



SAPIENZA
UNIVERSITÀ DI ROMA

Mathematical Analysis 1

Volume 5 — Derivatives

The fifth volume: differential calculus. The derivative function, derivatives of elementary functions, corner points and cusps, rules of differentiation, the mean value theorem, maxima and minima, L'Hôpital's rule, the second derivative, approximations and Taylor's formula, curve sketching, Newton's method. Definitions, theorems with proofs, graphs and worked examples; the exercises of the part, with solutions, are in the volume Exercises 5.

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Online version and downloadable material:
fabiofurini.github.io/mathematical-analysis-1/
github.com/fabiofurini/mathematical-analysis-1

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The derivative function

1.1 Line through two points

- Starting from two points

$$(x_1, y_1) \text{ and } (x_2, y_2)$$

it is possible to compute the function

$$y = m x + q$$

whose graph is the line through the two points (equation of the line in explicit form). The **slope** of the line is the value m and the **y-intercept** of the line is the value q .

- Requiring the line to pass through the two points we have:

$$\begin{cases} y_1 = m x_1 + q \\ y_2 = m x_2 + q \end{cases}$$

and we can determine m and q as follows (two equations in two unknowns):

$$\begin{cases} q = y_1 - m x_1 \\ q = y_2 - m x_2 \end{cases} \quad \begin{cases} q = y_1 - m x_1 \\ y_1 - m x_1 = y_2 - m x_2 \\ m x_2 - m x_1 = y_2 - y_1 \end{cases} \quad \text{which becomes}$$

$$\begin{cases} q = y_1 - m x_1 \\ m = \frac{y_2 - y_1}{x_2 - x_1} \end{cases} \quad \begin{cases} q = y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1 \\ m = \frac{y_2 - y_1}{x_2 - x_1} \end{cases}$$

And we obtain the line:

$$y = \underbrace{\frac{y_2 - y_1}{x_2 - x_1}}_m x + \underbrace{y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1}_q \quad \text{clearly} \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

The line through two points (x_1, y_1) and (x_2, y_2) :

$$y = \underbrace{\frac{y_2 - y_1}{x_2 - x_1}}_m x + y_1 - \underbrace{\frac{y_2 - y_1}{x_2 - x_1}}_m x_1 \text{ can be rewritten as } y = m x + y_1 - m x_1.$$

Hence the line can be written only in terms of the slope m and the point (x_1, y_1) as follows:

$$y = y_1 + m (x - x_1) \quad \text{with} \quad m = \frac{y_1 - y_2}{x_1 - x_2}$$

The same holds for any point $(\tilde{x}, \tilde{y})$ of the line; in this case we would have:

$$y = \tilde{y} + m(x - \tilde{x})$$

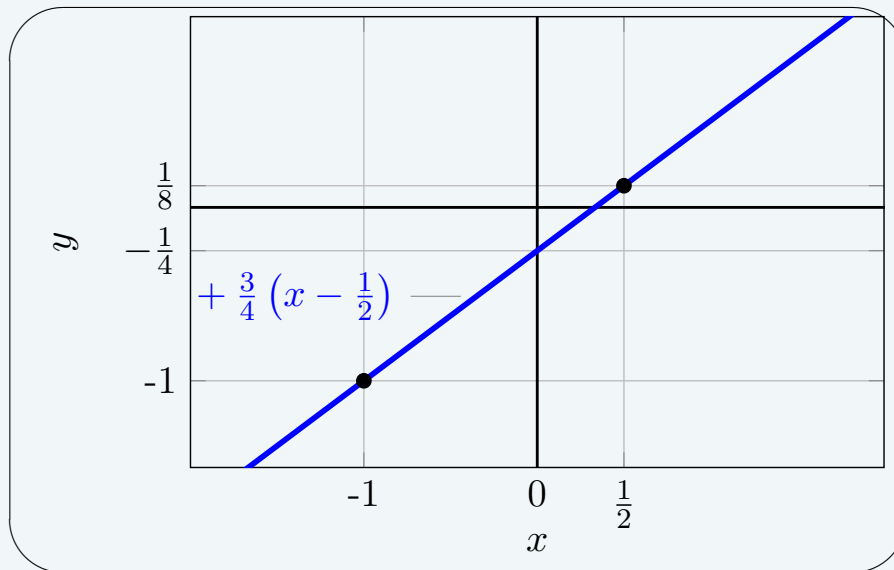
Example 1.1: Line through two points

We compute the line through the points: $(x_1, y_1) = (-1, -1)$ and $(x_2, y_2) = (\frac{1}{2}, \frac{1}{8})$. We compute the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{1}{8} + 1}{\frac{1}{2} + 1} = \frac{3}{4}$$

Using the point $(x_2, y_2) = (\frac{1}{2}, \frac{1}{8})$ we obtain:

$$y = \frac{1}{8} + \frac{3}{4} \left(x - \frac{1}{2} \right) = \frac{3}{4} x - \frac{1}{4}$$



Now using the point $(x_1, y_1) = (-1, -1)$ we obtain the same line:

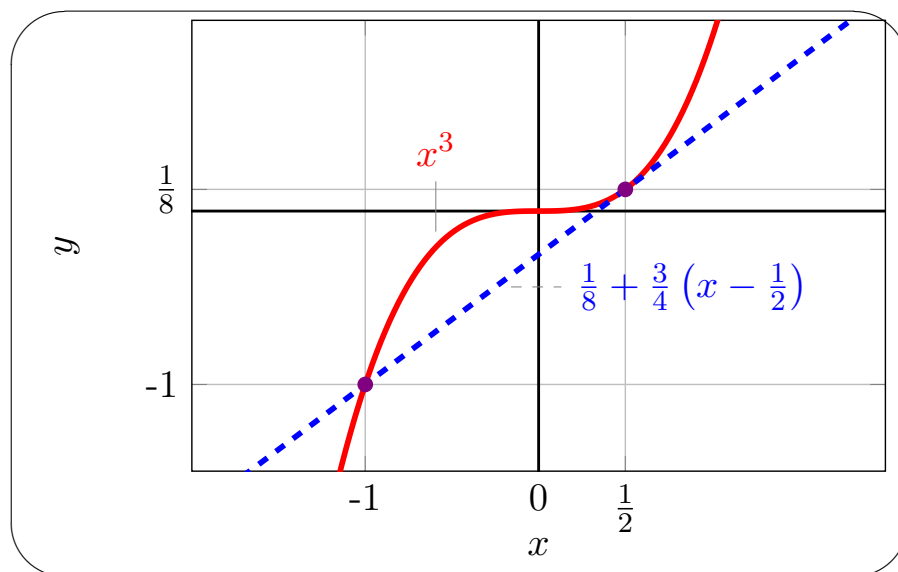
$$y = -1 + \frac{3}{4} (x + 1) = \frac{3}{4} x - \frac{1}{4}$$

1.2 Tangent line

What is the **tangent** line to the graph of a function at a point?

- The intuitive definition of a line tangent to the graph is that of a line that “touches” the graph without “cutting” or “crossing” it (imagining the curve as if it were an impenetrable physical object)
- This intuitive definition works, for example, for parabolas or hyperbolas. However, take for example the function $f(x) = x^3$

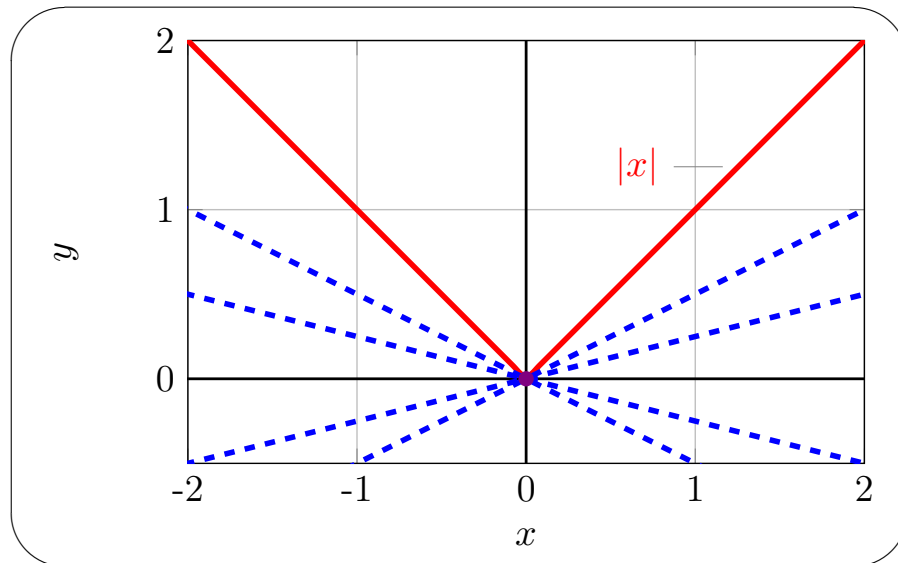
the graph $y = x^3$ and the line $y = \frac{1}{8} + \frac{3}{4} \left(x - \frac{1}{2} \right)$



the intuitive definition works at the point $(\frac{1}{2}, \frac{1}{8})$, but the line also intersects the graph of the function at the point $(-1, -1)$.

- Conversely, there are functions, such as $f(x) = |x|$, whose graph is

$$y = |x|$$



This graph has a point (the origin) at which there are infinitely many lines satisfying the intuitive definition, but clearly none of them can be called tangent.

Geometrically, the tangent line can be approximated by the **line through two points** very close to each other on the curve, the first one **fixed** (the point of tangency) and the other one “**moving**” (and chosen closer and closer to the point of tangency).

- Two points determine a line; the closer the two points get, the closer this line gets to the tangent. We therefore need to understand what happens to the line through the two points when the second, moving point gets closer and closer to the first one, without ever coinciding with it. The “**limit line**” - if it exists - will be called the **tangent** line.

A further way of looking at the concept of tangency is to think of the tangent (to a curve at a point) as the line that **best approximates** the curve in a neighborhood of the point.

1.2.1 Practical use of the tangent line

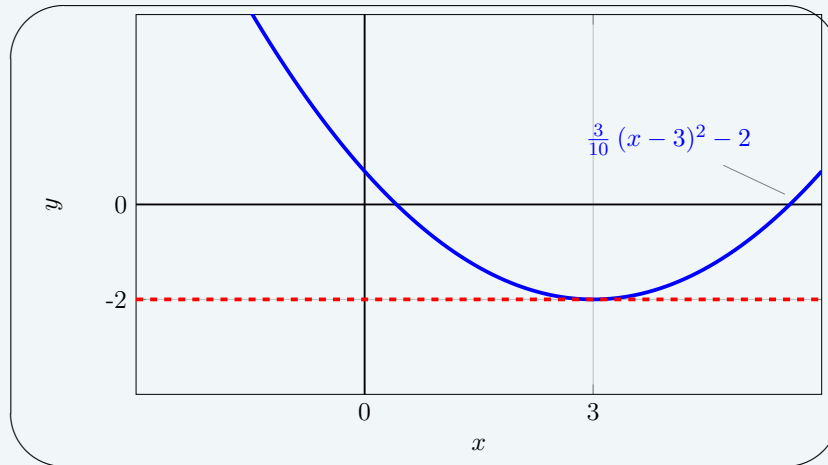
Computing the tangent is important to determine the points where the graph of a function has a **horizontal tangent** (local or global maximum and minimum points, and possibly other points as well).

- It is therefore useful to be able to write analytically the equation of the tangent line to the curve at a generic point, and then see at which points it is horizontal. This idea is due first to Fermat, who developed it around 1630.

Example 1.2: Graph of a function – point with horizontal tangent

Consider the function:

$$f(x) = \frac{3}{10} (x - 3)^2 - 2$$

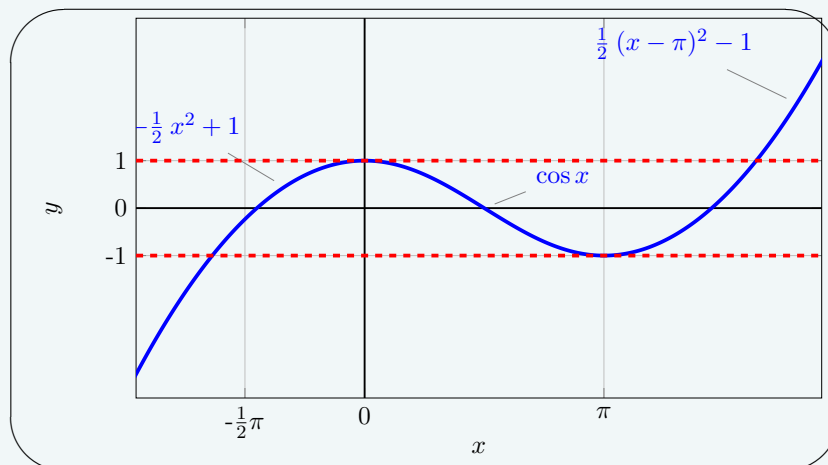


The function has a point with horizontal tangent, which is a global minimum point.

Example 1.3: Graph of a function – points with horizontal tangent

Consider the (piecewise-defined) function:

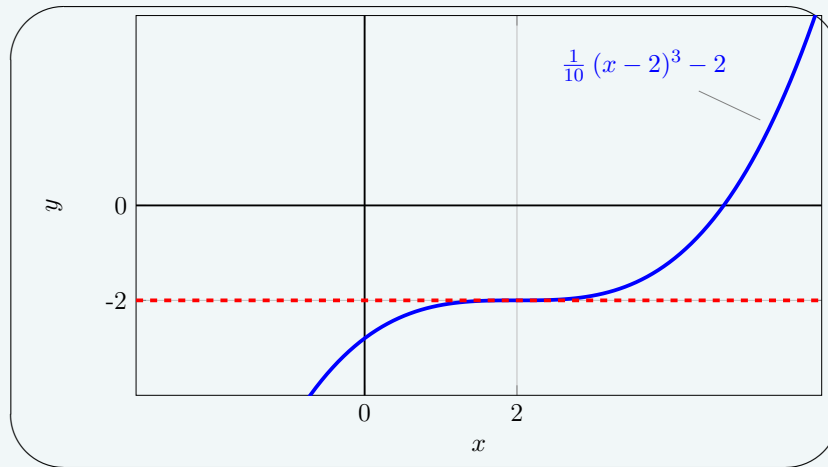
$$f(x) = \begin{cases} -\frac{1}{2}x^2 + 1 & x \leq 0 \\ \cos x & 0 < x \leq \pi \\ \frac{1}{2}(x - \pi)^2 - 1 & x > \pi \end{cases}$$



The function has two points with horizontal tangent, which are local minimum and maximum points, for example in the interval $[-\frac{1}{2}\pi, \pi]$.

Now consider the function:

$$f(x) = \frac{1}{10} (x - 2)^3 - 2$$



The function has a point with horizontal tangent which, however, is neither a maximum nor a minimum point.

1.3 Derivative of a function at a point and derivative function

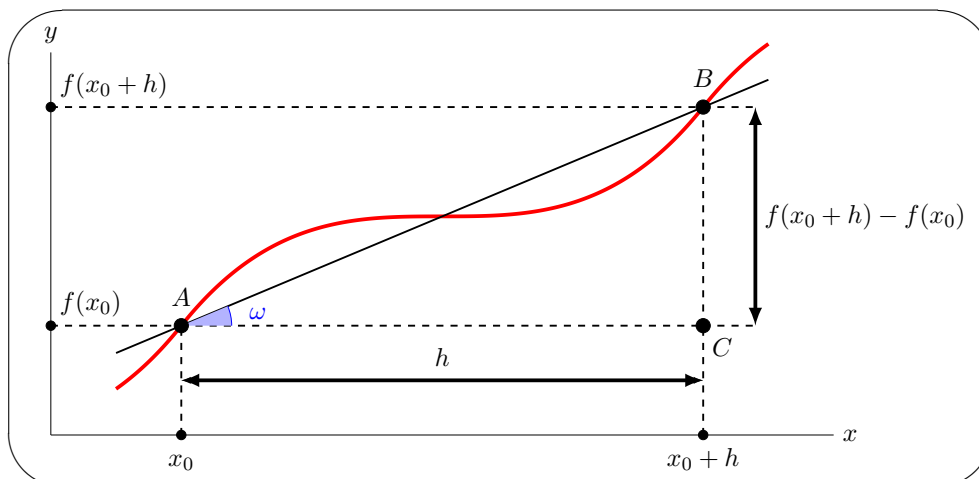
- **Differential calculus** is the study of the notion of derivative, and it is part of infinitesimal calculus.

1.3.1 Derivative and tangent line

We look for the **equation of the tangent line** to the graph of the function at a **generic point** A of its graph.

- Consider the graph of a generic function $y = f(x)$; the coordinates of the points A and B in the figure are respectively:

$$(x_0, f(x_0)) \quad \text{and} \quad (x_0 + h, f(x_0 + h))$$



- The **slope** of the line through the point A and the point B is:

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

Note that a positive slope indicates (on average) a rise, while a negative one indicates a fall (on average).

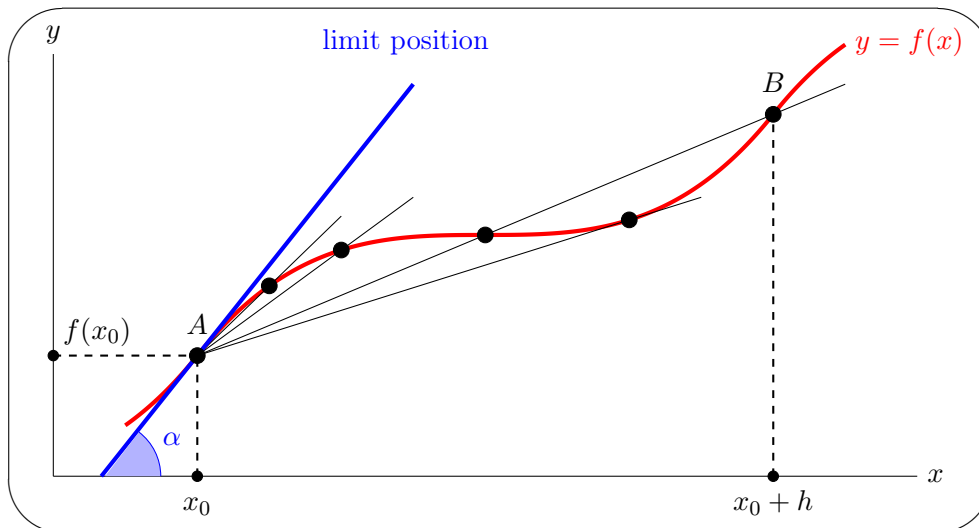
This ratio is called the **difference quotient** of the function f relative to the interval $[x_0, x_0 + h]$.

- Geometrically, since the triangle ABC is a right triangle, we have:

$$\frac{f(x_0 + h) - f(x_0)}{h} = \tan \omega \quad (\text{slope of the line through } A \text{ and } B)$$

- Consider the difference quotient and take the limit (assuming it exists) as $h \rightarrow 0$. Geometrically we have:

1. the point A , with coordinates $(x_0, f(x_0))$, stays fixed
2. while the point B , with coordinates $(x_0 + h, f(x_0 + h))$, moves towards A (staying on the graph of f).



- Moving B towards A along the graph of the function, the line AB changes its slope and settles into a **limit position**.

1. the **limit line** is called the **tangent line to the graph** of the function f at the point with abscissa x_0 ;
2. its **slope** is given by $\tan \alpha$ and is called the **first derivative** of the function f at the point x_0 .
3. The angle α is the angle between the tangent line and the x -axis.

Definition 1.1: of derivative

Let $f : (a, b) \rightarrow \mathbb{R}$; f is said to be differentiable at $x_0 \in (a, b)$ if the following limit exists and is finite

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

this limit is called the first derivative (or simply the derivative) of f at x_0 .

With $h = x - x_0$, we have $h \rightarrow 0 \Leftrightarrow x - x_0 \rightarrow 0 \Leftrightarrow x \rightarrow x_0$ and $x = x_0 + h$, hence:

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- To denote the derivative the following symbols are used:

$$\underbrace{f'(x_0)}_{\text{Lagrange's notation}} \quad \underbrace{\dot{f}(x_0)}_{\text{Newton's notation}} \quad \underbrace{\left. \frac{df}{dx} \right|_{x=x_0} \text{ and } \left. \frac{dy}{dx} \right|_{x=x_0}}_{\text{Leibniz's notation}}$$

Definition 1.2: tangent line

Given a function $f : (a, b) \rightarrow \mathbb{R}$ with f differentiable at $x_0 \in (a, b)$, the line with equation:

$$y = f(x_0) + f'(x_0)(x - x_0)$$

is called the **tangent line** to the graph of the function f at the point $(x_0, f(x_0))$.

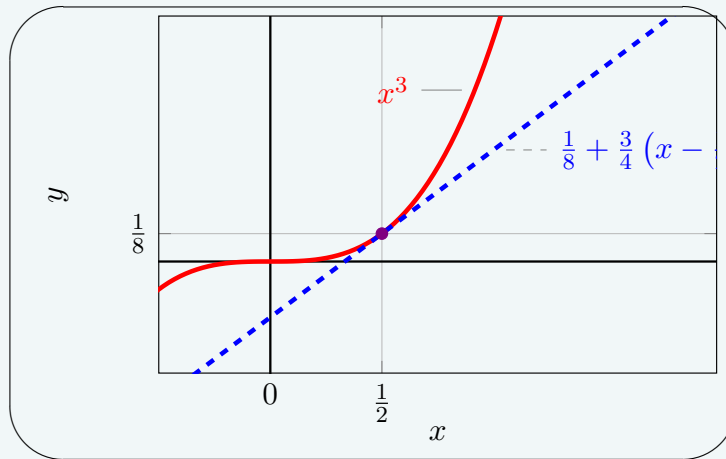
Example 1.4: Tangent line

We compute the equation of the tangent line to the graph of the function $f(x) = x^3$ at the point with abscissa $x_0 = \frac{1}{2}$. The derivative at the point $x_0 = \frac{1}{2}$ is:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} &= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{2} + h\right)^3 - \left(\frac{1}{2}\right)^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{8} + \frac{3}{4}h + \frac{3}{2}h^2 + h^3 - \frac{1}{8}}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{3}{4} + \frac{3}{2}h + h^2\right) = \frac{3}{4} \end{aligned}$$

We have $f\left(\frac{1}{2}\right) = \frac{1}{8}$, $f'\left(\frac{1}{2}\right) = \frac{3}{4}$ and the tangent line at the point $\left(\frac{1}{2}, \frac{1}{8}\right)$ is:

$$y = f(x_0) + f'(x_0)(x - x_0) = \frac{1}{8} + \frac{3}{4}\left(x - \frac{1}{2}\right)$$



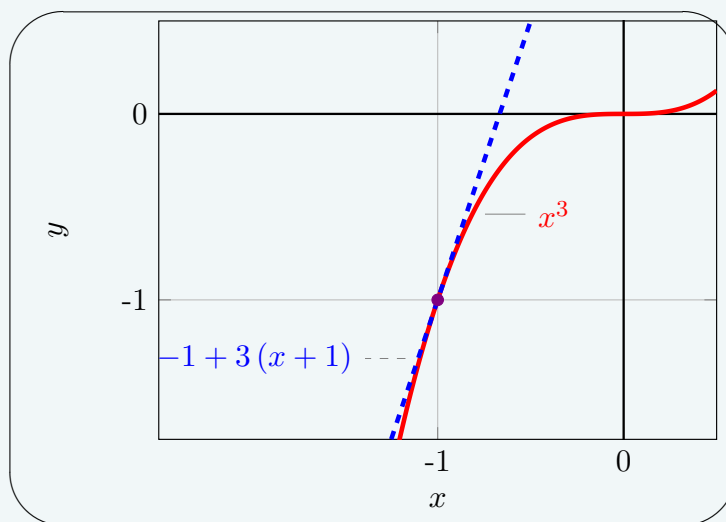
Example 1.5: Tangent line

We compute the equation of the tangent line to the graph of the function $f(x) = x^3$ at the point with abscissa $x_0 = -1$. The derivative at the point $x_0 = -1$ is:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} &= \lim_{h \rightarrow 0} \frac{(-1 + h)^3 - (-1)^3}{h} = \lim_{h \rightarrow 0} \frac{-1 + 3h - 3h^2 + h^3 + 1}{h} \\ &= \lim_{h \rightarrow 0} (3 - 3h + h^2) = 3 \end{aligned}$$

We have $f(-1) = -1$, $f'(-1) = 3$ and the tangent line at the point $(-1, -1)$ is:

$$y = f(x_0) + f'(x_0)(x - x_0) = -1 + 3(x + 1)$$



- We can now define a function that associates to each x the derivative of a function f at the point x (of course, if f is differentiable).

