$

► Clearly we have:

$$|a \sin(\omega t)| \leq a \quad \text{and} \quad |b \cos(\omega t)| \leq b$$

Amplitude, angular frequency and frequency

The elementary vibrations $a \sin(\omega t)$ and $b \cos(\omega t)$ are characterized by

amplitude a and b (respectively)

$$\text{angular frequency } \omega = \frac{2\pi}{T}$$

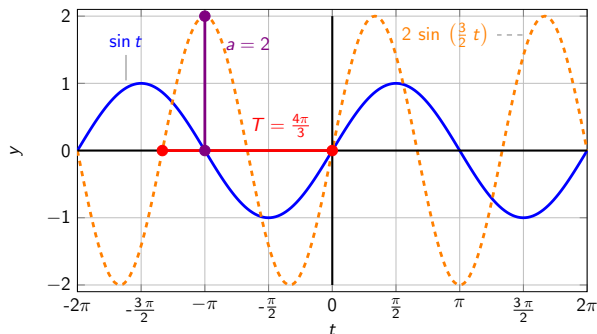
also called *angular velocity*, which indicates how many periods there are in an interval of length 2π . Moreover, they are characterized by the

$$\text{frequency } \nu = \frac{\omega}{2\pi}$$

which indicates how many times the function repeats itself in an interval of length 1.

Example: amplitude, angular frequency and frequency

$$t \mapsto 2 \sin \left(\frac{3}{2} t \right) \quad a = 2, \quad \omega = \frac{3}{2}, \quad T = \frac{4\pi}{3}$$



The angular frequency $\omega = \frac{3}{2}$ means that there are 1.5 periods in an interval of length 2π . The frequency ν is $\frac{3}{4\pi}$.

Superposition of vibrations (1/3)

- The function

$$h(t) = a \sin(\omega t) + b \cos(\omega t)$$

describes the **superposition** of the two elementary vibrations with the **same angular frequency** ω . The latter is again an elementary vibration, **phase-shifted** with respect to the previous ones.

- Indeed, setting $A = \sqrt{a^2 + b^2}$ we can write

$$h(t) = A \underbrace{\frac{a}{\sqrt{a^2 + b^2}}}_{\alpha} \sin(\omega t) + A \underbrace{\frac{b}{\sqrt{a^2 + b^2}}}_{\beta} \cos(\omega t)$$

Superposition of vibrations (2/3)

- Now observe that the numbers

$$\alpha = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \beta = \frac{b}{\sqrt{a^2 + b^2}}$$

satisfy the conditions

$$-1 \leq \alpha \leq 1 \quad -1 \leq \beta \leq 1 \quad \alpha^2 + \beta^2 = 1$$

Hence there exists a **unique angle** φ such that

$$\cos \varphi = \alpha \quad \sin \varphi = \beta$$

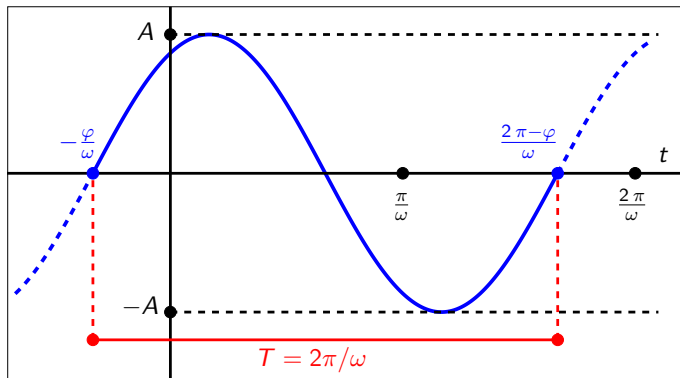
Superposition of vibrations (3/3)

- So that $h(t)$ can be rewritten in the following form

$$\begin{aligned}h(t) &= A \cos \varphi \sin(\omega t) + A \sin \varphi \cos(\omega t) \\&= A \sin(\omega t + \varphi)\end{aligned}$$

- Therefore $h(t)$ represents an elementary vibration with amplitude A , angular frequency ω , phase-shifted by an angle φ .

Superposition of vibrations: graph



In summary

$$a \sin(\omega t) + b \cos(\omega t) = A \sin(\omega t + \varphi)$$

$$A = \sqrt{a^2 + b^2} \quad \begin{cases} a = A \cos \varphi \\ b = A \sin \varphi \end{cases}$$

- Under fairly general conditions, a periodic natural phenomenon can be written as the superposition of a finite or infinite number of elementary vibrations with different frequencies (**Fourier series**).

