

SAPIENZA
UNIVERSITÀ DI ROMA

Mathematical Analysis 1

Volume 7 — Integrals

The seventh volume: integral calculus. The Riemann integral with lower and upper sums, the classes of integrable functions, the properties of the integral and the mean value theorem; antiderivatives and the fundamental theorem of calculus; integration by substitution, by parts and of rational functions, areas and mean values; improper integrals, convergence tests and the integral test for series. Definitions, theorems with proofs, figures and worked examples; the exercises of the part, with solutions, are in the volume Exercises 7.

Fabio Furini
fabiofurini.github.io
Department of Computer, Control and Management Engineering — Sapienza University of Rome

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fabiofurini.github.io/mathematical-analysis-1/
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Chapter 1

The Riemann integral

Prerequisites

- From this module, «Ordered fields, supremum/infimum and the completeness axiom»: supremum and infimum of a set and their existence in $\mathbb{R}$.
- From this module, «Summations, geometric and arithmetic progressions»: the summation symbol and the sum of the first n natural numbers.
- From this module, «Continuous functions» and «Weierstrass theorem and intermediate value theorem»: continuity, maximum and minimum on a closed and bounded interval, intermediate values.

Objectives

At the end of this chapter you will be able to:

- compute the lower and upper sums of a bounded function with respect to a partition;
- state the definition of the Riemann integral and interpret it as a signed area;
- prove that monotone functions and continuous functions on $[a, b]$ are integrable;
- use linearity, additivity, monotonicity and the inequality $\left|\int_a^b f\right| \leq \int_a^b |f|$;
- state and prove the integral mean value theorem.

1.1 Partitions and Riemann sums

Definition 1.1: partition of an interval

Given an interval $[a, b]$, with $a < b$, a ***partition*** *of $[a, b]$ is a finite set of points*

$$P = \{x_0, x_1, \ldots, x_n\} \quad \text{with} \quad a = x_0 < x_1 < \cdots < x_n = b$$

The points x_k divide $[a, b]$ into the n intervals $I_k = [x_{k-1}, x_k]$, with $k \in \{1, 2, \ldots, n\}$, of length $\Delta x_k = x_k - x_{k-1}$.

- The partition is called **uniform** when all the intervals have the same length $\Delta x_k = \frac{b-a}{n}$, that is, when $x_k = a + k\,\frac{b-a}{n}$.
- If $P \subseteq P'$, the partition P' is called **finer** than P (or a **refinement** of P): it is obtained from P by adding points.

Definition 1.2: lower sum and upper sum

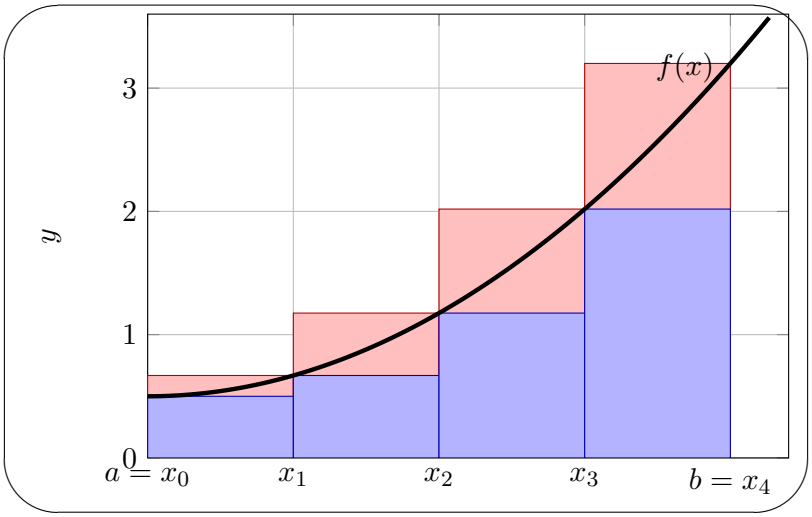
Let $f : [a,b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. Setting, for every $k \in \{1, 2, \dots, n\}$,

$$m_k = \inf_{x \in I_k} f(x) \quad \text{and} \quad M_k = \sup_{x \in I_k} f(x)$$

*the **lower sum** and the **upper sum** of f with respect to P are the numbers*

$$s(f,P) = \sum_{k=1}^{n} m_k\,\Delta x_k \quad \text{and} \quad S(f,P) = \sum_{k=1}^{n} M_k\,\Delta x_k$$

- The bounds m_k and M_k are real numbers because f is bounded. Since $m_k \le M_k$ for every k, we always have $s(f,P) \le S(f,P)$.
- If $f \ge 0$, the lower sum is the area of the rectangles **inscribed** in the region under the graph, the upper sum that of the **circumscribed** rectangles.



- In the figure, with f increasing, the blue rectangles form the lower sum and the blue rectangles together with the red ones the upper sum: the area under the graph lies in between.

Example 1.1: lower and upper sums of $f(x) = x$

Let $f(x) = x$ on $[0, 1]$ and let P_n be the uniform partition with $x_k = \frac{k}{n}$. Since f is increasing, on every interval I_k the minimum is at the left endpoint and the maximum at the right endpoint:

$$m_k = \frac{k-1}{n}, \qquad M_k = \frac{k}{n}, \qquad \Delta x_k = \frac{1}{n}$$

With the sum of the first n natural numbers, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, we get

$$S(f, P_n) = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{n+1}{2n}, \qquad s(f, P_n) = \frac{1}{n^2} \sum_{k=1}^{n} (k-1) = \frac{n-1}{2n}$$

The difference $S(f, P_n) - s(f, P_n) = \frac{1}{n}$ tends to 0 and both sums tend to $\frac{1}{2}$.

Proposition 1.3: sums and refinements

Let $f : [a, b] \to \mathbb{R}$ be bounded. If P' is finer than P, then

$$s(f, P) \le s(f, P') \le S(f, P') \le S(f, P)$$

Proof. It is enough to consider the case in which P' has one more point, $y \in (x_{k-1}, x_k)$: the general case follows by adding one point at a time. The two sums differ only on the interval I_k, which in P' is split into $[x_{k-1}, y]$ and $[y, x_k]$.

The infimum over a subset is greater than or equal to the infimum over the whole set, hence

$$m' = \inf_{[x_{k-1}, y]} f \ge m_k \quad \text{and} \quad m'' = \inf_{[y, x_k]} f \ge m_k$$

Consequently

$$m' (y - x_{k-1}) + m'' (x_k - y) \ge m_k (y - x_{k-1}) + m_k (x_k - y) = m_k \Delta x_k$$

that is, $s(f, P') \ge s(f, P)$. For the upper sums we argue in the same way with the suprema, which over a subset are smaller than or equal: $S(f, P') \le S(f, P)$. □

Corollary 1.4

Let $f : [a, b] \to \mathbb{R}$ be bounded. For every pair of partitions P_1 and P_2 of $[a, b]$ we have

$$s(f, P_1) \le S(f, P_2)$$

Proof. The partition $P = P_1 \cup P_2$ is finer than both P_1 and P_2. By the Proposition on sums and refinements

$$s(f, P_1) \le s(f, P) \le S(f, P) \le S(f, P_2)$$

□

- Every upper sum is therefore an upper bound of the set of lower sums. By the completeness axiom the following exist and are finite:

$$\sup_P s(f,P) \quad \leq \quad \inf_P S(f,P)$$

where supremum and infimum are taken over all partitions P of $[a,b]$.

1.2 The Riemann integral

Definition 1.5: Riemann integrable function

*A bounded function $f : [a,b] \to \mathbb{R}$ is called **Riemann integrable** on $[a,b]$ if*

$$\sup_P s(f,P) = \inf_P S(f,P)$$

*The common value is called the **integral** of f on $[a,b]$ and is denoted by*

$$\int_a^b f(x)\,dx$$

- The function f is called the **integrand**, the numbers a and b are the **limits of integration**.
- The variable x is a **dummy variable**: $\int_a^b f(x)\,dx$, $\int_a^b f(t)\,dt$ and $\int_a^b f(u)\,du$ denote the same number.
- The symbol $\int$ is an elongated S (sum) and dx recalls the widths Δx_k of the rectangles.

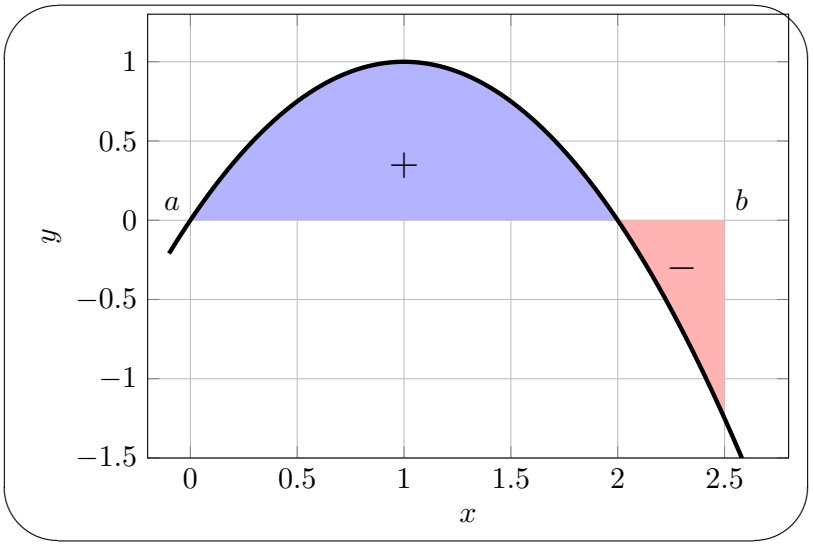
1.2.1 Geometric interpretation

- If $f \geq 0$ on $[a,b]$, the integral is the **area of the region under the graph** (the subgraph)

$$T_f = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b,\ 0 \leq y \leq f(x)\}$$

which lies between the inscribed rectangles (lower sums) and the circumscribed ones (upper sums).

- If f changes sign, the parts below the x-axis count with a minus sign: the integral is a **signed area**, that is, the area above the axis minus the area below the axis.



- In the figure the integral of f on $[a,b]$ is the blue area minus the red area.