Definition 1.3: of derivative function

If a function f is differentiable at every point of an interval (a, b) , the function:

$$f' : (a, b) \rightarrow \mathbb{R}, \quad f' : x \mapsto f'(x)$$

is called the **derivative function** of f .

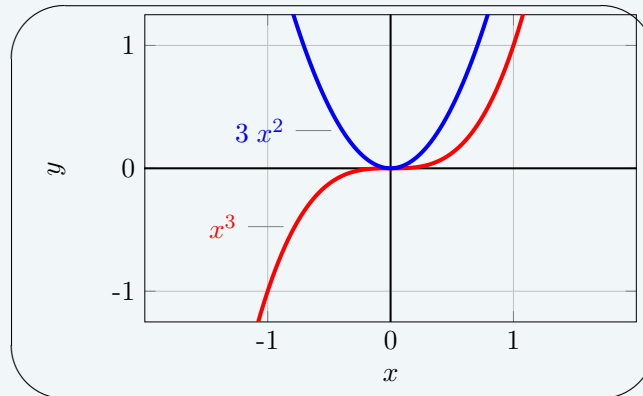
- To denote the first derivative function the following notations are used:

$$\underbrace{f'(x)}_{\text{Lagrange's notation}} \quad \underbrace{\dot{f}(x)}_{\text{Newton's notation}} \quad \underbrace{\frac{df}{dx}, \quad \frac{df(x)}{dx} \quad \text{and} \quad \frac{dy}{dx}}_{\text{Leibniz's notation}}$$

Example 1.6: derivative function

We compute the derivative function of the function $f(x) = x^3$. For every $x \in \mathbb{R}$ we have:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2 \quad \text{hence} \quad f'(x) = 3x^2 \end{aligned}$$



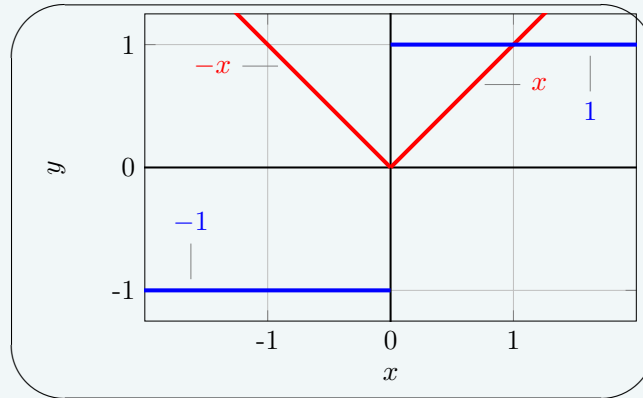
We compute the derivative function of the function $f(x) = |x|$. For $x \neq 0$, we have:

$$f(x) = \begin{cases} x & \text{if } x > 0 \quad \text{and} \\ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{x+h-x}{h} = 1 & \text{hence} \quad f'(x) = 1 \\ -x & \text{if } x < 0 \quad \text{and} \\ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{-(x+h)+x}{h} = -1 & \text{hence} \quad f'(x) = -1 \end{cases}$$

For $x = 0$ we have:

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{f(h) - f(0)}{h} = \frac{|h|}{h} \quad \text{hence} \\ \lim_{h \rightarrow 0^+} \frac{|h|}{h} &= \lim_{h \rightarrow 0^+} \frac{h}{h} = 1, \quad \lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1 \end{aligned}$$

since if $h \rightarrow 0^+$ then $|h| = h$ and if $h \rightarrow 0^-$ then $|h| = -h$. We conclude that, since the limit of the difference quotient does not exist, f is not differentiable at $x = 0$. The function is continuous at the origin but the tangent is not well defined.



1.3.2 Continuity and differentiability

Theorem 1.4

Given a function $f : [a, b] \rightarrow \mathbb{R}$, if f is differentiable at a point $x_0 \in (a, b)$ then it is continuous at x_0 .

Proof. The function is differentiable at $x_0 \in (a, b)$, so we write:

$$f(x_0 + h) - f(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} \cdot h \sim f'(x_0) \cdot h \text{ for } h \rightarrow 0$$

Moreover we have

$$f'(x_0) \cdot h \rightarrow 0 \text{ for } h \rightarrow 0$$

Therefore

$$\lim_{h \rightarrow 0} (f(x_0 + h) - f(x_0)) = 0 \quad \text{hence} \quad \lim_{h \rightarrow 0} f(x_0 + h) = f(x_0)$$

Setting $x_0 + h = x$ we have $h \rightarrow 0$ if and only if $x \rightarrow x_0$, and consequently we have:

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

□

- A possible alternative but equivalent proof is the following:

Proof. The function is differentiable at $x_0 \in (a, b)$, so we write:

$$\begin{aligned} \lim_{h \rightarrow 0} f(x_0 + h) &= \lim_{h \rightarrow 0} f(x_0 + h) - f(x_0) + f(x_0) \\ &= \lim_{h \rightarrow 0} \underbrace{\frac{f(x_0 + h) - f(x_0)}{h}}_{\substack{\rightarrow f'(x_0) \\ \rightarrow 0}} \cdot h + f(x_0) = f(x_0) \end{aligned}$$

Setting $x_0 + h = x$ we have $h \rightarrow 0$ if and only if $x \rightarrow x_0$, and consequently we have:

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

□

Given a function f , differentiability at a point x_0 in the interior of its domain implies continuity at the point x_0 :

$$f \text{ is differentiable at } x_0 \Rightarrow f \text{ is continuous at } x_0 \quad (1.1)$$

but the converse is not true:

$$f \text{ is continuous at } x_0 \nRightarrow f \text{ is differentiable at } x_0 \quad (1.2)$$

a counterexample is the function $f(x) = |x|$, which is continuous at $x_0 = 0$ but not differentiable at $x_0 = 0$. Consequently, **if a function is continuous at x_0 , it is not necessarily differentiable at x_0 .**

- The fact that f is differentiable at x_0 is a sufficient but not necessary condition for f to be continuous at x_0 . Moreover, the fact that f is continuous at x_0 is a necessary but not sufficient condition for f to be differentiable at x_0 .
- From the contrapositive of (1.1) we have:

$$f \text{ is not continuous at } x_0 \Rightarrow f \text{ is not differentiable at } x_0$$

that is, if a function is discontinuous at x_0 it cannot be differentiable at x_0 .

Derivatives of elementary functions

2.1 Derivatives of elementary functions

- In what follows we derive the derivative functions of some of the main elementary functions.

Observation 2.1

Given the function $f(x) = c$ with $c \in \mathbb{R}$ constant, the derivative function is $f'(x) = 0$.

Proof. We have:

$$\frac{f(x+h) - f(x)}{h} = \frac{c - c}{h} = 0$$

hence

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} 0 = 0$$

□

2.1.1 Derivatives of power functions

Observation 2.2

Given the function $f(x) = x^n$ with $n \in \mathbb{N}, n \geq 1$, the derivative function is $f'(x) = n x^{n-1}$.

Proof. From the binomial theorem (Newton's binomial formula) we have:

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^n - x^n}{h} = \frac{\sum_{k=0}^n \binom{n}{k} x^{n-k} h^k - x^n}{h} \\ &= \frac{x^n + n x^{n-1} h + \sum_{k=2}^n \binom{n}{k} x^{n-k} h^k - x^n}{h} \\ &= n x^{n-1} + \frac{\sum_{k=2}^n \binom{n}{k} x^{n-k} h^k}{h}\end{aligned}$$

hence

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} n x^{n-1} + \underbrace{\frac{\sum_{k=2}^n \binom{n}{k} x^{n-k} h^k}{h}}_{\rightarrow 0} = n x^{n-1}$$

□

Proof. With $n = 2$, we have:

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2 x h + h^2 - x^2}{h} = 2 x + h$$

hence

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} 2 x + h = 2 x$$

□

Observation 2.3

Given the function $f(x) = x^\alpha$ with $\alpha \in \mathbb{R}$, the derivative function is $f'(x) = \alpha x^{\alpha-1}$ for $x > 0$.

Proof. Let $x > 0$. We have:

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^\alpha - x^\alpha}{h} = \frac{\left(x \left(1 + \frac{h}{x}\right)\right)^\alpha - x^\alpha}{h} \\ &= x^\alpha \cdot \frac{\left(1 + \frac{h}{x}\right)^\alpha - 1}{h} \sim x^\alpha \cdot \frac{\alpha \frac{h}{x}}{h} = \alpha x^{\alpha-1} \quad \text{for } h \rightarrow 0\end{aligned}$$

where we used the fundamental limit

$$(1 + \varepsilon(h))^\alpha - 1 \sim \alpha \varepsilon(h) \quad \text{for } \varepsilon(h) \rightarrow 0$$

where

$$\varepsilon(h) = \frac{h}{x} \rightarrow 0 \quad \text{for } h \rightarrow 0.$$

Hence

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \alpha x^{\alpha-1} = \alpha x^{\alpha-1}$$

□

Example 2.1: Derivative function

$$f(x) = x^{10} \quad f'(x) = 10 x^9 \quad (\text{for } x \in \mathbb{R})$$

$$f(x) = \frac{1}{x} = x^{-1} \quad f'(x) = -\frac{1}{x^2} \quad (\text{for } x > 0)$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \quad f'(x) = \frac{1}{2\sqrt{x}} \quad (\text{for } x > 0)$$

2.1.2 Derivatives of elementary trigonometric functions

Observation 2.4

Given the function $f(x) = \sin x$, the derivative function is $f'(x) = \cos x$.

Proof. Using the addition formulas we have:

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\sin(x+h) - \sin x}{h} = \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} = \\ &= \sin x \frac{\cos h - 1}{h} + \frac{\sin h}{h} \cos x \end{aligned}$$

Using the fundamental limit

$$\frac{1 - \cos h}{h^2} \rightarrow \frac{1}{2} \quad \text{for } h \rightarrow 0$$

we have

$$\frac{\cos h - 1}{h} = h \cdot \left(\underbrace{-\frac{1 - \cos h}{h^2}}_{\rightarrow -\frac{1}{2} \text{ for } h \rightarrow 0} \right) \sim -\frac{1}{2} h \quad \text{for } h \rightarrow 0.$$

Using the fundamental limit

$$\frac{\sin h}{h} \rightarrow 1 \quad \text{for } h \rightarrow 0$$

we have

$$\frac{\sin h}{h} \cos x \sim \cos x \quad \text{for } h \rightarrow 0.$$

Then

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \sin x \frac{\cos h - 1}{h} + \frac{\sin h}{h} \cos x \\ &= \lim_{h \rightarrow 0} \sin x \left(-\frac{1}{2} h \right) + \cos x = \cos x \end{aligned}$$

□

Observation 2.5

Given the function $f(x) = \cos x$, the derivative function is $f'(x) = -\sin x$.

Proof. Using the addition formulas we have:

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\cos(x+h) - \cos x}{h} = \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} = \\ &= \cos x \frac{\cos h - 1}{h} - \frac{\sin h}{h} \sin x \end{aligned}$$

and, using the arguments of the previous proof, we have:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \cos x \frac{\cos h - 1}{h} - \frac{\sin h}{h} \sin x \\ &= \lim_{h \rightarrow 0} \cos x \left(-\frac{1}{2} h \right) - \sin x = -\sin x \end{aligned}$$

□

2.1.3 Derivatives of the exponential and logarithmic functions with base e

Observation 2.6

Given the function $f(x) = e^x$, the derivative function is $f'(x) = e^x$.

Proof. We have

$$\frac{f(x+h) - f(x)}{h} = \frac{e^{x+h} - e^x}{h} = e^x \cdot \frac{e^h - 1}{h} \sim e^x \text{ for } h \rightarrow 0$$

using the fundamental limit

$$\frac{e^h - 1}{h} \rightarrow 1 \text{ for } h \rightarrow 0.$$

hence

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} e^x = e^x$$

□

Observation 2.7

Given the function $f(x) = \log x$, the derivative function is $f'(x) = \frac{1}{x}$.

Proof. We have:

$$\frac{f(x+h) - f(x)}{h} = \frac{\log(x+h) - \log x}{h} = \frac{\log\left(1 + \frac{h}{x}\right)}{h} \sim \frac{h}{x} \cdot \frac{1}{h} = \frac{1}{x} \quad \text{for } h \rightarrow 0$$

where we used the fundamental limit

$$\log(1 + \varepsilon(h)) \sim \varepsilon(h) \quad \text{for } \varepsilon(h) \rightarrow 0,$$

and

$$\varepsilon(h) = \frac{h}{x} \rightarrow 0 \quad \text{for } h \rightarrow 0.$$

hence

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{x} = \frac{1}{x}$$

□

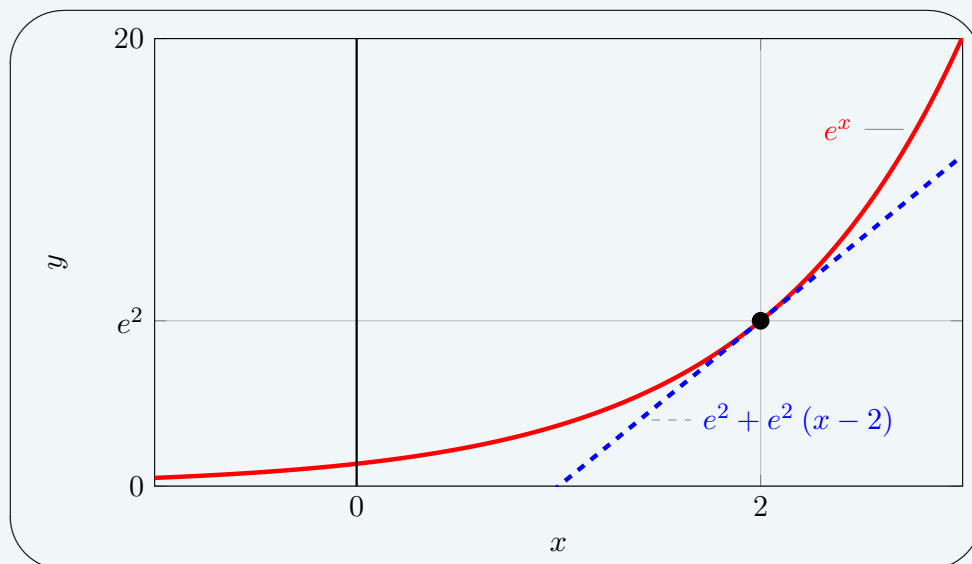
Example 2.2: Tangent line

We compute the equation of the tangent line to the graph of the function

$$f(x) = e^x \quad \text{at the point with abscissa } x = 2$$

We have $f(2) = e^2$, $f'(x) = e^x$, $f'(2) = e^2$, hence the tangent line at the point $(2, e^2)$ is:

$$y = f(2) + f'(2)(x - 2) = e^2 + e^2(x - 2)$$



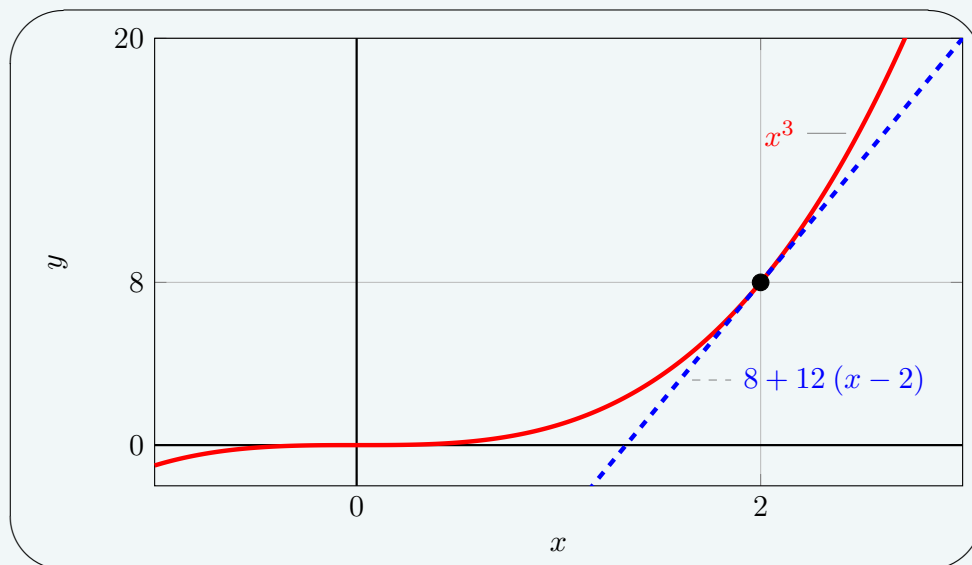
Example 2.3: Tangent line

We compute the equation of the tangent line to the graph of the function

$$f(x) = x^3 \text{ at the point with abscissa } x = 2$$

We have $f(2) = 8$, $f'(x) = 3x^2$, $f'(2) = 12$, hence the tangent line at the point $(2, 8)$ is:

$$y = f(2) + f'(2)(x - 2) = 8 + 12(x - 2)$$



Corner points, cusps, points with vertical/horizontal tangent

3.1 Right derivative and left derivative

Definition 3.1: of right (left) derivative

Let $f : (a, b) \rightarrow \mathbb{R}$; the function f is said to be differentiable at $x_0 \in (a, b)$ from the right (from the left) if the following limit exists and is finite

$$\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \quad \left(\lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} \right)$$

then f is differentiable from the right (from the left) and the limit is called the **right derivative** (**left derivative**).

- The right derivative is denoted by the symbol $f'_+(x_0)$, while the left derivative is denoted by the symbol $f'_-(x_0)$

3.2 Corner points

Definition 3.2: of corner point

If f is continuous and differentiable from the right and from the left (but not differentiable) at x_0 , we say that f has a **corner point** at $x = x_0$.

Example 3.1: Corner points

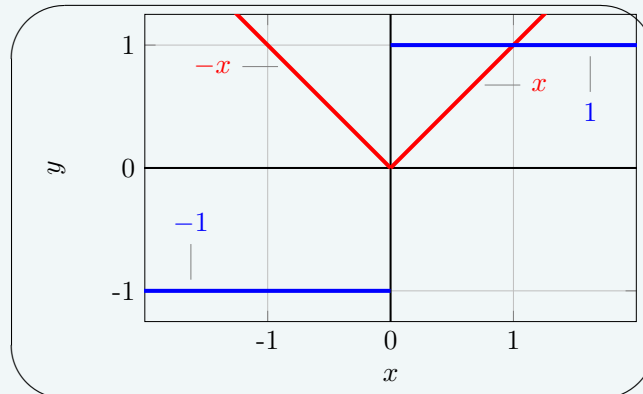
Consider the function $f(x) = |x|$ and the point $x_0 = 0$. We have:

$$f'_+(0) = \lim_{h \rightarrow 0^+} \frac{|h|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1 \quad \text{and} \quad f'_-(0) = \lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1$$

Since the limit of the difference quotient does not exist, f is not differentiable at $x = 0$. The function is continuous at $x = 0$ (the origin) since

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} -x = 0 \quad \text{hence} \quad \lim_{x \rightarrow 0} f(x) = f(0) = 0$$

Since the right and left limits of the difference quotient at $x = 0$ exist and are finite, the graph therefore has a corner point at $x = 0$.



The formula that concisely expresses the derivative of the absolute value function (away from the origin) is the following:

$$f(x) = |x|, \quad f'(x) = \operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

3.3 Points with vertical/horizontal tangent

- If f is continuous at a point x_0 and

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \pm\infty$$

then f is not differentiable at x_0 but, geometrically, the graph of f has a well-defined tangent line parallel to the y -axis.

In this case we will allow the notation

$$f'(x_0) = +\infty, \quad f'(x_0) = -\infty$$

and we will speak of a **point with vertical tangent**.

Definition 3.3: of point with vertical tangent

If

$$f'(x_0) = +\infty \quad \text{or} \quad f'(x_0) = -\infty$$

we say that f has a **point with vertical tangent** at $x = x_0$.

- Similarly, if f is continuous at a point x_0 and

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = 0$$

the graph of f has a well-defined tangent line parallel to the x -axis. In this case we will speak of a **point with horizontal tangent**.

Definition 3.4: of point with horizontal tangent

If

$$f'(x_0) = 0$$

we say that f has a **point with horizontal tangent** at $x = x_0$.

