



SAPIENZA
UNIVERSITÀ DI ROMA

Mathematical Analysis 1

Volume 6 — Series

The sixth volume: series. Numerical series, series with non-negative terms and convergence tests, series with terms of variable sign, series of functions. Definitions, theorems with proofs and worked examples; the exercises on series and on antiderivatives and integrals, with solutions, are in the volume Exercises 6.

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Online version and downloadable material:
fabiofurini.github.io/mathematical-analysis-1/
github.com/fabiofurini/mathematical-analysis-1

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Numerical series

1.1 Numerical series

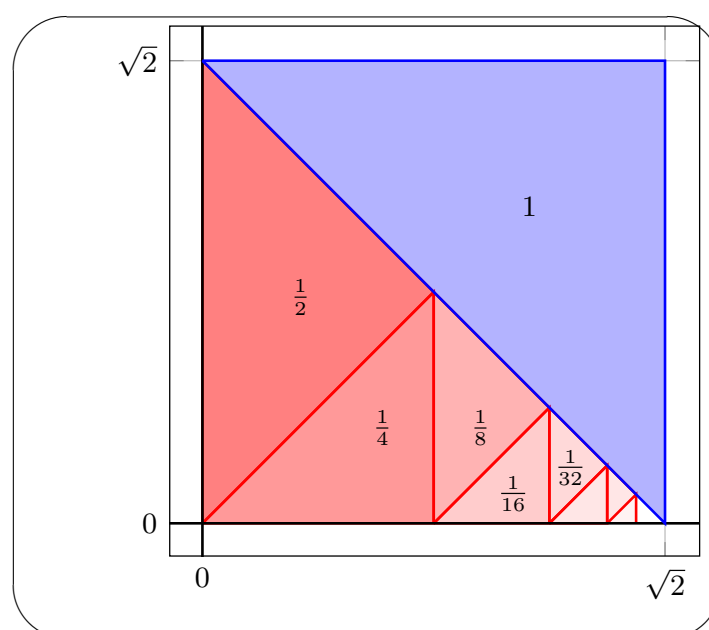
- We now introduce numerical series, which extend the operation of addition to an infinite number of terms.

The sum of infinitely many terms, even if they are all positive, can give a finite result.

- Imagine measuring a square of area 2 using the following procedure: we divide the square in half along the diagonal and measure the first right triangle: we obtain 1; then we divide the second right triangle in half and measure the first of the two resulting right triangles: we obtain $\frac{1}{2}$; the remaining right triangle is again divided in half... and so on indefinitely. We obtain the infinite sum:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots + \frac{1}{2^k} + \dots = \sum_{k=0}^{\infty} \frac{1}{2^k}$$

which, by the way it was constructed, must give 2 as its result.



Definition 1.1: of numerical series

Given a sequence $\{a_k\}_{k \in \mathbb{N}}$, we call **numerical series** of the terms a_k the expression:

$$\sum_{k=0}^{\infty} a_k$$

- It is read “series (or also sum) for k from 0 to $+\infty$ of a_k ”. The values a_k are called the **general terms of the series**.

Definition 1.2: of sequence of partial sums

The numbers

$$s_n = \sum_{k=0}^n a_k = a_0 + a_1 + \dots + a_n, \quad \forall n \in \mathbb{N}$$

are called the **n -th partial sums** of the series and they define **the sequence $\{s_n\}$ of partial sums**.

The **behavior (character) of the series** is determined by the limit of the sequence $\{s_n\}$ as n tends to infinity. We say that a series is **convergent**, **divergent**, **irregular**, if the **sequence $\{s_n\}$ of partial sums** is **convergent**, **divergent** or **irregular**, respectively.

Definition 1.3: of sum of the series

If the sequence $\{s_n\}$ of partial sums is **convergent**, that is, if

$$s_n \rightarrow s \in \mathbb{R} \quad \text{as} \quad n \rightarrow +\infty$$

we say that s is **the sum of the series**, and we write $\sum_{k=0}^{\infty} a_k = s$.

Hence, if the sequence $\{s_n\}$ is **convergent**, we have:

$$\sum_{k=0}^{\infty} a_k = \lim_{n \rightarrow +\infty} \sum_{k=0}^n a_k = \lim_{n \rightarrow +\infty} s_n = s$$

The series makes precise the idea of a sum of infinitely many terms, that is, we compute the limit as $n \rightarrow +\infty$ of the finite sum of the first n terms.

- If, instead of summing starting from 0, we start from an index $n_0 > 0$, we write $\sum_{k=n_0}^{\infty} a_k$
- To denote a generic numerical series without specifying the starting index n_0 we will use the symbol $\sum a_k$

A generic numerical series $\sum a_k$ always involves **two different sequences**:

1. the sequence $\{a_k\}$ of the **terms of the series**
2. the sequence $\{s_n\}$ of **partial sums**

Example 1.1: behavior of a series

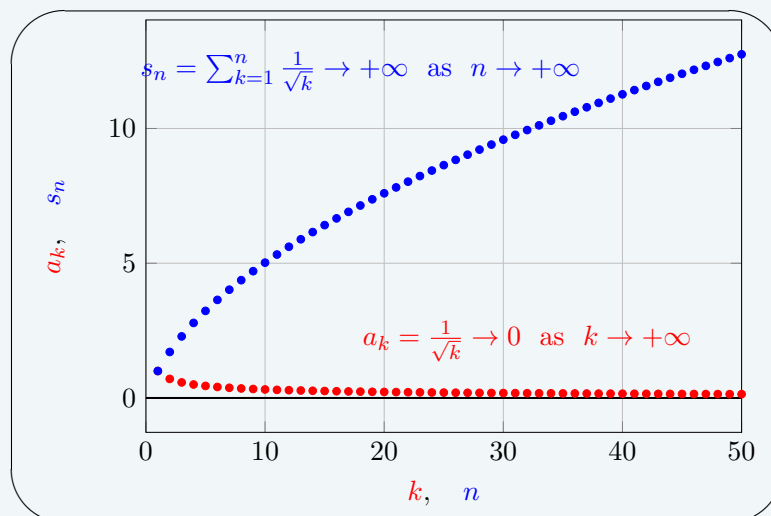
Let us determine the behavior of the series:

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

We have:

$$s_n = \sum_{k=1}^n \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} \geq \underbrace{n \cdot \frac{1}{\sqrt{n}}}_{n \text{ terms, each } \geq \frac{1}{\sqrt{n}}} = \sqrt{n} \rightarrow +\infty \quad \text{as } n \rightarrow +\infty$$

Hence the sequence of partial sums $\{s_n\}$ is divergent by the comparison theorem for sequences and, consequently, the series is divergent.



1.1.1 Main properties of numerical series

Observation 1.4

If a sequence $\{a_k\}$ has non-negative terms, that is, $a_k \geq 0, \forall k$, then the sequence of partial sums $\{s_n\}$ is increasing and regular.

Proof. Given a sequence $\{a_k\}$ with non-negative terms, the sequence of partial sums $\{s_n\}$ is increasing since:

$$s_{n+1} = s_n + \underbrace{a_{n+1}}_{\geq 0} \geq s_n, \quad \forall n, \quad \text{hence} \quad \lim_{n \rightarrow +\infty} s_n = \sup_{n \in \mathbb{N}} \{s_n\}$$

by the monotonicity theorem for sequences. Consequently the sequence $\{s_n\}$ cannot be irregular. It is therefore regular, that is, it either converges or diverges. $\square$

Given a sequence $\{a_k\}$ with non-negative terms:

- If $\{s_n\}$ is **increasing and bounded**, then it has a finite limit and $\sum a_k$ converges.
- If $\{s_n\}$ is **increasing and unbounded**, then $\sum a_k$ diverges to $+\infty$.
- This remark also holds for sequences with eventually non-negative terms, positive terms, or eventually positive terms.

Theorem 1.5

If a series $\sum a_k$ is convergent, then $\lim_{k \rightarrow +\infty} a_k = 0$

Proof. By definition of convergent series, the sequence of partial sums $\{s_n\}$ converges to a real number s , i.e., $s_n \rightarrow s \in \mathbb{R}$ as $n \rightarrow +\infty$. Without loss of generality we consider $n_0 = 0$.

We observe that the sequence $\{s_n\}$ can be defined recursively:

$$s_0 = a_0, \quad s_n = s_{n-1} + a_n, \quad \forall n \geq 1$$

Consequently we have:

$$a_n = s_n - s_{n-1} \quad \text{and hence} \quad \lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow +\infty} (s_n - s_{n-1}) = s - s = 0$$

$\square$

The convergence of a series therefore implies that $a_k \rightarrow 0$ as $k \rightarrow +\infty$, that is:

$$\sum a_k \text{ convergent} \quad \Rightarrow \quad \lim_{k \rightarrow +\infty} a_k = 0 \quad (1.1)$$

but the converse is not true (Example 1 is a counterexample):

$$\lim_{k \rightarrow +\infty} a_k = 0 \quad \nRightarrow \quad \sum a_k \text{ convergent} \quad (1.2)$$

Hence the fact that $a_k \rightarrow 0$ as $k \rightarrow +\infty$ is a necessary but not sufficient condition for the convergence of a series.

- From the contrapositive of (1.1) we have:

$$\lim_{k \rightarrow +\infty} a_k \neq 0 \Rightarrow \sum a_k \text{ is divergent or irregular}$$

that is, if the limit is not equal to zero, then the series is not convergent.

1.1.2 Tails of series

If we modify a finite number of terms of a series, the value of the sum may change, but the behavior of the series remains unchanged.

- If we modify the value of one term, for example a_{n_0} , for $n \geq n_0$ the sequences of partial sums, the original and the modified one, differ only by that term: hence both converge, both diverge, or both are irregular. The same holds if we modify a finite number of terms.

The results on the behavior of series therefore also hold if the hypotheses are satisfied “eventually”, i.e., from a certain index n_0 onward.

Definition 1.6: of tail of a series

Given a series $\sum_{k=0}^{\infty} a_k$ and a value $m \in \mathbb{N}$, the series $\sum_{k=m+1}^{\infty} a_k$ is called a **tail of the series**.

- Given a series $\sum_{k=0}^{\infty} a_k$ and $m \in \mathbb{N}$ we have:

$$\sum_{k=m+1}^n a_k = \left(\sum_{k=0}^n a_k \right) - \left(\sum_{k=0}^m a_k \right) = s_n - s_m, \quad \forall n > m \quad (1.3)$$

hence for every tail (that is, for every m) the associated sequence of partial sums is equal to that of the original series minus a constant. Consequently we have the following remark:

Every tail of a series has the same behavior as the original series.

- Given a convergent series, taking the limit as $n \rightarrow +\infty$ in (1.3) we obtain the relation:

$$\sum_{k=m+1}^{\infty} a_k = \left(\sum_{k=0}^{\infty} a_k \right) - \left(\sum_{k=0}^m a_k \right) = s - s_m \quad \text{as } n \rightarrow +\infty \quad (1.4)$$

Consequently we have the following remark:

Given $m \in \mathbb{N}$, every tail $\sum_{k=m+1}^{\infty} a_k$ of a convergent series can be interpreted as the **error** that we make when approximating the sum s with the partial sum s_m .