Example 1.2: integral of a constant

Let $f(x)=c$ on $[a,b]$, with $c\in\mathbb{R}$. For every partition P we have $m_k=M_k=c$, hence

$$s(f,P)=S(f,P)=\sum_{k=1}^{n} c\,\Delta x_k = c\,(b-a)$$

The constant function is integrable and

$$\int_a^b c\,dx = c\,(b-a)$$

If $c>0$ this is the area of the rectangle with base $b-a$ and height c.

Example 1.3: integral of $f(x)=x$

Let us go back to $f(x)=x$ on $[0,1]$ and to the uniform partitions P_n of the Example on the lower and upper sums of $f(x)=x$. For every n

$$\frac{n-1}{2n} = s(f,P_n) \le \sup_P s(f,P) \le \inf_P S(f,P) \le S(f,P_n) = \frac{n+1}{2n}$$

Letting $n\to+\infty$ both ends tend to $\frac{1}{2}$, hence

$$\sup_P s(f,P) = \inf_P S(f,P) = \frac{1}{2} \quad \text{that is} \quad \int_0^1 x\,dx = \frac{1}{2}$$

This is the area of the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

Example 1.4: integral of $f(x) = x^2$

Let $f(x) = x^2$ on $[0,1]$ and P_n the uniform partition. The function is increasing on $[0,1]$, so $m_k = \left(\frac{k-1}{n}\right)^2$ and $M_k = \left(\frac{k}{n}\right)^2$. Using the formula $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ (it is proved by induction) we get

$$S(f,P_n) = \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{(n+1)(2n+1)}{6n^2}, \qquad s(f,P_n) = \frac{1}{n^3}\sum_{k=1}^{n-1} k^2 = \frac{(n-1)(2n-1)}{6n^2}$$

Both tend to $\frac{2}{6} = \frac{1}{3}$ as $n \to +\infty$; arguing as in the previous example

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

Example 1.5: a non-integrable function

The **Dirichlet function** on $[0,1]$ is

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Every interval I_k contains both rational and irrational numbers (density of $\mathbb{Q}$ and of $\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$), so $m_k = 0$ and $M_k = 1$ for every k. For every partition

$$s(D,P) = 0 \quad \text{and} \quad S(D,P) = \sum_{k=1}^{n} \Delta x_k = 1$$

Hence $\sup_P s(D,P) = 0 \neq 1 = \inf_P S(D,P)$: the Dirichlet function is bounded but it is **not** Riemann integrable.

1.2.2 Integrability criterion

Theorem 1.6: Riemann's integrability criterion

A bounded function $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that

$$S(f,P) - s(f,P) < \varepsilon$$

Proof. Let $I_- = \sup_P s(f,P)$ and $I^+ = \inf_P S(f,P)$; we know that $I_- \leq I^+$.

- ($\Leftarrow$) For every $\varepsilon > 0$ there is P with $s(f,P) \leq I_- \leq I^+ \leq S(f,P)$ and $S(f,P) - s(f,P) < \varepsilon$. Hence $0 \leq I^+ - I_- < \varepsilon$ for every $\varepsilon > 0$, that is, $I^+ = I_-$.
- ($\Rightarrow$) Let $I = I_- = I^+$ and let $\varepsilon > 0$. By the properties of supremum and infimum

there exist P_1 and P_2 with

$$s(f, P_1) > I - \frac{\varepsilon}{2} \quad \text{and} \quad S(f, P_2) < I + \frac{\varepsilon}{2}$$

With $P = P_1 \cup P_2$, by the Proposition on sums and refinements, $S(f, P) - s(f, P) \leq S(f, P_2) - s(f, P_1) < \varepsilon$.

□

1.2.3 Riemann sums

- Choosing a point t_k in each interval I_k, the sum $\sigma(f, P) = \sum_{k=1}^{n} f(t_k)\,\Delta x_k$ is called a **Riemann sum**.
- Since $m_k \leq f(t_k) \leq M_k$, we have $s(f, P) \leq \sigma(f, P) \leq S(f, P)$: if f is integrable and $S(f, P) - s(f, P) \to 0$, the Riemann sums tend to the integral too, whatever the choice of the points t_k.
- For continuous functions with uniform partitions, t_k can be the left endpoint, the right endpoint or the midpoint of I_k: this is how an integral is approximated on a computer.

1.3 Classes of integrable functions

Theorem 1.7: integrability of monotone functions

If $f : [a, b] \to \mathbb{R}$ is monotone, then f is integrable on $[a, b]$.

Proof. Assume f non-decreasing (the non-increasing case is analogous). The function is bounded, because $f(a) \leq f(x) \leq f(b)$. Let P_n be the uniform partition, with $\Delta x_k = \frac{b-a}{n}$. On every interval I_k we have $m_k = f(x_{k-1})$ and $M_k = f(x_k)$, hence

$$S(f, P_n) - s(f, P_n) = \frac{b-a}{n} \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1})\big) = \frac{(b-a)\,\big(f(b) - f(a)\big)}{n}$$

because the sum telescopes. Given $\varepsilon > 0$, it is enough to take $n > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$ to get $S(f, P_n) - s(f, P_n) < \varepsilon$. By Riemann's integrability criterion, f is integrable. □

- A monotone function may have discontinuity points: for example the integer part $[x]$ is integrable on every interval $[a, b]$, even though it has a jump at every integer.

1.3.1 Uniform continuity

Definition 1.8: uniformly continuous function

A function $f : I \to \mathbb{R}$ is called ***uniformly continuous*** *on the interval I if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$x, y \in I, \quad |x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon$$

- In continuity at every point the number δ may depend on the point; in uniform continuity the same δ works for all the points of I.
- A function that is uniformly continuous on I is continuous at every point of I.

Example 1.6: uniform continuity

The function $f(x) = x^2$ is uniformly continuous on $[0, 1]$: for $x, y \in [0, 1]$

$$|x^2 - y^2| = |x + y|\,|x - y| \le 2\,|x - y|$$

and it is enough to choose $\delta = \frac{\varepsilon}{2}$.

The function $g(x) = \frac{1}{x}$ is continuous on $(0, 1]$ but not uniformly continuous: the points $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$ have distance $\frac{1}{n(n+1)} \to 0$, while $|g(x_n) - g(y_n)| = 1$ for every n.

Theorem 1.9: Heine–Cantor theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on the closed and bounded interval $[a, b]$, then f is uniformly continuous on $[a, b]$.

Proof. By contradiction: there exists $\varepsilon > 0$ such that for every $n \ge 1$ there are $x_n, y_n \in [a, b]$ with

$$|x_n - y_n| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(y_n)| \ge \varepsilon$$

The sequence $\{x_n\}$ is bounded. By the Bolzano–Weierstrass theorem (every bounded sequence has a convergent subsequence: it is proved with the same bisection used for the Weierstrass theorem) there exists a subsequence $x_{n_j} \to c$, with $c \in [a, b]$ because the interval is closed.

Since $|x_{n_j} - y_{n_j}| < \frac{1}{n_j}$, also $y_{n_j} \to c$. By the continuity of f at c

$$f(x_{n_j}) - f(y_{n_j}) \to f(c) - f(c) = 0$$

which contradicts $|f(x_{n_j}) - f(y_{n_j})| \ge \varepsilon$ for every j. □

1.3.2 Integrability of continuous functions

Theorem 1.10: integrability of continuous functions

If $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable on $[a, b]$.

Proof. By the Weierstrass theorem f is bounded. Let $\varepsilon > 0$. By the Heine–Cantor theorem there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{b - a}$$

Take a partition P with all intervals of length $\Delta x_k < \delta$. On every I_k, by the Weierstrass theorem, f has a minimum $m_k = f(\xi_k)$ and a maximum $M_k = f(\eta_k)$, with $|\xi_k - \eta_k| \leq \Delta x_k < \delta$. Hence $M_k - m_k < \frac{\varepsilon}{b-a}$ and

$$S(f, P) - s(f, P) = \sum_{k=1}^{n} (M_k - m_k)\, \Delta x_k < \frac{\varepsilon}{b - a} \sum_{k=1}^{n} \Delta x_k = \varepsilon$$

By Riemann's integrability criterion, f is integrable. □

Observation 1.11

One can prove that a function which is bounded on $[a, b]$ and continuous except at finitely many points is integrable as well. Moreover, changing an integrable function at finitely many points gives an integrable function with the same integral.

1.4 Properties of the integral

- It is convenient to extend the integral symbol to the case of equal or swapped limits:

$$\int_a^a f(x)\, dx = 0 \quad \text{and} \quad \int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$$

1.4.1 Linearity

Theorem 1.12: linearity of the integral

If f and g are integrable on $[a, b]$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is integrable on $[a, b]$ and

$$\int_a^b \big(\alpha f(x) + \beta g(x)\big)\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx$$

Proof. It is enough to treat separately the sum $f+g$ and the multiple αf.

- **Sum.** On every interval I_k we have $\inf(f+g) \geq \inf f + \inf g$ and $\sup(f+g) \leq \sup f + \sup g$, hence

$$s(f,P)+s(g,P) \leq s(f+g,P) \leq S(f+g,P) \leq S(f,P)+S(g,P)$$

  Given $\varepsilon > 0$, by Riemann's integrability criterion and the Proposition on sums and refinements there is a partition P (the union of those that work for f and for g) with $S-s < \frac{\varepsilon}{2}$ both for f and for g. Then $S(f+g,P)-s(f+g,P) < \varepsilon$ and $f+g$ is integrable. Moreover $\int(f+g)$ and $\int f + \int g$ both lie between $s(f,P)+s(g,P)$ and $S(f,P)+S(g,P)$: their distance is less than ε for every $\varepsilon > 0$, so they coincide.

- **Multiple.** If $\alpha \geq 0$, then $\inf(\alpha f) = \alpha \inf f$ and $\sup(\alpha f) = \alpha \sup f$, hence $s(\alpha f,P) = \alpha\, s(f,P)$ and $S(\alpha f,P) = \alpha\, S(f,P)$. If $\alpha < 0$, infimum and supremum are swapped: $s(\alpha f,P) = \alpha\, S(f,P)$ and $S(\alpha f,P) = \alpha\, s(f,P)$. In both cases $\int \alpha f = \alpha \int f$.