Example 3.2: Point with vertical tangent

Let

$$f(x) = \sqrt[3]{x}$$

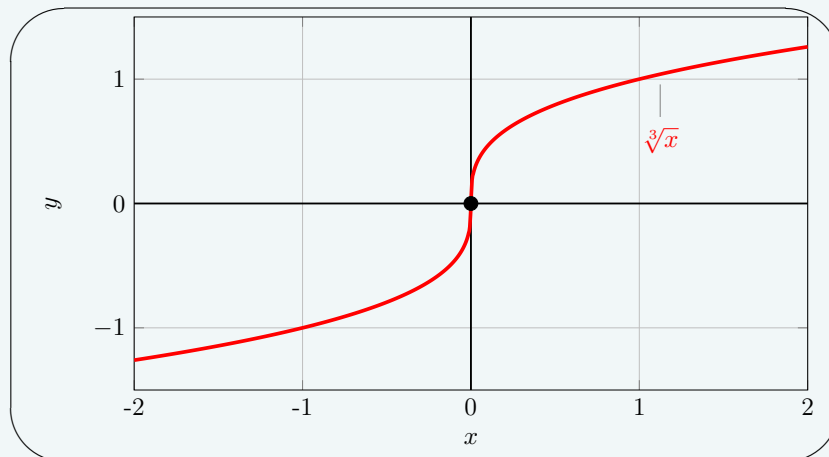
then for $x_0 = 0$

$$\frac{f(h) - f(0)}{h} = \frac{\sqrt[3]{h}}{h} = \frac{1}{h^{2/3}}$$

and the limits are:

$$\lim_{h \rightarrow 0^-} \frac{1}{h^{2/3}} = +\infty \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{1}{h^{2/3}} = +\infty \quad \text{hence}$$

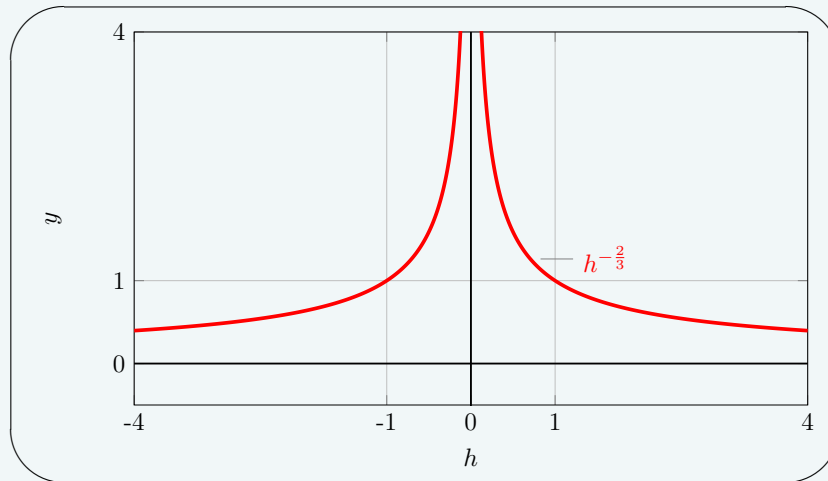
$$\lim_{h \rightarrow 0} \frac{1}{h^{2/3}} = +\infty \quad \text{and} \quad f'(0) = +\infty.$$



The function has a point with vertical tangent at $x_0 = 0$. The function:

$$f(h) = \frac{1}{h^{2/3}} = h^{-\frac{2}{3}}$$

is a power with negative rational exponent $\frac{m}{n}$ with n odd (hence defined on $\mathbb{R} \setminus \{0\}$) and m even (hence an even function). Its graph is:


Example 3.3: Point with vertical tangent

Let

$$f(x) = -\sqrt[3]{x-2} + 1$$

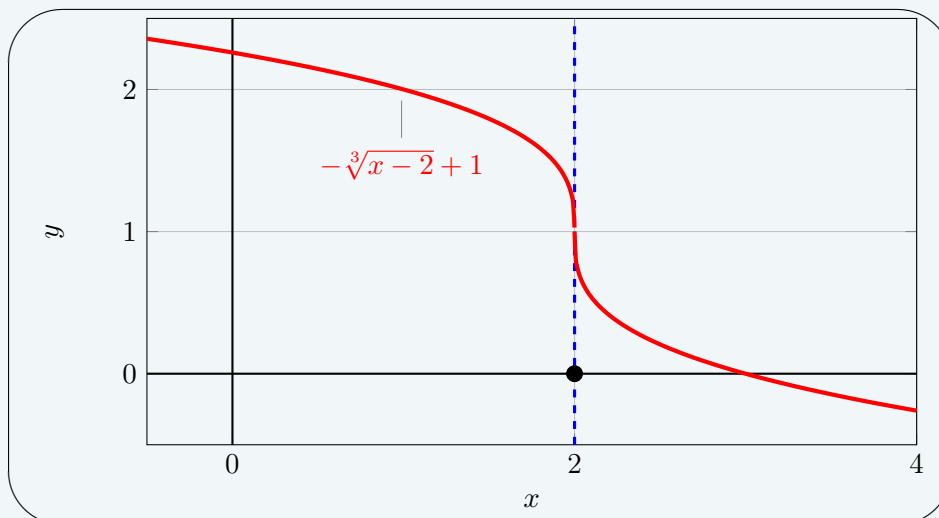
then for $x_0 = 2$

$$\frac{f(2+h) - f(2)}{h} = \frac{-\sqrt[3]{(2+h)-2} + 1 - 1}{h} = -\frac{\sqrt[3]{h}}{h} = -\frac{1}{h^{2/3}}$$

and the limits are

$$\lim_{h \rightarrow 0^-} -\frac{1}{h^{2/3}} = -\infty \quad \text{and} \quad \lim_{h \rightarrow 0^+} -\frac{1}{h^{2/3}} = -\infty \quad \text{hence}$$

$$\lim_{h \rightarrow 0} -\frac{1}{h^{2/3}} = -\infty \quad \text{and} \quad f'(2) = -\infty.$$



The function has a point with vertical tangent at $x_0 = 2$.

3.4 Cusps

Definition 3.5: of cusp

Let f be a function continuous at x_0 ; if

$$f'_+(x_0) = \pm\infty \quad \text{and} \quad f'_-(x_0) = \mp\infty$$

we say that f has a cusp at x_0 .

Example 3.4: Cusp

Let

$$f(x) = \sqrt[3]{|x|}$$

then for $x_0 = 0$

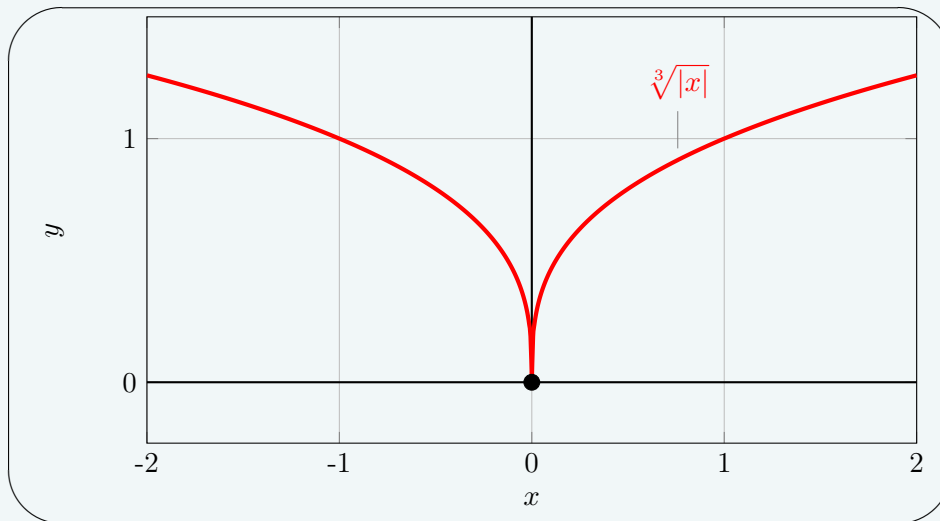
$$\frac{f(h) - f(0)}{h} = \frac{\sqrt[3]{|h|}}{h}$$

and the limits are:

$$\lim_{h \rightarrow 0^-} \frac{\sqrt[3]{-h}}{h} = \lim_{h \rightarrow 0^-} -\frac{1}{h^{2/3}} = -\infty \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{\sqrt[3]{h}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{h^{2/3}} = +\infty,$$

hence:

$$f'_-(0) = -\infty \quad \text{and} \quad f'_+(0) = +\infty.$$



The function has a cusp at $x_0 = 0$.

- In the mixed case where one of the two derivatives is finite and the other is infinite (with f continuous) we still speak of a corner point.

- Finally, if the function is defined only for $x \geq x_0$ and at that point has an infinite (right) derivative, we will simply say that at that point it has a vertical tangent, without speaking of either a cusp or a point with vertical tangent.

Example 3.5: Point with vertical tangent

Let

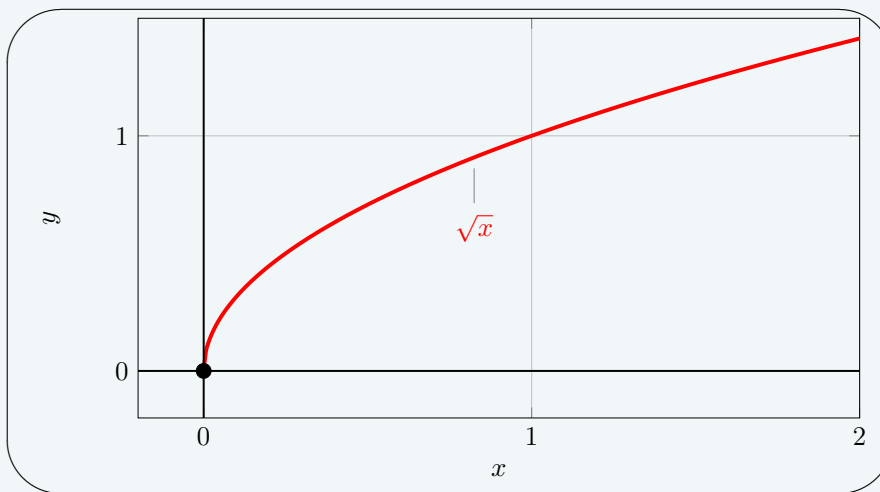
$$f(x) = \sqrt{x}$$

then for $x_0 = 0$

$$\frac{f(h) - f(0)}{h} = \frac{\sqrt{h}}{h} = \frac{1}{h^{1/2}}$$

we only have the right limit

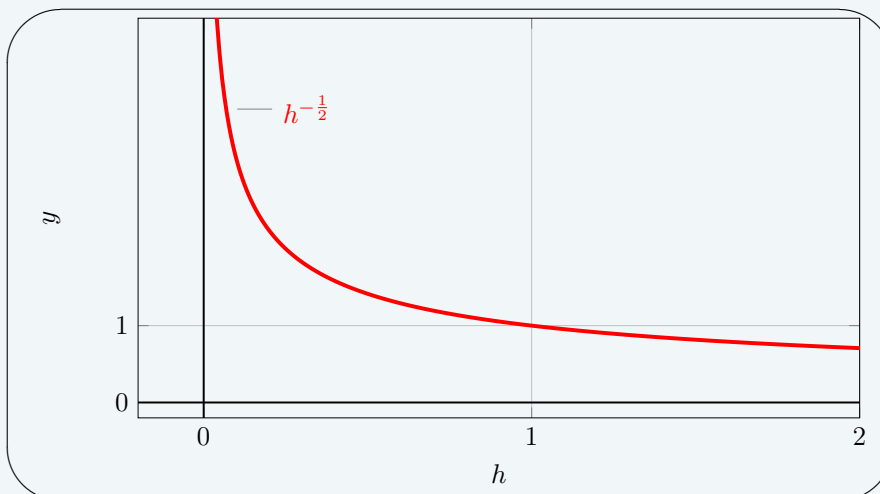
$$\lim_{h \rightarrow 0^+} \frac{1}{h^{1/2}} = +\infty \quad \text{and} \quad f'_+(0) = +\infty.$$



The function has a point with vertical tangent at $x_0 = 0$. The function:

$$f(h) = \frac{1}{h^{1/2}} = h^{-\frac{1}{2}}$$

is a power with negative rational exponent $\frac{m}{n}$ with n even (hence defined only on $\mathbb{R}_+$). Its graph is:



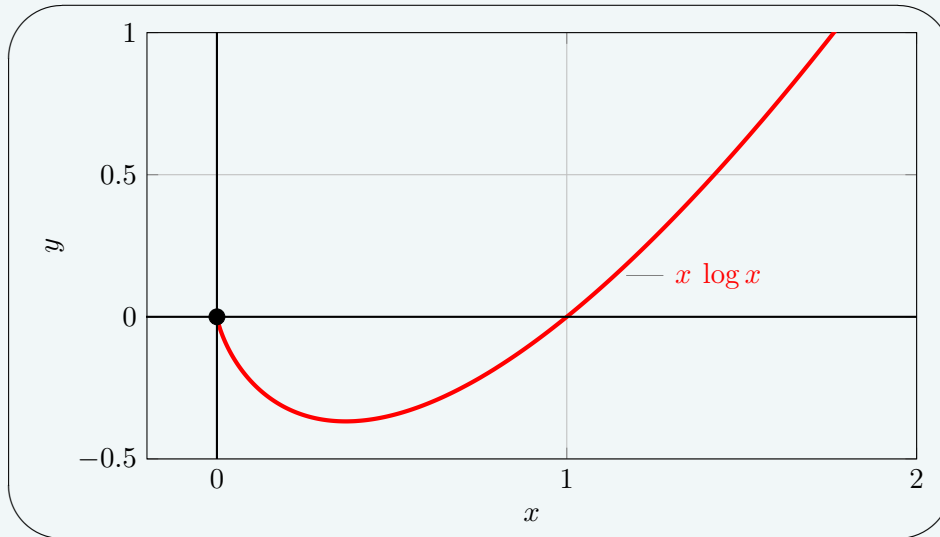
Example 3.6: Continuous extension from the right and behavior at the origin

Let

$$f(x) = x \log x, \text{ for } x > 0 \text{ we have } \lim_{x \rightarrow 0^+} x \log x = 0$$

hence the function can be extended by continuity from the right at $x = 0$, by setting:

$$f(x) = \begin{cases} x \log x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$



We compute the right derivative at $x_0 = 0$ of $f(x) = x \log x$ extended by continuity from the right at $x = 0$. We have, for $x_0 = 0$:

$$\frac{f(x_0 + h) - \overbrace{f(x_0)}^{=0}}{h} = \frac{f(x_0 + h)}{h} = \frac{f(h)}{h}$$

and consequently

$$\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{h \log h}{h} = \lim_{h \rightarrow 0^+} \log h = -\infty \quad \text{hence} \quad f'_+(x_0) = -\infty$$

The function therefore has a point with vertical tangent at $x_0 = 0$.

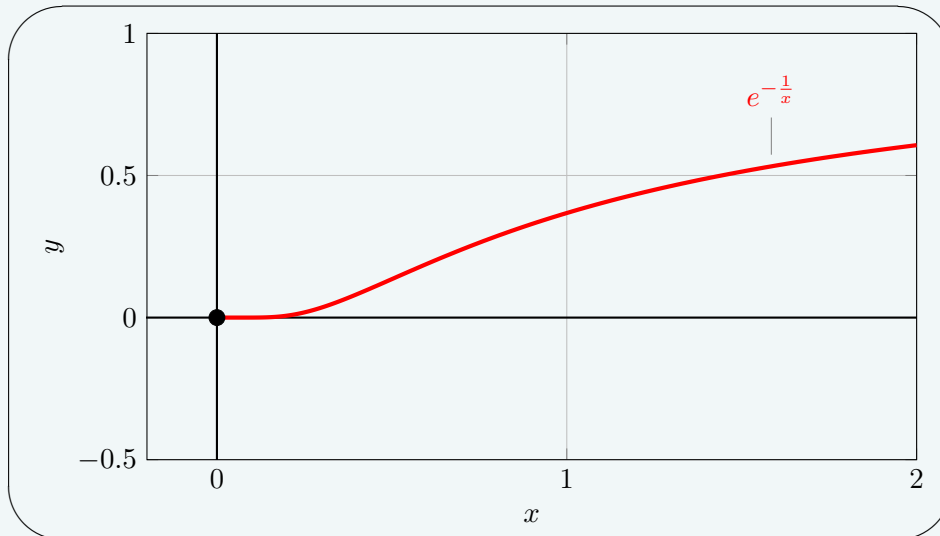
Example 3.7: Continuous extension from the right and behavior at the origin

Let

$$f(x) = e^{-\frac{1}{x}}, \text{ for } x > 0 \text{ we have } \lim_{x \rightarrow 0^+} e^{-\frac{1}{x}} = 0$$

hence the function can be extended by continuity from the right at $x = 0$, by setting:

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$



We compute the right derivative at $x_0 = 0$ of $f(x) = e^{-\frac{1}{x}}$ extended by continuity from the right at $x = 0$.

We have:

$$\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{e^{-\frac{1}{h}}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{e^{\frac{1}{h}} h}$$

Changing the variable:

$$y = \frac{1}{h}, \text{ if } h \rightarrow 0^+ \text{ then } y \rightarrow +\infty$$

We obtain:

$$\lim_{h \rightarrow 0^+} \frac{1}{e^{\frac{1}{h}} h} = \lim_{y \rightarrow +\infty} \frac{y}{e^y} = 0 \quad \text{hence} \quad f'_+(x_0) = 0$$

and the function has a point with horizontal tangent (from the right) at $x_0 = 0$.

Rules for computing derivatives

4.1 Rules for computing derivatives

- We now look at the relation between the derivative operation and the main operations on functions that we already know; in particular we will show the relation between:
 1. differentiation and algebraic operations
 2. differentiation and composition
 3. differentiation and inversion

4.1.1 Algebra of derivatives

Theorem 4.1: algebra of derivatives

Let $f, g: (a, b) \rightarrow \mathbb{R}$ be two functions differentiable in (a, b) .

Then $f \pm g$, $f \cdot g$, f/g ($g \neq 0$) are differentiable in (a, b) and the following formulas hold:

$$(f \pm g)' = f' \pm g' \quad (4.1)$$

$$(f \cdot g)' = f' \cdot g + f \cdot g' \quad (4.2)$$

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2} \quad (g \neq 0) \quad (4.3)$$

- From rule (4.2) we deduce

$$(k \cdot g)' = k \cdot g' \quad \text{with } k \in \mathbb{R} \text{ constant} \quad (4.4)$$

since the derivative of a constant is equal to zero.

- From rule (4.3) we deduce

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2} \quad (g \neq 0) \quad (4.5)$$

- Rule (4.2) is called the **Leibniz rule** (product rule) and it extends to the product of n functions:

$$(f_1 f_2 \cdots f_n)' = f_1' f_2 \cdots f_n + f_1 f_2' \cdots f_n + \cdots + f_1 f_2 \cdots f_n' \quad (4.6)$$

- The theorem actually has a pointwise character: that is, if f and g are differentiable at a point $x_0 \in (a, b)$, then at that point $f \pm g$, $f \cdot g$, f/g are also differentiable, and the formulas above hold (and they extend to intervals).

Proof. We prove rule (4.2). Fixing $x \in (a, b)$, we have

$$\begin{aligned} & f(x+h)g(x+h) - f(x)g(x) = \\ & = f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x), \end{aligned}$$

and hence the difference quotient can be written as:

$$\begin{aligned} & \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \\ & = f(x+h) \underbrace{\frac{g(x+h) - g(x)}{h}}_{=g'(x) \text{ for } h \rightarrow 0} + g(x) \underbrace{\frac{f(x+h) - f(x)}{h}}_{=f'(x) \text{ for } h \rightarrow 0} \end{aligned}$$

We have

$$f(x+h) \rightarrow f(x) \quad \text{for } h \rightarrow 0,$$

since f is continuous, being differentiable.

Hence

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \rightarrow f(x)g'(x) + f'(x)g(x) \quad \text{for } h \rightarrow 0.$$

□

Proof. We prove rule (4.5). Fixing $x \in (a, b)$, the difference quotient can be written as:

$$\begin{aligned} & \frac{1}{h} \left[\frac{1}{g(x+h)} - \frac{1}{g(x)} \right] = \frac{g(x) - g(x+h)}{h g(x) g(x+h)} = \\ & = -\frac{g(x+h) - g(x)}{h} \cdot \frac{1}{g(x) g(x+h)} \end{aligned}$$

We have

$$g(x+h) \rightarrow g(x) \quad \text{for } h \rightarrow 0,$$

since g is continuous, being differentiable.

Hence

$$\frac{1}{h} \left[\frac{1}{g(x+h)} - \frac{1}{g(x)} \right] \rightarrow -\frac{g'(x)}{g^2(x)} \text{ for } h \rightarrow 0.$$

□

Proof. Note that rule (4.3) follows from rules (4.2) and (4.5); indeed:

$$\begin{aligned} \left(\frac{f(x)}{g(x)} \right)' &= \left(f(x) \cdot \frac{1}{g(x)} \right)' = \underbrace{f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \left(\frac{1}{g(x)} \right)'}_{\text{rule (4.2)}} = \\ &= f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \underbrace{\left(-\frac{g'(x)}{g^2(x)} \right)}_{\text{rule (4.5)}} = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}. \end{aligned}$$

□

Observation 4.2

Given the function $f(x) = \tan x$, the derivative function is $f'(x) = \frac{1}{\cos^2 x} = 1 + \tan^2 x$.

Proof. Using formula (4.3) we have

$$f'(x) = \left(\frac{\sin x}{\cos x} \right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = 1 + \tan^2 x$$

□

Observation 4.3

Given the function $f(x) = \cot x$, the derivative function is $f'(x) = -\frac{1}{\sin^2 x} = -(1 + \cot^2 x)$.

Proof. Using formula (4.3) we have

$$\begin{aligned} f'(x) &= \left(\frac{\cos x}{\sin x} \right)' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} \\ &= -\frac{1}{(\sin^2 x)} = -(1 + \cot^2 x) \end{aligned}$$

□

4.1.2 Derivative of a composite function

Theorem 4.4: chain rule

Let $g \circ f$ be the composite function of two functions f and g . If f is differentiable at a point x and g is differentiable at $y = f(x)$, then $g \circ f$ is differentiable at x and the following formula holds:

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \quad (4.7)$$

Proof. We have

$$(g \circ f)(x+h) - (g \circ f)(x) = g(f(x+h)) - g(f(x))$$

If we set

$$k = f(x+h) - f(x) \quad \text{and} \quad y = f(x) \quad \text{then} \quad f(x+h) = y + k$$

By the continuity of f , $h \rightarrow 0$ implies $k \rightarrow 0$. With the new notation we have:

$$g(f(x+h)) - g(f(x)) = g(y+k) - g(y)$$

We now observe that the definition of derivative:

$$g'(y) = \lim_{k \rightarrow 0} \frac{g(y+k) - g(y)}{k}$$

can be rewritten, for $k \neq 0$, as

$$\frac{g(y+k) - g(y)}{k} = g'(y) + \varepsilon(k)$$

where $\varepsilon(k)$ denotes a quantity that tends to zero as $k \rightarrow 0$.

Multiplying both sides of the previous equation by k we find

$$g(y+k) - g(y) = k \cdot g'(y) + k \cdot \varepsilon(k)$$

a relation that also holds for $k = 0$. Therefore:

$$g(f(x+h)) - g(f(x)) = k \cdot g'(y) + k \cdot \varepsilon(k)$$

Dividing by h , and observing that

$$\frac{k}{h} \rightarrow f'(x) \quad \text{for} \quad h \rightarrow 0$$

which implies, as seen above, also $k \rightarrow 0$, we obtain the claim. □

- As the name “chain rule” suggests, (4.7) can be generalized to the composition of any number of functions, composed one with another. For example, for three functions we have:

$$\left(f \left(g \left(h(x) \right) \right) \right)' = f' \left(g \left(h(x) \right) \right) \cdot g' \left(h(x) \right) \cdot h'(x)$$

Example 4.1: Derivative of a composite function

We compute the derivative function of the composite function:

$$h(x) = \underbrace{\sin^3 x}_{=(g \circ f)(x)}$$

The two functions are:

$$f(x) = \sin x \quad \text{and} \quad g(y) = y^3 \quad \text{with} \quad y = \sin x.$$

Both functions are differentiable on all of $\mathbb{R}$. We have

$$g'(y) = 3y^2 \quad \text{and} \quad f'(x) = \cos x$$

hence, using the chain rule (4.7) and substituting back, we obtain:

$$h'(x) = 3 \sin^2 x \cdot \cos x.$$

Consider the composite function $g(f(x))$, equivalently written $(g \circ f)(x)$. Setting $y = f(x)$ and $w = g(y)$, and using the (Leibniz) notation

$$\frac{df}{dx} \quad \text{and} \quad \frac{dg}{dy} \quad \text{for the derivatives of } f \text{ and } g \text{ with respect to } x$$

the (chain) rule (4.7) can be rewritten as:

$$\frac{dw}{dx} = \frac{dw}{dy} \cdot \frac{dy}{dx} \quad (4.8)$$

Rule (4.8) expresses the fact that the rate of change of w with respect to x is the product of the “intermediate” rates of change, of w with respect to y and of y with respect to x .