Theorem 1.7

If a series $\sum a_k$ is convergent, then

$$\underbrace{s - s_m}_{\sum_{k=m+1}^{\infty} a_k} \rightarrow 0 \quad \text{as } m \rightarrow +\infty$$

Proof. For every $n > m$, the n -th partial sum of the series is:

$$s_n = s_m + \sum_{k=m+1}^n a_k$$

Taking the limit as $n \rightarrow +\infty$, we have:

$$\begin{aligned} s &= s_m + \lim_{n \rightarrow +\infty} \sum_{k=m+1}^n a_k \\ &= \sum_{k=m+1}^{\infty} a_k \end{aligned}$$

Now taking the limit as $m \rightarrow +\infty$, we have:

$$s = s + \lim_{m \rightarrow +\infty} \sum_{k=m+1}^{\infty} a_k \Rightarrow \lim_{m \rightarrow +\infty} \sum_{k=m+1}^{\infty} a_k = 0$$

that is, the claim of the theorem. $\square$

This theorem tells us that, if a series is convergent, then the value $s - s_m$, that is, the error that we make when approximating the sum s with the partial sum s_m , tends to zero as $m \rightarrow +\infty$. In other words, the tail of a convergent series tends to zero as m tends to infinity.

1.1.3 Harmonic series

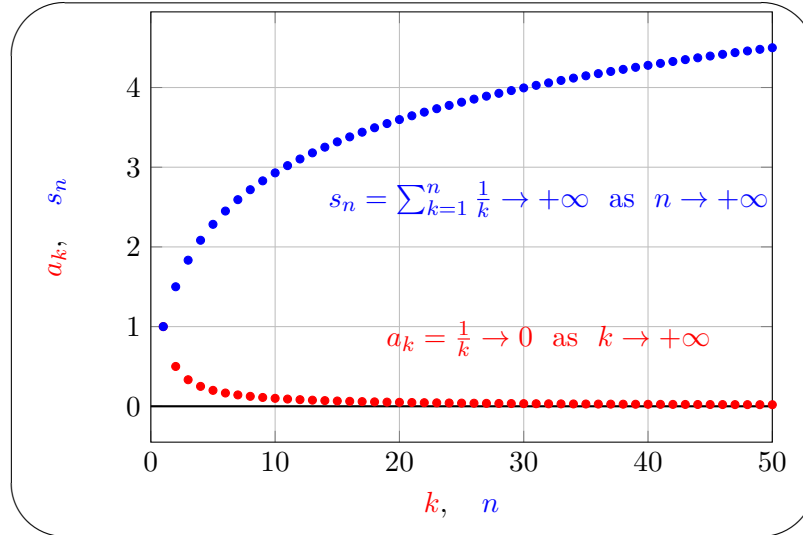
Definition 1.8: of harmonic series

The **harmonic series** is the series $\sum_{k=1}^{\infty} \frac{1}{k}$

Theorem 1.9: behavior of the harmonic series

The harmonic series diverges to $+\infty$

- Graphically we have:



The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ is a counterexample that proves (1.2), since:

$$\lim_{k \rightarrow +\infty} \frac{1}{k} = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges to } +\infty$$

- Before proving the theorem, we observe that for certain values of n the partial sums are:

$$\begin{aligned} \underbrace{s_1}_{=s_{2^0}} &= 1 + \frac{0}{2}, & \underbrace{s_2}_{=s_{2^1}} &= s_1 + \frac{1}{2} = 1 + \frac{0}{2} + \frac{1}{2} = 1 + \frac{1}{2}, \\ \underbrace{s_4}_{=s_{2^2}} &= s_2 + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}} \geq 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}, \\ \underbrace{s_8}_{=s_{2^3}} &= s_4 + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}} \geq 1 + \frac{2}{2} + \frac{1}{2} = 1 + \frac{3}{2}, \\ \underbrace{s_{16}}_{=s_{2^4}} &= s_8 + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right)}_{\geq \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2}} \geq 1 + \frac{3}{2} + \frac{1}{2} = 1 + \frac{4}{2} \end{aligned}$$

Proof. We prove by induction on 2^n that

$$s_{2^n} \geq 1 + \frac{n}{2}, \quad \forall n \in \mathbb{N}$$

Base case. Let $n = 0$. Then the statement becomes $s_{2^0} \geq 1 + \frac{0}{2}$, i.e., $1 \geq 1$, which is clearly true.

Inductive step. Assume that it is true for 2^{n-1} and let us prove it for 2^n . By the inductive hypothesis, we have:

$$s_{2^{n-1}} \geq 1 + \frac{n-1}{2}$$

Moreover, we have:

$$\begin{aligned} s_{2^n} &= s_{2^{n-1}} + \underbrace{\sum_{k=2^{n-1}+1}^{2^n} \left(\frac{1}{k}\right)}_{\geq 2^{n-1} \cdot \frac{1}{2^n} = \frac{1}{2}} \quad \text{hence} \quad s_{2^n} \geq 1 + \frac{n-1}{2} + \frac{1}{2} = 1 + \frac{n}{2} \end{aligned}$$

which is exactly the desired statement for 2^n .

We therefore have:

$$s_{2^n} \geq 1 + \frac{n}{2} \rightarrow +\infty \quad \text{as} \quad n \rightarrow +\infty$$

and $\{s_n\}$ is unbounded above and also monotonically increasing, since $a_k = 1/k > 0, \forall k > 1$. Consequently $\{s_n\}$ diverges to $+\infty$ by the monotonicity theorem for sequences, and the harmonic series diverges to $+\infty$. □

The sequence $a_k = 1/k, \forall k \geq 1$, of the harmonic series has positive terms, hence the sequence of partial sums $\{s_n\}$ is increasing, and it is divergent since it is unbounded.

1.1.4 Geometric series

Definition 1.10: of geometric series

Given $q \in \mathbb{R}$, the **geometric series** with **ratio** q is the series $\sum_{k=0}^{\infty} q^k$

Theorem 1.11: behavior and sum of the geometric series

Given the ratio $q \in \mathbb{R}$,

$$\text{the geometric series } \sum_{k=0}^{\infty} q^k \quad \text{is} \quad \begin{cases} \text{convergent} & \text{if } |q| < 1 \\ \text{divergent to } +\infty & \text{if } q \geq 1 \\ \text{irregular} & \text{if } q \leq -1 \end{cases}$$

If the geometric series is convergent, its sum s is $\frac{1}{1-q}$

Proof. Given $q \in \mathbb{R}$ and $n \in \mathbb{N}$, the n -th partial sum of the geometric series is:

$$\sum_{k=0}^n q^k = \begin{cases} \frac{1 - q^{n+1}}{1 - q} & \text{if } q \neq 1 \\ n + 1 & \text{if } q = 1 \end{cases}$$

since it equals the sum of the first $n + 1$ terms of the geometric progression. Moreover, we have:

$$\lim_{n \rightarrow +\infty} q^n = \begin{cases} +\infty & \text{if } q > 1 \\ 1 & \text{if } q = 1 \\ 0 & \text{if } |q| < 1 \\ \text{does not exist} & \text{if } q \leq -1 \end{cases}$$

Hence, if $q \neq 1$:

$$\lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q} \lim_{n \rightarrow +\infty} (1 - q^{n+1}) = \begin{cases} \frac{1}{1 - q} & \text{if } |q| < 1 \\ +\infty & \text{if } q > 1 \\ \text{does not exist} & \text{if } q \leq -1 \end{cases}$$

and if $q = 1$ we have:

$$\lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} (n + 1) = +\infty$$

□

- Given the ratio $q \in \mathbb{R}$ and a first term $a \in \mathbb{R}$, the result extends to the series $\sum_{k=0}^{\infty} a q^k$ of the terms of the geometric progression:

$$\text{the series } \sum_{k=0}^{\infty} a q^k \text{ is } \begin{cases} \text{convergent, with sum } 0 & \text{if } a = 0 \\ \text{convergent, with sum } \frac{a}{1-q} & \text{if } a \neq 0 \text{ and } |q| < 1 \\ \text{divergent to } +\infty & \text{if } a > 0 \text{ and } q \geq 1 \\ \text{divergent to } -\infty & \text{if } a < 0 \text{ and } q \geq 1 \\ \text{irregular} & \text{if } a \neq 0 \text{ and } q \leq -1 \end{cases} \quad (1.5)$$

Indeed, if $a = 0$ all the terms of the series are zero and therefore $s_n = 0$ for every n . If instead $a \neq 0$, by the product by a constant of summations the n -th partial sum is:

$$s_n = \sum_{k=0}^n a q^k = a \sum_{k=0}^n q^k$$

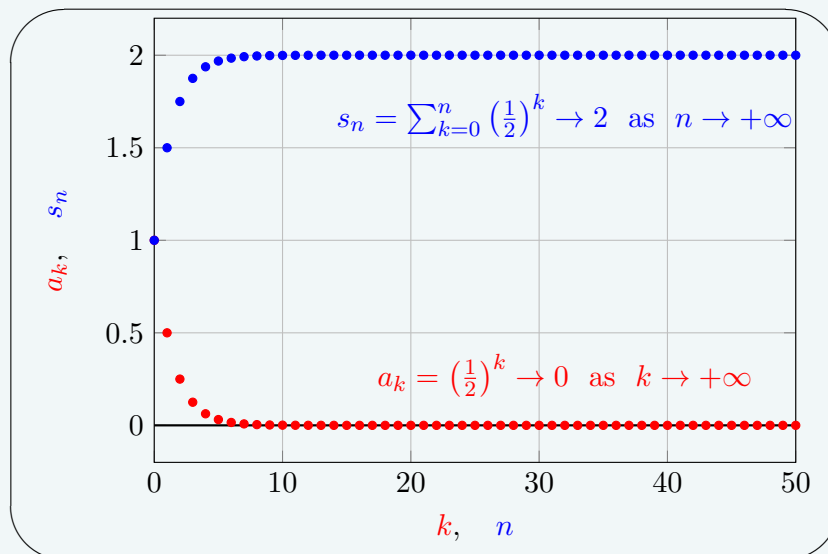
and therefore, by the theorem on the limit of the product of a constant and a sequence, $\{s_n\}$ has the same behavior as the sequence of the partial sums of $\sum_{k=0}^{\infty} q^k$, with the limit multiplied by a (in the divergent cases the sign of a decides between $+\infty$ and $-\infty$).

Example 1.2: geometric series

Let us determine the behavior of the series:

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

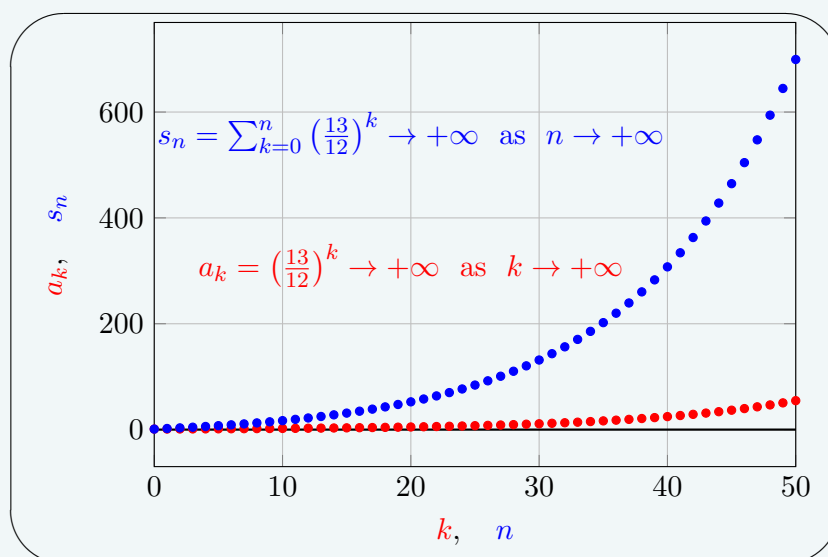
It is a geometric series with ratio $q = 1/2$, hence it is convergent and its sum s is $1/(1 - 1/2) = 2$.