Summary

Vibration phenomena:

- ▶ **Elementary vibrations:** $a \sin(\omega t)$ and $b \cos(\omega t)$, with period $T = \frac{2\pi}{\omega}$
- ▶ **Amplitude** a (or b), **angular frequency** $\omega = \frac{2\pi}{T}$, **frequency** $\nu = \frac{\omega}{2\pi}$
- ▶ $\sin(t + \varphi)$: **phase shift** φ
- ▶ $a \sin(\omega t) + b \cos(\omega t) = A \sin(\omega t + \varphi)$ with $A = \sqrt{a^2 + b^2}$
- ▶ General periodic phenomena: superposition of elementary vibrations (Fourier series)

Damping and amplification effects

Introduction: Damping and amplification effects

- ▶ By multiplying an elementary vibration by **powers** or **exponentials**, we can model **damping** or **amplification** effects.
- ▶ We see two examples: an **amplified vibration** $t \sin(\omega t)$ and a **damped vibration** $e^{-\alpha t} \sin(\omega t)$
- ▶ In both cases the graph lies between two curves that act as its “envelope”

First example (1/3)

- For example, consider the function

$$h(t) = t \sin(\omega t)$$

which models an **amplified vibration**.

- Since $-1 \leq \sin(\omega t) \leq 1$, we have

$$-t \leq t \sin(\omega t) \leq t$$

and hence the graph of $h(t)$ lies between the graphs of the lines with equations $y = -t$, $y = t$.

First example (2/3)

- At the points where $\sin(\omega t) = 1$, i.e.,

$$t = \frac{\pi}{2\omega} + k \frac{2\pi}{\omega} \quad (k = 0, 1, 2, \dots)$$

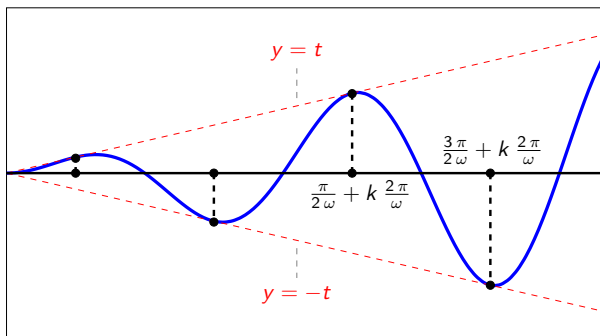
the graph of $h(t)$ touches that of $y = t$.

- At the points where $\sin(\omega t) = -1$, i.e.,

$$t = \frac{3\pi}{2\omega} + k \frac{2\pi}{\omega} \quad (k = 0, 1, 2, \dots)$$

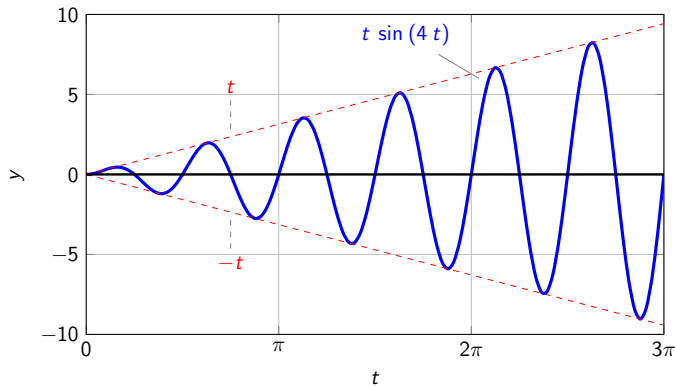
the graph of $h(t)$ touches that of $y = -t$.

First example (3/3)

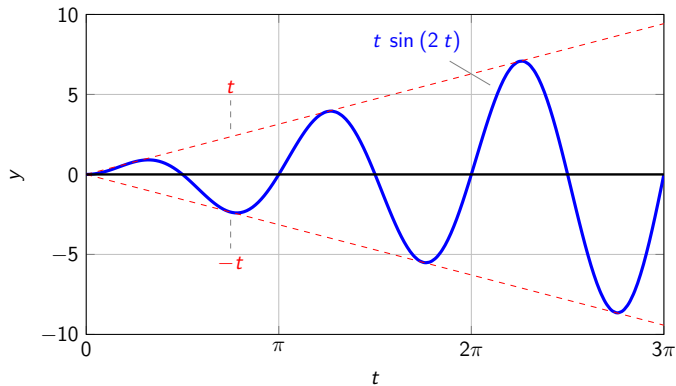


- From the graph we can see that multiplication by t has the effect of amplifying the vibration as t increases.