Example 4.2: Derivative of a composite function

Consider the composite function of the previous exercise. Setting

$$y = \sin x \quad (y = f(x)) \quad \text{and} \quad w = y^3 \quad (w = g(y))$$

and using Leibniz’s notation, rule (4.8) would be written:

$$\frac{dw}{dx} = \frac{dw}{dy} \cdot \frac{dy}{dx} = 3y^2 \cdot \cos x.$$

And hence, substituting $y = \sin x$, we obtain:

$$\frac{dw}{dx} = 3 \sin^2 x \cdot \cos x.$$

Example 4.3: Derivative of a composite function

We compute the derivative function of the composite function (multiplied by a constant):

$$h(x) = A \cdot \underbrace{\sin(\omega x + \varphi)}_{=(g \circ f)(x)} \quad \text{with} \quad A, \omega, \varphi \in \mathbb{R}.$$

The two functions are:

$$f(x) = \omega x + \varphi \quad \text{and} \quad g(y) = \sin y \quad \text{with} \quad y = \omega x + \varphi.$$

Both functions are differentiable on all of $\mathbb{R}$. We have:

$$f'(x) = \omega \quad \text{and} \quad g'(y) = \cos y$$

hence, using rule (4.7), rule (4.4) and substituting back, we obtain:

$$h'(x) = A \cdot \left(\cos(\omega x + \varphi) \cdot \omega \right)$$

Example 4.4: Derivative of a product of composite functions

We compute the derivative function of the product of two composite functions times a constant:

$$h(x) = A \cdot \underbrace{e^{-\alpha x}}_{=(r \circ s)(x)} \cdot \underbrace{\cos(\omega x + \varphi)}_{=(g \circ f)(x)} \quad \text{with} \quad A, \omega, \varphi \in \mathbb{R}, \alpha \in \mathbb{R}_+.$$

The functions are:

$$\begin{aligned} s(x) &= -\alpha x & \text{and} & & r(z) &= e^z & \text{with} & & z &= -\alpha x; \\ f(x) &= \omega x + \varphi & \text{and} & & g(y) &= \cos y & \text{with} & & y &= \omega x + \varphi. \end{aligned}$$

The functions are differentiable on all of $\mathbb{R}$. We have:

$$s'(x) = -\alpha \quad \text{and} \quad r'(z) = e^z; \quad f'(x) = \omega \quad \text{and} \quad g'(y) = -\sin y$$

consequently, using the chain rule (4.7) and substituting back, we obtain:

$$(r \circ s)'(x) = -\alpha e^{-\alpha x}; \quad (g \circ f)'(x) = -\sin(\omega x + \varphi) \cdot \omega$$

Hence, using rule (4.2) and rule (4.4):

$$h'(x) = A \cdot \left(\underbrace{-\alpha e^{-\alpha x}}_{(r \circ s)'(x)} \cdot \underbrace{\cos(\omega x + \varphi)}_{(g \circ f)(x)} + \underbrace{e^{-\alpha x}}_{(r \circ s)(x)} \cdot \underbrace{\left(-\sin(\omega x + \varphi) \cdot \omega \right)}_{(g \circ f)'(x)} \right)$$

Example 4.5: Derivative of a composite function

Let $f(x) > 0$ be differentiable; we compute the derivative of the function

$$h(x) = \log(f(x))$$

using the chain rule (4.7) we have:

$$h'(x) = \frac{1}{f(x)} \cdot f'(x)$$

Observation 4.5

Given the function $f(x) = a^x$, with $a > 0$, the derivative function is $f'(x) = a^x \cdot \log a$.

Proof. We have

$$f'(x) = \left(\exp(x \cdot \log a) \right)'$$

hence, using the formulas for the derivative of e^y (with $y = x \log a$) and for the derivative of a composite function, we have

$$f'(x) = \exp(x \cdot \log a) \cdot \log a = a^x \cdot \log a$$

□

Observation 4.6

Given the function $f(x) = \log_a x$, with $a > 0, a \neq 1$, the derivative function is $f'(x) = \frac{1}{x \log a}$.

Proof. We have

$$f'(x) = \left(\frac{\log x}{\log a} \right)'$$

hence, using the formulas for the derivative of $\log x$ and for the derivative of $k f(x)$, we have:

$$f'(x) = \frac{1}{\log a} \cdot \frac{1}{x} = \frac{1}{x \log a}$$

□

Observation 4.7

Given the function $f(x) = \sinh x$, the derivative function is $f'(x) = \cosh x$.

Proof. We have:

$$f'(x) = \left(\frac{e^x - e^{-x}}{2} \right)' = \frac{1}{2} \left(e^x - (-e^{-x}) \right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

□

Observation 4.8

Given the function $f(x) = \cosh x$, the derivative function is $f'(x) = \sinh x$.

Proof. We have:

$$f'(x) = \left(\frac{e^x + e^{-x}}{2} \right)' = \frac{1}{2} \left(e^x + (-e^{-x}) \right) = \frac{e^x - e^{-x}}{2} = \sinh x$$

□

The derivatives of functions of the type:

$$h(x) = f(x)^{g(x)}$$

are based on rewriting the function (with $f(x) > 0$) in the following form:

$$f(x)^{g(x)} = \exp \left(g(x) \cdot \log(f(x)) \right)$$

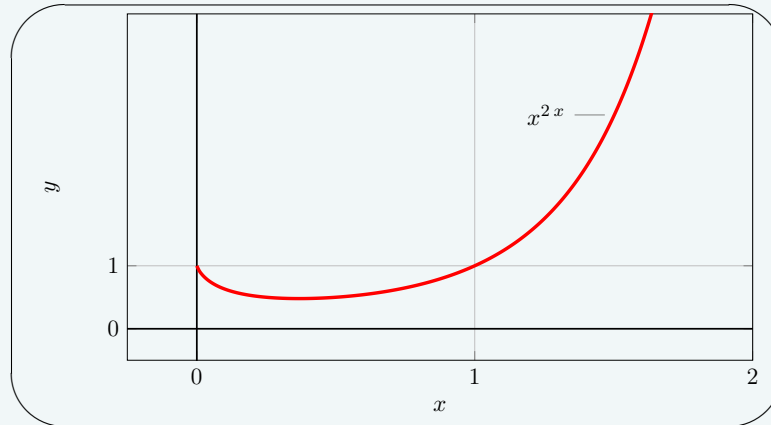
We have:

$$\begin{aligned} \left(f(x)^{g(x)} \right)' &= \left(\exp \left(g(x) \cdot \log(f(x)) \right) \right)' \\ &= \exp \left(g(x) \cdot \log(f(x)) \right) \cdot \left(g(x) \cdot \log(f(x)) \right)' \\ &= f(x)^{g(x)} \cdot \left(g'(x) \cdot \log(f(x)) + g(x) \cdot \frac{f'(x)}{f(x)} \right) \end{aligned}$$

Example 4.6: Derivative of a function raised to another function

We compute the derivative of the function

$$h(x) = x^{2x} = \exp(2x \cdot \log x) \quad \text{with } x > 0$$



The two functions are:

$$f(x) = x \quad \text{and} \quad g(x) = 2x$$

Hence:

$$\begin{aligned} h'(x) &= \left(\exp(2x \cdot \log x) \right)' = \exp(2x \cdot \log x) \cdot (2x \cdot \log x)' \\ &= x^{2x} \cdot \left(2 \log x + 2x \frac{1}{x} \right) = x^{2x} (2 \log x + 2) \end{aligned}$$

Consider the absolute value of a function: $|f(x)|$. At the points where $f(x) \neq 0$, the differentiation of a composite function gives:

$$(|f(x)|)' = \operatorname{sgn}(f(x)) \cdot f'(x) = \begin{cases} f'(x) & \text{for } f(x) > 0 \\ -f'(x) & \text{for } f(x) < 0 \end{cases}$$

In general, we expect the function $|f(x)|$ to have corner points at the points where $f(x)$ vanishes.

Example 4.7: Derivative of a function with absolute value

We compute the derivative function of the function:

$f(x) = |x^2 - 4x + 3|$, with $x \neq 1$ and $x \neq 3$, we have $f'(x) = \operatorname{sgn}(x^2 - 4x + 3) \cdot (2x - 4)$ equivalently:

$$f'(x) = \begin{cases} 2x - 4, & x < 1 \text{ or } x > 3 \quad \text{i.e. for } f(x) > 0 \\ -(2x - 4) & 1 < x < 3 \quad \text{i.e. for } f(x) < 0 \end{cases}$$

At $x_0 = 1$, we need to use the limit of the difference quotient:

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{f(1 + h) - f(1)}{h} = \frac{|(1 + h)^2 - 4(1 + h) + 3|}{h} = \frac{|h^2 - 2h|}{h}$$

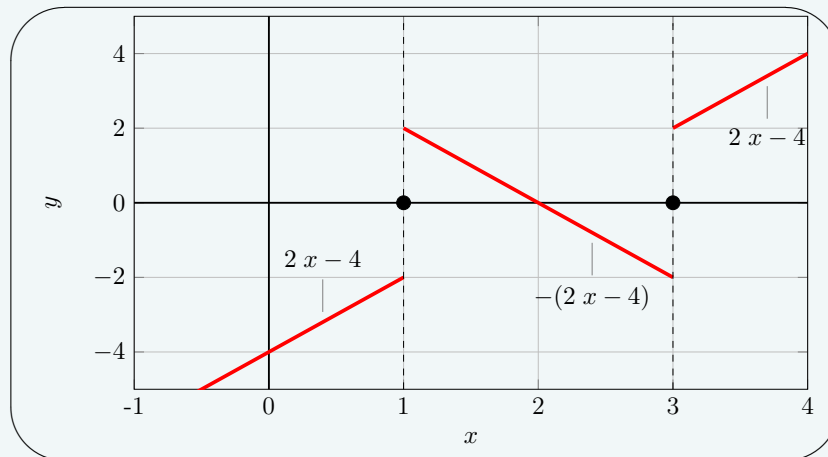
If $h \rightarrow 0^-$ we have $|h^2 - 2h| = h^2 - 2h$, hence:

$$\lim_{h \rightarrow 0^-} \frac{h^2 - 2h}{h} = -2$$

If $h \rightarrow 0^+$ we have $|h^2 - 2h| = -h^2 + 2h$, hence:

$$\lim_{h \rightarrow 0^+} \frac{-h^2 + 2h}{h} = 2$$

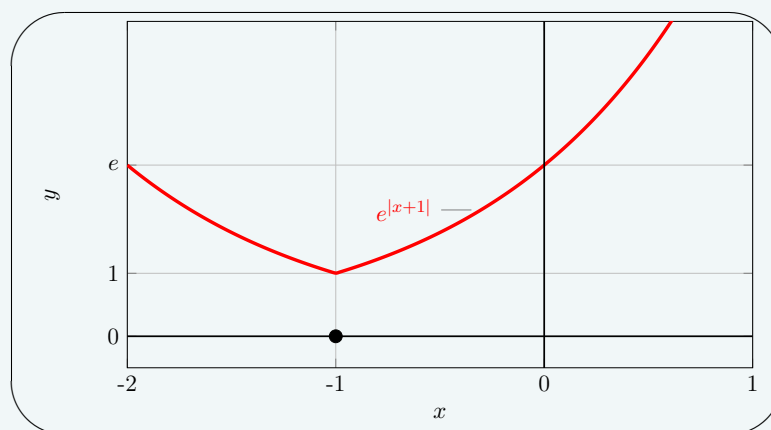
The function is not differentiable at $x_0 = 1$ and $x_0 = 3$ (by arguments analogous to those for $x_0 = 1$), where it has corner points.



Example 4.8: Derivative of a function with absolute value

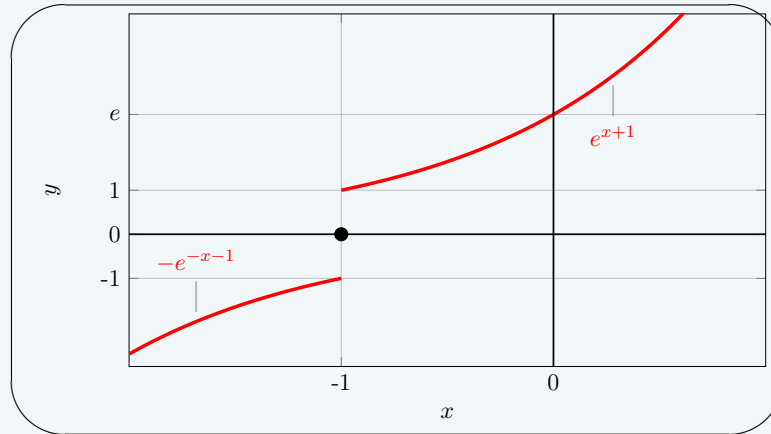
We compute the derivative of the function:

$$f(x) = e^{|x+1|}$$



For $x \neq -1$, we have:

$$f(x) = \begin{cases} e^{x+1}, & x > -1 \\ e^{-x-1}, & x < -1 \end{cases} \quad \text{hence} \quad f'(x) = \begin{cases} e^{x+1}, & x > -1 \\ -e^{-x-1}, & x < -1 \end{cases}$$



At $x_0 = -1$, we need to use the limit of the difference quotient:

$$\frac{f(x_0 + h) - f(x_0)}{h} = \frac{f(-1 + h) - f(-1)}{h} = \frac{e^{|-1+h+1|} - 1}{h} = \frac{e^{|h|} - 1}{h}$$

We have:

$$\begin{aligned} \text{if } h \rightarrow 0^+, e^{|h|} &= e^h \text{ and } \lim_{h \rightarrow 0^+} \frac{e^h - 1}{h} = 1; \\ \text{if } h \rightarrow 0^-, e^{|h|} &= e^{-h} \text{ and } \lim_{h \rightarrow 0^-} \frac{e^{-h} - 1}{h} = -1 \end{aligned}$$

We conclude that, since the limit of the difference quotient does not exist, the function is not differentiable at $x_0 = -1$. The function has a corner point at $x_0 = -1$.

We now consider the derivatives of some logarithmic functions. We compute:

$$\left(\log(|x|) \right)' = \frac{1}{|x|} \cdot \text{sgn}(x) = \frac{1}{x}$$

Moreover, by the differentiation of composite functions we have:

$$\left(\log(|f(x)|) \right)' = \frac{1}{|f(x)|} \cdot \text{sgn}(f(x)) \cdot f'(x) = \frac{f'(x)}{f(x)}$$

Sometimes it is convenient to use the properties of logarithms to transform a logarithmic function before computing its derivative function:

$$(\log(cx))' = (\log(c) + \log(x))' = \frac{1}{x} \quad \text{with } c \in \mathbb{R}$$

$$\left(\log\left(\frac{ax+b}{cx+d}\right) \right)' = (\log(ax+b) - \log(cx+d))' = \frac{a}{ax+b} - \frac{c}{cx+d} \quad \text{with } a, b, c, d \in \mathbb{R}$$

4.1.3 Derivative of the inverse function

Theorem 4.9: derivative of the inverse function

Let $f : (a, b) \rightarrow \mathbb{R}$ be a continuous and invertible function in (a, b) and $g = f^{-1}$ its inverse function, defined in $f((a, b))$.

Assume moreover that $f'(x_0) \neq 0$ exists for some $x_0 \in (a, b)$.

Then g is differentiable at $y_0 = f(x_0)$ and

$$g'(y_0) = \frac{1}{f'(x_0)} \quad (4.9)$$

Proof. Let:

$$f(x_0) = y_0; \quad g(y_0) = x_0$$

$$f(x_0 + h) = y_0 + k; \quad g(y_0 + k) = x_0 + h$$

Consider the difference quotient of g at y_0 :

$$\frac{g(y_0 + k) - g(y_0)}{k} = \frac{h}{f(x_0 + h) - f(x_0)}$$

If $k \neq 0$, then $f(x_0 + h) - f(x_0) \neq 0$ and hence also $h \neq 0$; therefore the last quotient can also be rewritten in the form:

$$\frac{1}{\frac{f(x_0 + h) - f(x_0)}{h}}$$

Moreover, for $k \rightarrow 0$ we have $g(y_0 + k) \rightarrow g(y_0)$ because g is continuous, being the inverse of a continuous function on an interval (theorem on the continuity of inverse functions); on the other hand $h = g(y_0 + k) - g(y_0)$, hence for $k \rightarrow 0$ also $h \rightarrow 0$, and by hypothesis

$$\frac{1}{\frac{f(x_0 + h) - f(x_0)}{h}} \rightarrow \frac{1}{f'(x_0)} \quad \text{for } h \rightarrow 0$$

hence the following limit exists

$$\lim_{k \rightarrow 0} \frac{g(y_0 + k) - g(y_0)}{k} = \frac{1}{f'(x_0)}$$

□

- We observe that, assuming the differentiability of f^{-1} , (4.9) follows immediately from the identity

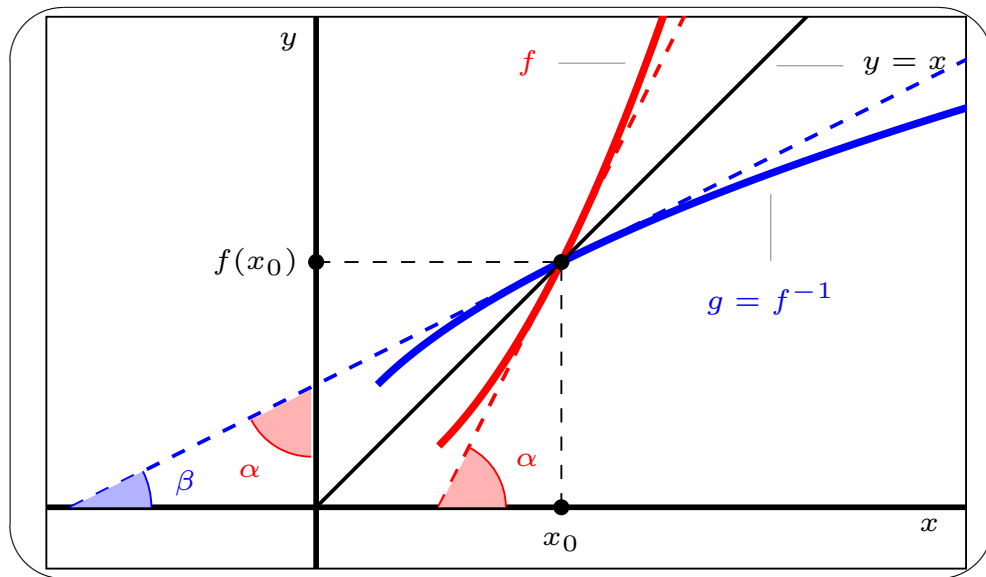
$$g(f(x)) = x$$

and from the chain rule:

$$g'(f(x)) \cdot f'(x) = 1$$

from which, if $f'(x) \neq 0$, (4.9) follows. The proof we have given, however, is necessary in order to deduce the differentiability of g from our hypotheses on f .

- (4.9) has a simple geometric meaning, recalling that the graphs of f and $g = f^{-1}$ are symmetric with respect to the bisector $y = x$. Geometrically we have:



hence the angles α and β are complementary, that is:

$$\left(\alpha + \beta = \frac{\pi}{2}\right)$$

and hence

$$f'(x) = \tan \alpha = \tan \left(\frac{\pi}{2} - \beta\right) = \frac{1}{\tan \beta} = \frac{1}{g'(y)}$$

With Leibniz's notation, setting $y = f(x)$ and $x = g(y)$, (4.9) is written in the form

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

- Pay attention to the fact that in the formula for the derivative of the inverse function, the derivatives f' and g' are computed at two different points: this is the main thing to be careful about when applying this theorem.

Observation 4.10

Given the function $f(y) = \arctan y$, the derivative function is $f'(y) = \frac{1}{1+y^2}$.

Proof. We set $y = \tan x$, $x = \arctan y$ with $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $y \in \mathbb{R}$. We have:

$$(\arctan y)' = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + y^2}$$

□

Observation 4.11

Given the function $f(y) = \arcsin y$, the derivative function is $f'(y) = \frac{1}{\sqrt{1-y^2}}$

Proof. We set $y = \sin x$, $x = \arcsin y$, with $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $y \in [-1, 1]$. Since for those values of x we have

$$\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - y^2}$$

we have:

$$(\arcsin y)' = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - y^2}}$$

□

Observation 4.12

Given the function $f(y) = \arccos y$, the derivative function is $f'(y) = -\frac{1}{\sqrt{1-y^2}}$

Proof. Now we set $y = \cos x$, $x = \arccos y$, with $x \in [0, \pi]$, $y \in [-1, 1]$. Since for those values of x we have

$$\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - y^2}$$

we have

$$(\arccos y)' = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{-\sin x} = -\frac{1}{\sqrt{1 - y^2}}$$

□

- Note that the functions $\arcsin x$, $\arccos x$, although defined and continuous on $[-1, 1]$, are not differentiable at the endpoints of the interval: precisely, at these points they have a vertical tangent.

The usefulness of the theorem on the derivative of the inverse function lies in the fact that it allows us to compute the derivative of the inverse function g even in situations where we do not know how to write g explicitly.