Let us determine the behavior of the series:

$$\sum_{k=0}^{\infty} \left(\frac{13}{12}\right)^k$$

It is a geometric series with ratio $q = 13/12$, hence it is divergent.



1.1.5 Arithmetic series

Definition 1.12: of arithmetic series

Given the first term $a \in \mathbb{R}$ and the common difference $d \in \mathbb{R}$, the **arithmetic series** is the series $\sum_{k=0}^{\infty} (a + k d)$

- The terms of the arithmetic series are the terms of the arithmetic progression with first term a and common difference d , that is, $a, a + d, a + 2d, \dots$. As for the geometric series, the first term is in position $k = 0$.

Theorem 1.13: behavior and sum of the arithmetic series

Given the first term $a \in \mathbb{R}$ and the common difference $d \in \mathbb{R}$,

$$\text{the arithmetic series } \sum_{k=0}^{\infty} (a + k d) \quad \text{is} \quad \begin{cases} \text{divergent to } +\infty & \text{if } d > 0 \\ \text{divergent to } -\infty & \text{if } d < 0 \\ \text{divergent to } +\infty & \text{if } d = 0 \text{ and } a > 0 \\ \text{divergent to } -\infty & \text{if } d = 0 \text{ and } a < 0 \\ \text{convergent} & \text{if } d = 0 \text{ and } a = 0 \end{cases}$$

If the arithmetic series is convergent, its sum s is 0

Proof. Given $a, d \in \mathbb{R}$ and $n \in \mathbb{N}$, the n -th partial sum of the arithmetic series is:

$$s_n = \sum_{k=0}^n (a + k d) = a(n+1) + d \left(\frac{n(n+1)}{2} \right) = \frac{d}{2} n^2 + \left(a + \frac{d}{2} \right) n + a$$

since it equals the sum of the first $n+1$ terms of the arithmetic progression. Therefore, if $d \neq 0$, collecting the term of highest degree we have:

$$\lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} \frac{d}{2} n^2 \left(1 + \underbrace{\frac{2a+d}{dn}}_{\rightarrow 0} + \underbrace{\frac{2a}{dn^2}}_{\rightarrow 0} \right) = \begin{cases} +\infty & \text{if } d > 0 \\ -\infty & \text{if } d < 0 \end{cases}$$

and if $d = 0$ we have $s_n = a(n+1)$ and therefore:

$$\lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} a(n+1) = \begin{cases} +\infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0 \\ 0 & \text{if } a = 0 \end{cases}$$

□

The arithmetic series is therefore convergent only in the trivial case $a = d = 0$, in which all its terms are zero, and it is never irregular (unlike the geometric series, which is irregular for $q \leq -1$).

- The fact that the arithmetic series cannot converge, except in the trivial case, also follows from the necessary condition for convergence (1.1). The general term is $a_k = a + k d$ and we have:

$$\lim_{k \rightarrow +\infty} (a + k d) = \begin{cases} +\infty & \text{if } d > 0 \\ -\infty & \text{if } d < 0 \\ a & \text{if } d = 0 \end{cases}$$

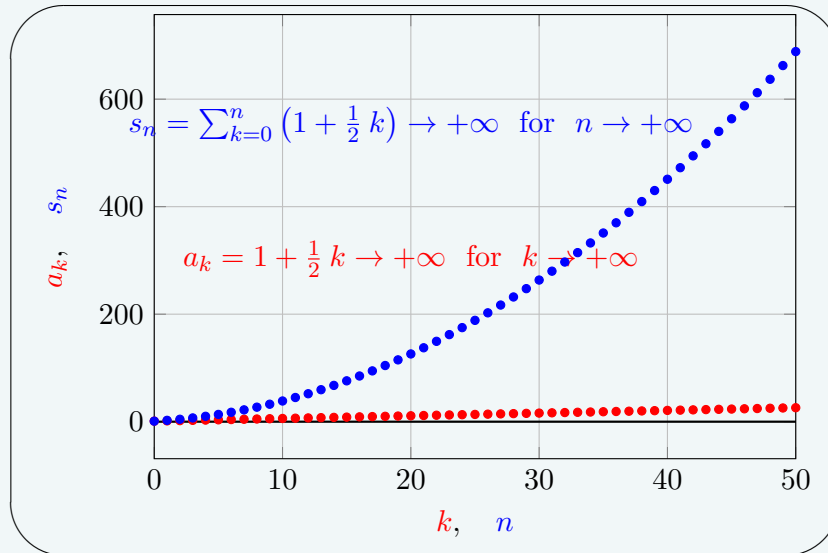
that is, $\lim_{k \rightarrow +\infty} a_k = 0$ only if $d = 0$ and $a = 0$. In all the other cases the contrapositive of (1.1) guarantees that the series is not convergent. The necessary condition does not say, however, *which* the behavior is: it is the computation of the partial sums carried out in the proof that establishes that the series is divergent, and with which sign.

Example 1.3: arithmetic series

Let us determine the behavior of the series:

$$\sum_{k=0}^{\infty} \left(1 + \frac{1}{2} k \right)$$

It is an arithmetic series with first term $a = 1$ and common difference $d = 1/2$. Since $d > 0$, the series is divergent to $+\infty$.



1.1.6 Telescoping series

Definition 1.14: of telescoping series

A **telescoping series** is a series of the form:

$$\sum_{k=n_0}^{\infty} (b_k - b_{k+1}) \quad \text{where } \{b_k\} \text{ is a sequence}$$

Theorem 1.15: behavior and sum of the telescoping series

A telescoping series converges, diverges or is irregular according to whether the sequence $\{b_k\}$ converges, diverges or is irregular, respectively.

$$\text{If } \lim_{k \rightarrow +\infty} b_k = \ell \in \mathbb{R}^* \quad \text{then} \quad \sum_{k=n_0}^{\infty} (b_k - b_{k+1}) = b_{n_0} - \underbrace{\lim_{k \rightarrow +\infty} b_k}_{=\ell \in \mathbb{R}^*}$$

Moreover, if $\ell \in \mathbb{R}$, then the telescoping series is convergent and its sum s is $b_{n_0} - \ell$

Proof. We have

$$\begin{aligned} s_n &= \sum_{k=n_0}^{n_0+n} (b_k - b_{k+1}) \\ &= (b_{n_0} - b_{n_0+1}) + (b_{n_0+1} - b_{n_0+2}) + \dots + (b_{n_0+n} - b_{n_0+n+1}) \\ &= b_{n_0} - b_{n_0+n+1} \end{aligned}$$

hence:

$$\sum_{k=n_0}^{\infty} (b_k - b_{k+1}) = \lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} (b_{n_0} - b_{n_0+n+1}) = b_{n_0} - \lim_{n \rightarrow +\infty} b_n$$

□

Observation 1.16

The series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$, called **Mengoli's series**, is convergent and its sum s is equal to 1.

Proof. We have:

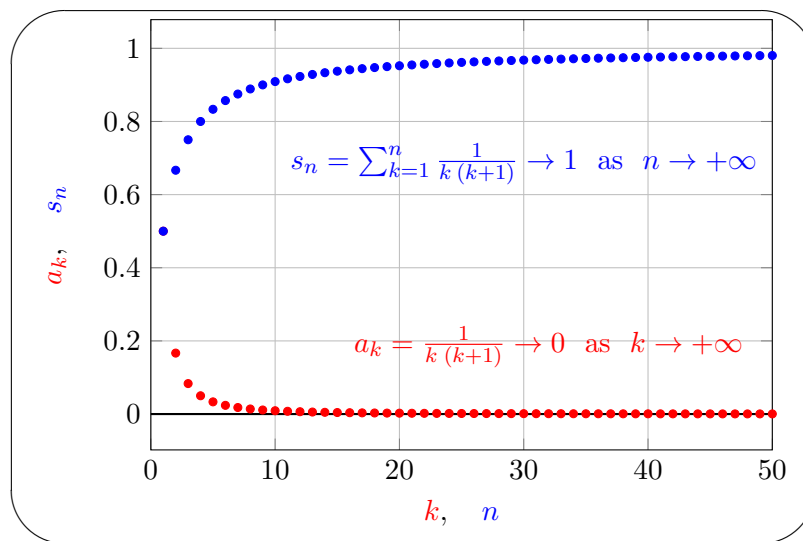
$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \frac{(k+1) - k}{k(k+1)} = \sum_{k=1}^{\infty} \underbrace{\frac{1}{k}}_{=b_k} - \underbrace{\frac{1}{k+1}}_{=b_{k+1}}$$

Mengoli's series therefore has the form:

$$b_k - b_{k+1} \quad \text{with} \quad b_k = \frac{1}{k} \quad \text{and moreover} \quad \lim_{k \rightarrow +\infty} b_k = \lim_{k \rightarrow +\infty} \frac{1}{k} = 0$$

hence it is a convergent telescoping series ($n_0 = 1$) and moreover $s = b_1 = 1$. $\square$

- Graphically we have:



Proof. Alternative proof

For every $n \geq 1$, the value of the partial sum of Mengoli's series is:

$$\begin{aligned} s_n &= \sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{n+1} \end{aligned}$$

Hence we have

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} 1 - \frac{1}{n+1} = 1$$

Consequently Mengoli's series is convergent and its sum s is 1. $\square$

Example 1.4: telescoping series

Let us determine the behavior of the series:

$$\sum_{k=1}^{\infty} \frac{1}{(3k+2)(3k+5)}$$

We have:

$$\begin{aligned} \frac{1}{(3k+2)(3k+5)} &= \frac{\alpha}{3k+2} - \frac{\beta}{3k+5} = \frac{\alpha(3k+5) - \beta(3k+2)}{(3k+2)(3k+5)} \\ &= \frac{3(\alpha - \beta)k + 5\alpha - 2\beta}{(3k+2)(3k+5)} \end{aligned}$$

$$\Rightarrow \begin{cases} 3(\alpha - \beta) = 0 \\ 5\alpha - 2\beta = 1 \end{cases} \Rightarrow \alpha = \beta, \quad 3\alpha = 1 \quad \text{hence} \quad \alpha = \beta = \frac{1}{3}$$

consequently:

$$\frac{1}{(3k+2)(3k+5)} = \frac{1}{3} \left(\frac{1}{3k+2} - \frac{1}{3k+5} \right) = \frac{1}{3} \left(\underbrace{\frac{1}{3k+2}}_{=b_k} - \underbrace{\frac{1}{3(k+1)+2}}_{=b_{k+1}} \right)$$

The series therefore has the form:

$$b_k - b_{k+1} \quad \text{with} \quad b_k = \frac{1}{3k+2} \quad \text{and moreover} \quad \lim_{k \rightarrow +\infty} b_k = \lim_{k \rightarrow +\infty} \frac{1}{3k+2} = 0$$

that is, a telescoping series ($n_0 = 1$), and we have:

$$\sum_{k=1}^{\infty} \frac{1}{(3k+2)(3k+5)} = \frac{1}{3} \left(b_1 - \underbrace{\lim_{k \rightarrow +\infty} b_k}_{\rightarrow 0} \right) = \frac{1}{3} b_1 = \frac{1}{15}$$

Hence it is a convergent telescoping series and its sum s is $\frac{1}{15}$.