□

1.4.2 Additivity with respect to the interval

Theorem 1.13: additivity with respect to the interval

Let $a < c < b$. A function f is integrable on $[a,b]$ if and only if it is integrable on $[a,c]$ and on $[c,b]$, and in that case

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Proof. By the Proposition on sums and refinements we may use only partitions P of $[a,b]$ containing c (adding c does not increase the difference $S-s$). Such a P is the union of a partition P_1 of $[a,c]$ and a partition P_2 of $[c,b]$, and

$$s(f,P) = s(f,P_1) + s(f,P_2), \qquad S(f,P) = S(f,P_1) + S(f,P_2)$$

Hence $S(f,P)-s(f,P)$ is the sum of the two differences, both non-negative: it is smaller than ε if and only if both are (up to halving ε). By Riemann's integrability criterion the two integrability conditions are equivalent. Finally $\int_a^b f$ and $\int_a^c f + \int_c^b f$ both lie between $s(f,P)$ and $S(f,P)$, whose distance is less than ε: they coincide. □

- With the convention on swapped limits, the additivity formula holds for a, b, c in any order, provided that f is integrable on the largest interval.

Example 1.7: additivity

Let $f(x) = x$ for $x \in [0,1]$ and $f(x) = 1$ for $x \in [1,3]$. The function is continuous, hence integrable, and by additivity

$$\int_0^3 f(x)\,dx = \int_0^1 x\,dx + \int_1^3 1\,dx = \frac{1}{2} + 2 = \frac{5}{2}$$

where we used the Example on the integral of $f(x) = x$ and the one on the integral of a constant.

1.4.3 Monotonicity and absolute value

Theorem 1.14: monotonicity of the integral

If f and g are integrable on $[a,b]$ and $f(x) \le g(x)$ for every $x \in [a,b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Proof. On every interval I_k we have $\inf f \le \inf g$, hence $s(f,P) \le s(g,P) \le \int_a^b g$ for every partition P. Taking the supremum over P, $\int_a^b f = \sup_P s(f,P) \le \int_a^b g$. □

Corollary 1.15: estimates of the integral

Let f be integrable on $[a,b]$.

- *If $f(x) \ge 0$ for every $x \in [a,b]$, then $\int_a^b f(x)\,dx \ge 0$.*
- *If $m \le f(x) \le M$ for every $x \in [a,b]$, then*

$$m\,(b-a) \le \int_a^b f(x)\,dx \le M\,(b-a)$$

Proof. Apply the theorem on the monotonicity of the integral comparing f with the constants 0, m and M, whose integrals are given by the Example on the integral of a constant. □

Example 1.8: an estimate without computing the integral

On $[0,1]$ the function $f(x) = e^{-x^2}$ is decreasing, so $e^{-1} \le f(x) \le 1$. By the Corollary on the estimates of the integral

$$\frac{1}{e} \le \int_0^1 e^{-x^2}\,dx \le 1$$

even though e^{-x^2} has no elementary antiderivative.

Theorem 1.16: integral of the absolute value

If f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$ too and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

Proof.

- **Integrability of $|f|$.** For $x, y \in I_k$ we have $\big||f(x)|-|f(y)|\big| \leq |f(x)-f(y)| \leq M_k - m_k$. Taking the supremum in x and the infimum in y, the difference between the supremum and the infimum of $|f|$ on I_k is at most $M_k - m_k$. Hence $S(|f|,P) - s(|f|,P) \leq S(f,P) - s(f,P)$ and we conclude with Riemann's integrability criterion.
- **Inequality.** From $-|f(x)| \leq f(x) \leq |f(x)|$, by the linearity and the monotonicity of the integral,

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

which is the claim.

□

Example 1.9: the inequality can be strict

Let $f(x) = x$ on $[-1,1]$. The signed areas of the two triangles cancel out, while for $|x|$ they add up:

$$\int_{-1}^1 x\,dx = -\frac{1}{2} + \frac{1}{2} = 0, \qquad \int_{-1}^1 |x|\,dx = \frac{1}{2} + \frac{1}{2} = 1$$

Hence $\left|\int_{-1}^1 x\,dx\right| = 0 < 1 = \int_{-1}^1 |x|\,dx$.

1.5 Integral mean value theorem

Definition 1.17: integral mean

*Let f be integrable on $[a,b]$. The **integral mean** (or mean value) of f on $[a,b]$ is the number*

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

- If $f \geq 0$, the integral mean is the height of the rectangle with base $[a,b]$ having the same area as the region under the graph of f.

Theorem 1.18: integral mean value theorem

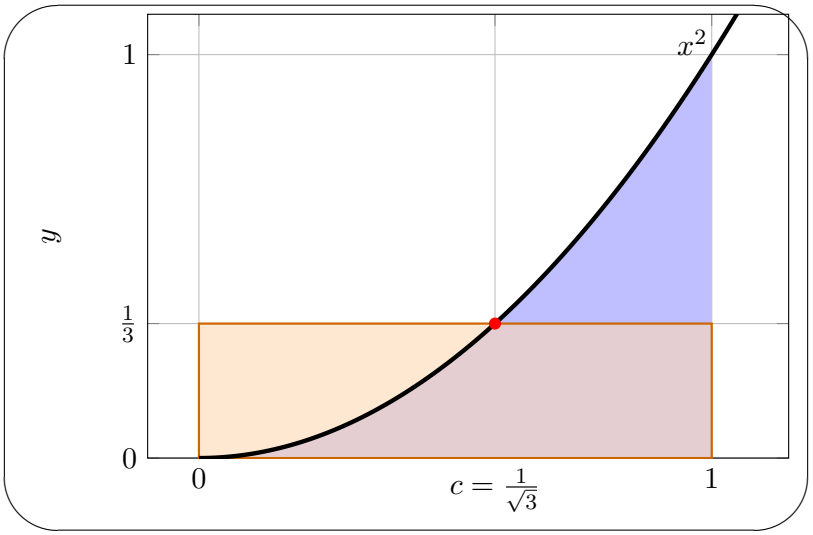
If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Proof. By the Weierstrass theorem f has a minimum m and a maximum M on $[a, b]$. By the Corollary on the estimates of the integral

$$m\,(b-a) \leq \int_a^b f(x)\, dx \leq M\,(b-a) \quad \Longrightarrow \quad m \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq M$$

The integral mean is therefore a value between the minimum and the maximum of f. By the intermediate value theorem f takes all values between m and M: there exists $c \in [a, b]$ at which $f(c)$ equals the integral mean. □



Example 1.10: integral mean of x^2

By the Example on the integral of $f(x) = x^2$, the integral mean of x^2 on $[0, 1]$ is

$$\frac{1}{1-0} \int_0^1 x^2\, dx = \frac{1}{3}$$

The point of the integral mean value theorem is found by solving $c^2 = \frac{1}{3}$ with $c \in [0, 1]$: $c = \frac{1}{\sqrt{3}}$. In the figure the orange rectangle of height $\frac{1}{3}$ has the same area as the blue region under the parabola.

Observation 1.19

The continuity assumption cannot be dropped. The function $f(x) = 0$ for $x \in [0, 1)$ and $f(x) = 1$ for $x \in [1, 2]$ is integrable (it is monotone) and, by additivity and the Remark on changes at finitely many points, its integral mean is $\frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}$, a value that f does not take.

1.6 In summary

In summary

- For a bounded function and a partition of $[a, b]$, the lower sum uses the infima and the upper sum the suprema; refining the partition the former increases and the latter decreases.
- f is integrable if the supremum of the lower sums equals the infimum of the upper sums; equivalently, if $S(f, P) - s(f, P)$ can be made arbitrarily small.
- If $f \geq 0$ the integral is the area under the graph; in general it is a signed area.
- Monotone functions and continuous functions on $[a, b]$ are integrable (for continuous ones we need the Heine–Cantor theorem); the Dirichlet function is not.
- The integral is linear, additive with respect to the interval and monotone, and $\left|\int_a^b f\right| \leq \int_a^b |f|$.
- A continuous function takes its integral mean at least at one point of $[a, b]$.

Chapter 2

Fundamental theorem of calculus

Prerequisites

- From this module, «The Riemann integral»: definition and properties of the integral, integral mean value theorem.
- From this module, «The derivative function», «Derivatives of elementary functions» and «Rules for computing derivatives»: derivatives and the chain rule.
- From this module, «Mean value theorem, maxima and minima»: a function with zero derivative on an interval is constant.
- From this module, «L'Hôpital's rule and differentiability»: L'Hôpital's rule.

Objectives

At the end of this chapter you will be able to:

- recognize an antiderivative and describe all the antiderivatives of a function on an interval;
- state and prove the two parts of the fundamental theorem of calculus;
- compute definite integrals with an antiderivative and the table of basic antiderivatives;
- differentiate integral functions with variable limits and use them in limits and in the study of functions.

2.1 Antiderivatives

Definition 2.1: antiderivative

Let $f : I \to \mathbb{R}$, with I an interval. A function $F : I \to \mathbb{R}$ is called an ***antiderivative*** *(or primitive) of f on I if F is differentiable on I and*

$$F'(x) = f(x), \quad \forall x \in I$$

- Finding an antiderivative is the inverse process of differentiation: we look for a function knowing its derivative.

- If I contains an endpoint, the derivative at that point is one-sided.

Example 2.1: antiderivatives

- $F(x) = \frac{x^3}{3}$ is an antiderivative of $f(x) = x^2$ on $\mathbb{R}$, because $F'(x) = x^2$; so is $\frac{x^3}{3} + 5$.
- $F(x) = \sin x$ is an antiderivative of $f(x) = \cos x$ on $\mathbb{R}$.
- $F(x) = \log x$ is an antiderivative of $f(x) = \frac{1}{x}$ on $(0, +\infty)$; $F(x) = \log(-x)$ is one on $(-\infty, 0)$.

Theorem 2.2: characterization of antiderivatives

Let F be an antiderivative of f on the interval I. Then G is an antiderivative of f on I if and only if there exists a constant $c \in \mathbb{R}$ such that

$$G(x) = F(x) + c, \quad \forall x \in I$$

Proof.