Example 4.9: Derivative of the inverse function

Let

$$f(x) = x + e^x$$

The function is strictly increasing on all of $\mathbb{R}$, hence invertible; let g be its inverse. We compute, for example, $g'(y_0)$ for $y_0 = f(0) = 1$. We have:

$$f'(x) = 1 + e^x, \quad f'(0) = 2 \neq 0$$

Hence:

$$g'(1) = \frac{1}{f'(0)} = \frac{1}{2}.$$

Observation 4.13

Given the function $f(y) = \text{SettSh } y$ (the inverse hyperbolic sine), the derivative function is $f'(y) = \frac{1}{\sqrt{y^2+1}}$

Proof. Now we set $y = \text{Sh } x$, $x = \text{SettSh } y$, with $x \in \mathbb{R}$, $y \in \mathbb{R}$. Since for those values of x we have

$$\text{Ch } x = \sqrt{\text{Sh}^2 x + 1} = \sqrt{y^2 + 1}$$

we have:

$$(\text{SettSh } y)' = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\text{Ch } x} = \frac{1}{\sqrt{y^2 + 1}}$$

□

Observation 4.14

Given the function $f(y) = \text{SettCh } y$ (the inverse hyperbolic cosine), the derivative function is $f'(y) = \frac{1}{\sqrt{y^2-1}}$

Proof. Now we set $y = \text{Ch } x$, $x = \text{SettCh } y$, with $x \in [0, +\infty)$, $y \in [1, +\infty)$. Since for those values of x we have

$$\text{Sh } x = \sqrt{\text{Ch}^2 x - 1} = \sqrt{y^2 - 1}$$

we have:

$$(\text{SettCh } y)' = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\text{Sh } x} = \frac{1}{\sqrt{y^2 - 1}}$$

□

4.1.4 Logarithmic derivative and elasticity

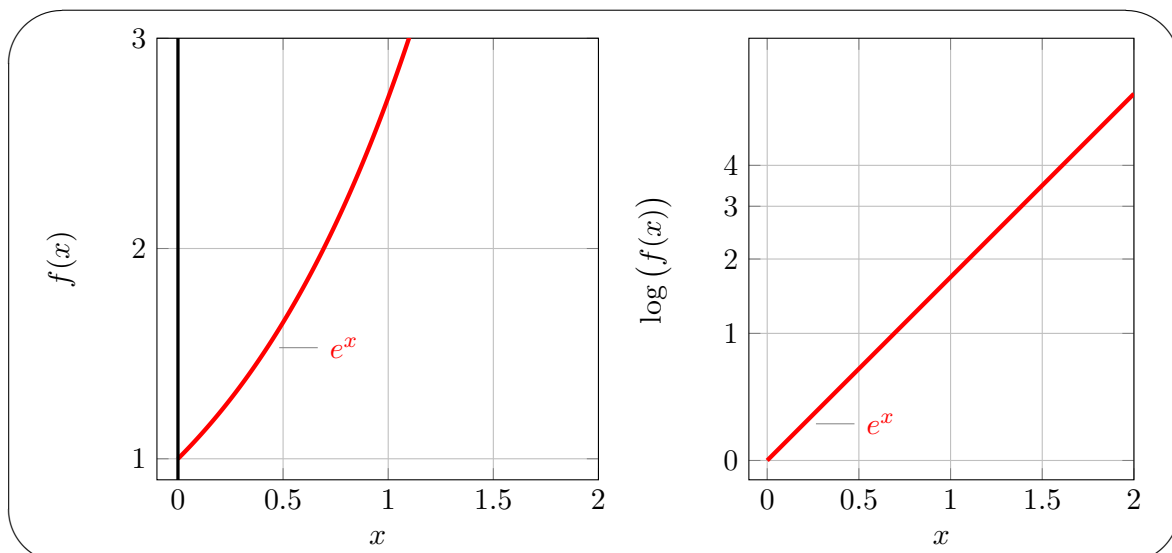
Definition 4.15: of logarithmic derivative

Given $f > 0$, the logarithmic derivative function of f is the derivative function of $\log f$; in formulas:

$$\frac{d}{dx} \log(f(x)) = \frac{f'(x)}{f(x)}$$

- The logarithmic derivative has the meaning of **relative growth rate** of f with respect to x .
- In some cases the relative growth rate $\frac{f'(x)}{f(x)}$ is often more meaningful than the absolute rate $f'(x)$.
- For example, a yearly capital increase of 1 billion out of a total of 10 has a very different effect from an increase of 1 billion out of 1000 billion. In the first case the relative rate of change is $1/10$, in the second $1/1000$, while the absolute increases are equal.

When we are interested in visualizing relative increases, we use graphs on a **semi-logarithmic scale**: the values of x are placed on the x -axis, while the values of $\log(f(x))$ are placed on the y -axis



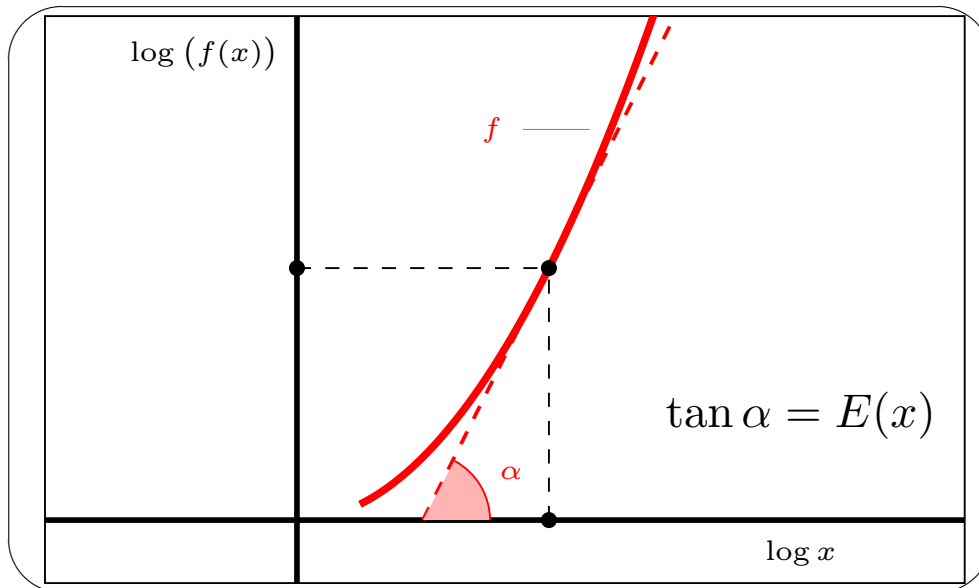
- This kind of representation is also used when $f(x)$ grows so rapidly that it would require an excessive compression of the scale (unit of measure) on the y -axis (or on the x -axis).

A **log-log graph** (graph on a logarithmic scale) is a graph in which, instead of x , the values of $\log x$ are placed on the x -axis and, instead of $f(x)$, the values of $\log f(x)$ are placed on the y -axis

Definition 4.16: of elasticity

The slope of the tangent line to a graph on a logarithmic scale is called the elasticity of f and is denoted by $E(x)$.

- Elasticity represents the relative rate of change of f with respect to relative changes of x , and it equals the derivative of $\log f$ with respect to $\log x$



- To find the analytic expression of $E(x)$, we observe that, by the theorem on the differentiation of composite functions, setting $u = \log x$ we have:

$$\frac{d \log f}{dx} = \frac{d \log f}{du} \frac{du}{dx} \quad \text{i.e.} \quad \frac{f'(x)}{f(x)} = E(x) \cdot \frac{1}{x}$$

from which we obtain:

$$E(x) = x \cdot \frac{f'(x)}{f(x)}$$

Example 4.10: elasticity

We compute the elasticity of the power function $f(x) = x^\alpha$, $x > 0$, $\alpha \in \mathbb{R}$; we have:

$$E(x) = x \cdot \frac{\alpha x^{\alpha-1}}{x^\alpha} = \alpha \quad (\text{constant elasticity})$$

Mean value theorem, maxima and minima

5.1 Local and global maxima/minima

- One of the uses of differential calculus is the **search for maxima and minima**, that is, the **optimization** of a function defined on an interval I (closed/open, bounded/unbounded).

Definition 5.1: of maximum point and maximum

Given a function $f : I \rightarrow \mathbb{R}$, if there exists a point $\tilde{x}_M \in I$ such that:

$$f(\tilde{x}_M) \geq f(x), \quad \forall x \in I$$

then $\tilde{x}_M$ is a (global) **maximum point** and $f(\tilde{x}_M)$ is the (global) **maximum** of f in I .

Definition 5.2: of minimum point and minimum

Given a function $f : I \rightarrow \mathbb{R}$, if there exists a point $\tilde{x}_m \in I$ such that:

$$f(\tilde{x}_m) \leq f(x), \quad \forall x \in I$$

then $\tilde{x}_m$ is a (global) **minimum point** and $f(\tilde{x}_m)$ is the (global) **minimum** of f in I .

We call an **extremum** a maximum or a minimum, and an **extremum point** a maximum or minimum point. An extremum, if it exists, is unique, while there may be several extremum points (even infinitely many).

Definition 5.3: of local maximum point and local maximum

Given a function $f : I \rightarrow \mathbb{R}$, if there exist a point $\bar{x}_M \in I$ and a neighborhood $(\bar{x}_M - \delta, \bar{x}_M + \delta)$ with $\delta > 0$, such that:

$$f(\bar{x}_M) \geq f(x), \quad \forall x \in (\bar{x}_M - \delta, \bar{x}_M + \delta) \cap I$$

then $\bar{x}_M$ is a **local maximum point** and $f(\bar{x}_M)$ is the **local maximum** of f in $(\bar{x}_M - \delta, \bar{x}_M + \delta) \cap I$.

Definition 5.4: of local minimum point and local minimum

Given a function $f : I \rightarrow \mathbb{R}$, if there exist a point $\bar{x}_m \in I$ and a neighborhood $(\bar{x}_m - \delta, \bar{x}_m + \delta)$ with $\delta > 0$, such that:

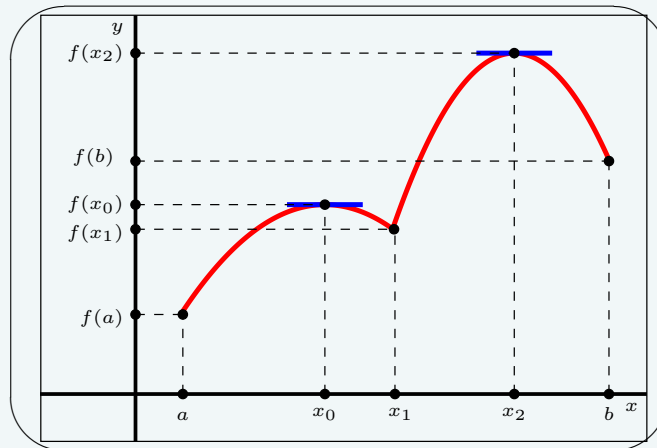
$$f(\bar{x}_m) \leq f(x), \quad \forall x \in (\bar{x}_m - \delta, \bar{x}_m + \delta) \cap I$$

then $\bar{x}_m$ is a **local minimum point** and $f(\bar{x}_m)$ is the **local minimum** of f in $(\bar{x}_m - \delta, \bar{x}_m + \delta) \cap I$.

- Global extremum points are also local extremum points, and global extrema are also local extrema.

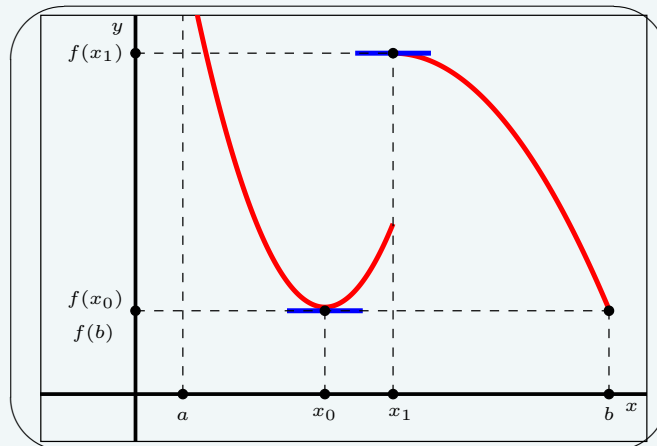
Example 5.1: extrema and extremum points (global and local)

Consider the function f with the following graph:



- the global maximum is $f(x_2)$ and x_2 is a global maximum point (it is also unique); $f(x_0)$ is a local (not global) maximum and x_0 is a local (not global) maximum point
- the global minimum is $f(a)$ and a is a global minimum point (it is also unique); $f(b)$ and $f(x_1)$ are two local (not global) minima and b and x_1 are two local (not global) minimum points

Now consider the function f with the following graph:



- the global maximum does not exist ($\lim_{x \rightarrow a^+} f(x) = +\infty$); $f(x_1)$ is a local (not global) maximum and x_1 is a local (not global) maximum point
- the global minimum is $f(x_0)$, which is also equal to $f(b)$; x_0 and b are two global minimum points

5.2 Fermat's theorem and stationary points

- At a local or global extremum point the function may fail to be differentiable and may even be discontinuous. However, the following theorem tells us that if a function is differentiable at a local extremum point, then at that point the derivative vanishes and hence the tangent to the graph is horizontal.

Theorem 5.5: Fermat's theorem

Given a function $f : (a, b) \rightarrow \mathbb{R}$ differentiable at $x_0 \in (a, b)$, if f has a local extremum at x_0 then $f'(x_0) = 0$.

Proof. Consider the case where x_0 is a local maximum point. Then:

$$\exists(x_0 - \delta, x_0 + \delta) \text{ with } \delta > 0 : f(x_0) \geq f(x), \quad \forall x \in (x_0 - \delta, x_0 + \delta) \cap (a, b)$$

Therefore for every $x \in (x_0 - \delta, x_0 + \delta) \cap (a, b)$ we have:

$$x < x_0 \Rightarrow \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \quad \text{and hence} \quad f'_-(x_0) = \lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

where for the non-negativity of the limit we used the sign-preservation theorem. On the other hand:

$$x > x_0 \Rightarrow \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \quad \text{and hence} \quad f'_+(x_0) = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

Since f is differentiable at x_0 , we have:

$$f'(x_0) = f'_-(x_0) = f'_+(x_0) = 0$$

The case where x_0 is a local minimum point is handled in the same way. $\square$

Proof. Suppose, by contradiction, that $f'(x_0) > 0$; then by the sign-preservation theorem we would have $\frac{f(x) - f(x_0)}{x - x_0} > 0$ eventually as $x \rightarrow x_0$. Taking into account the sign of $x - x_0$, this implies that $f(x) > f(x_0)$ eventually as $x \rightarrow x_0^+$ and $f(x) < f(x_0)$ eventually as $x \rightarrow x_0^-$. But this contradicts the hypothesis that x_0 is a local extremum point of f . Similarly, we rule out the case $f'(x_0) < 0$. Therefore we must have $f'(x_0) = 0$. $\square$

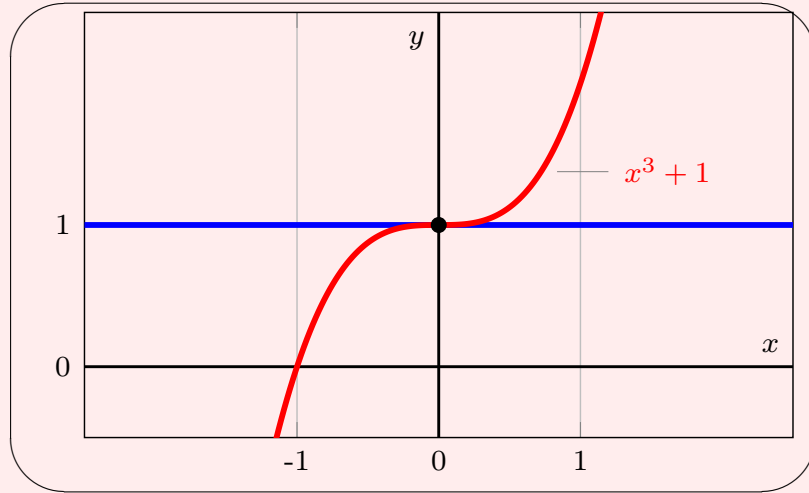
Definition 5.6: stationary point

A point x_0 is called a stationary point of f if f is differentiable at x_0 and $f'(x_0) = 0$.

Fermat's theorem says that for a function $f : (a, b) \rightarrow \mathbb{R}$ differentiable at $x_0 \in (a, b)$ we have:

$$x_0 \text{ is a local extremum point} \Rightarrow \underbrace{x_0 \text{ is a stationary point}}_{f'(x_0)=0} \quad (5.1)$$

Hence “ x_0 is a local extremum point” is a sufficient condition for “ x_0 is a stationary point”. But the converse is not true. A counterexample is given by the function $f(x) = x^3 + 1, \forall x \in \mathbb{R}$, whose derivative function is $f'(x) = 3x^2, \forall x \in \mathbb{R}$. With $x_0 = 0$ we have $f'(0) = 0$ but $x_0 = 0$ is not a local extremum point.



Hence we have:

$$\underbrace{x_0 \text{ is a stationary point}}_{f'(x_0)=0} \nRightarrow x_0 \text{ is a local extremum point}$$

Finally, from the contrapositive of (5.1), we have:

$$x_0 \text{ is not a stationary point} \Rightarrow x_0 \text{ is not a local extremum point}$$

5.3 Mean value theorem (Lagrange's theorem)

Theorem 5.7: mean value theorem (Lagrange)

Given a function f differentiable in (a, b) and continuous in $[a, b]$, then:

$$\exists c \in (a, b) : \frac{f(b) - f(a)}{b - a} = f'(c) \quad (5.2)$$

In the case $f(b) = f(a)$, the theorem tells us that there exists $c \in (a, b)$ such that $f'(c) = 0$, that is, there exists at least one point where the derivative vanishes. This corollary of Lagrange's theorem is called **Rolle's theorem**.

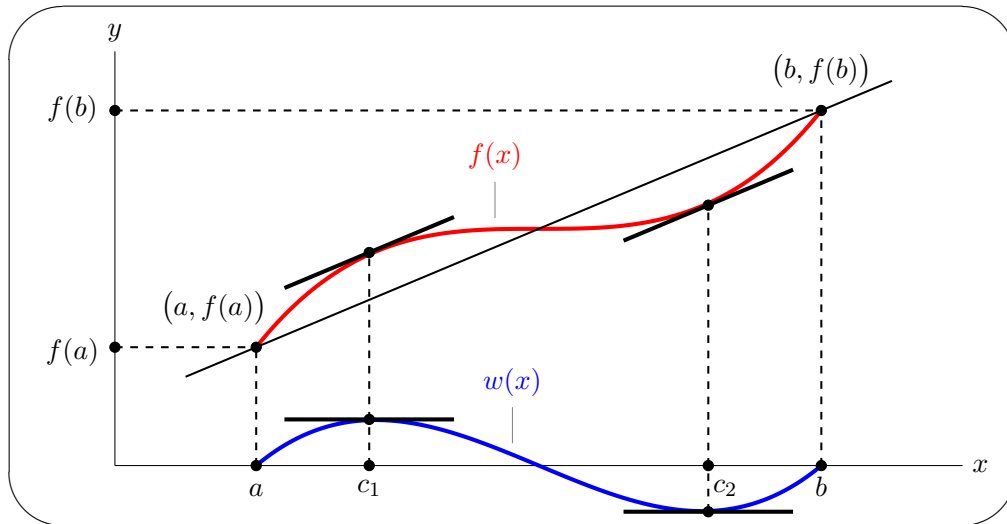
- Differentiability is required only on the open interval (a, b) in order to include also functions that are continuous at one of the endpoints but not differentiable there. For example, with $a = 0$:

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) x & \text{if } x \in (0, b) \\ 0 & \text{if } x = 0 \end{cases} \quad \text{or} \quad g(x) = \sqrt{x}$$

- Geometrically, considering the graph of f we have:

1. $\frac{f(b)-f(a)}{b-a}$ is the slope of the line through $(a, f(a))$ and $(b, f(b))$
2. $f'(c)$ is the slope of the tangent line to the graph of f at the point $(c, f(c))$

The mean value theorem therefore expresses the fact that at the point $(c, f(c))$ the tangent to the graph of f is parallel to the line through $(a, f(a))$ and $(b, f(b))$. There may be more than one point with this property, as the following figure shows:



- The figure also shows the function

$$w(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right)$$

given by the difference between the values of the function and the line through the points $(a, f(a))$ and $(b, f(b))$, which has equation:

$$y = f(a) + \frac{f(b) - f(a)}{b - a} (x - a)$$

The function $w(x)$ is the key to the proof of the theorem.

Proof. Consider the function:

$$w(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right)$$

Clearly $w(a) = w(b) = 0$; moreover w is continuous in $[a, b]$ and differentiable in

(a, b) . Since

$$w'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \quad \text{then} \quad \frac{f(b) - f(a)}{b - a} = f'(c) \iff \exists c \in (a, b) : w'(c) = 0$$

Since w is continuous in $[a, b]$, by the Weierstrass theorem there exist two points x_1 and x_2 in $[a, b]$ such that:

$w(x_1) = M$, the maximum of w in $[a, b]$; $w(x_2) = m$, the minimum of w in $[a, b]$.

If $M = m$, then $w(x)$ is constant in $[a, b]$, and hence $w'(x) = 0, \forall x \in (a, b)$.

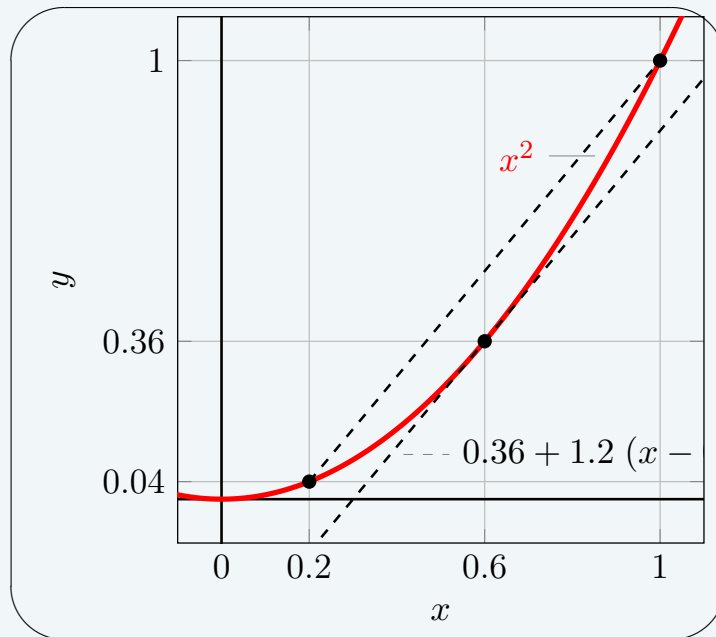
If $M > m$, at least one of the two points x_1 or x_2 is not at the endpoints of the interval, since $w(a) = w(b) = 0$. Fermat's theorem then implies that at the maximum or minimum point that lies in the interior (possibly both) the derivative of w vanishes, and the theorem is thus proved. $\square$

Example 5.2: using the mean value theorem

Let $f(x) = x^2$. Then $f'(x) = 2x$ and the mean value theorem states that in every interval $[a, b]$ there exists a number c such that:

$$\frac{b^2 - a^2}{b - a} = 2c \quad \text{from which} \quad c = \frac{a + b}{2} \quad (\text{arithmetic mean})$$

That is, every chord AB of the parabola $y = x^2$ is parallel to the tangent at the point whose abscissa equals the arithmetic mean of the abscissas of A and B . Taking for example the interval $[0.2, 1]$ ($a = 0.2, b = 1$ and $c = 0.6$), we have:

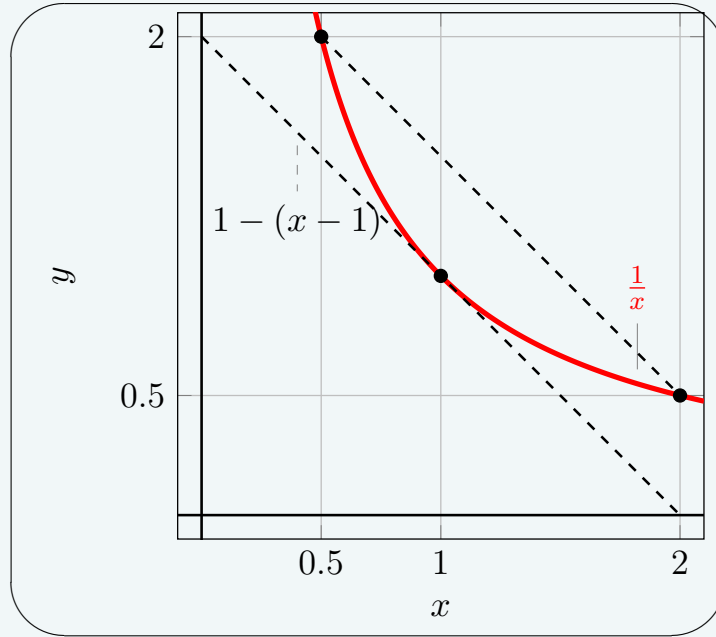


Example 5.3: using the mean value theorem

Let $f(x) = \frac{1}{x}$. Then $f'(x) = -\frac{1}{x^2}$ and the mean value theorem states that in every interval $[a, b]$ there exists a number c such that:

$$\frac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\frac{1}{c^2} \quad \text{from which} \quad c = \sqrt{a \cdot b} \quad (\text{geometric mean})$$

That is, every chord AB of the hyperbola $y = \frac{1}{x}$ is parallel to the tangent at the point whose abscissa equals the geometric mean of the abscissas of A and B . Taking for example the interval $[0.5, 2]$ ($a = 0.5, b = 2$ and $c = 1$), we have:



5.4 Differential monotonicity test theorem

Theorem 5.8: Differential Monotonicity Test

Given a function $f : I \rightarrow \mathbb{R}$, continuous in I and differentiable at the interior points of I , then:

$$f'(x) \geq 0, \forall x \text{ in the interior of } I \iff f \text{ is non-decreasing in } I \quad (5.3)$$

$$f'(x) \leq 0, \forall x \text{ in the interior of } I \iff f \text{ is non-increasing in } I \quad (5.4)$$

$$f'(x) > 0, \forall x \text{ in the interior of } I \implies f \text{ is increasing in } I \quad (5.5)$$

$$f'(x) < 0, \forall x \text{ in the interior of } I \implies f \text{ is decreasing in } I \quad (5.6)$$

Proof.