Series with non-negative terms

2.1 Series with non-negative terms

A series $\sum a_k$ with **non-negative terms** (or positive terms) is **regular**, that is, it is either convergent or divergent to $+\infty$ (it cannot be irregular, as we have proved). Such a series converges if and only if the sequence of its partial sums is bounded.

2.1.1 Comparison test

Theorem 2.1: Comparison test

Let $\{a_k\}$ and $\{b_k\}$ be two sequences with non-negative terms such that $a_k \leq b_k$ eventually; then:

$$\begin{aligned} i) \quad \sum b_k \text{ convergent} &\Rightarrow \sum a_k \text{ convergent} \\ ii) \quad \sum a_k \text{ divergent} &\Rightarrow \sum b_k \text{ divergent} \end{aligned}$$

- The series $\sum b_k$ is called the **dominating series** (majorant), while the series $\sum a_k$ is called the **dominated series** (minorant).

Proof. Since $a_k \leq b_k$ eventually, there exists $m \in \mathbb{N}$ such that:

$$a_k \leq b_k, \quad \forall k \geq m$$

Let us now consider the n -th partial sums of the tails of the two sequences $\{a_k\}$ and $\{b_k\}$, with $n > m$:

$$s_n^a = \sum_{k=m+1}^n a_k \quad \text{and} \quad s_n^b = \sum_{k=m+1}^n b_k$$

Since $0 \leq a_k \leq b_k, \forall k \geq m$, we have

$$0 \leq s_n^a \leq s_n^b, \quad \forall n \geq m \quad (2.1)$$

The sequences $\{s_n^a\}$ and $\{s_n^b\}$ are regular, since $\{a_k\}$ and $\{b_k\}$ have non-negative terms. Hence the claims *i)* and *ii)* are logically equivalent, so it suffices to prove the second one.

Saying that $\sum a_n$ is divergent means, by the definition of divergent series, that $s_n \rightarrow +\infty$ as $n \rightarrow +\infty$. Moreover, since every tail has the same behavior as the original series, we have $s_n^a \rightarrow +\infty$ as $n \rightarrow +\infty$.

From (2.1), by the comparison theorem for sequences, also $s_n^b \rightarrow +\infty$ as $n \rightarrow +\infty$. Hence the tail of $\{b_k\}$ is divergent and consequently $\sum b_n$ is divergent. $\square$

Observation 2.2

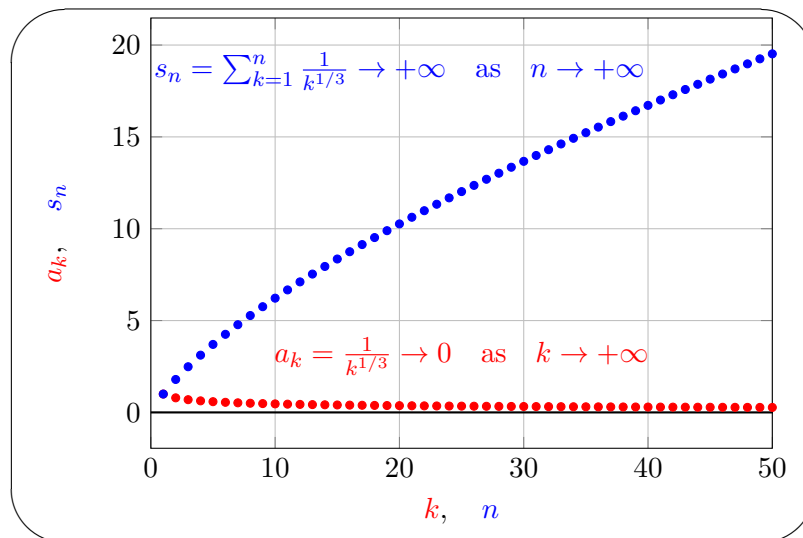
Given $\alpha \leq 1$, the series $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ is divergent to $+\infty$.

Proof. With $\alpha = 1$ we have the harmonic series, which diverges to $+\infty$. With $\alpha < 1$, the series dominates the harmonic series, since:

$$\frac{1}{k} \leq \frac{1}{k^\alpha}, \quad \forall k \in \mathbb{N}, k \geq 1 \quad \text{and} \quad \alpha < 1$$

hence $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ is divergent by the comparison test. $\square$

- Graphically, for example with $\alpha = \frac{1}{3}$, we have:



2.1.2 Limit comparison test

Theorem 2.3: Limit comparison test

Let $\{a_k\}$ and $\{b_k\}$ be two sequences with positive terms. If the sequences are asymptot-

ically equivalent, that is, if:

$$a_k \sim b_k \quad \text{as } k \rightarrow +\infty$$

then the corresponding series $\sum a_k$ and $\sum b_k$ are regular and have the same behavior, i.e., either they are both convergent or they are both divergent.

Proof. The series $\sum a_k$ and $\sum b_k$ are regular because the sequences $\{a_k\}$ and $\{b_k\}$ have positive terms. Since $a_k \sim b_k$ as $k \rightarrow +\infty$, we have:

$$\frac{a_k}{b_k} \rightarrow 1 \quad \text{as } k \rightarrow +\infty$$

Hence, for every $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that $\forall k \geq m$ we have:

$$1 - \varepsilon < \frac{a_k}{b_k} < 1 + \varepsilon \quad \text{and hence} \quad (1 - \varepsilon) b_k < a_k < (1 + \varepsilon) b_k \quad \text{since } b_k > 0, \forall k$$

We have thus proved that $(1 - \varepsilon) b_k < a_k < (1 + \varepsilon) b_k$ eventually. Hence, by the comparison test, the series $\sum a_k$ and $\sum b_k$ have the same behavior.

The first of the two inequalities implies that if $\sum a_k$ is convergent then $\sum b_k$ is also convergent, while the second one implies that if $\sum a_k$ is divergent then $\sum b_k$ is also divergent. □

Observation 2.4

Given $\alpha \geq 2$, the series $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ is convergent.

Proof. With $\alpha = 2$ we have:

$$\frac{1}{k^2} \sim \frac{1}{k(k+1)} \quad \text{as } k \rightarrow +\infty$$

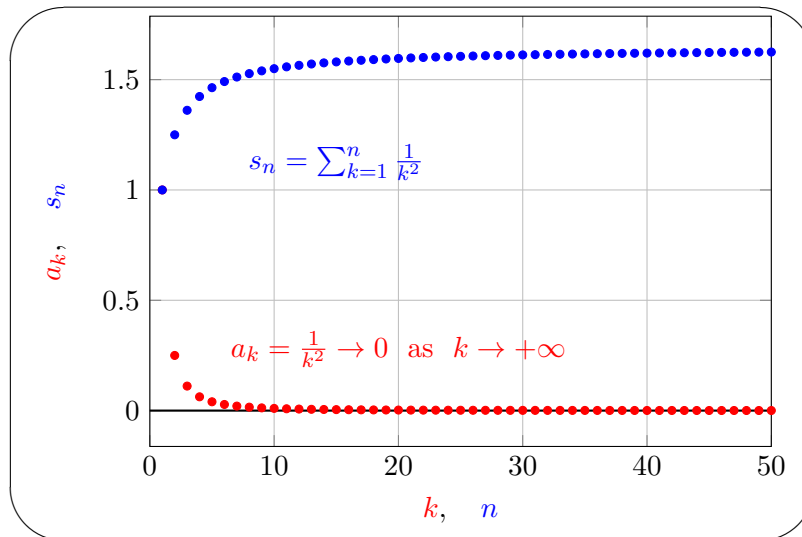
and the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges (Mengoli's series). Hence, by the limit comparison test, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ also converges.

With $\alpha > 2$, the series is dominated by the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, since:

$$\frac{1}{k^\alpha} \leq \frac{1}{k^2}, \quad \forall k \in \mathbb{N}, k \geq 1 \quad \text{and} \quad \alpha > 2$$

hence $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ is convergent by the comparison test. □

- Graphically, for example with $\alpha = 2$, we have:



Example 2.1: limit comparison test

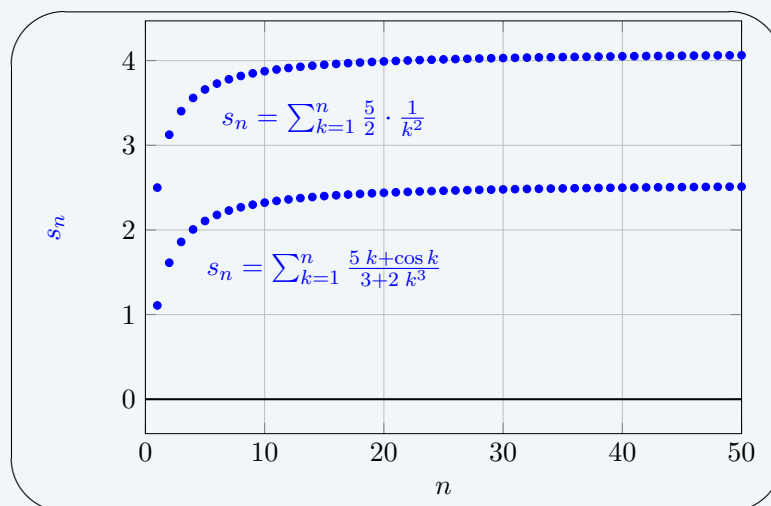
Let us determine the behavior of the series:

$$\sum_{k=1}^{\infty} \frac{5k + \cos k}{3 + 2k^3}$$

We have:

$$\frac{5k + \cos k}{3 + 2k^3} \sim \frac{5}{2} \cdot \frac{1}{k^2} \quad \text{as } k \rightarrow +\infty \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{5}{2} \cdot \frac{1}{k^2} \quad \text{converges}$$

hence the series is convergent by the limit comparison test.



However, even if the sequences are asymptotically equivalent, the corresponding series may not have the same sum.

2.1.3 Condensation test

Theorem 2.5: Condensation test

If $\{a_k\}$ is an eventually decreasing sequence with non-negative terms, then the series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=0}^{\infty} 2^k a_{2^k}$ are regular and have the same behavior, i.e., either they are both convergent or they are both divergent.