- ($\Leftarrow$) If $G = F + c$, then $G' = F' + 0 = f$ on I.
- ($\Rightarrow$) The function $H = G - F$ has derivative $H' = f - f = 0$ on the whole interval I. By the corollary of Lagrange's theorem, a function with zero derivative on an interval is constant: $H(x) = c$ for every $x \in I$.

□

Observation 2.3

The assumption that I is an interval is essential. On $\mathbb{R} \setminus \{0\}$ the functions

$$F(x) = \log|x| \quad \text{and} \quad G(x) = \begin{cases} \log|x| + 1 & \text{if } x > 0 \\ \log|x| & \text{if } x < 0 \end{cases}$$

both have derivative $\frac{1}{x}$, but $G - F$ is not constant: it equals 1 for $x > 0$ and 0 for $x < 0$.

Definition 2.4: indefinite integral

The set of all antiderivatives of f on an interval I is called the ***indefinite integral*** *of f and is denoted by $\int f(x)\,dx$. If F is an antiderivative, we write*

$$\int f(x)\,dx = F(x) + c, \quad c \in \mathbb{R}$$

- From the linearity of the derivative follows the **linearity of the indefinite integral**: $\int(\alpha f + \beta g)\,dx = \alpha \int f\,dx + \beta \int g\,dx$.

- The indefinite integral, which is a set of functions, must not be confused with the definite integral $\int_a^b f(x)\,dx$, which is a number. The fundamental theorem of calculus shows the link between the two.

2.2 The integral function

Definition 2.5: integral function

Let f be integrable on $[a,b]$ and let $x_0 \in [a,b]$. The function

$$F(x) = \int_{x_0}^{x} f(t)\,dt, \quad x \in [a,b]$$

*is called the **integral function** of f with initial point x_0.*

- The integration variable is called t so as not to confuse it with the upper limit x.
- For $x < x_0$ the convention $\int_{x_0}^{x} f = -\int_{x}^{x_0} f$ applies; moreover $F(x_0) = 0$.

Example 2.2: an integral function

Let $f(t) = t$ and $x_0 = 0$. For $x > 0$ the integral $\int_0^x t\,dt$ is the area of the triangle with base and height x, that is, $\frac{x^2}{2}$. For $x < 0$

$$\int_0^x t\,dt = -\int_x^0 t\,dt = -\left(-\frac{x^2}{2}\right) = \frac{x^2}{2}$$

because between x and 0 the graph lies below the axis. Hence $F(x) = \frac{x^2}{2}$ for every $x \in \mathbb{R}$: the integral function of $f(t) = t$ is an antiderivative of it.

Proposition 2.6: continuity of the integral function

If f is integrable on $[a,b]$ and $|f(t)| \le L$ for every $t \in [a,b]$, then the integral function F satisfies

$$|F(x) - F(y)| \le L\,|x - y|, \quad \forall x, y \in [a,b]$$

In particular F is continuous on $[a,b]$.

Proof. Let $y < x$. By additivity with respect to the interval, $F(x) - F(y) = \int_y^x f(t)\,dt$. By the inequality for the absolute value and the Corollary on the estimates of the integral

$$|F(x) - F(y)| \le \int_y^x |f(t)|\,dt \le L\,(x - y)$$

As $y \to x$ the right-hand side tends to 0, so F is continuous. □

2.3 Fundamental theorem of calculus

Theorem 2.7: fundamental theorem of calculus, first part

Let $f : [a, b] \to \mathbb{R}$ be continuous and let $x_0 \in [a, b]$. The integral function

$$F(x) = \int_{x_0}^{x} f(t)\, dt$$

is differentiable on $[a, b]$ and

$$F'(x) = f(x), \quad \forall x \in [a, b]$$

that is, F is an antiderivative of f on $[a, b]$.

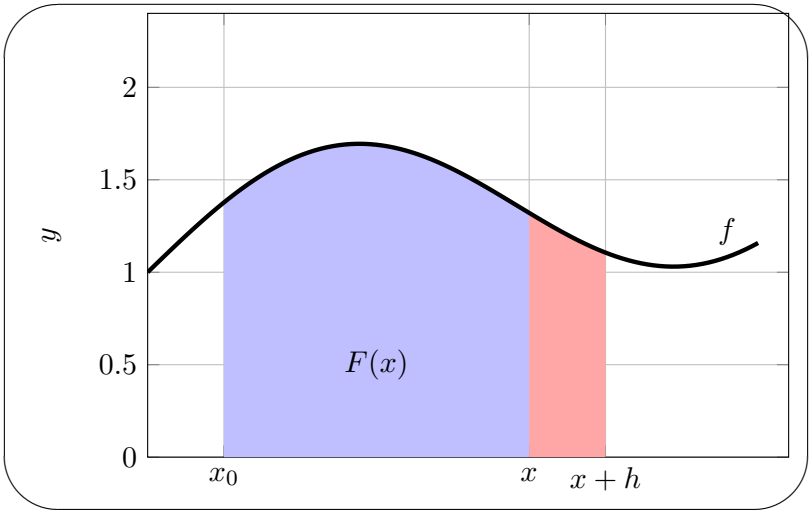
Proof. Let $x \in [a, b]$ and let $h \neq 0$ with $x + h \in [a, b]$. By additivity with respect to the interval

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_{x}^{x+h} f(t)\, dt$$

The right-hand side is the integral mean of f on the interval with endpoints x and $x + h$ (also for $h < 0$, swapping the limits). By the integral mean value theorem there exists c_h between x and $x + h$ such that

$$\frac{F(x+h) - F(x)}{h} = f(c_h)$$

As $h \to 0$ we have $c_h \to x$ and, by the continuity of f, $f(c_h) \to f(x)$. The limit of the difference quotient exists and equals $f(x)$. □



- In the figure the blue area is $F(x)$ and the red strip is $F(x + h) - F(x)$: for small h it is almost a rectangle with base h and height $f(x)$. Dividing by h we get about $f(x)$.

- Hence every continuous function on an interval has an antiderivative, even when it cannot be written with elementary functions (for example e^{-x^2}).

Theorem 2.8: fundamental theorem of calculus, second part

Let $f : [a, b] \to \mathbb{R}$ be continuous and let G be any antiderivative of f on $[a, b]$. Then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

Proof. By the fundamental theorem of calculus, first part, the function $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f on $[a, b]$. By the theorem on the characterization of antiderivatives there exists $c \in \mathbb{R}$ with $G = F + c$ on $[a, b]$. Then

$$G(b) - G(a) = \big(F(b) + c\big) - \big(F(a) + c\big) = \int_a^b f(t)\,dt - 0$$

□

- We write $G(b) - G(a) = \big[G(x)\big]_a^b$. The second part is also called the **fundamental formula of integral calculus**: to compute a definite integral an antiderivative is enough.

Example 2.3: definite integrals with an antiderivative

- $\displaystyle\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$, as we found with Riemann sums.
- $\displaystyle\int_0^\pi \sin x\,dx = \big[-\cos x\big]_0^\pi = -\cos\pi + \cos 0 = 2.$
- $\displaystyle\int_1^e \frac{1}{x}\,dx = \big[\log x\big]_1^e = 1 - 0 = 1.$
- $\displaystyle\int_0^{2\pi} \sin x\,dx = \big[-\cos x\big]_0^{2\pi} = 0$, while $\displaystyle\int_0^{2\pi} |\sin x|\,dx = 2\int_0^\pi \sin x\,dx = 4.$

2.4 Basic antiderivatives

- Reading the table of the derivatives of elementary functions backwards we get the **basic antiderivatives**. Each row is checked by differentiating the right-hand side.

$f(x)$	$\int f(x)\,dx$
$x^\alpha, \quad \alpha \neq -1$	$\dfrac{x^{\alpha+1}}{\alpha+1} + c$
$\dfrac{1}{x}$	$\log\lvert x\rvert + c$
e^x	$e^x + c$
$a^x, \quad a > 0,\ a \neq 1$	$\dfrac{a^x}{\log a} + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\dfrac{1}{\cos^2 x} = 1 + \tan^2 x$	$\tan x + c$
$\dfrac{1}{1+x^2}$	$\arctan x + c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$
$\operatorname{Sh} x$	$\operatorname{Ch} x + c$
$\operatorname{Ch} x$	$\operatorname{Sh} x + c$

- Each formula holds on an interval where the function is defined: for example $\int \frac{1}{x}\,dx = \log|x| + c$ on $(0, +\infty)$ or on $(-\infty, 0)$.
- By the chain rule, if g is differentiable, the table also holds in **composite** form: for example

$$\int g'(x)\, g(x)^\alpha\, dx = \frac{g(x)^{\alpha+1}}{\alpha+1} + c, \quad \int \frac{g'(x)}{g(x)}\, dx = \log|g(x)| + c, \quad \int g'(x)\, e^{g(x)}\, dx = e^{g(x)} + c$$

Example 2.4: basic antiderivatives and linearity

- $\displaystyle\int \left(3x^2 - \frac{4}{x} + 2e^x\right) dx = x^3 - 4\log|x| + 2e^x + c.$
- $\displaystyle\int \sqrt{x}\, dx = \int x^{1/2}\, dx = \frac{2}{3}\, x^{3/2} + c$, on $(0, +\infty)$.
- $\displaystyle\int \tan x\, dx = -\int \frac{-\sin x}{\cos x}\, dx = -\log|\cos x| + c$, with $g(x) = \cos x$.
- $\displaystyle\int \frac{x}{1+x^2}\, dx = \frac{1}{2}\int \frac{2x}{1+x^2}\, dx = \frac{1}{2}\log(1+x^2) + c.$
- $\displaystyle\int_0^1 \frac{1}{1+x^2}\, dx = \big[\arctan x\big]_0^1 = \frac{\pi}{4}.$

2.5 Derivative of an integral function

Theorem 2.9: differentiation of integral functions

Let f be continuous on the interval I and let α and β be differentiable functions on an interval J, with values in I. Then the function

$$G(x) = \int_{\alpha(x)}^{\beta(x)} f(t)\,dt, \quad x \in J$$

is differentiable on J and

$$G'(x) = f\big(\beta(x)\big)\,\beta'(x) - f\big(\alpha(x)\big)\,\alpha'(x)$$

Proof. Let F be an antiderivative of f on I, which exists by the fundamental theorem of calculus, first part. By the second part

$$G(x) = F\big(\beta(x)\big) - F\big(\alpha(x)\big)$$

The chain rule and $F' = f$ give the claim. □

Example 2.5: derivatives of integral functions

- If $G(x) = \displaystyle\int_1^{x^2} e^{t^2}\,dt$, then $G'(x) = e^{(x^2)^2} \cdot 2x = 2x\,e^{x^4}$.
- If $G(x) = \displaystyle\int_x^{3x} \frac{1}{1+t^2}\,dt$, then $G'(x) = \frac{3}{1+9x^2} - \frac{1}{1+x^2}$.