($\Rightarrow$) Taking two points $x_1 < x_2$ in the interval I , we can apply Lagrange's theorem on the interval $[x_1, x_2]$, hence there exists $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

If $f'(c) \geq 0$ (resp. > 0), then $f(x_2) \geq f(x_1)$ (resp. $f(x_2) > f(x_1)$). Since x_1 and x_2 are arbitrary, we have proved that f is non-decreasing (resp. increasing) in I . If $f'(c) \leq 0$ (resp. < 0) the argument is analogous.

($\Leftarrow$) If f is non-decreasing, then we have

$$\frac{f(z) - f(x)}{z - x} \geq 0 \quad \text{for all } z, x \in I, z \neq x$$

and, by the sign-preservation theorem, the limit of the ratio as $z \rightarrow x$ is also non-negative; by hypothesis this limit exists and equals $f'(x)$ for every x in the interior of I . If f is non-increasing the argument is analogous. □

The Differential Monotonicity Test theorem says that for a function $f : I \rightarrow \mathbb{R}$ differentiable at the interior points of I we have:

$$f'(x) \geq 0, \forall x \text{ in the interior of } I \iff f \text{ is non-decreasing in } I$$

Hence " $f'(x) \geq 0, \forall x$ in the interior of I " is a necessary and sufficient condition for " f is non-decreasing in I ". The same holds for the case of non-increasing functions.

The Differential Monotonicity Test theorem says that for a function $f : I \rightarrow \mathbb{R}$ differentiable at the interior points of I we have:

$$f'(x) > 0, \forall x \text{ in the interior of } I \implies f \text{ is increasing in } I$$

Hence " $f'(x) > 0, \forall x$ in the interior of I " is a sufficient condition for " f is increasing in I ". But the converse is not true. A counterexample is given by the function $f(x) = x^3 + 1, \forall x \in \mathbb{R}$, which is increasing and differentiable in $\mathbb{R}$, but $f'(x) = 3x^2$ vanishes at $x_0 = 0$.

Hence we have:

$$f \text{ is increasing in } I \not\Rightarrow f'(x) > 0, \forall x \text{ in the interior of } I$$

Finally, from the contrapositive of (5.5), we have:

$$f \text{ is not increasing in } I \Rightarrow \exists x \text{ in the interior of } I : f'(x) \leq 0,$$

To be checked. The same holds for the case of decreasing functions.

- It follows immediately from the theorem that with $I = (a, b)$ we have:

$$f'(x) = 0, \forall x \in (a, b) \implies f \text{ constant in } (a, b)$$

The converse implication is obvious. We then have

$$f'(x) = 0, \forall x \in (a, b) \iff f \text{ constant in } (a, b) \quad (5.7)$$

Hence “ $f'(x) = 0, \forall x \in (a, b)$ ” is a necessary and sufficient condition for “ f constant in (a, b) ”.

Example 5.4: functions with zero derivative

Consider the function:

$$f(x) = \arctan x + \arctan \frac{1}{x}, \quad \forall x \neq 0$$

we have

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \left(-\frac{1}{x^2}\right) = 0, \quad \forall x \neq 0$$

Implication (5.7) allows us to say that f is constant on the interval $(-\infty, 0)$ and on the interval $(0, \infty)$. To find its value, it is enough to compute f at one point of each interval, for example:

$$f(1) = \arctan(1) + \arctan(1) = 2 \frac{\pi}{4} = \frac{\pi}{2}$$

$$f(-1) = \arctan(-1) + \arctan(-1) = -\frac{\pi}{2}$$

We have therefore proved that:

$$\arctan x + \arctan \frac{1}{x} = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}$$

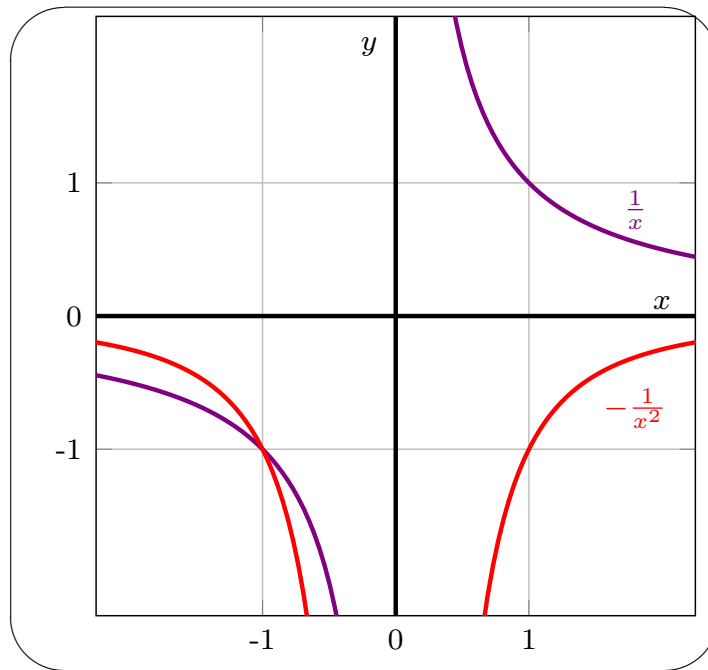
- Both in the Differential Monotonicity Test theorem and in implication (5.7) it is essential that the set I is an interval. For example, the function

$$f(x) = \frac{1}{x}, \quad \forall x \in \mathbb{R} \setminus \{0\}$$

has derivative

$$f'(x) = -\frac{1}{x^2} < 0, \quad \forall x \in \mathbb{R} \setminus \{0\}$$

but the function is not decreasing on its domain. The function is decreasing on the interval $(-\infty, 0)$ and on the interval $(0, \infty)$, as the figure shows:



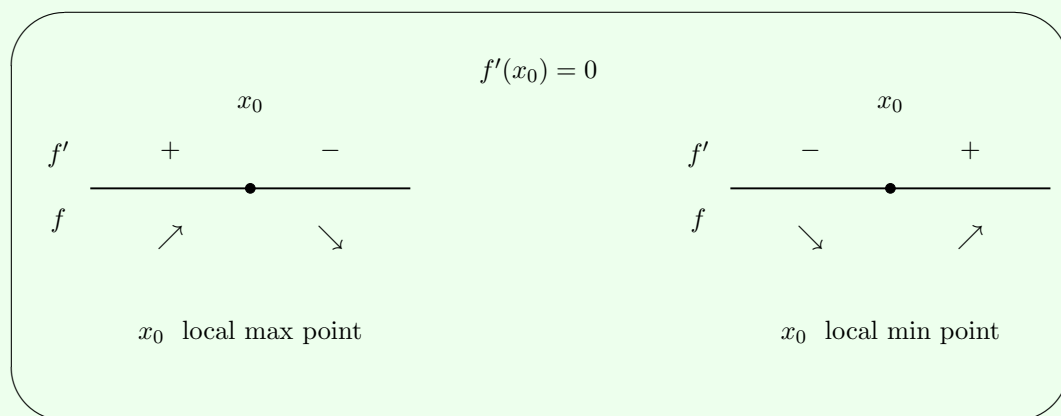
5.5 Finding maxima and minima

Suppose we have a function $f : [a, b] \rightarrow \mathbb{R}$ and we want to find its local and global maxima and minima in $[a, b]$. If f is differentiable in (a, b) we can proceed as follows:

Step 1 Compute $f(a)$ and $f(b)$.

Step 2 Compute $f'(x)$ and solve the equation $f'(x) = 0$. In this way we find the stationary points, among which are the possible local extremum points in the interval (a, b) .

Step 3 If there are no stationary points, $f(a)$ and $f(b)$ are the global extrema and a and b are global extremum points. Otherwise we need to determine the nature of the stationary points found. The function f has a local extremum at a stationary point if and only if the sign of f' changes there, which we check by studying the sign of f' in a neighborhood of the point. We have the following cases:



Step 4 Having found the possible local extremum points, compute the value of f at these points and compare it with $f(a)$ and $f(b)$ to understand whether or not they are global extremum points; in this way the global extrema are determined.

Example 5.5: finding maxima and minima

Consider the function:

$$f(x) = x e^{-x^2}, \quad \text{with } x \in [0, 2]$$

Step 1:

$$f(0) = 0, \quad f(2) = 2 e^{-4}$$

Example 5.6: finding maxima and minima

Step 2:

$$f'(x) = e^{-x^2} + x e^{-x^2} (-2x) = e^{-x^2} (1 - 2x^2)$$

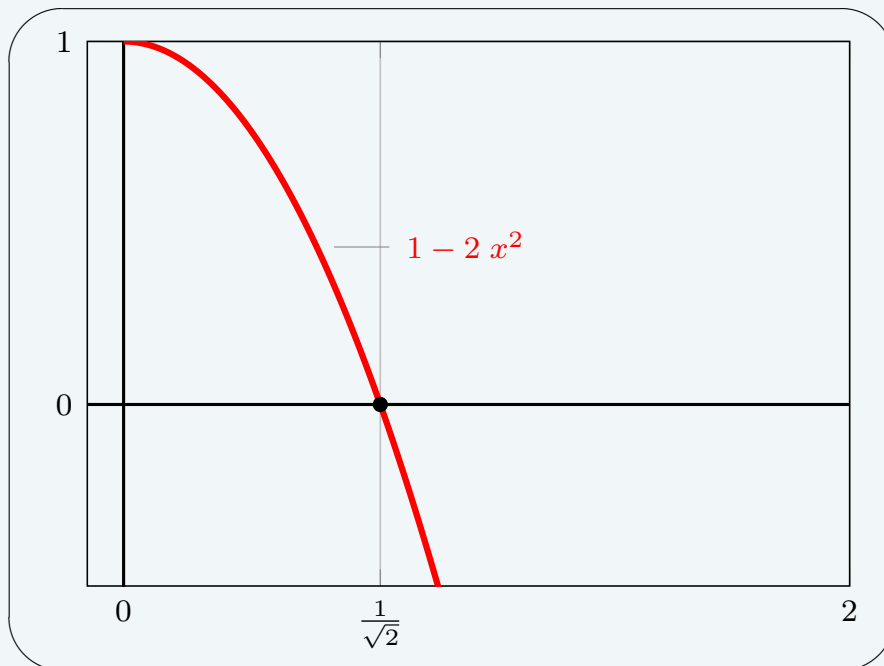
We have $e^{-x^2} > 0, \forall x \in \mathbb{R}$, hence:

$$f'(x) = 0 \iff (1 - 2x^2) = 0 \iff x = \pm \frac{1}{\sqrt{2}}$$

Only $\frac{1}{\sqrt{2}} \in [0, 2]$, hence we have the stationary point $x_0 = \frac{1}{\sqrt{2}}$.

Step 3:

We study the sign of f' near $x_0 = \frac{1}{\sqrt{2}}$. We have $f'(x) \geq 0$ for $2x^2 \leq 1$, that is, for $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$



We therefore conclude that:

$\frac{1}{\sqrt{2}}$ is a local maximum point and $f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} e^{-1/2} = \frac{1}{\sqrt{2}e}$ is a local maximum

We also note that:

$$\lim_{x \rightarrow 0^+} \frac{x e^{-x^2}}{x} = \lim_{x \rightarrow 0^+} \frac{1}{e^{x^2}} = 1 \quad \text{hence} \quad f(x) \sim x \quad \text{for } x \rightarrow 0^+$$

Example 5.7: finding maxima and minima

Step 4:

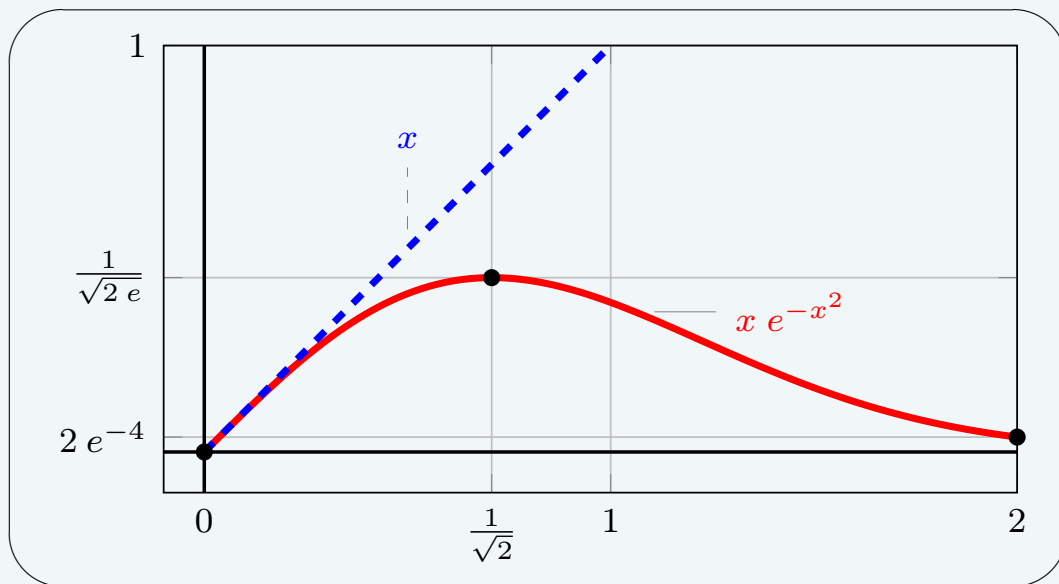
$$\frac{1}{\sqrt{2}e} > f(0) = 0, \quad \frac{1}{\sqrt{2}e} > f(2) = 2e^{-4}$$

We therefore conclude that:

$\frac{1}{\sqrt{2}}$ is the (unique) global maximum point and

0 is the (unique) global minimum point

$\frac{1}{\sqrt{2}e}$ is the global maximum and 0 is the global minimum



5.5.1 From the discrete to the continuous

Passing “from the discrete to the continuous” for sequences (i.e., from the natural numbers to the reals) is a way to make the tools of differential calculus available

Example 5.8: proving monotonicity of sequences by passing to the continuous

Consider the sequence

$$a_n = \frac{\log n}{n}, \quad \text{we have } a_n \geq 0, \forall n \in \mathbb{N}_{>0} \text{ and } a_n \rightarrow 0 \text{ for } n \rightarrow +\infty$$

To prove that the sequence is eventually monotone non-increasing, one way is to use the definition and prove that:

$$a_n \geq a_{n+1}, \quad \forall n \in \mathbb{N}_{>0} \quad \text{i.e.} \quad \frac{\log n}{n} \geq \frac{\log(n+1)}{n+1}, \quad \forall n \in \mathbb{N}_{>0}$$

Since both the numerator and the denominator grow as n grows, it is not easy to prove this inequality algebraically.

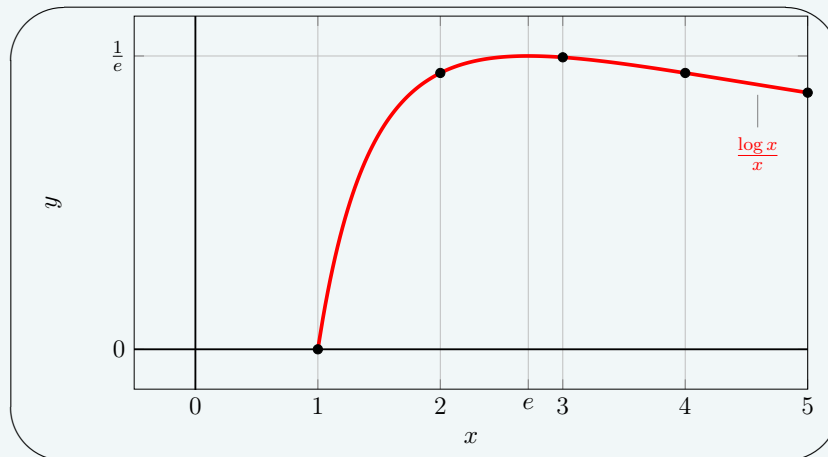
We pass from the discrete to the continuous and define the function

$$f(x) = \frac{\log x}{x}, \quad \forall x > 1$$

We have

$$f'(x) = \frac{1 - \log x}{x^2} \leq 0, \quad \forall x \geq e$$

and it follows that f is non-increasing $\forall x \geq e$. Consequently the sequence a_n is non-increasing for $n \geq 3$ (the first natural number $> e$).



- Be careful not to use the passage from the discrete to the continuous indiscriminately. For example, it cannot be done for the sequences $\frac{n!}{2^n}$ or $\frac{n^2 + (-1)^n n}{n^3 + 1}$, since they are defined only for natural numbers.

L'Hôpital's rule and differentiability

6.1 L'Hôpital's rule

- A remarkable application of differential calculus is the computation of limits that appear in the indeterminate forms:

$$\left[\frac{0}{0}\right] \quad \text{and} \quad \left[\frac{\infty}{\infty}\right]$$

Theorem 6.1: L'Hôpital's rule

Let f, g be differentiable functions on an interval (a, b) with $g, g' \neq 0$ in (a, b) . If

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x) = 0 \quad (\text{or } \pm \infty) \quad \text{and} \quad \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = \ell \in \mathbb{R}^*$$

then:

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \ell$$

The theorem still holds if $a = -\infty$ or if we consider the limit as $x \rightarrow b^-$, with $b \leq +\infty$ (a case we do not prove). Hence, if the hypotheses hold for a^- and a^+ , it can clearly be applied also at an interior point of an interval.

Example 6.1: using L'Hôpital's rule

We want to compute:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \left[\frac{0}{0}\right] \quad \text{and} \quad f(x) = \sin x, f'(x) = \cos x, \quad g(x) = x, g'(x) = 1$$

applying L'Hôpital's rule, for example with $x \in (-1, 0) \cup (0, 1)$, we obtain:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$$

We want to compute:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} = \left[\frac{0}{0}\right] \quad \text{and} \quad f(x) = 1 - \sin x, f'(x) = -\cos x, \quad g(x) = \cos x, g'(x) = -\sin x$$

applying L'Hôpital's rule, for example with $x \in (\frac{\pi}{4}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{4})$, we obtain:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} \stackrel{H}{=} \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\cos x}{-\sin x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\sin x} = \frac{0}{1} = 0$$

Proof. We consider the case:

$$f(x), g(x) \rightarrow 0, \text{ as } x \rightarrow a^+$$

Let $\{x_n\}$ be a sequence such that $x_n \rightarrow a^+$ as $n \rightarrow +\infty$ and $x_n \neq a, \forall n$, and let us extend f and g by continuity at a by setting $f(a) = g(a) = 0$.

We now define the following function:

$$h(x) = f(x_n) g(x) - g(x_n) f(x)$$

The function h satisfies the hypotheses of Lagrange's theorem on the interval $[a, x_n]$, hence there exists $t_n \in (a, x_n)$ such that

$$h'(t_n) = \frac{h(x_n) - h(a)}{x_n - a} = 0 \quad \text{since} \quad h(a) = h(x_n) = 0$$

Computing

$$h'(x) = f(x_n) g'(x) - g(x_n) f'(x), \quad \text{we have} \quad \underbrace{f(x_n) g'(t_n) - g(x_n) f'(t_n)}_{=h'(t_n)} = 0$$

Hence for every x_n there exists a point $t_n \in (a, x_n)$ such that:

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(t_n)}{g'(t_n)} \tag{6.1}$$

As $n \rightarrow +\infty$ we have $t_n \rightarrow a^+$, since $a^+ < t_n < x_n$ and $x_n \rightarrow a^+$ as $n \rightarrow +\infty$, therefore by the hypothesis of the theorem we have:

$$\frac{f'(t_n)}{g'(t_n)} \rightarrow \ell \quad \text{and consequently} \quad \frac{f(x_n)}{g(x_n)} \rightarrow \ell$$

which is what we wanted to prove. □

With L'Hôpital's rule we can easily prove the theorem on the hierarchy of infinities for functions.

Proof. We prove:

$$\lim_{x \rightarrow +\infty} \frac{x^\alpha}{b^\lambda x} = 0, \quad \text{with} \quad \alpha > 0, \lambda > 0, b > 1$$

Setting $f(x) = x$, $g(x) = b^{\mu x}$ ($\mu > 0$), we have $f'(x) = 1$, $g'(x) = \mu \log b \, b^{\mu x}$ and since:

$$\lim_{x \rightarrow +\infty} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow +\infty} \frac{1}{\mu \log b \, b^{\mu x}} = 0 \quad \text{then} \quad \lim_{x \rightarrow +\infty} \frac{x}{b^{\mu x}} = 0$$

Observing now that

$$\begin{aligned} \frac{x^\alpha}{b^{\lambda x}} &= \left(\frac{x}{b^{(\lambda/\alpha) x}} \right)^\alpha = \left(\frac{x}{b^{\mu x}} \right)^\alpha \quad \text{with} \quad \mu = \frac{\lambda}{\alpha} > 0 \\ \text{then} \quad \lim_{x \rightarrow +\infty} \frac{x^\alpha}{b^{\lambda x}} &= \left(\lim_{x \rightarrow +\infty} \frac{x}{b^{\mu x}} \right)^\alpha = 0 \end{aligned}$$

We prove:

$$\lim_{x \rightarrow +\infty} \frac{(\log_a x)^\beta}{x^\alpha} = 0, \quad \text{with} \quad \beta > 0, \alpha > 0, a > 1$$

it suffices to perform the change of variable

$$t = \log_a x, \quad x = a^t, \quad \text{we have that as } x \rightarrow +\infty \quad \text{also } t \rightarrow +\infty$$

hence

$$\lim_{x \rightarrow +\infty} \frac{(\log_a x)^\beta}{x^\alpha} = \lim_{t \rightarrow +\infty} \frac{t^\beta}{a^{\alpha t}} = 0$$

since the second limit has the same form as the one just proved. □

- L'Hôpital's rule can be useful for limits that cannot be solved using only fundamental limits or the asymptotic expansions derived from them.