- Before proving the theorem, let us consider the sequence $\{\tilde{s}_n\}$ of the partial sums of the sequence $\{2^k a_{2^k}\}$, that is, the sequence:

$$\tilde{s}_n = \sum_{k=0}^n 2^k a_{2^k} \quad \forall n \geq 0$$

and let us derive two important relations with the sequence $\{s_n\}$ of the partial sums of the sequence $\{a_k\}$, that is, the sequence:

$$s_n = \sum_{k=1}^n a_k \quad \forall n \geq 1$$

- We observe that, for certain values of n , the partial sums s_n are:

$$\begin{aligned} \underbrace{s_1}_{=s_{2^1-1}} &= a_1, & \underbrace{s_3}_{=s_{2^2-1}} &= s_1 + a_2 + \underbrace{a_3}_{\leq a_2} \leq a_1 + 2 a_2 \\ \underbrace{s_7}_{=s_{2^3-1}} &= s_3 + a_4 + \underbrace{a_5}_{\leq a_4} + \underbrace{a_6}_{\leq a_4} + \underbrace{a_7}_{\leq a_4} \leq a_1 + 2 a_2 + 4 a_4 \\ \underbrace{s_{15}}_{=s_{2^4-1}} &= s_7 + a_8 + \underbrace{a_9}_{\leq a_8} + \underbrace{a_{10}}_{\leq a_8} + \underbrace{a_{11}}_{\leq a_8} \\ &\quad + \underbrace{a_{12}}_{\leq a_8} + \underbrace{a_{13}}_{\leq a_8} + \underbrace{a_{14}}_{\leq a_8} + \underbrace{a_{15}}_{\leq a_8} \leq a_1 + 2 a_2 + 4 a_4 + 8 a_8 \end{aligned}$$

Observation 2.6

Given an eventually decreasing sequence $\{a_k\}$ with non-negative terms, we have:

$$s_{2^n-1} \leq \underbrace{\sum_{k=0}^{n-1} 2^k a_{2^k}}_{=\tilde{s}_{n-1}} \quad \forall n \geq 1 \quad (2.2)$$

Proof. We prove by induction on 2^n that

$$s_{2^n-1} \leq \sum_{k=0}^{n-1} 2^k a_{2^k} \quad \forall n \geq 1$$

Base case. Let $n = 1$. Then the claim becomes $s_{2^1-1} \leq 2^0 a_1$, i.e., $a_1 \leq a_1$, which is clearly true.

Inductive step. Assume that it is true for 2^{n-1} and let us prove it for 2^n . By the inductive hypothesis, we have $s_{2^{n-1}-1} \leq \sum_{k=0}^{n-2} 2^k a_{2^k}$. Moreover, we have:

$$s_{2^n-1} = s_{2^{n-1}-1} + \underbrace{\sum_{i=2^{n-1}}^{2^n-1} a_i}_{\leq 2^{n-1} a_{2^{n-1}}} \quad \text{hence} \quad s_{2^n-1} \leq \sum_{k=0}^{n-2} 2^k a_{2^k} + 2^{n-1} a_{2^{n-1}} = \sum_{k=0}^{n-1} 2^k a_{2^k}$$

which is exactly the desired claim for 2^n . □

- We now observe that the values of $\tilde{s}_n$ are:

$$\tilde{s}_0 = a_1 \leq 2 a_1 = 2 \underbrace{s_1}_{= s_{2^0}}, \quad \tilde{s}_1 = \tilde{s}_0 + 2 a_2 \leq 2 s_1 + 2 a_2 = 2 \underbrace{s_2}_{= s_{2^1}}$$

$$\tilde{s}_2 = \tilde{s}_1 + 4 a_4 \leq 2 s_2 + 2 a_3 + 2 a_4 = 2 \underbrace{s_4}_{= s_{2^2}} \quad (\text{since } a_4 \leq a_3)$$

$$\tilde{s}_3 = \tilde{s}_2 + 8 a_8 \leq 2 s_4 + 2 a_5 + 2 a_6 + 2 a_7 + 2 a_8 = 2 \underbrace{s_8}_{= s_{2^3}} \quad (\text{since } a_8 \leq a_7 \leq a_6 \leq a_5)$$

Observation 2.7

Given an eventually decreasing sequence $\{a_k\}$ with non-negative terms, we have:

$$\underbrace{\sum_{k=0}^n 2^k a_{2^k}}_{= \tilde{s}_n} \leq 2 s_{2^n} \quad \forall n \geq 0 \quad (2.3)$$

Proof. We prove by induction on 2^n that

$$\tilde{s}_n \leq 2 s_{2^n} \quad \forall n \geq 0$$

Base case. Let $n = 0$. Then the claim becomes $\tilde{s}_0 \leq 2 s_{2^0} = 2 s_1$, i.e., $a_1 \leq 2 a_1$, which is clearly true.

Inductive step. Assume that it is true for $n - 1$ and let us prove it for n . By the

inductive hypothesis, we have $\tilde{s}_{n-1} \leq 2 s_{2^{n-1}}$. Moreover, we have:

$$\begin{aligned} \tilde{s}_n &= \tilde{s}_{n-1} + \underbrace{2^n a_{2^n}}_{2^n} \quad \text{hence} \quad \tilde{s}_n \leq 2 s_{2^{n-1}} + \sum_{i=2^{n-1}+1}^{2^n} 2 a_i = 2 s_{2^n} \\ &\leq \sum_{i=2^{n-1}+1}^{2^n} 2 a_i \end{aligned}$$

which is exactly the desired claim for n . $\square$

- We are now able to prove the theorem.

Proof. From relation (2.2) we have:

$$s_{2^n-1} \leq \tilde{s}_{n-1}, \quad \forall n \geq 1$$

Hence, if $\{\tilde{s}_n\}$ is convergent, that is, if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ is convergent, the subsequence $\{s_{2^n-1}\}$ is bounded. Since $\{s_n\}$ is monotone, the whole sequence $\{s_n\}$ is bounded and convergent by the monotone sequence theorem. Consequently, if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ is convergent, then $\sum_{k=1}^{\infty} a_k$ is also convergent.

From relation (2.3) we have:

$$\tilde{s}_n \leq 2 s_{2^n}, \quad \forall n \geq 0$$

Hence, if $\{s_n\}$ is convergent, that is, if $\sum_{k=1}^{\infty} a_k$ is convergent, then the subsequence $\{s_{2^n}\}$ is convergent (since $\{s_n\}$ is monotone). Consequently, by comparison, $\{\tilde{s}_n\}$ is also convergent, that is, $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Moreover, since $\{a_k\}$ has non-negative terms, the series cannot be irregular; consequently, we also have:

$$\sum_{k=1}^{\infty} a_k \text{ divergent} \iff \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ divergent}$$

which completes the proof of the theorem. $\square$

Given an eventually decreasing sequence $\{a_k\}$ with non-negative terms, we have proved that:

$$\sum_{k=1}^{\infty} a_k \text{ convergent/divergent} \iff \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ convergent/divergent}$$

Hence the convergence/divergence of $\sum_{k=0}^{\infty} 2^k a_{2^k}$ is a necessary and sufficient condition for the convergence/divergence of $\sum_{k=1}^{\infty} a_k$.

2.1.4 Generalized harmonic series

Definition 2.8: Generalized harmonic series

Given $\alpha \in \mathbb{R}$, the series $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ is called the **generalized harmonic series**

Theorem 2.9: Behavior of the generalized harmonic series

Given $\alpha \in \mathbb{R}$,

the generalized harmonic series $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ is $\begin{cases} \text{divergent to } +\infty & \text{if } \alpha \leq 1 \\ \text{convergent} & \text{if } \alpha > 1 \end{cases}$

Proof. We have already established that it diverges for $\alpha \leq 1$. For $\alpha > 1$, the sequence $\{\frac{1}{k^\alpha}\}$ is a decreasing sequence with positive terms. We have:

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^\alpha} = \sum_{k=0}^{\infty} (2^{1-\alpha})^k$$

that is, a geometric series with ratio $2^{1-\alpha}$, which converges if and only if $2^{1-\alpha} < 1$, that is, if $1 - \alpha < 0$. Hence, for $\alpha > 1$, by the condensation test, the series converges. $\square$

2.1.5 Passing from the discrete to the continuous

- The tools used to establish asymptotic estimates of functions can also be used to obtain asymptotic estimates of sequences (**passing from the discrete to the continuous**), and they provide useful tools for studying the behavior of a series with positive terms.

Example 2.2: passing from the discrete to the continuous

Let us determine the behavior of the series:

$$\sum_{k=1}^{\infty} \frac{e^{1/k} - 1}{k}$$

From the asymptotic equivalence

$$e^{\varepsilon(x)} - 1 \sim \varepsilon(x) \quad \text{as} \quad \varepsilon(x) \rightarrow 0$$

we have

$$\frac{e^{1/k} - 1}{k} \sim \frac{1}{k^2} \quad \text{as} \quad k \rightarrow +\infty$$

Therefore the series, which has positive terms, converges by limit comparison with the series of $1/k^2$.

Example 2.3: passing from the discrete to the continuous

Let us determine the behavior of the series:

$$\sum_{k=1}^{\infty} \log \left(\frac{k+3}{k+2} \right)$$

We have

$$\frac{k+3}{k+2} = 1 + \frac{1}{k+2} = 1 + \frac{1}{k+2}$$

From the asymptotic equivalence

$$\log(1 + \varepsilon(x)) \sim \varepsilon(x) \quad \text{as} \quad \varepsilon(x) \rightarrow 0$$

we have

$$\log \left(1 + \frac{1}{k+2} \right) \sim \frac{1}{k+2} \sim \frac{1}{k} \quad \text{as} \quad k \rightarrow +\infty$$

Thus it is a series with positive terms whose general term is asymptotically equivalent to $1/k$. By comparison with the harmonic series, this series diverges to $+\infty$.

Example 2.4: passing from the discrete to the continuous

Let us determine the behavior of the series:

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \sin \frac{1}{k} \right)$$

From the third-order Maclaurin expansion

$$\sin(\varepsilon(x)) = \varepsilon(x) - \frac{(\varepsilon(x))^3}{3!} + o\left((\varepsilon(x))^3\right) \quad \text{as} \quad \varepsilon(x) \rightarrow 0$$

we have

$$\frac{1}{k} - \sin \frac{1}{k} = \frac{1}{k} - \left(\frac{1}{k} - \frac{1}{6k^3} + o\left(\frac{1}{k^3}\right) \right) \sim \frac{1}{6k^3} \quad \text{as} \quad k \rightarrow +\infty$$

Therefore the series, which has positive terms, converges by limit comparison with the series of $1/(6k^3)$.

2.1.6 Root test

Theorem 2.10: Root test

Let $\{a_k\}$ be a sequence with non-negative terms. If the following limit exists:

$$\lim_{k \rightarrow +\infty} \sqrt[k]{a_k} = \ell \in \mathbb{R}^*$$

then the series $\sum a_k$ is $\begin{cases} \text{convergent} & \text{if } \ell < 1 \text{ or } \ell = -\infty \\ \text{divergent to } +\infty & \text{if } \ell > 1 \text{ or } \ell = +\infty \end{cases}$

Proof. Suppose that $\lim_{k \rightarrow +\infty} \sqrt[k]{a_k} = \ell < 1$. Since $\sqrt[k]{a_k} \rightarrow \ell \in \mathbb{R}$, for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\forall k \geq m$:

$$\sqrt[k]{a_k} \leq \ell + \frac{\varepsilon}{2}$$

Moreover, since $\ell < 1$, we have $\ell < 1 - \varepsilon$ for a suitable $\varepsilon > 0$. For this ε we therefore have, eventually:

$$\sqrt[k]{a_k} \leq \ell + \frac{\varepsilon}{2} < (1 - \varepsilon) + \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2} \quad \text{hence} \quad a_k < \left(1 - \frac{\varepsilon}{2}\right)^k, \quad \forall k \geq m$$

We have thus proved that $a_k < \left(1 - \frac{\varepsilon}{2}\right)^k$ eventually. The geometric series:

$$\sum_{k=0}^{\infty} \left(1 - \frac{\varepsilon}{2}\right)^k \quad \text{is convergent since } 1 - \frac{\varepsilon}{2} < 1$$

Hence, by the comparison test, the original series converges.