Example 2.6: a limit with an integral function

Let us compute

$$\lim_{x\to 0} \frac{1}{x^3} \int_0^x \sin(t^2)\,dt$$

The function $\sin(t^2)$ has no elementary antiderivative, but the numerator $F(x) = \int_0^x \sin(t^2)\,dt$ tends to $F(0) = 0$ and is differentiable with $F'(x) = \sin(x^2)$. We have a form $\frac{0}{0}$ and, by L'Hôpital's rule and the fundamental limit of the sine,

$$\lim_{x\to 0} \frac{\sin(x^2)}{3x^2} = \frac{1}{3} \quad \text{hence} \quad \lim_{x\to 0} \frac{1}{x^3} \int_0^x \sin(t^2)\,dt = \frac{1}{3}$$

Example 2.7: study of an integral function

Let $F(x) = \int_0^x e^{-t^2}\,dt$, $x \in \mathbb{R}$.

- **Monotonicity.** $F'(x) = e^{-x^2} > 0$: F is increasing on $\mathbb{R}$ and $F(0) = 0$, hence $F(x) > 0$ for $x > 0$ and $F(x) < 0$ for $x < 0$.
- **Concavity.** $F''(x) = -2x\,e^{-x^2}$: F is convex on $(-\infty, 0]$, concave on $[0, +\infty)$ and has an inflection point at $x = 0$, with tangent line $y = x$.
- **Symmetry.** The function $H(x) = F(x)+F(-x)$ has derivative $H'(x) = e^{-x^2} - e^{-x^2} = 0$, so it is constant and equals $H(0) = 0$: F is odd.

2.6 In summary

In summary

- An antiderivative of f on an interval is a function whose derivative is f; two antiderivatives on the same interval differ by a constant.
- The integral function $F(x) = \int_{x_0}^x f(t)\,dt$ is continuous; if f is continuous, F is differentiable and $F' = f$ (first part of the fundamental theorem).
- If G is an antiderivative of the continuous function f, then $\int_a^b f(x)\,dx = G(b) - G(a)$ (second part).
- Basic antiderivatives are obtained by reading the table of derivatives backwards, also in composite form with $g'(x)$.
- $\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(t)\,dt = f(\beta(x))\,\beta'(x) - f(\alpha(x))\,\alpha'(x)$: it is used in limits (with L'Hôpital's rule) and in the study of integral functions.

Chapter 3

Integration techniques and applications

Prerequisites

- From this module, «Fundamental theorem of calculus»: antiderivatives, fundamental formula of integral calculus, basic antiderivatives.
- From this module, «Rules for computing derivatives»: derivative of a product and of a composite function.
- From this module, «Limits of polynomials and rational functions»: polynomials, roots and rational functions.
- From this module, «The Riemann integral»: area under a graph and integral mean.

Objectives

At the end of this chapter you will be able to:

- prove and use the formulas of integration by substitution and by parts, for indefinite and definite integrals;
- integrate a rational function with polynomial division and partial fraction decomposition;
- compute the area of a region between two graphs;
- compute the mean value of a function on an interval and interpret it.

3.1 Integration by substitution

Theorem 3.1: integration by substitution

Let f be continuous on the interval I, let F be an antiderivative of it and let $g : J \to I$ be differentiable with continuous derivative on the interval J. Then

$$i) \qquad \int f\big(g(x)\big)\, g'(x)\, dx = F\big(g(x)\big) + c$$

ii) $$\int_a^b f\big(g(x)\big)\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt, \quad \forall a,b \in J$$

Proof.
- *i)* By the chain rule $\big(F(g(x))\big)' = F'\big(g(x)\big)\,g'(x) = f\big(g(x)\big)\,g'(x)$: the function $F \circ g$ is an antiderivative of the integrand.
- *ii)* The integrand on the left is continuous. By the fundamental formula of integral calculus and by *i)* both sides equal $F\big(g(b)\big) - F\big(g(a)\big)$.

□

- In practice we set $t = g(x)$ and formally replace $dt = g'(x)\,dx$. In definite integrals the limits change too: $x = a$ becomes $t = g(a)$, $x = b$ becomes $t = g(b)$.

Example 3.1: substitution

- $\displaystyle\int x\,e^{x^2}\,dx$: with $t = x^2$, $dt = 2x\,dx$, we get $\displaystyle\frac{1}{2}\int e^t\,dt = \frac{1}{2}\,e^{x^2} + c$.
- $\displaystyle\int \cos x\,\sin^3 x\,dx$: with $t = \sin x$, $dt = \cos x\,dx$, we get $\displaystyle\int t^3\,dt = \frac{\sin^4 x}{4} + c$.
- $\displaystyle\int_0^1 x\sqrt{1+x^2}\,dx$: with $t = 1 + x^2$, $dt = 2x\,dx$, the limits become $t = 1$ and $t = 2$:

$$\int_0^1 x\sqrt{1+x^2}\,dx = \frac{1}{2}\int_1^2 \sqrt{t}\,dt = \frac{1}{2}\left[\frac{2}{3}\,t^{3/2}\right]_1^2 = \frac{2\sqrt{2}-1}{3}$$

Observation 3.2: change of variable

Formula ii) is also used from right to left: to compute $\int_\alpha^\beta f(x)\,dx$ *we set* $x = \varphi(t)$*, with* φ *differentiable with continuous derivative and invertible, and*

$$\int_\alpha^\beta f(x)\,dx = \int_{\varphi^{-1}(\alpha)}^{\varphi^{-1}(\beta)} f\big(\varphi(t)\big)\,\varphi'(t)\,dt$$

For an indefinite integral, at the end we go back to the variable x *with* $t = \varphi^{-1}(x)$.

Example 3.2: change of variable

- $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$, for $x > 0$: with $x = t^2$, $t = \sqrt{x} > 0$, $dx = 2t\,dt$,

$$\int \frac{2t^2}{1+t^2}\,dt = 2\int\left(1 - \frac{1}{1+t^2}\right)dt = 2t - 2\arctan t + c = 2\sqrt{x} - 2\arctan\sqrt{x} + c$$

- $\int_0^1 \sqrt{1-x^2}\,dx$: with $x = \sin t$, $t \in \left[0, \frac{\pi}{2}\right]$, we have $dx = \cos t\,dt$ and $\sqrt{1-\sin^2 t} = \cos t$ (because $\cos t \geq 0$). Using $\cos^2 t = \frac{1+\cos 2t}{2}$,

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \cos^2 t\,dt = \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{\pi/2} = \frac{\pi}{4}$$

  This is the area of a quarter of the circle of radius 1.

3.2 Integration by parts

Theorem 3.3: integration by parts

Let f and g be differentiable with continuous derivative on the interval I. Then

$$i) \quad \int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

$$ii) \quad \int_a^b f(x)\,g'(x)\,dx = \left[f(x)\,g(x)\right]_a^b - \int_a^b f'(x)\,g(x)\,dx, \quad \forall a, b \in I$$

Proof. By the product rule $(f\,g)' = f'\,g + f\,g'$: the function $f\,g$ is an antiderivative of $f'\,g + f\,g'$. By the linearity of the indefinite integral

$$f(x)\,g(x) + c = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx$$

which is $i)$. For $ii)$ we integrate $(f\,g)' = f'\,g + f\,g'$ between a and b with the fundamental formula of integral calculus and linearity. □

- f is called the **finite factor** and g' the **differential factor**. It is convenient to choose as f the factor that becomes simpler when differentiated (a polynomial, a logarithm, an arctangent) and as g' one whose antiderivative is known.

Example 3.3: by parts

- $\int x\,e^x\,dx$: with $f = x$, $g' = e^x$, we get $x\,e^x - \int e^x\,dx = (x-1)\,e^x + c$.
- $\int \log x\,dx$, for $x > 0$: with $f = \log x$, $g' = 1$, we get $x\log x - \int x \cdot \frac{1}{x}\,dx = x\log x - x + c$.
- $\int \arctan x\,dx$: with $f = \arctan x$, $g' = 1$, we get $x\arctan x - \int \frac{x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}\log(1+x^2) + c$.

- $\int_0^\pi x \sin x \, dx$: with $f = x$, $g' = \sin x$,

$$\int_0^\pi x \sin x \, dx = \big[-x \cos x \big]_0^\pi + \int_0^\pi \cos x \, dx = \pi + \big[\sin x \big]_0^\pi = \pi$$

Example 3.4: an integral that comes back

Let $I = \int e^x \sin x \, dx$. Integrating by parts twice, with $g' = e^x$ both times,

$$I = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right) = e^x (\sin x - \cos x) - I$$

Solving the equation for I (up to the constant):

$$\int e^x \sin x \, dx = \frac{e^x \, (\sin x - \cos x)}{2} + c$$

3.3 Integration of rational functions

- We want to compute $\int \frac{P(x)}{Q(x)} \, dx$ with P and Q polynomials. If $\deg P \geq \deg Q$, we divide: $P = A\,Q + R$ with $\deg R < \deg Q$, and

$$\frac{P(x)}{Q(x)} = A(x) + \frac{R(x)}{Q(x)}$$

  The polynomial A is integrated at once; what remains is a **proper** rational function, with the degree of the numerator smaller than that of the denominator.