Example 6.2: L'Hôpital's rule and fundamental limits

We compute

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = \left[\frac{0}{0} \right]$$

Fundamental limits are not sufficient to resolve the indeterminate form, since:

$$\frac{x - \sin x}{x^3} = \frac{1}{x^2} \left(1 - \frac{\sin x}{x} \right) \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{1}{x^2} \underbrace{\left(1 - \frac{\sin x}{x} \right)}_{\rightarrow 1} = [\infty \cdot 0]$$

L'Hôpital's rule gives:

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{(x - \sin x)'}{(x^3)'} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} = \frac{1}{3} \cdot \lim_{x \rightarrow 0} \underbrace{\frac{1 - \cos x}{x^2}}_{\rightarrow \frac{1}{2}} = \frac{1}{6}$$

Example 6.3: L'Hôpital's rule and asymptotic expansions

We now treat the same limit with the first-order asymptotic expansion:

$$\sin x = x + o(x) \quad \text{as } x \rightarrow 0$$

we obtain:

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = \lim_{x \rightarrow 0} \frac{x - (x + o(x))}{x^3} = \lim_{x \rightarrow 0} \frac{o(x)}{x^3}$$

That is, we still have an indeterminate form, since:

$$\lim_{x \rightarrow 0} \frac{o(x)}{x^3} = \lim_{x \rightarrow 0} \frac{1}{x^2} \frac{o(x)}{x} = \lim_{x \rightarrow 0} \frac{1}{x^2} o(1) = [+\infty \cdot 0]$$

where $o(1)$ denotes a generic function that tends to 0 as $x \rightarrow 0$.

We reach the same conclusion also with the following reasoning. The symbol $o(x)$ as $x \rightarrow 0$ denotes the set of functions that, divided by x , tend to 0 as $x \rightarrow 0$. Hence, for example, $5x^2 = o(x)$ but also $x^3 = o(x)$, and the value of the limit cannot be determined with the first-order asymptotic expansion.

As we will see later, to determine this limit with asymptotic expansions we need an asymptotic expansion of order higher than the first.

Example 6.4: L'Hôpital's rule and asymptotic estimates

We compute

$$\lim_{x \rightarrow +\infty} \frac{x - \sin x}{x + \sin x} = \left[\frac{+\infty}{+\infty} \right]$$

L'Hôpital's rule gives:

$$\lim_{x \rightarrow +\infty} \frac{x - \sin x}{x + \sin x} \stackrel{H}{=} \lim_{x \rightarrow +\infty} \frac{(x - \sin x)'}{(x + \sin x)'} = \lim_{x \rightarrow +\infty} \frac{1 - \cos x}{1 + \cos x} \quad \text{does not exist}$$

the fact that this limit does not exist does not allow us to conclude that the limit we are looking for does not exist.

Let us proceed instead with asymptotic estimates:

$$\frac{x - \sin x}{x + \sin x} \sim \frac{x}{x} \quad \text{as } x \rightarrow +\infty$$

Since

$$\lim_{x \rightarrow +\infty} \frac{x - \sin x}{x} = \lim_{x \rightarrow +\infty} 1 - \underbrace{\frac{\sin x}{x}}_{\rightarrow 0} \quad \text{and} \quad \lim_{x \rightarrow +\infty} \frac{x + \sin x}{x} = \lim_{x \rightarrow +\infty} 1 + \underbrace{\frac{\sin x}{x}}_{\rightarrow 0}$$

hence:

$$\lim_{x \rightarrow +\infty} \frac{x - \sin x}{x + \sin x} = \lim_{x \rightarrow +\infty} \frac{x}{x} = 1$$

- The theorem is useful when its application simplifies the limit instead of complicating it, that is, when the order of the infinitesimal (or of the infinity) in the numerator or in the denominator decreases.

Example 6.5: L'Hôpital's rule

We compute

$$\lim_{x \rightarrow 0^+} \frac{e^{-1/x^2}}{x} = \left[\frac{0}{0} \right]$$

L'Hôpital's rule gives:

$$\lim_{x \rightarrow 0^+} \frac{e^{-1/x^2}}{x} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{(e^{-1/x^2})'}{(x)'} = \lim_{x \rightarrow 0^+} \frac{2 e^{-1/x^2}}{x^3} = \left[\frac{0}{0} \right]$$

Applying instead the change of variable:

$$x^2 = \frac{1}{t}, \quad x \rightarrow 0^+ \text{ is equivalent to } t \rightarrow +\infty$$

substituting we have:

$$\lim_{x \rightarrow 0^+} \frac{e^{-1/x^2}}{(x^2)^{1/2}} = \lim_{t \rightarrow +\infty} \frac{e^{-t}}{(\frac{1}{t})^{1/2}} = \lim_{t \rightarrow +\infty} \frac{e^{-t}}{t^{-1/2}} = \lim_{t \rightarrow +\infty} \frac{\sqrt{t}}{e^t} = 0$$

where the last limit is obtained thanks to the theorem on the hierarchy of infinities.

- Sometimes the theorem must be applied several times consecutively to resolve the indeterminate form. Also in this case, already after the first application, one should notice that the order of the infinitesimal (or of the infinity) in the numerator or in the denominator has decreased.

Example 6.6: L'Hôpital's rule

We compute

$$\lim_{x \rightarrow 0} \frac{1 - \cos^3 x}{x^3 - x^2} = \left[\frac{0}{0} \right]$$

L'Hôpital's rule gives:

$$\lim_{x \rightarrow 0} \frac{1 - \cos^3 x}{x^3 - x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{3 \sin x \cos^2 x}{3x^2 - 2x} = \left[\frac{0}{0} \right]$$

L'Hôpital's rule applied twice gives:

$$\lim_{x \rightarrow 0} \frac{1 - \cos^3 x}{x^3 - x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{3 \sin x \cos^2 x}{3x^2 - 2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{3 \overset{\rightarrow 1}{\cos x} (\overset{\rightarrow 1}{\cos^2 x} - 2 \overset{\rightarrow 0}{\sin^2 x})}{\underset{\rightarrow 0}{6x} - 2} = -\frac{3}{2}$$

- The theorem is used for quotients, not for products
- The theorem is used for quotients that are actual indeterminate forms:

$$\left[\frac{0}{0}\right] \quad \text{and} \quad \left[\frac{\infty}{\infty}\right]$$

- The theorem prescribes computing the quotient of the derivatives, not the derivative of the quotient
- If the limit of f'/g' does not exist, nothing can be said about the limit of f/g .

6.2 Limit of the derivative and differentiability

Theorem 6.2: limit of the derivative

If $f : [a, b) \rightarrow \mathbb{R}$ is continuous at a , differentiable in (a, b) , and $\lim_{x \rightarrow a^+} f'(x) = m \in \mathbb{R}^*$, then $f'_+(a) = m$.

If the function is continuous at a and the right-hand limit of the derivative exists, then the right derivative exists and coincides with that limit. An analogous statement of the theorem holds for the left derivative and hence for the derivative.

Proof. Let $h > 0$. Applying Lagrange's theorem to f on the interval $[a, a+h]$, we obtain that there exists $t_h \in [a, a+h]$ such that

$$\frac{f(a+h) - f(a)}{h} = f'(t_h)$$

As $h \rightarrow 0^+$ we have $t_h \rightarrow a^+$, hence by the hypothesis of the theorem $f'(t_h) \rightarrow m$ as $t_h \rightarrow a^+$. Consequently, there exists

$$f'_+(a) = \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h} = m$$

□

Under the hypotheses of the theorem it is therefore possible to compute the right derivative, the left derivative or the derivative without using the limit of the difference quotient

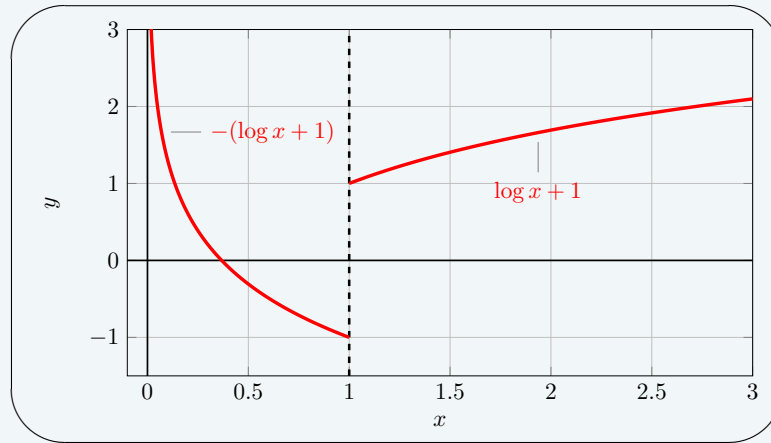
Example 6.7: differentiability

Suppose we want to study whether or not the following function is differentiable:

$$f(x) = x \cdot |\log x|$$

defined for $x > 0$. For $x \neq 1$, where $\log x \neq 0$, we have:

$$f(x) = \begin{cases} x \cdot \log x, & x > 1 \\ -x \cdot \log x, & 0 < x < 1 \end{cases} \quad \text{hence} \quad f'(x) = \begin{cases} \log x + 1, & x > 1 \\ -(\log x + 1), & 0 < x < 1 \end{cases}$$

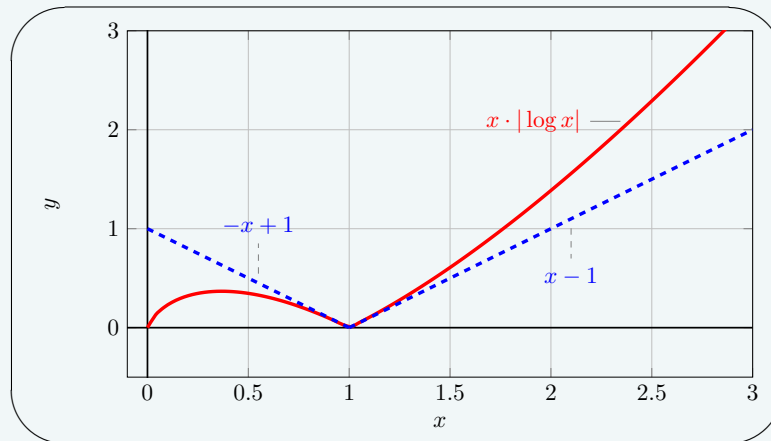


At $x = 1$ the function $f(x)$ is continuous, and we have:

$$\lim_{x \rightarrow 1^+} f'(x) = \lim_{x \rightarrow 1^+} (\log x + 1) = 1 \quad \text{and} \quad \lim_{x \rightarrow 1^-} f'(x) = \lim_{x \rightarrow 1^-} (-\log x - 1) = -1$$

hence the theorem is applicable (from the right and from the left) and we have:

$$f'_+(1) = 1, f'_-(1) = -1, \text{ so that } x = 1 \text{ is a corner point and } f'(1) \text{ does not exist}$$



Example 6.8: differentiability

Suppose we want to study whether or not the following function is differentiable:

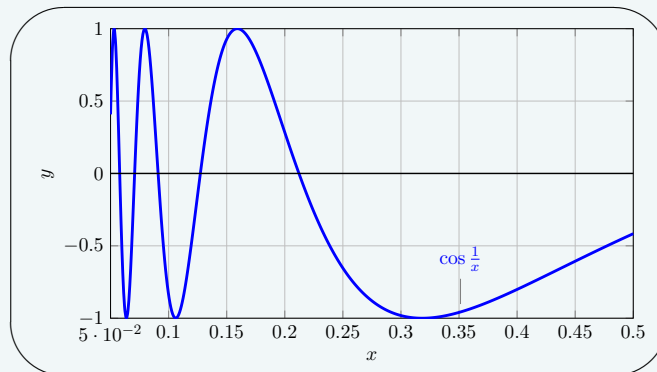
$$f(x) = x^2 \cdot \sin\left(\frac{1}{x}\right)$$

defined for $x \neq 0$. We know that

$$\lim_{x \rightarrow 0} x^2 \cdot \sin\left(\frac{1}{x}\right) = 0$$

and we can extend f by continuity by defining $f(0) = 0$, and f turns out to be continuous also at 0. For $x \neq 0$, we have:

$$f'(x) = 2x \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = \underbrace{2x \cdot \sin\left(\frac{1}{x}\right)}_{\rightarrow 0 \text{ as } x \rightarrow 0} - \cos\left(\frac{1}{x}\right)$$

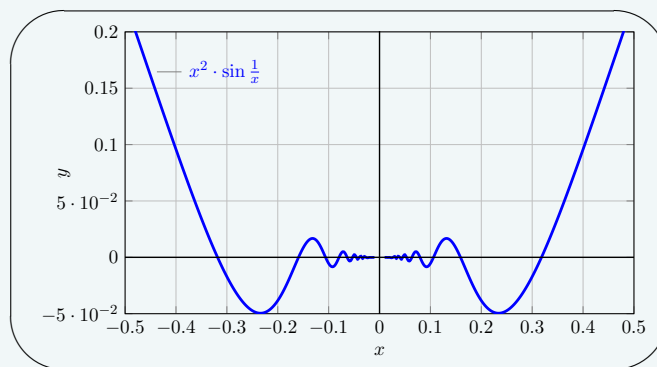


Near zero, the function $\cos \frac{1}{x}$, like the function $\sin \frac{1}{x}$, cannot be drawn since it has infinitely many oscillations. Hence the hypotheses of the theorem are not satisfied, since

$$\lim_{x \rightarrow 0} f'(x) \text{ does not exist, } \lim_{x \rightarrow 0^-} f'(x) \text{ does not exist, } \lim_{x \rightarrow 0^+} f'(x) \text{ does not exist}$$

However, this does not allow us to conclude that the function is not differentiable at $x_0 = 0$. To compute the derivative we need to use the limit of the difference quotient. We have

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - 0}{h} = \lim_{h \rightarrow 0} h \cdot \sin\left(\frac{1}{h}\right) = 0 \text{ hence the function is differentiable at } x = 0$$



Second derivative

7.1 Second derivative and second derivative function

- We can now ask whether the function $f'(x)$ is in turn differentiable (at a point or on an interval).

Definition 7.1: second derivative

Let $f : (a, b) \rightarrow \mathbb{R}$, $x_0 \in (a, b)$ and $f' : (x_0 - \delta, x_0 + \delta) \rightarrow \mathbb{R}$ with $\delta > 0$; f is said to be twice differentiable at x_0 if the following limit exists and is finite

$$\lim_{h \rightarrow 0} \frac{f'(x_0 + h) - f'(x_0)}{h}$$

this limit is called the second derivative of f at x_0 .

- To denote the second derivative function the following notations are used:

$$\underbrace{f''(x_0)}_{\text{Lagrange's notation}} \quad \underbrace{\ddot{f}(x_0)}_{\text{Newton's notation}} \quad \underbrace{\left. \frac{d^2 f}{dx^2} \right|_{x=x_0} \quad \text{and} \quad \left. \frac{d^2 y}{dx^2} \right|_{x=x_0}}_{\text{Leibniz's notation}}$$

Definition 7.2: second derivative function

If a derivative function f' is differentiable at every point of the interval (a, b) , the function

$$f'' : (a, b) \rightarrow \mathbb{R}, \quad f'' : x \mapsto f''(x)$$

is called the **second derivative function** of f .

- To denote the second derivative function the following notations are used:

$$\underbrace{f''(x)}_{\text{Lagrange's notation}} \quad \underbrace{\ddot{f}(x)}_{\text{Newton's notation}} \quad \underbrace{\frac{d^2 f}{dx^2}, \quad \frac{d^2 f(x)}{dx^2} \quad \text{and} \quad \frac{d^2 y}{dx^2}}_{\text{Leibniz's notation}}$$

7.1.1 Derivative function of order n

- In a completely analogous way one defines the derivative of order n , or n -th derivative, and the n -th derivative function, which are denoted by the symbols:

$$\underbrace{f^{(n)}(x_0), f^{(n)}(x)}_{\text{Lagrange's notation}} \quad \underbrace{\frac{d^n f}{dx^n} \Big|_{x=x_0} \text{ and } \frac{d^n y}{dx^n} \Big|_{x=x_0}, \frac{d^n f}{dx^n}, \frac{d^n f(x)}{dx^n} \text{ and } \frac{d^n y}{dx^n}}_{\text{Leibniz's notation}}$$

7.1.2 Geometric meaning of the second derivative

While the first derivative has, as its geometric meaning, the slope of the graph, the second derivative represents the **rate of change of this slope** and therefore is a measure of the **degree to which the graph deviates from a straight line**.

- We start by considering the family of functions that satisfy the conditions:

$$f(0) = f'(0) = 0, \quad f''(0) \geq 0$$

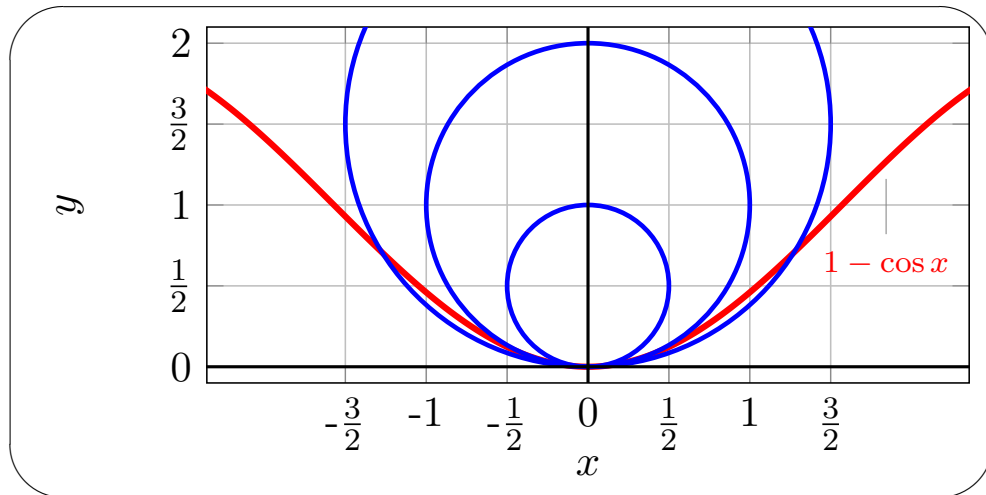
and the family of semicircles with center on the y -axis, tangent to the graph of f at the origin

$$c_r(x) = r - \sqrt{r^2 - x^2}, \quad \text{where } r > 0 \text{ is the radius}$$

For every $r \in \mathbb{R}_+$, we have:

$$c'_r(0) = c_r(0) = 0 \quad \text{since} \quad c'_r(x) = \frac{x}{\sqrt{r^2 - x^2}}$$

For example, the function $f(x) = 1 - \cos x$ with $f(0) = f'(0) = 0$ and the circles with radii $r \in \{0.5, 1, 1.5\}$ are:



- Among these semicircles we want to select the one that not only has the same tangent at $x = 0$ but also the same rate of change of the slope at $x = 0$. That is, we want to choose r so that:

$$c''_r(0) = f''(0) \tag{7.1}$$

We have:

$$c'_r(x) = \frac{x}{\sqrt{r^2 - x^2}} \quad \text{and} \quad c''_r(x) = \frac{r^2}{(r^2 - x^2)^{3/2}}$$

hence, if we want (7.1) to be satisfied, we need to choose r so that:

$$\underbrace{\frac{1}{r}}_{=c''_r(0)} = f''(0) \quad (7.2)$$

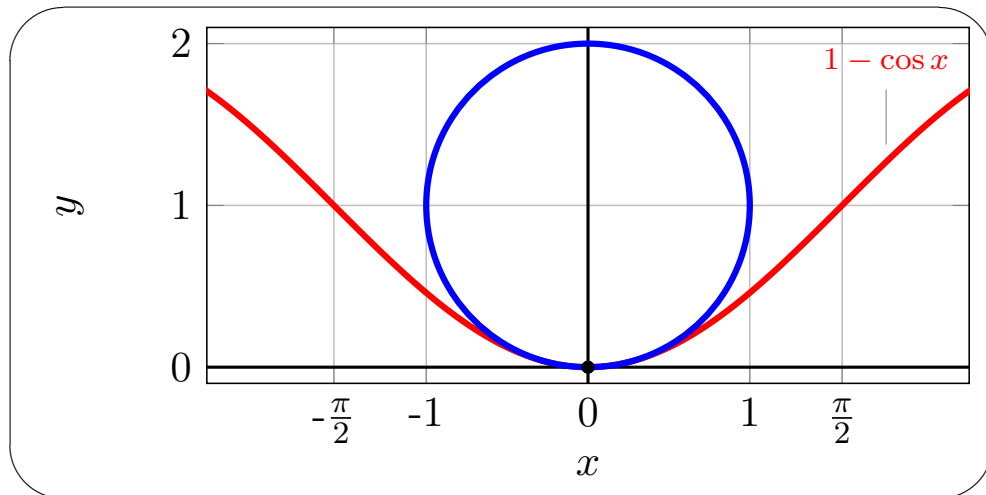
Equation (7.2) expresses the geometric meaning of the second derivative at $x = 0$ for the family of functions that satisfy the conditions $f(0) = f'(0) = 0$, $f''(0) \geq 0$. That is, $f''(0)$ represents the reciprocal of the radius of the semicircle that best approximates f at $x = 0$.

- Let us go back to the function $f(x) = 1 - \cos x$; we have

$$f'(x) = \sin x \quad \text{and} \quad f''(x) = \cos x$$

hence

$$f''(0) = 1 \quad \text{and} \quad \frac{1}{r} = 1 \quad \text{hence} \quad r = 1$$



Repeating an analogous construction at a generic point $(x, f(x))$ of the graph of f (without assuming a priori $f(x) = f'(x) = 0$) one finds that the **circle that best approximates the graph of the function at the point** has radius $r(x)$ given by:

$$\frac{1}{r(x)} = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}} \quad (7.3)$$

The value $\frac{1}{r(x)}$ is called the **curvature** (of the graph) of f at x and the value $r(x)$ is the **radius of curvature**.