Amplified vibration: graphs (1/2)



Amplified vibration: graphs (2/2)



Second example

- ▶ For example, consider the function

$$k(t) = e^{-\alpha t} \sin(\omega t) \quad (\alpha > 0)$$

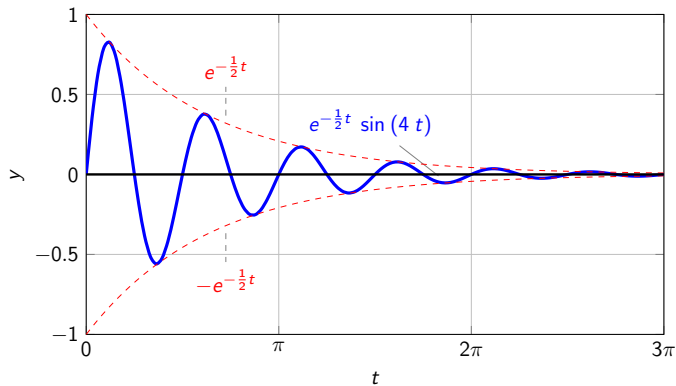
which models a **damped vibration**.

- ▶ Recalling that $e^{-\alpha t} = \left(\frac{1}{e^\alpha}\right)^t$ is an exponential with base less than 1, considerations analogous to those made for the function $h(t)$ show that the graph of k lies between the graphs of the functions

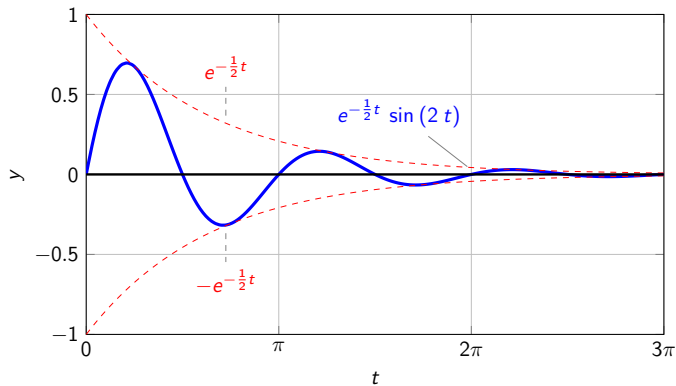
$$y_1 = e^{-\alpha t} \quad \text{and} \quad y_2 = -e^{-\alpha t}$$

as shown in the figure.

Damped vibration: graphs (1/2)



Damped vibration: graphs (2/2)



Summary

Damping and amplification:

- ▶ $h(t) = t \sin(\omega t)$: **amplified vibration**, with graph between $y = -t$ and $y = t$
- ▶ $k(t) = e^{-\alpha t} \sin(\omega t)$, $\alpha > 0$: **damped vibration**, with graph between $-e^{-\alpha t}$ and $e^{-\alpha t}$
- ▶ The graph touches the envelope curves at the points where $\sin(\omega t) = \pm 1$

Summary: oscillations

- ▶ **Elementary vibrations** $a \sin(\omega t)$, $b \cos(\omega t)$ are characterized by amplitude, angular frequency, period and frequency
- ▶ The superposition of two vibrations with the same angular frequency is a **phase-shifted** elementary vibration:
 $A \sin(\omega t + \varphi)$
- ▶ Multiplying by powers or exponentials gives **amplified** or **damped** vibrations

Final summary

- ▶ **Sine** and **cosine**: periodic with period 2π , odd and even respectively, with $\sin^2 x + \cos^2 x = 1$
- ▶ **Tangent**, **cotangent** (period π), **secant** and **cosecant** (period 2π) are built from sine and cosine
- ▶ Tables of **notable values** and of **related angles**
- ▶ The **trigonometric formulas** allow us to transform and simplify expressions
- ▶ An application: **oscillations**, described with sine and cosine (elementary, phase-shifted, damped and amplified vibrations)