3.3.1 Denominator of first and second degree

- **First degree**: $\displaystyle\int \frac{1}{x - x_1} \, dx = \log |x - x_1| + c$.
- **Second degree**, $Q(x) = x^2 + px + q$ with discriminant $\Delta = p^2 - 4q$:
  - $\Delta > 0$: $Q(x) = (x - x_1)(x - x_2)$ and we look for A, B with $\dfrac{R(x)}{Q(x)} = \dfrac{A}{x - x_1} + \dfrac{B}{x - x_2}$;
  - $\Delta = 0$: $Q(x) = (x - x_1)^2$ and $\dfrac{R(x)}{Q(x)} = \dfrac{A}{x - x_1} + \dfrac{B}{(x - x_1)^2}$, with $\displaystyle\int \frac{dx}{(x - x_1)^2} = -\frac{1}{x - x_1} + c$;
  - $\Delta < 0$: we complete the square, $Q(x) = \left(x + \frac{p}{2}\right)^2 + k^2$ with $k^2 = -\frac{\Delta}{4}$, and write the numerator as a multiple of $Q'(x) = 2x + p$ plus a constant.
- In the case $\Delta < 0$ we need the two antiderivatives

$$\int \frac{2x + p}{x^2 + px + q} \, dx = \log(x^2 + px + q) + c, \qquad \int \frac{dx}{(x + p/2)^2 + k^2} = \frac{1}{k} \arctan \frac{x + p/2}{k} + c$$

Observation 3.4: partial fraction decomposition

One can prove that every real polynomial Q is the product of a constant, of first-degree factors $(x-a)$ and of second-degree factors x^2+px+q with $\Delta<0$, possibly repeated. A proper rational function $\frac{R}{Q}$ is the sum of ***partial fractions****: for every factor $(x-a)^m$ the terms $\frac{A_1}{x-a}+\cdots+\frac{A_m}{(x-a)^m}$, for every (non-repeated) factor x^2+px+q one term $\frac{Mx+N}{x^2+px+q}$. The constants are found by equating the numerators.*

Example 3.5: rational functions

- **Division.** $x^3+x+1 = x\,(x^2+1)+1$, hence $\displaystyle\int \frac{x^3+x+1}{x^2+1}\,dx = \int\left(x+\frac{1}{x^2+1}\right)dx = \frac{x^2}{2}+\arctan x+c$.
- $\Delta>0$. $\displaystyle\frac{1}{x^2-1} = \frac{A}{x-1}+\frac{B}{x+1}$ gives $1=A(x+1)+B(x-1)$: with $x=1$ and $x=-1$ we get $A=\frac12$, $B=-\frac12$, and
$$\int\frac{dx}{x^2-1} = \frac12\log\left|\frac{x-1}{x+1}\right|+c$$
- $\Delta=0$. $2x+3=2(x+1)+1$, hence $\displaystyle\int\frac{2x+3}{(x+1)^2}\,dx = \int\left(\frac{2}{x+1}+\frac{1}{(x+1)^2}\right)dx = 2\log|x+1|-\frac{1}{x+1}+c$.
- $\Delta<0$. $x^2+2x+5=(x+1)^2+4$ and $x+3=\frac12(2x+2)+2$, hence
$$\int\frac{x+3}{x^2+2x+5}\,dx = \frac12\log(x^2+2x+5)+\arctan\frac{x+1}{2}+c$$

Example 3.6: a repeated factor and a definite integral

- $\displaystyle\frac{1}{x(x+1)^2} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$ gives $1=A(x+1)^2+Bx(x+1)+Cx$. With $x=0$: $A=1$; with $x=-1$: $C=-1$; comparing the coefficients of x^2: $A+B=0$, that is, $B=-1$. Hence
$$\int\frac{dx}{x(x+1)^2} = \log|x|-\log|x+1|+\frac{1}{x+1}+c$$
- With the antiderivative of the previous example,
$$\int_2^3\frac{dx}{x^2-1} = \frac12\left[\log\frac{x-1}{x+1}\right]_2^3 = \frac12\left(\log\frac12-\log\frac13\right) = \frac12\log\frac32$$

3.4 Areas of plane regions

Proposition 3.5: area between two graphs

Let f and g be continuous on $[a,b]$ with $f(x) \le g(x)$ for every $x \in [a,b]$. The area of the region

$$D = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ f(x) \le y \le g(x)\}$$

is

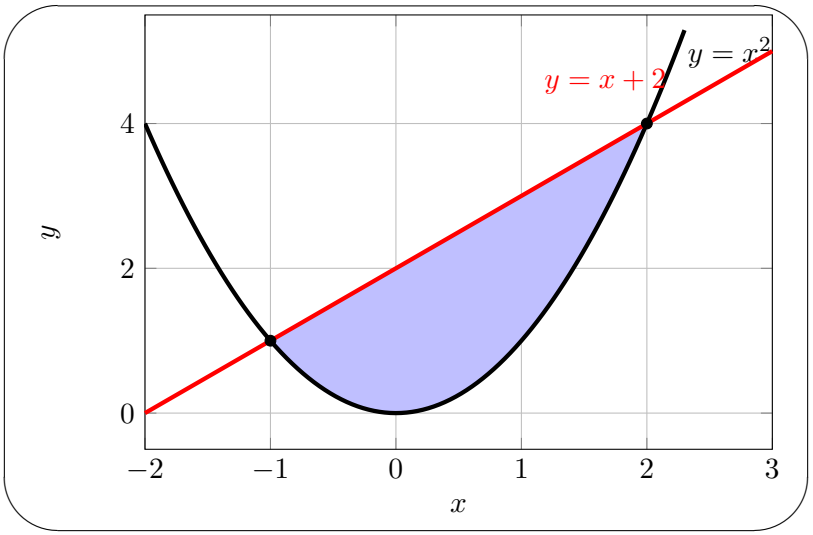
$$\text{Area}(D) = \int_a^b \big(g(x) - f(x)\big)\,dx$$

Proof. By the Weierstrass theorem f is bounded below: there is a constant c with $f+c \ge 0$ on $[a,b]$. Shifting up by c does not change the area, and the shifted region is the region under the graph of $g+c$ minus the region under the graph of $f+c$. By the geometric interpretation of the integral and by linearity

$$\text{Area}(D) = \int_a^b (g+c)\,dx - \int_a^b (f+c)\,dx = \int_a^b (g-f)\,dx$$

□

- The area of the region between the graph of f and the x-axis is $\int_a^b |f(x)|\,dx$: we split $[a,b]$ into the intervals where f has constant sign.
- If the two graphs cross, the area between them is $\int_a^b |g(x)-f(x)|\,dx$: first we look for the intersection points.



Example 3.7: area between a parabola and a line

Let us compute the area of the region between $y = x^2$ and $y = x + 2$ (blue in the figure). The intersections come from $x^2 = x + 2$: $x = -1$ and $x = 2$. On $[-1, 2]$ the line lies above the parabola, hence

$$\text{Area} = \int_{-1}^{2} (x + 2 - x^2)\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{10}{3} - \left(-\frac{7}{6}\right) = \frac{9}{2}$$

Example 3.8: graphs that cross

Let us compute the area between $y = \sin x$ and $y = \cos x$ for $x \in \left[0, \frac{\pi}{2}\right]$. The graphs cross at $x = \frac{\pi}{4}$: first the cosine lies above, then the sine. Hence

$$\text{Area} = \int_{0}^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$$

$$= \big[\sin x + \cos x\big]_{0}^{\pi/4} + \big[-\cos x - \sin x\big]_{\pi/4}^{\pi/2} = (\sqrt{2} - 1) + (\sqrt{2} - 1) = 2\sqrt{2} - 2$$

Example 3.9: area under the logarithm

On $[1, e]$ we have $\log x \geq 0$. With the antiderivative found by parts,

$$\text{Area} = \int_{1}^{e} \log x\, dx = \big[x \log x - x\big]_{1}^{e} = (e - e) - (0 - 1) = 1$$

3.5 Mean value of a function

- The mean value of f on $[a, b]$ is the integral mean $\frac{1}{b-a} \int_a^b f(x)\, dx$: now we can compute it with an antiderivative.
- It is the natural extension of the arithmetic mean $\frac{1}{n} \sum_{k=1}^{n} f(x_k)$: with the uniform partition, $\frac{1}{n} \sum_{k=1}^{n} f(x_k) = \frac{1}{b-a} \sum_{k=1}^{n} f(x_k)\, \frac{b-a}{n}$ is a Riemann sum divided by $b - a$.

Example 3.10: mean values

- The mean value of $\sin x$ on $[0, \pi]$ is $\dfrac{1}{\pi} \displaystyle\int_0^{\pi} \sin x\, dx = \frac{2}{\pi} \approx 0.64$.
- The mean value of x^2 on $[0, 3]$ is $\dfrac{1}{3} \left[\dfrac{x^3}{3}\right]_0^3 = 3$; as the integral mean value theorem predicts, it is taken at $c = \sqrt{3} \in [0, 3]$.
- If a point moves with velocity $v(t) = 3t^2$ (metres per second) for $t \in [0, 2]$, the distance covered is $\int_0^2 3t^2\, dt = 8$ metres and the mean velocity is $\frac{8}{2} = 4$ metres per second.

3.6 In summary

In summary

- **Substitution**: $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt$; with $t = g(x)$ the integrand, the differential and the limits change. Read backwards it is the change of variable $x = \varphi(t)$.
- **By parts**: $\int f\,g' = f\,g - \int f'\,g$; we differentiate the factor that becomes simpler. Sometimes the integral comes back and is found from an equation.
- **Rational functions**: division if needed, then partial fractions according to the factors of the denominator; second-degree factors with $\Delta < 0$ give a logarithm and an arctangent.
- The area between two graphs is $\int_a^b |g - f|$, after finding the intersections.
- The mean value $\frac{1}{b-a}\int_a^b f$ extends the arithmetic mean: it is the mean velocity if f is a velocity.

Chapter 4

Improper integrals

Prerequisites

- From this module, «Fundamental theorem of calculus» and «Integration techniques and applications»: computing definite integrals with antiderivatives.
- From this module, «Fundamental limits and asymptotic estimates» and «Monotone functions on an interval and invertibility»: asymptotic equivalences and limits of monotone functions.
- From this module, «Series with non-negative terms»: comparison tests and the generalized harmonic series.