Suppose that $\lim_{k \rightarrow +\infty} \sqrt[k]{a_k} = \ell > 1$. Since $\sqrt[k]{a_k} \rightarrow \ell \in \mathbb{R}$, for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\forall k \geq m$:

$$\sqrt[k]{a_k} \geq \ell - \frac{\varepsilon}{2}$$

Moreover, since $\ell > 1$, we have $\ell > 1 + \varepsilon$ for a suitable $\varepsilon > 0$. For this ε we therefore have, eventually:

$$\sqrt[k]{a_k} \geq \ell - \frac{\varepsilon}{2} > (1 + \varepsilon) - \frac{\varepsilon}{2} = 1 + \frac{\varepsilon}{2} \quad \text{hence} \quad a_k > \left(1 + \frac{\varepsilon}{2}\right)^k, \quad \forall k \geq m$$

We have thus proved that $a_k > \left(1 + \frac{\varepsilon}{2}\right)^k$ eventually. The geometric series:

$$\sum_{k=0}^{\infty} \left(1 + \frac{\varepsilon}{2}\right)^k \quad \text{is divergent since } 1 + \frac{\varepsilon}{2} > 1$$

Hence, by the comparison test, the original series diverges. □

Clearly, these arguments remain valid also if $\ell = -\infty$ or $+\infty$, respectively.

Observation 2.11

The series $\sum_{k=1}^{\infty} k^{\beta} \cdot b^k$ with $\beta \in \mathbb{R}, b \geq 0$ is

convergent	if $b < 1$
divergent	if $b > 1$
convergent	if $b = 1$ and $\beta < -1$
divergent	if $b = 1$ and $\beta \geq -1$

Proof. It is a series with non-negative terms, and we have:

$$\sqrt[k]{k^{\beta} \cdot b^k} = b \cdot k^{\beta/k} \quad \text{and} \quad \lim_{k \rightarrow +\infty} k^{\beta/k} = \lim_{k \rightarrow +\infty} \exp \left(\underbrace{\frac{\beta}{k} \cdot \log k}_{\rightarrow 0} \right) = 1$$

hence, by the root test, it converges for $b < 1$ and diverges for $b > 1$. If $b = 1$, the series becomes:

$$\sum_{k=1}^{\infty} k^{\beta} = \sum_{k=1}^{\infty} \frac{1}{k^{-\beta}}$$

that is, a generalized harmonic series; it is convergent for $\beta < -1$, while it is divergent for $\beta \geq -1$. $\square$

Observation 2.12

The series $\sum_{k=1}^{\infty} b^k/k^k$ with $b \geq 0$ is convergent

Proof. It is a series with non-negative terms, and we have:

$$\sqrt[k]{\frac{b^k}{k^k}} = \frac{b}{k} \rightarrow 0 \quad \text{as} \quad k \rightarrow +\infty$$

hence the series converges by the root test. $\square$

2.1.7 Ratio test

Theorem 2.13: Ratio test

Let $\{a_k\}$ be a sequence with positive terms. If the following limit exists:

$$\lim_{k \rightarrow +\infty} \frac{a_{k+1}}{a_k} = \ell \in \mathbb{R}^*$$

then the series $\sum a_k$ is $\begin{cases} \text{convergent} & \text{if } \ell < 1 \text{ or } \ell = -\infty \\ \text{divergent to } +\infty & \text{if } \ell > 1 \text{ or } \ell = +\infty \end{cases}$

Proof. Suppose that $\lim_{k \rightarrow +\infty} a_{k+1}/a_k = \ell < 1$. Reasoning as in the proof of the root test, there exists $m \in \mathbb{N}$ such that $\forall k \geq m$:

$$\frac{a_{k+1}}{a_k} < \left(1 - \frac{\varepsilon}{2}\right)$$

for a suitable $\varepsilon > 0$. Reasoning iteratively, this implies that:

$$a_{k+1} < \left(1 - \frac{\varepsilon}{2}\right) a_k < \left(1 - \frac{\varepsilon}{2}\right) \left(1 - \frac{\varepsilon}{2}\right) a_{k-1} < \dots < \left(1 - \frac{\varepsilon}{2}\right)^{k-m+1} a_m$$

We have thus proved that $a_k < \left(1 - \frac{\varepsilon}{2}\right)^{k-m} a_m$ eventually. The geometric series:

$$\sum_{k=m}^{\infty} \left(1 - \frac{\varepsilon}{2}\right)^{k-m} a_m \text{ is convergent since } 1 - \frac{\varepsilon}{2} < 1$$

Hence, by the comparison test, the original series converges.

Suppose that $\lim_{k \rightarrow +\infty} a_{k+1}/a_k = \ell > 1$. Reasoning as in the proof of the root test, there exists $m \in \mathbb{N}$ such that $\forall k \geq m$:

$$\frac{a_{k+1}}{a_k} > \left(1 + \frac{\varepsilon}{2}\right)$$

for a suitable $\varepsilon > 0$. Reasoning iteratively, this implies that:

$$a_{k+1} > \left(1 + \frac{\varepsilon}{2}\right) a_k > \left(1 + \frac{\varepsilon}{2}\right) \left(1 + \frac{\varepsilon}{2}\right) a_{k-1} > \dots > \left(1 + \frac{\varepsilon}{2}\right)^{k-m+1} a_m$$

We have thus proved that $a_k > \left(1 + \frac{\varepsilon}{2}\right)^{k-m} a_m$ eventually. The geometric series:

$$\sum_{k=m}^{\infty} \left(1 + \frac{\varepsilon}{2}\right)^{k-m} a_m \text{ is divergent since } 1 + \frac{\varepsilon}{2} > 1$$

Hence, by the comparison test, the original series diverges. $\square$

Clearly, these arguments remain valid also if $\ell = -\infty$ or $+\infty$, respectively.

Observation 2.14

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e$$

Proof. We have

$$\frac{1/(k+1)!}{1/k!} = \frac{1}{k+1} \rightarrow 0 \quad \text{as } k \rightarrow +\infty$$

Therefore the series, which has non-negative terms, converges by the ratio test. The proof that the sum of the series is equal to e will be given later using series of functions. $\square$

2.1.8 Series with non-positive terms

- We know that the behavior of a series does not change if we alter a finite number of its terms. Consequently, the tests for series with non-negative terms also apply to series whose terms are eventually non-negative.
- Moreover, by factoring a minus sign out of the whole series, we see that these tests can also be applied to series with non-positive terms, and hence to series whose terms are eventually non-positive.

In summary, therefore, the tests we have seen apply to series all of whose terms (except for a finite number) have the same sign.

Series with terms of variable sign

3.1 Series with terms of variable sign

Definition 3.1: absolutely convergent series

A series $\sum a_k$ is absolutely convergent if the series $\sum |a_k|$ converges.

Theorem 3.2

If a series $\sum a_k$ converges absolutely, then it converges.

Proof. Without loss of generality, let $n_0 = 0$ and consider the following series:

$$\sum_{k=0}^{\infty} (|a_k| - a_k)$$

It is a series with nonnegative terms since, for every $k \in \mathbb{N}$, we have:

$$\begin{cases} |a_k| - a_k = -2a_k \geq 0 & \text{if } a_k < 0 \\ |a_k| - a_k = 0 & \text{if } a_k \geq 0 \end{cases}$$

Moreover, by the triangle inequality, for every $k \in \mathbb{N}$, we have:

$$\underbrace{|a_k| - a_k}_{\geq 0} = ||a_k| - a_k| \leq |a_k| + |a_k| = 2|a_k|$$

Hence, by the comparison test for series with nonnegative terms, we have

$$\sum_{k=0}^{\infty} |a_k| \text{ convergent} \implies \sum_{k=0}^{\infty} (|a_k| - a_k) \text{ convergent}$$

Since for every $k \in \mathbb{N}$ we have $a_k = |a_k| - (|a_k| - a_k)$ and the series $\sum_{k=0}^{\infty} |a_k|$ and $\sum_{k=0}^{\infty} (|a_k| - a_k)$ converge, we have:

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} |a_k| - \sum_{k=0}^{\infty} (|a_k| - a_k)$$

that is, the series is the difference of two convergent series. Consequently, the series $\sum_{k=0}^{\infty} a_k$ is convergent as well. $\square$

- The following is an alternative proof.

Proof. Without loss of generality, let $n_0 = 0$ and split the sequence $\{s_n\}$ of partial sums into two sequences, the first containing only the positive terms and the second only the negative terms:

$$s_n^+ = \sum_{k \in \{0,1,\dots,n\}: a_k > 0} a_k \quad \text{and} \quad s_n^- = \sum_{k \in \{0,1,\dots,n\}: a_k < 0} -a_k \quad \text{hence} \quad s_n = s_n^+ - s_n^-$$

Consequently, it is enough to show that the sequences $\{s_n^+\}$ and $\{s_n^-\}$ are convergent to conclude that $\{s_n\}$ converges, and hence that the series $\sum a_k$ converges.

We observe that $\{s_n^+\}$ and $\{s_n^-\}$ are monotone non-decreasing sequences and moreover, by the triangle inequality, we have:

$$s_n^+ \leq \sum_{k=0}^n |a_k| \quad \text{and} \quad s_n^- \leq \sum_{k=0}^n |a_k|$$

On the other hand, by hypothesis the series $\sum a_k$ converges absolutely, i.e., $\sum |a_k|$ converges, and hence the quantity $\sum_{k=0}^n |a_k|$ is bounded; therefore the sequences $\{s_n^+\}$ and $\{s_n^-\}$ are bounded above and non-decreasing, and hence they converge, by the monotone sequence theorem. $\square$

Example 3.1: absolute convergence

Let us determine the behavior of the series:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^\alpha} \quad \text{with} \quad \alpha > 1$$

We have:

$$\left| \frac{(-1)^k}{k^\alpha} \right| = \frac{1}{k^\alpha}, \quad \forall k \in \mathbb{N}, k > 1 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^\alpha} \quad \text{is convergent for} \quad \alpha > 1$$

Therefore the series converges absolutely and hence it converges.

Absolute convergence implies ordinary convergence, also called simple convergence:

$$\sum |a_k| \text{ convergent} \quad \Rightarrow \quad \sum a_k \text{ convergent} \quad (3.1)$$

but the converse is not true (we will give a counterexample later):

$$\sum a_k \text{ convergent} \quad \nRightarrow \quad \sum |a_k| \text{ convergent} \quad (3.2)$$

Hence absolute convergence is a sufficient but not necessary condition for ordinary convergence.

3.1.1 Alternating series and the Leibniz test

- Among series with terms of variable sign, a particularly simple case is given by alternating series, for which the following convergence test holds.