Objectives

At the end of this chapter you will be able to:

- define the improper integral on an unbounded interval and of an unbounded function, and compute it with the limit;
- determine the behaviour of the standard integrals $\int_1^{+\infty} \frac{dx}{x^p}$ and $\int_0^1 \frac{dx}{x^p}$;
- state, prove and use the comparison test, the limit comparison test and absolute convergence;
- use the integral test to study a series with positive terms.

- The Riemann integral is defined for bounded functions on closed and bounded intervals. In this chapter we extend it, by means of a limit, to unbounded intervals and to unbounded functions.

4.1 Integrals on unbounded intervals

Definition 4.1: improper integral on an unbounded interval

Let f be continuous on $[a, +\infty)$. The ***improper integral*** *of f on $[a, +\infty)$ is the limit*

$$\int_a^{+\infty} f(x)\,dx = \lim_{\omega \to +\infty} \int_a^{\omega} f(x)\,dx$$

If the limit is finite the integral is called ***convergent****, if it is* $+\infty$ *or* $-\infty$ *it is called* ***divergent****; if the limit does not exist the integral* ***does not exist****.*

- In the same way $\int_{-\infty}^{b} f(x)\,dx = \lim_{\omega\to-\infty}\int_{\omega}^{b} f(x)\,dx$ for f continuous on $(-\infty, b]$.
- For f continuous on $\mathbb{R}$ we fix $c \in \mathbb{R}$ and set $\int_{-\infty}^{+\infty} f = \int_{-\infty}^{c} f + \int_{c}^{+\infty} f$: the integral converges if **both** integrals converge, and the value does not depend on c.
- If $\int_{a}^{+\infty} f$ converges we also say that f is **integrable in the improper sense** on $[a, +\infty)$.

Example 4.1: improper integrals on unbounded intervals

- $\displaystyle\int_0^{+\infty} e^{-x}\,dx = \lim_{\omega\to+\infty}\left[-e^{-x}\right]_0^{\omega} = \lim_{\omega\to+\infty}\left(1-e^{-\omega}\right) = 1$: it converges.
- $\displaystyle\int_0^{+\infty} \cos x\,dx$: $\int_0^{\omega}\cos x\,dx = \sin\omega$ has no limit as $\omega\to+\infty$, so the integral does not exist.
- $\displaystyle\int_{-\infty}^{+\infty}\frac{dx}{1+x^2} = \lim_{\omega\to-\infty}\left[\arctan x\right]_{\omega}^{0} + \lim_{\omega\to+\infty}\left[\arctan x\right]_0^{\omega} = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$

Theorem 4.2: standard improper integral at infinity

Let $p \in \mathbb{R}$*. The integral* $\displaystyle\int_1^{+\infty}\frac{dx}{x^p}$ *converges if and only if* $p > 1$*, and in that case*

$$\int_1^{+\infty}\frac{dx}{x^p} = \frac{1}{p-1}$$

For $p \le 1$ *it diverges to* $+\infty$*.*

Proof.

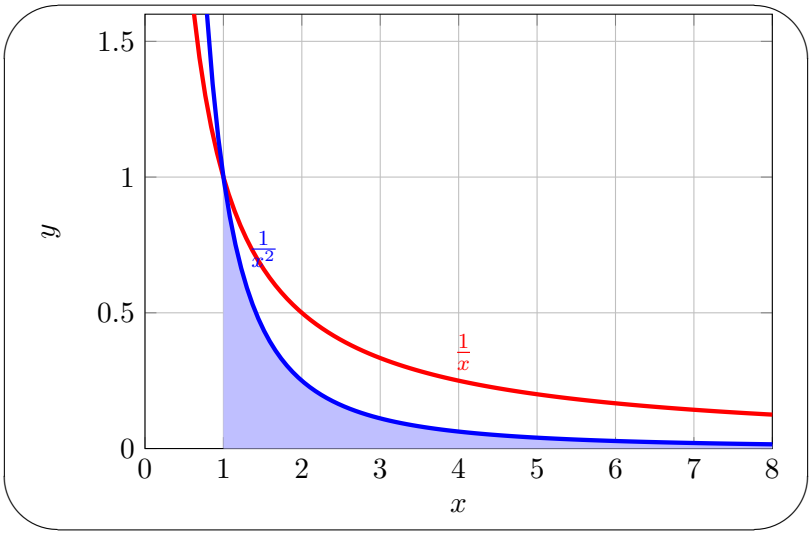
- For $p \neq 1$:

$$\int_1^{\omega}\frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^{\omega} = \frac{\omega^{1-p}-1}{1-p}$$

  If $p > 1$, the exponent $1-p$ is negative and $\omega^{1-p}\to 0$: the limit is $\frac{-1}{1-p} = \frac{1}{p-1}$. If $p < 1$, $\omega^{1-p}\to+\infty$ and the integral diverges.

- For $p = 1$: $\int_1^{\omega}\frac{dx}{x} = \log\omega \to +\infty$.

□



- In the figure the blue area under $\frac{1}{x^2}$ on $[1, +\infty)$ is finite and equals 1, while the area under $\frac{1}{x}$ is infinite: the two graphs look alike, but $\frac{1}{x^2}$ tends to zero fast enough.

4.2 Integrals of unbounded functions

Definition 4.3: improper integral of an unbounded function

*Let f be continuous on $(a, b]$. The **improper integral** of f on $(a, b]$ is the limit*

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f(x)\,dx$$

with the same terminology: convergent, divergent, does not exist.

- The interesting case is when f is unbounded near a (if f has a continuous extension at a, the limit is the Riemann integral of the extension).
- In the same way, for f continuous on $[a, b)$ we set $\int_a^b f = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f$. If there are several critical points (both endpoints, an interior point, an infinite endpoint) we split the interval: the integral converges if all the pieces converge.

Theorem 4.4: standard improper integral at zero

Let $p \in \mathbb{R}$. The integral $\displaystyle\int_0^1 \frac{dx}{x^p}$ converges if and only if $p < 1$, and in that case

$$\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}$$

For $p \geq 1$ it diverges to $+\infty$.

Proof.

- For $p \neq 1$:
$$\int_\varepsilon^1 \frac{dx}{x^p} = \frac{1-\varepsilon^{1-p}}{1-p}$$
If $p<1$, $\varepsilon^{1-p} \to 0$ as $\varepsilon \to 0^+$ and the limit is $\frac{1}{1-p}$. If $p>1$, $\varepsilon^{1-p} \to +\infty$ and the integral diverges.
- For $p=1$: $\int_\varepsilon^1 \frac{dx}{x} = -\log\varepsilon \to +\infty$.

□

- The two standard integrals behave in opposite ways: at infinity we need $p>1$, at zero we need $p<1$. With the change of variable $t=x-a$ we see that $\int_a^b \frac{dx}{(x-a)^p}$ converges if and only if $p<1$.

Example 4.2: integrals of unbounded functions

- $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \frac{1}{1-\frac{1}{2}} = 2$ (theorem on the standard improper integral at zero with $p=\frac{1}{2}$).
- $\displaystyle\int_0^1 \log x\,dx = \lim_{\varepsilon\to 0^+} \big[x\log x - x\big]_\varepsilon^1 = -1 - \lim_{\varepsilon\to 0^+}(\varepsilon\log\varepsilon - \varepsilon) = -1$, because $\varepsilon\log\varepsilon \to 0$.
- $\displaystyle\int_0^{1/2} \frac{dx}{x\log^2 x}$: with $t=\log x$ an antiderivative is $-\frac{1}{\log x}$, hence
$$\int_0^{1/2} \frac{dx}{x\log^2 x} = \lim_{\varepsilon\to 0^+}\left(-\frac{1}{\log\frac{1}{2}} + \frac{1}{\log\varepsilon}\right) = \frac{1}{\log 2}$$

Beware of points where the function is unbounded **inside** the interval. The computation $\int_{-1}^1 \frac{dx}{x^2} = \big[-\frac{1}{x}\big]_{-1}^1 = -2$ is wrong: the integrand is positive and it is not defined at 0. Splitting at 0, $\int_0^1 \frac{dx}{x^2}$ diverges (with $p=2$), hence $\int_{-1}^1 \frac{dx}{x^2} = +\infty$.

4.3 Convergence tests

- Often we do not need the value of an improper integral, only whether it converges, even when an antiderivative cannot be written. As for series, for non-negative functions there are comparison tests.

Observation 4.5: non-negative functions

*If f is continuous and non-negative on $[a,+\infty)$, the function $\omega \mapsto \int_a^\omega f(x)\,dx$ is non-decreasing. By the monotonicity theorem for functions its limit as $\omega\to+\infty$ exists: the integral **either converges or diverges to** $+\infty$, and it converges if and only if $\int_a^\omega f$ is bounded.*

Theorem 4.6: comparison test for improper integrals

Let f and g be continuous on $[a, +\infty)$ with $0 \le f(x) \le g(x)$ for every $x \ge a$. Then:

$$i) \quad \int_a^{+\infty} g(x)\,dx \ \text{ convergent} \quad \Rightarrow \quad \int_a^{+\infty} f(x)\,dx \ \text{ convergent}$$

$$ii) \quad \int_a^{+\infty} f(x)\,dx \ \text{ divergent} \quad \Rightarrow \quad \int_a^{+\infty} g(x)\,dx \ \text{ divergent}$$

Proof. By the monotonicity of the integral, for every $\omega \ge a$

$$\int_a^{\omega} f(x)\,dx \le \int_a^{\omega} g(x)\,dx \le \int_a^{+\infty} g(x)\,dx$$

where the last inequality holds because the integral function of $g \ge 0$ is non-decreasing. If $\int_a^{+\infty} g$ converges, $\int_a^{\omega} f$ is bounded and by the Remark on non-negative functions $\int_a^{+\infty} f$ converges: this is $i)$. Statement $ii)$ is the contrapositive of $i)$, because both integrals either converge or diverge to $+\infty$. □

Theorem 4.7: limit comparison test for improper integrals

Let f and g be continuous and positive on $[a, +\infty)$ and let

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \ell \quad \text{with} \quad 0 < \ell < +\infty$$

(in particular if $f(x) \sim g(x)$ as $x \to +\infty$). Then $\int_a^{+\infty} f(x)\,dx$ and $\int_a^{+\infty} g(x)\,dx$ are both convergent or both divergent.

Proof. By the definition of limit with $\varepsilon = \frac{\ell}{2}$, there exists $x_1 \ge a$ such that

$$\frac{\ell}{2} \le \frac{f(x)}{g(x)} \le \frac{3\ell}{2} \quad \Longrightarrow \quad \frac{\ell}{2}\,g(x) \le f(x) \le \frac{3\ell}{2}\,g(x), \quad \forall x \ge x_1$$

By the comparison test applied on $[x_1, +\infty)$: if $\int g$ converges, $\int f$ converges (second inequality); if $\int g$ diverges, $\int f$ diverges (first inequality). Finally $\int_a^{x_1} f$ and $\int_a^{x_1} g$ are integrals of continuous functions on a closed and bounded interval, hence finite: they do not change the behaviour. □

- If the limit is $\ell = 0$ only one implication holds: if $\int_a^{+\infty} g$ converges, then $\int_a^{+\infty} f$ converges too (because eventually $f \le g$).
- The same tests hold for unbounded functions on $(a, b]$, with the limits as $x \to a^+$; the typical comparison is with $\frac{1}{(x-a)^p}$.

Example 4.3: limit comparison at infinity

- $\displaystyle\int_1^{+\infty} \frac{x+1}{x^3+2}\,dx$: as $x \to +\infty$, $\frac{x+1}{x^3+2} \sim \frac{1}{x^2}$, and $\int_1^{+\infty} \frac{dx}{x^2}$ converges: the integral converges.
- $\displaystyle\int_1^{+\infty} \frac{dx}{\sqrt{x^2+x}}$: as $x \to +\infty$, $\frac{1}{\sqrt{x^2+x}} \sim \frac{1}{x}$: the integral diverges.
- $\displaystyle\int_0^{+\infty} e^{-x^2}\,dx$: for $x \geq 1$ we have $x^2 \geq x$ and hence $0 < e^{-x^2} \leq e^{-x}$; since $\int_1^{+\infty} e^{-x}\,dx$ converges, by comparison $\int_1^{+\infty} e^{-x^2}\,dx$ converges, and $\int_0^1 e^{-x^2}\,dx$ is finite: the integral converges.

Example 4.4: limit comparison at zero

- $\displaystyle\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$: as $x \to 0^+$, $\sin x \sim x$ and $\frac{\sin x}{x^{3/2}} \sim \frac{1}{x^{1/2}}$; since $\frac{1}{2} < 1$, the integral converges.
- $\displaystyle\int_0^1 \frac{dx}{e^x-1}$: as $x \to 0^+$, $e^x - 1 \sim x$ and $\frac{1}{e^x-1} \sim \frac{1}{x}$: the integral diverges.

Example 4.5: an integral with a parameter

For which $\alpha \in \mathbb{R}$ does $\displaystyle\int_0^{+\infty} \frac{x^\alpha}{1+x^2}\,dx$ converge? The integrand is continuous and positive on $(0,+\infty)$; we split at $x = 1$.

- On $(0,1]$: $\frac{x^\alpha}{1+x^2} \sim x^\alpha = \frac{1}{x^{-\alpha}}$ as $x \to 0^+$; it converges if and only if $-\alpha < 1$, that is, $\alpha > -1$.
- On $[1,+\infty)$: $\frac{x^\alpha}{1+x^2} \sim \frac{1}{x^{2-\alpha}}$ as $x \to +\infty$; it converges if and only if $2-\alpha > 1$, that is, $\alpha < 1$.

The integral converges if and only if $-1 < \alpha < 1$.

Theorem 4.8: absolute convergence for improper integrals

Let f be continuous on $[a,+\infty)$. If $\int_a^{+\infty} |f(x)|\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ converges too and

$$\left|\int_a^{+\infty} f(x)\,dx\right| \leq \int_a^{+\infty} |f(x)|\,dx$$

Proof. We have $0 \leq f(x) + |f(x)| \leq 2\,|f(x)|$. By the comparison test $\int_a^{+\infty} \big(f + |f|\big)$

converges. Then

$$\int_a^\omega f(x)\,dx = \int_a^\omega \big(f(x) + |f(x)|\big)\,dx - \int_a^\omega |f(x)|\,dx$$

converges as well, because both terms have a finite limit as $\omega \to +\infty$. The inequality follows by taking the limit in $\left|\int_a^\omega f\right| \le \int_a^\omega |f|$. □

Example 4.6: absolute convergence

The integrand of $\displaystyle\int_1^{+\infty} \frac{\sin x}{x^2}\,dx$ changes sign infinitely many times, so the comparison tests do not apply directly. But $\left|\frac{\sin x}{x^2}\right| \le \frac{1}{x^2}$ and $\int_1^{+\infty} \frac{dx}{x^2}$ converges: the integral converges absolutely, hence it converges.

4.4 Integral test for series

Theorem 4.9: integral test

Let $f : [1, +\infty) \to \mathbb{R}$ be continuous, non-negative and non-increasing. Then the series $\sum_{k=1}^\infty f(k)$ and the integral $\int_1^{+\infty} f(x)\,dx$ are both convergent or both divergent. Moreover, for every $n \ge 2$,

$$\sum_{k=2}^{n} f(k) \le \int_1^n f(x)\,dx \le \sum_{k=1}^{n-1} f(k)$$

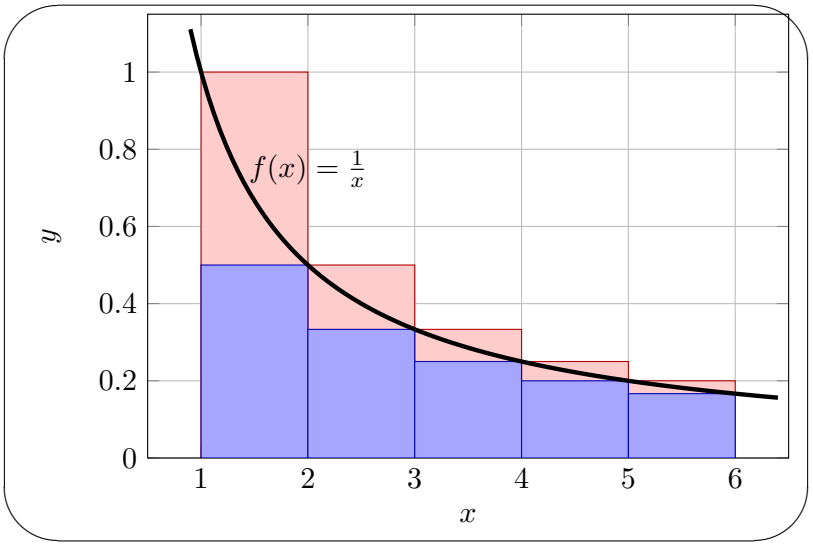
Proof. For $x \in [k, k+1]$, since f is non-increasing, $f(k+1) \le f(x) \le f(k)$. By the Corollary on the estimates of the integral

$$f(k+1) \le \int_k^{k+1} f(x)\,dx \le f(k)$$

Summing over $k \in \{1, 2, \dots, n-1\}$ and using additivity we get the inequalities of the statement.

- If the integral converges, the partial sums $\sum_{k=1}^n f(k) \le f(1) + \int_1^{+\infty} f$ are bounded: the series, with non-negative terms, converges.
- If the series converges, $\int_1^n f \le \sum_{k=1}^\infty f(k)$ for every n; since $\omega \mapsto \int_1^\omega f$ is non-decreasing, it is bounded and the integral converges by the Remark on non-negative functions.

□



- In the figure, with $f(x) = \frac{1}{x}$: the blue rectangles (heights $f(2), f(3), \dots$) lie below the graph, the red ones (heights $f(1), f(2), \dots$) above. The series and the integral differ at most by the first term.
- The test also applies starting from an integer k_0 other than 1: we compare $\sum_{k=k_0}^{\infty} f(k)$ and $\int_{k_0}^{+\infty} f$.

Example 4.7: series and integrals

- **Generalized harmonic series.** For $p > 0$ the function $\frac{1}{x^p}$ is positive and decreasing: by the theorem on the standard improper integral at infinity, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$ (for $p \le 0$ the general term does not tend to 0). We recover the behaviour of the generalized harmonic series.
- $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \log k}$ diverges: with $t = \log x$, $\int_2^{\omega} \frac{dx}{x \log x} = \log(\log \omega) - \log(\log 2) \to +\infty$.
- $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \log^2 k}$ converges: $\int_2^{\omega} \frac{dx}{x \log^2 x} = \frac{1}{\log 2} - \frac{1}{\log \omega} \to \frac{1}{\log 2}$.

Example 4.8: how fast the harmonic series grows

With $f(x) = \frac{1}{x}$ the inequalities of the integral test give, for $n \ge 2$,

$$\sum_{k=2}^{n} \frac{1}{k} \le \log n \quad \text{and} \quad \log(n+1) = \int_1^{n+1} \frac{dx}{x} \le \sum_{k=1}^{n} \frac{1}{k}$$

that is

$$\log(n+1) \le 1 + \frac{1}{2} + \cdots + \frac{1}{n} \le 1 + \log n$$

The partial sums of the harmonic series grow like $\log n$: to exceed 10 more than 8000 terms are needed, because $1 + \log n < 10$ as long as $n < e^9 \approx 8103$.

4.5 In summary

In summary

- An improper integral is a limit of Riemann integrals: $\int_a^{+\infty} f = \lim_{\omega\to+\infty} \int_a^{\omega} f$, $\int_a^b f = \lim_{\varepsilon\to 0^+} \int_{a+\varepsilon}^b f$ if f is unbounded near a. With several critical points we split the interval.
- $\int_1^{+\infty} \frac{dx}{x^p}$ converges if and only if $p > 1$; $\int_0^1 \frac{dx}{x^p}$ converges if and only if $p < 1$.
- For non-negative functions the integral converges or diverges to $+\infty$; the comparison test and the limit comparison test hold (with $f \sim g$ same behaviour).
- If $\int |f|$ converges, then $\int f$ converges too (absolute convergence).
- For f non-negative and non-increasing, $\sum f(k)$ and $\int_1^{+\infty} f$ have the same behaviour (integral test).