Theorem 3.3: Leibniz (alternating series) test

Consider the series

$$\sum_{k=n_0}^{\infty} (-1)^k a_k \quad \text{with} \quad a_k \geq 0, \forall k$$

If the sequence $\{a_k\}$ is decreasing and $a_k \rightarrow 0$ as $k \rightarrow +\infty$, then the series is convergent. Moreover:

$$s_{2n} = \sum_{k=n_0}^{2n} (-1)^k a_k \downarrow s \quad \text{and} \quad s_{2n+1} = \sum_{k=n_0}^{2n+1} (-1)^k a_k \uparrow s \quad \text{as } n \rightarrow +\infty$$

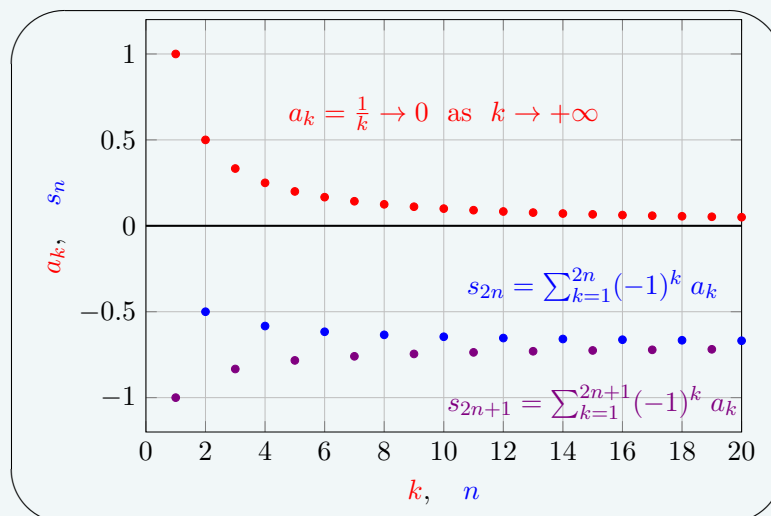
- The partial sums with even index approximate the sum s from above and those with odd index from below.
- The Leibniz test can clearly be applied also if the terms eventually have alternating signs and the sequence $\{a_k\}$ is eventually decreasing.

Example 3.2: Leibniz test

Let us determine the behavior of the series:

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$$

The sequence $a_k = \frac{1}{k}$ is decreasing and non-negative. Moreover $\frac{1}{k} \rightarrow 0$ as $k \rightarrow +\infty$, hence it satisfies the two conditions of the Leibniz test theorem. Consequently, the series is convergent.



The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges and is a counterexample to (3.2), that is, a series that converges but does not converge absolutely, since:

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges (harmonic series)}$$

Proof. Consider the sequence of partial sums $\{s_n\}$ with $n_0 = 0$ and the two subsequences extracted from it, $\{s_{2n}\}$ and $\{s_{2n+1}\}$.

For $\{s_{2n}\}$ we have:

$$s_0 = a_0, \quad s_2 = s_0 - a_1 + \underbrace{a_2}_{\leq a_1} \leq s_0, \quad s_4 = s_2 - a_3 + \underbrace{a_4}_{\leq a_3} \leq s_2, \quad \dots$$

hence the sequence $\{s_{2n}\}$ is monotone decreasing.

For $\{s_{2n+1}\}$ we have:

$$s_1 = a_0 - a_1, \quad s_3 = s_1 + a_2 - \underbrace{a_3}_{\leq a_2} \geq s_1, \quad s_5 = s_3 + a_4 - \underbrace{a_5}_{\leq a_4} \geq s_3, \quad \dots$$

hence the sequence $\{s_{2n+1}\}$ is monotone increasing.

Moreover, we have:

$$s_1 \leq s_{2n+1} = s_{2n} - a_{2n+1} \leq s_{2n} \leq s_0$$

therefore $\{s_{2n+1}\}$ is bounded above and $\{s_{2n}\}$ is bounded below. The two sequences are hence convergent, by the monotone sequence theorem.

The two sequences converge to the same limit, because

$$0 \leq s_{2n} - s_{2n+1} \leq a_{2n+1} \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty$$

Calling s this limit, since $\{s_{2n}\}$ is monotone decreasing and $\{s_{2n+1}\}$ is monotone increasing, we have:

$$s_{2n} = \sum_{k=n_0}^{2n} (-1)^k a_k \downarrow s \quad \text{and} \quad s_{2n+1} = \sum_{k=n_0}^{2n+1} (-1)^k a_k \uparrow s \quad \text{as} \quad n \rightarrow +\infty$$

Hence the series is convergent since $s_{2n+1} \rightarrow s$ and $s_{2n} \rightarrow s$ as $n \rightarrow +\infty$, and consequently we have:

$$s_n \rightarrow s \quad \text{as} \quad n \rightarrow +\infty$$

□

Corollary 3.4: of the Leibniz test theorem

Given a series satisfying the hypotheses of the Leibniz test theorem, we have:

$$\underbrace{|s - s_m|}_{=|\sum_{k=m+1}^{\infty} (-1)^k a_k|} \leq a_{m+1}, \quad \forall m \in \mathbb{N}$$

- For every m , the error made by approximating s with s_m is, in absolute value, bounded above by the value of the first omitted term. In other words, the tail of the series tends to a value less than or equal to a_{m+1} .

Proof. Since

$$s_{2n+1} \uparrow s \quad \text{and} \quad s_{2n} \downarrow s \quad \text{as } n \rightarrow +\infty$$

we therefore have, for every $n \in \mathbb{N}$:

$$s_{2n-1} \leq s \leq s_{2n} \quad \text{and} \quad s_{2n+1} \leq s \leq s_{2n}$$

from which we deduce

$$0 \leq s - s_{2n-1} \leq s_{2n} - s_{2n-1} = a_{2n} \quad \text{and} \quad 0 \leq s_{2n} - s \leq s_{2n} - s_{2n+1} = a_{2n+1}$$

Therefore, for every m , whether even or odd, we have:

$$|s - s_m| = \left| \sum_{k=m+1}^{\infty} (-1)^k a_k \right| \leq a_{m+1}$$

□

Example 3.3: Leibniz test

Let us determine the behavior of the series:

$$\sum_{k=1}^{\infty} (-1)^k \frac{k-1}{k^2+k}$$

We have:

$$a_k = \frac{k-1}{k^2+k} = \frac{k-1}{k(k+1)} \geq 0, \forall k \in \mathbb{N}, k \geq 1 \quad \text{and} \quad \frac{k-1}{k(k+1)} \sim \frac{1}{k} \rightarrow 0 \quad \text{as } k \rightarrow +\infty$$

hence the series does not converge absolutely. However, the series is decreasing for $k \geq 2$ since:

$$\underbrace{\frac{k}{(k+1)(k+2)}}_{=a_{k+1}} \leq \underbrace{\frac{k-1}{k(k+1)}}_{=a_k} \iff k^2 \leq k^2 + k - 2 \iff k \geq 2$$

hence the series converges by the Leibniz test, since it is eventually decreasing.

Example 3.4: Leibniz test

Let us determine the behavior of the series:

$$\sum_{k=2}^{\infty} (-1)^k \frac{\log k}{k}$$

We have:

$$a_k = \frac{\log k}{k} \geq 0, \forall k \geq 2 \quad \text{and} \quad \frac{\log k}{k} \rightarrow 0 \quad \text{as } k \rightarrow +\infty$$

moreover

$$\frac{\log k}{k} > \frac{1}{k} \quad \text{for } k \geq 3 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k} \quad \text{diverges}$$

hence the series does not converge absolutely. Proving algebraically that the sequence $\{a_k\}$ is decreasing is complicated; instead, we pass from the discrete to the continuous setting. We have, for $x \in \mathbb{R}$ and $x \geq 2$:

$$f(x) = \frac{\log x}{x} \quad \text{and} \quad f'(x) = \frac{1 - \log x}{x^2} \leq 0 \quad \text{for } x \geq e$$

It follows that f is decreasing for $x \geq e$; consequently the sequence $a_k = f(k)$ is decreasing for $k \geq 3$ (the first integer $> e$). Hence the series converges by the Leibniz test.

if $\sum a_k$ converges and $\sum b_k$ converges then $\sum (a_k + b_k)$ converges

if $\sum a_k$ converges and $\sum b_k$ diverges then $\sum (a_k + b_k)$ diverges

This can be checked by viewing the series as the limit of the sequence of partial sums and applying the theorem on the limit of a sum.

Example 3.5: Leibniz test

Let us determine the behavior of the series:

$$\sum_{k=1}^{\infty} \frac{k+1 + (-1)^k k^2}{k^3}$$

We split it into the sum of two series:

$$\sum_{k=1}^{\infty} \frac{k+1 + (-1)^k k^2}{k^3} = \sum_{k=1}^{\infty} \frac{k+1}{k^3} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

For the first series we have:

$$\frac{k+1}{k^3} \geq 0, \forall k \geq 1 \quad \text{and} \quad \frac{k+1}{k^3} \rightarrow 0 \quad \text{as } k \rightarrow +\infty$$

hence it converges by the limit comparison test, since:

$$\frac{k+1}{k^3} \sim \frac{1}{k^2} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{converges}$$

The second series converges by the Leibniz test; hence the original series converges.

Example 3.6: Leibniz test

Let us determine the behavior of the series:

$$\sum_{k=1}^{\infty} (-1)^k \left(\frac{\sqrt{k} + (-1)^k}{k} \right)$$

We split it into the sum of two series:

$$\sum_{k=1}^{\infty} (-1)^k \left(\frac{\sqrt{k} + (-1)^k}{k} \right) = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} + \sum_{k=1}^{\infty} \frac{1}{k}$$

The first series converges by the Leibniz test; the second diverges (harmonic series); hence the original series diverges.

Example 3.7: Leibniz test

Let us determine the behavior of the series:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

We have:

$$\frac{(-1)^{k+1}}{k} = -\frac{(-1)^k}{k}, \quad \forall k \in \mathbb{N}, k > 1$$

moreover, the sequence $a_k = \frac{1}{k}$ is decreasing and $\frac{1}{k} \rightarrow 0$ as $k \rightarrow +\infty$, hence it satisfies the two conditions of the theorem. Consequently, the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges.

Series of functions

4.1 Series of functions

- **Series of functions** are numerical series depending on a parameter x : when these series converge for every value x belonging to a certain interval, they represent **functions of a new kind**
- There are two important classes of series of functions, namely trigonometric series and power series. We will only deal with **power series**, also called **Taylor series**, which can be seen as a natural extension of Taylor expansions.

4.1.1 Taylor series of the elementary transcendental functions

- Taylor's expansion/formula with Lagrange remainder can be rewritten in the form:

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + E_n(x) \quad (4.1)$$

where the Lagrange approximation error is

$$E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1} \quad (4.2)$$

and c is a suitable point between x_0 and x .

Definition 4.1: Taylor series

If a function f has derivatives of every order, the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

is called the **Taylor series** of the function f centered at x_0 .

At a given point $x \in (a, b)$, if the error

$$E_n(x) \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

then the Taylor series is convergent and its sum equals $f(x)$. This is equivalent to saying

that the n -th partial sum of the series has a finite limit and this limit is precisely $f(x)$. In formulas:

$$\underbrace{T_{n,x_0}(x)}_{\text{Taylor polynomial}} = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \rightarrow f(x) \quad \text{as } n \rightarrow +\infty$$

Definition 4.2: function expandable in Taylor series on an interval

Given a function $f : D \rightarrow \mathbb{R}$ with derivatives of every order, if

$$E_n(x) \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \quad \forall x \in I \subseteq D$$

we say that $f(x)$ is **expandable in Taylor series** on the interval I .

- There are (infinitely differentiable) functions that are expandable in Taylor series on the whole real line, others that are expandable only on a bounded interval, and others for which the interval reduces to a single point (the point x_0).
- We now deal with the three elementary transcendental functions:

$$f(x) = e^x, \quad f(x) = \sin x \quad \text{and} \quad f(x) = \cos x$$

which we will show to be expandable in Taylor series on the whole of $\mathbb{R}$. They are three transcendental (not algebraic) functions.

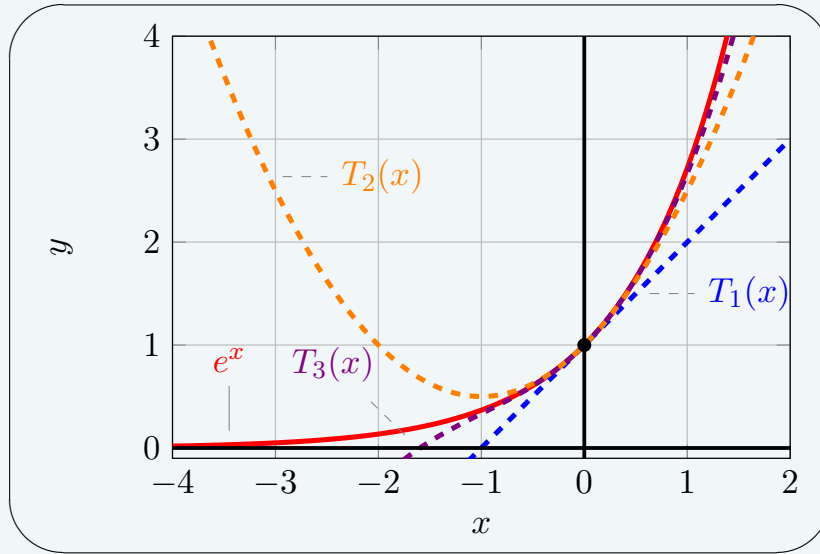
- Real functions of a real variable are classified into **algebraic functions and transcendental functions**. Algebraic functions are built through a finite number of applications of the four arithmetic operations, of raising to a power and of extracting the n -th root. Transcendental functions are all the functions that are not algebraic.

4.1.1.1 The Taylor series of the exponential function

Example 4.1: Maclaurin formula/expansion of the exponential

Maclaurin formula/expansion of order n of the exponential with Lagrange remainder:

$$e^x = T_n(x) + \frac{e^c}{(n+1)!} x^{n+1} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!} x^{n+1} \quad \text{with } c \in [0, x]$$


Observation 4.3: Taylor series of the exponential function

$$\forall x \in \mathbb{R}, \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Proof. From the Maclaurin expansion of order n of e^x with Lagrange remainder, we have that for every integer n and $x \in \mathbb{R}$ there exists a point c , lying between 0 and x , such that:

$$e^x = \sum_{k=0}^n \frac{x^k}{k!} + \frac{e^c}{(n+1)!} x^{n+1}$$

We fix x and let n tend to $+\infty$. The point c may vary with n but, since it always lies between 0 and x , we have:

$$e^c \leq \begin{cases} e^x & \text{if } x > 0 \\ 1 & \text{if } x < 0 \end{cases}$$

and $e^0 = 1$, hence e^c stays bounded. From the hierarchy of infinities theorem we have:

$$\frac{x^{n+1}}{(n+1)!} \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

hence the error tends to zero, that is:

$$\frac{x^{n+1}}{(n+1)!} e^c \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

since it is the product of an infinitesimal sequence and a bounded one. We have therefore proved that the function e^x can be written as the sum of a power series, its Taylor series, convergent for every $x \in \mathbb{R}$. $\square$

Observation 4.4

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} \quad (4.3)$$

Proof. It suffices to set x equal to 1 in the Taylor series of the exponential function. $\square$

- We have derived a second definition of Euler's number e (Napier's constant); the first one was:

$$e = \lim_{k \rightarrow +\infty} \left(1 + \frac{1}{k}\right)^k \quad (4.4)$$

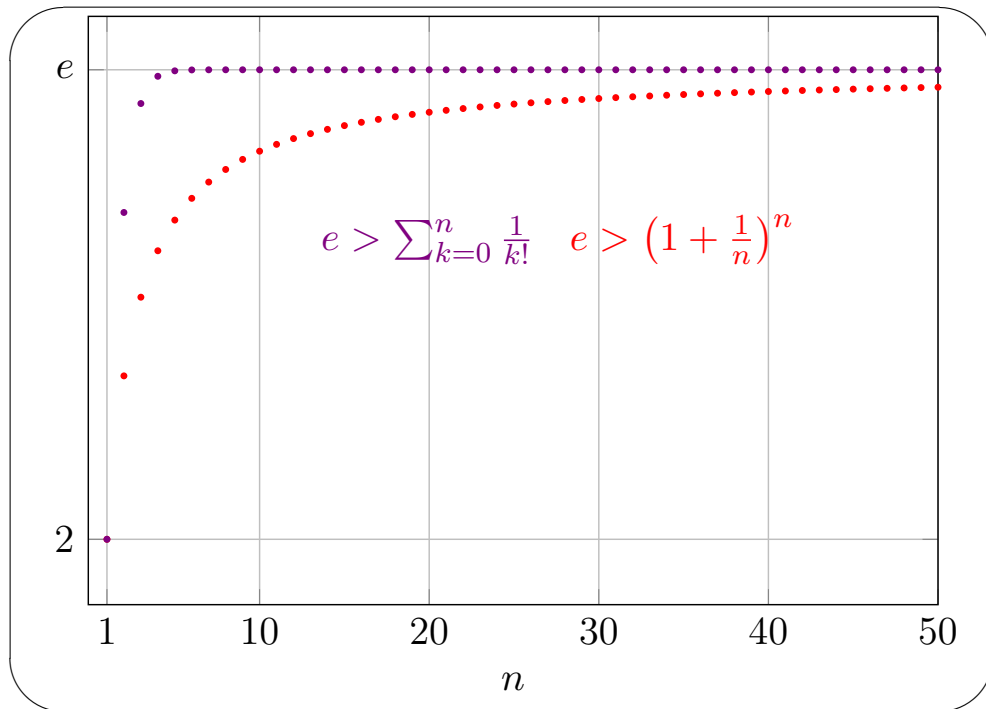
- Consequently, we have two methods for computing an approximation of the value of e :
 1. The first method is based on formula (4.4), and by fixing $k = n$ we obtain the approximation:

$$e \approx \left(1 + \frac{1}{n}\right)^n$$

2. The second method is based on formula (4.3), and by computing the n -th partial sum we obtain the approximation:

$$e \approx \sum_{k=0}^n \frac{1}{k!}$$

	$\left(1 + \frac{1}{n}\right)^n$	$\sum_{k=0}^n \frac{1}{k!}$	e
$n = 1$	2.0000000000...	2.0000000000...	2.7182818284...
$n = 2$	2.2500000000...	2.5000000000...	2.7182818284...
$n = 3$	2.3703703704...	2.6666666666...	2.7182818284...
$n = 4$	2.4414062500...	2.7083333333...	2.7182818284...
$n = 5$	2.4883200000...	2.7166666666...	2.7182818284...



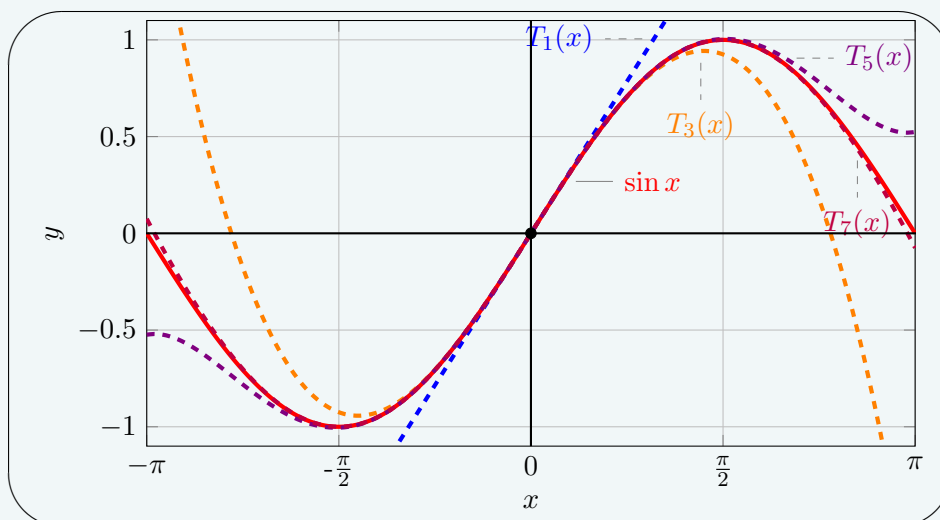
- The second method is therefore much more efficient at approximating the value of e

4.1.1.2 The Taylor series of the elementary trigonometric functions

Example 4.2: Maclaurin formula/expansion of sine and cosine

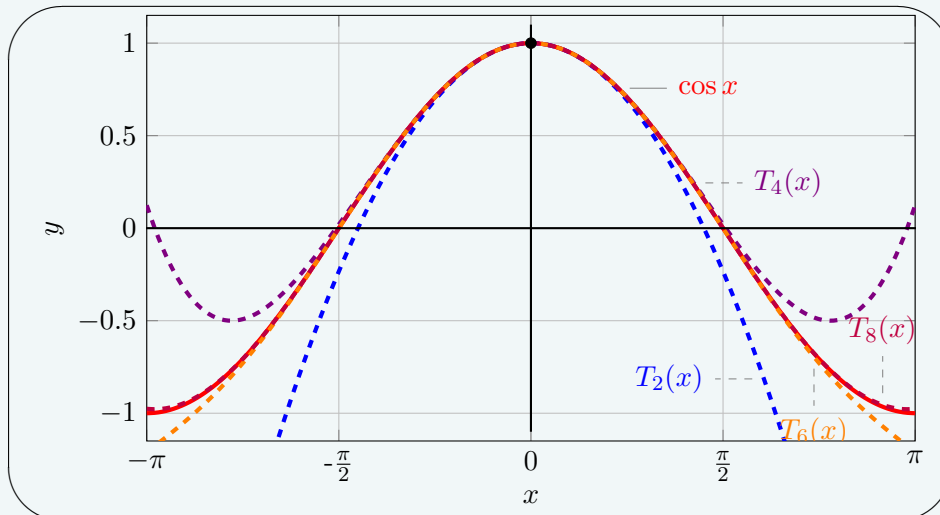
Maclaurin formula/expansion of odd order of the sine with Lagrange remainder:

$$\begin{aligned} \sin x &= T_{2k+1}(x) + \frac{\sin^{(2k+2)}(c)}{(2k+2)!} x^{2k+2} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \frac{\sin^{(2k+2)}(c)}{(2k+2)!} x^{2k+2} \quad \text{with } c \in [0, x] \end{aligned}$$



Maclaurin formula/expansion of even order of the cosine with Lagrange remainder:

$$\begin{aligned}\cos x &= T_{2k}(x) + \frac{\cos^{(2k+1)}(c)}{(2k+1)!} x^{2k+1} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \frac{\cos^{(2k+1)}(c)}{(2k+1)!} x^{2k+1} \quad \text{with } c \in [0, x]\end{aligned}$$



Observation 4.5: Taylor series of the elementary trigonometric functions

$$\forall x \in \mathbb{R}, \quad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Proof.

□

4.2 Series in the complex field