Example 7.1: radius of curvature and curvature

Consider the (parametric) parabola:

$$f(x) = a x^2 \quad \text{with } a \in \mathbb{R} \quad \text{we have } f'(x) = 2 a x \quad \text{and} \quad f''(x) = 2 a$$

and the following curvature:

$$\frac{1}{r(x)} = \frac{2 |a|}{(1 + 4 a^2 x^2)^{3/2}}$$

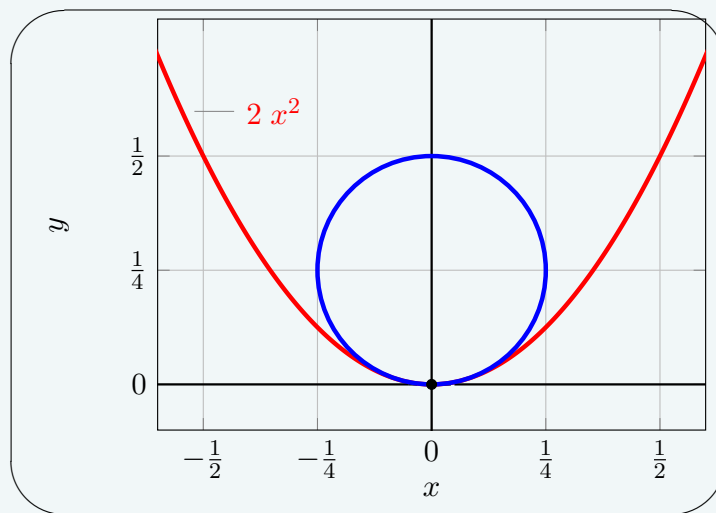
Note that the curvature is maximum at $x = 0$, i.e., at the vertex.

Let us now consider the parabola

$$y = 2 x^2$$

and compute the radius $r(x)$ of the circle that best approximates the graph at the points with abscissa $x = 0$ and $x = \frac{1}{2}$. We have:

$$\frac{1}{r(0)} = \frac{2 |2|}{(1 + 4 \cdot 2^2 \cdot 0^2)^{3/2}} = 4 \quad \text{hence} \quad r(0) = \frac{1}{4}$$



$$\frac{1}{r\left(\frac{1}{2}\right)} = \frac{2 |2|}{\left(1 + 4 \cdot 2^2 \left(\frac{1}{2}\right)^2\right)^{3/2}} = \frac{4}{5^{3/2}} \approx 0.3577 \quad \text{hence} \quad r\left(\frac{1}{2}\right) \approx 2.7950$$

7.2 Second derivative, concavity and convexity

- We will now see how to make this idea of curvature of the graph more precise, qualitatively and quantitatively, through the concept of **convexity**.

7.2.1 Convex sets

Definition 7.3: convex set

A set $F \subseteq \mathbb{R}^2$ (a portion of the Euclidean space) is **convex** if for every pair of points $P_a, P_b \in F$ the segment joining P_a to P_b (called a **chord**) is entirely contained in F .

Given two points P_a and P_b and a value λ between 0 and 1, i.e., $0 \leq \lambda \leq 1$, the point:

$$P_c = \lambda P_a + (1 - \lambda) P_b$$

is called a **convex (linear) combination** of the points P_a and P_b . As λ varies, the points P_c move along the segment joining P_a to P_b .

Definition 7.4: convex set – equivalent definition

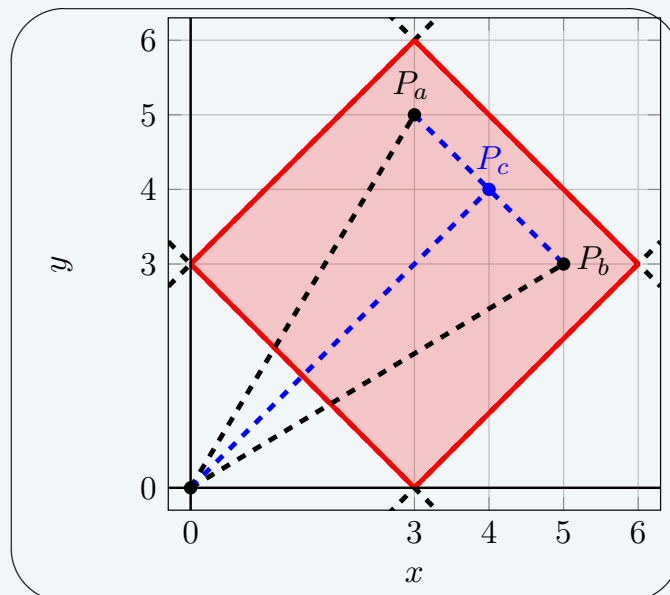
A set $F \subseteq \mathbb{R}^2$ is **convex** if:

$$\forall P_a, P_b \in F, \quad 0 \leq \lambda \leq 1, \quad P_c = \lambda P_a + (1 - \lambda) P_b \in F$$

Example 7.2: convex set

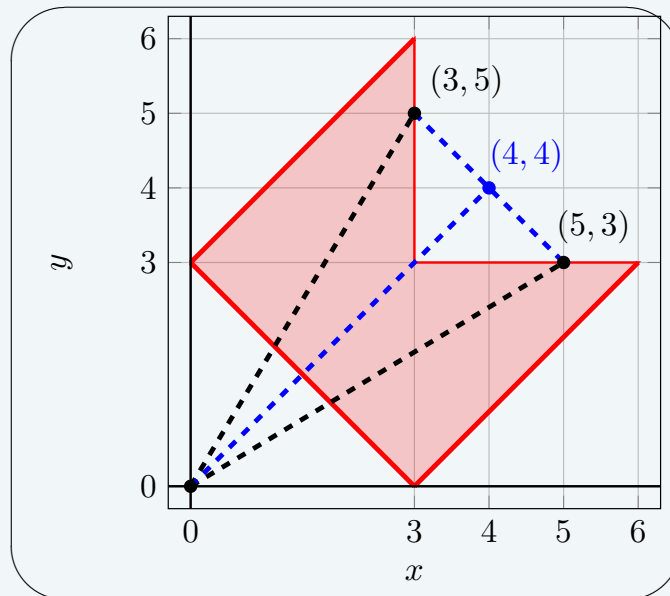
Consider, for example, the following convex set given by the intersection of 4 half-planes:

$$F = \left\{ (x, y) \in \mathbb{R}^2 : \begin{array}{ll} x + y \geq 3, & x + y \leq 9, \\ -x + y \leq 3, & -x + y \geq -3 \end{array} \right\}$$



Example 7.3: non-convex set

Let us now consider the following set:



The set is not convex since, for example, the convex combination of the points $(5, 3)$ and $(3, 5)$ with $\lambda = \frac{1}{2}$, that is, the point:

$$\left(\frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 3, \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot 5 \right) = (4, 4)$$

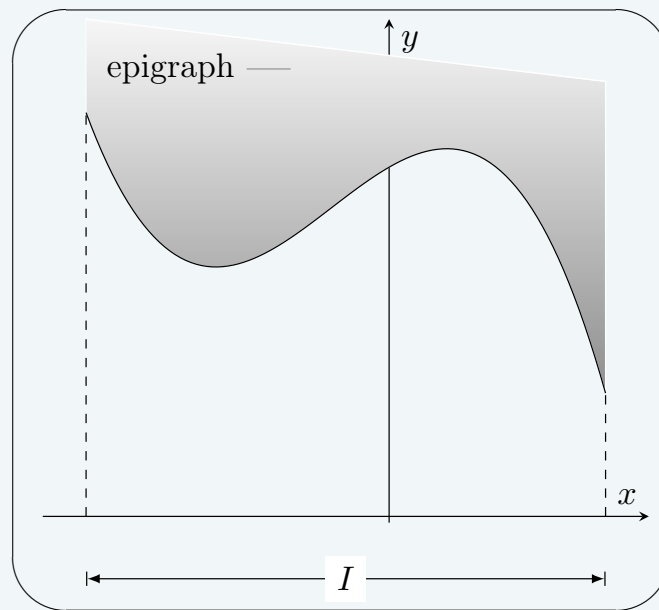
does not belong to the set.

7.2.2 Convex/concave functions**Definition 7.5: epigraph**

Consider a function $f : I \rightarrow \mathbb{R}$. The **epigraph** of f is the set:

$$\text{epi } f = \{(x, y) \in \mathbb{R}^2 : x \in I \text{ and } y \geq f(x)\}$$

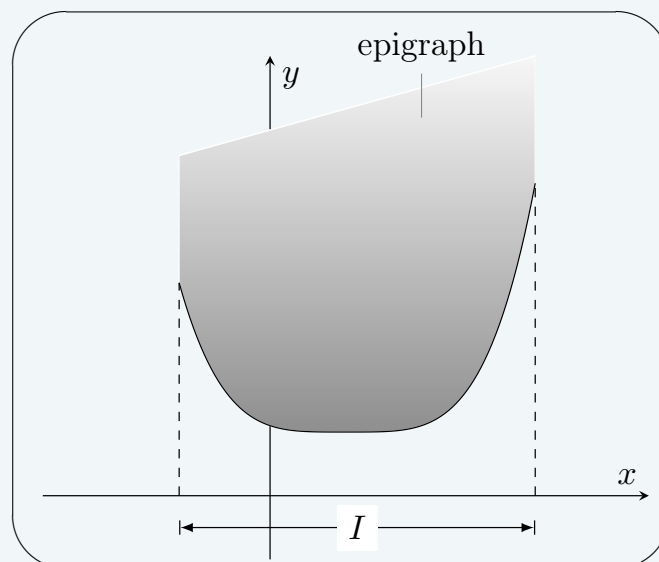
Example 7.4: epigraph

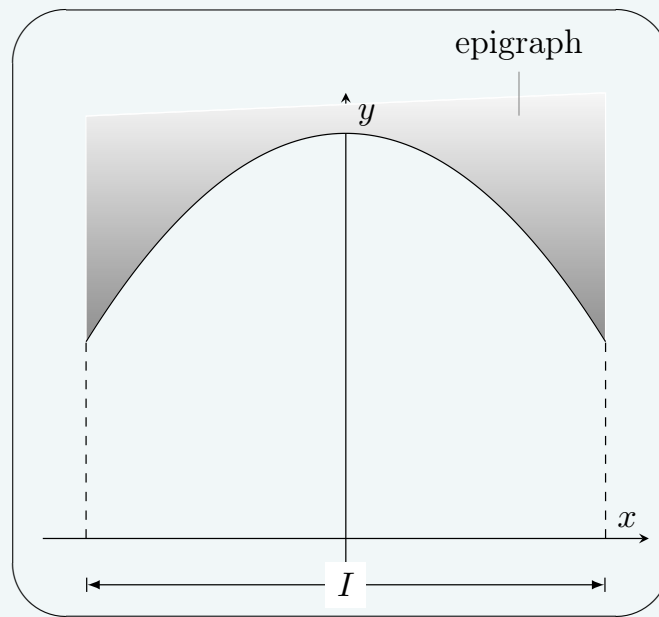


Definition 7.6: convex (concave) function

A function $f : I \rightarrow \mathbb{R}$ is **convex** on I if its epigraph is a convex set. A function is **concave** on I if $-f$ is convex on I .

Example 7.5: convex function

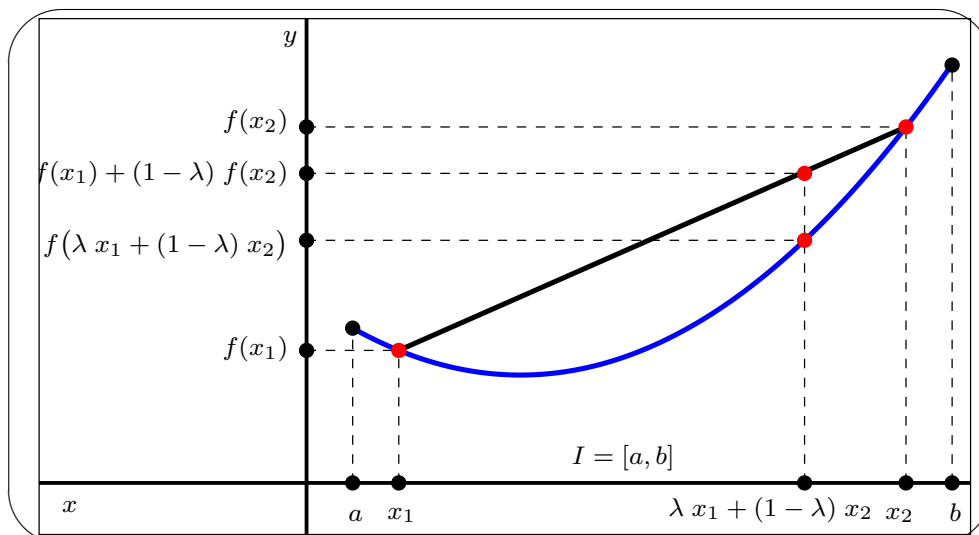


Example 7.6: concave function

Definition 7.7: convex (concave) function – equivalent definition

A function $f : I \rightarrow \mathbb{R}$ is **convex (concave)** on I if for every pair of points $x_1, x_2 \in I$ the segment (“chord”) with endpoints $(x_1, f(x_1))$, $(x_2, f(x_2))$ has no points below (above) the graph of f .

The definition of convex function translates into the analytic inequality:

$$f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2), \quad \forall x_1, x_2 \in I, 0 \leq \lambda \leq 1$$



- If in the previous inequality the strict $<$ always holds (with $\lambda \neq 0, 1$), the function is called **strictly convex**. For concave (or strictly concave) functions an analogous inequality holds with the sign $\geq$ (or $>$).

- Observe that as $0 \leq \lambda \leq 1$ varies: the point $\lambda x_1 + (1 - \lambda) x_2$ runs along the segment $[x_1, x_2]$ on the x -axis; the point $\lambda f(x_1) + (1 - \lambda) f(x_2)$ runs along the segment $[f(x_1), f(x_2)]$ on the y -axis; the point:

$$\left(\lambda x_1 + (1 - \lambda) x_2, f(\lambda x_1 + (1 - \lambda) x_2) \right)$$

runs along the graph of the function; the point:

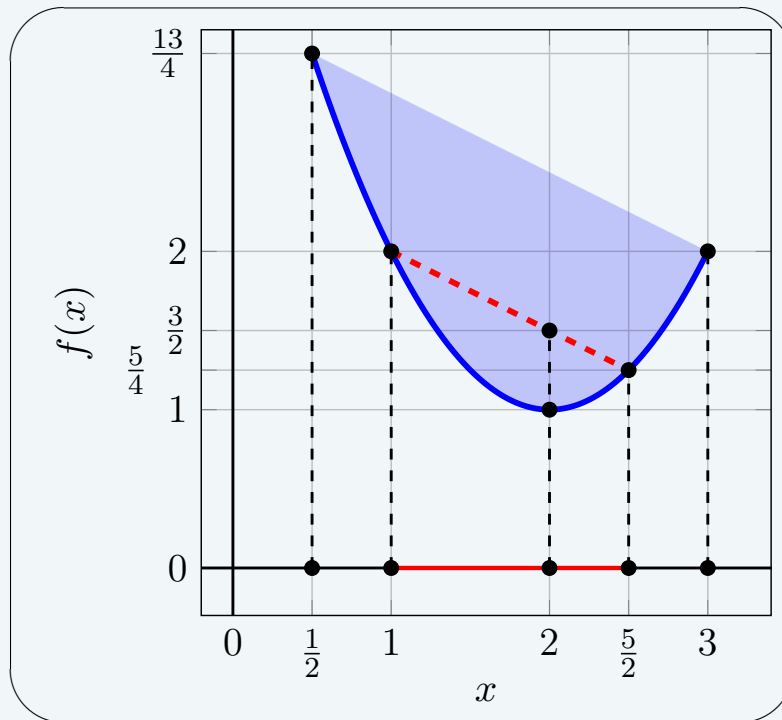
$$(\lambda x_1 + (1 - \lambda) x_2, \lambda f(x_1) + (1 - \lambda) f(x_2))$$

runs along the segment with endpoints $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

Example 7.7: graph of a convex function

Consider the convex function $f(x) = (x - 2)^2 + 1$ with domain $[\frac{1}{2}, 3]$ and take, for example, $x_1 = 1$ and $x_2 = \frac{5}{2}$ and $\lambda = \frac{1}{3}$. We have the following convex (linear) combination:

$$\left(\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{5}{2}, \frac{1}{3} \cdot 2 + \frac{2}{3} \cdot \frac{5}{4} \right) = \left(2, \frac{3}{2} \right)$$



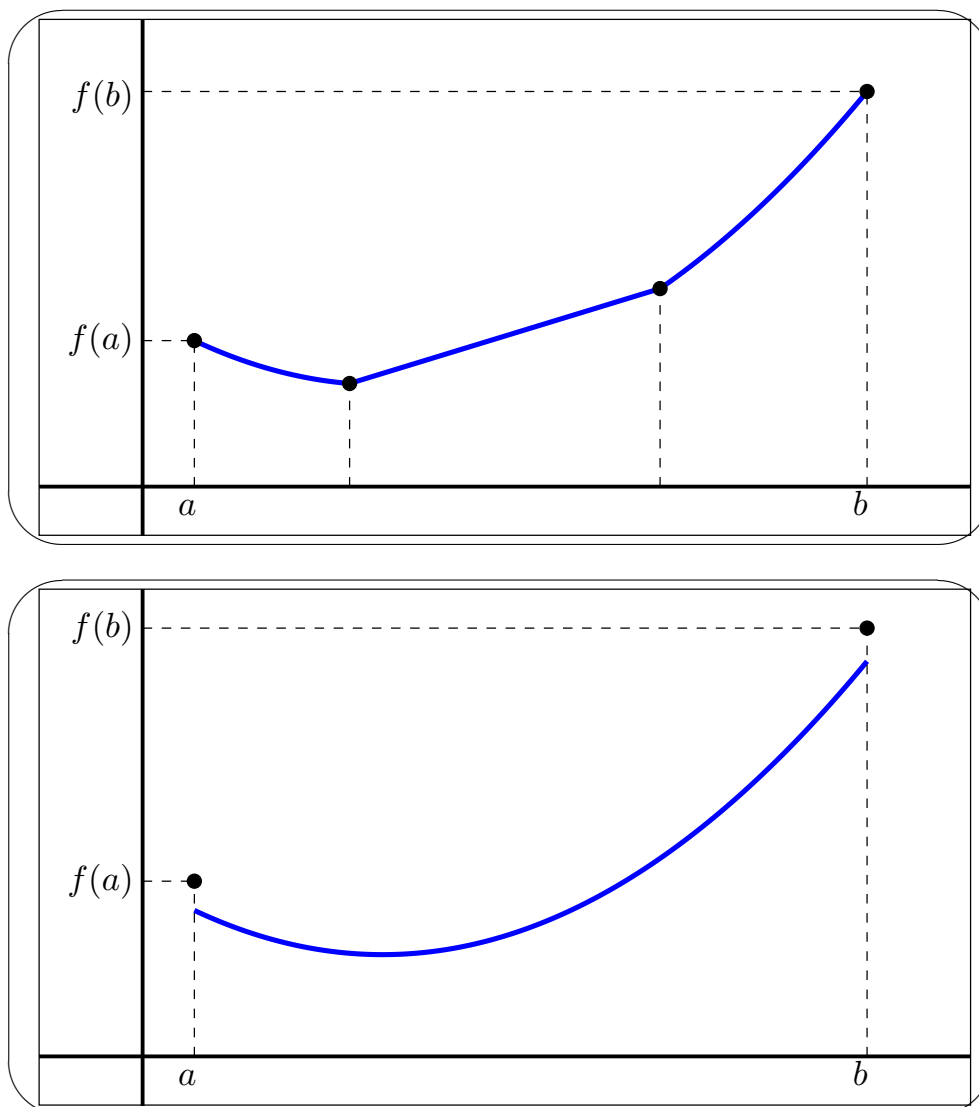
- Note that the definition of convex function does not require a priori that the function be continuous or differentiable on an interval.

Theorem 7.8

A convex (or concave) function on an interval I is continuous on I , except possibly at the endpoints of the interval. Moreover, it has a right and a left derivative at every interior point of the interval.

Proof. Omitted □

- Corner points inside the interval and points of discontinuity at the endpoints of the interval are the only irregular behaviors allowed for a convex or concave function, as shown by the following examples:



7.3 Convexity and derivatives

- If we know a priori that the function is differentiable once or twice on the interval considered, then convexity is related to the first and second derivative of the function.

Theorem 7.9

Let $f : (a, b) \rightarrow \mathbb{R}$.

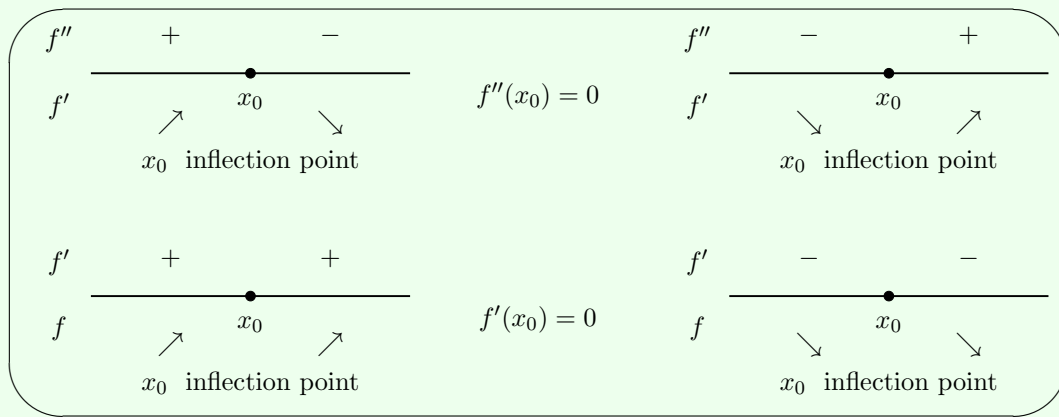
- a) If f is differentiable in (a, b) , then f is convex (concave) on (a, b) if and only if f' is increasing (decreasing) on (a, b) .
- b) If f is twice differentiable in (a, b) , then f is convex (concave) on (a, b) if and only if:

$$f''(x) \geq 0 \quad (\leq 0), \quad \forall x \in (a, b)$$

- The theorem is modified in the obvious way for strictly convex or concave functions.

Proof. We do not prove part (a). Part (b) follows from (a) by the monotonicity test applied to f' . $\square$

As a consequence of this theorem, the study of the sign of the second derivative allows us to decide on the convexity or concavity of a function (**check**).


Example 7.8: convexity of exponential functions

The exponential functions

$$f(x) = a^x$$

are convex on $\mathbb{R}$, for any base $a > 0, a \neq 1$, since:

$$f'(x) = a^x \log a; \quad f''(x) = a^x \log^2 a > 0, \quad \forall x \in \mathbb{R}, \forall a > 0, a \neq 1$$

Example 7.9: convexity/concavity of logarithmic functions

The logarithmic functions

$$f(x) = \log_a x$$

are concave on $(0, +\infty)$ if $a > 1$ and convex on $(0, +\infty)$ if $0 < a < 1$, since:

$$f'(x) = \frac{1}{x \log a}; \quad f''(x) = -\frac{1}{x^2 \log a} \quad \begin{cases} < 0, \forall x > 0, & \text{if } a > 1 \\ > 0, \forall x > 0, & \text{if } 0 < a < 1 \end{cases}$$

Example 7.10: convexity/concavity of power functions

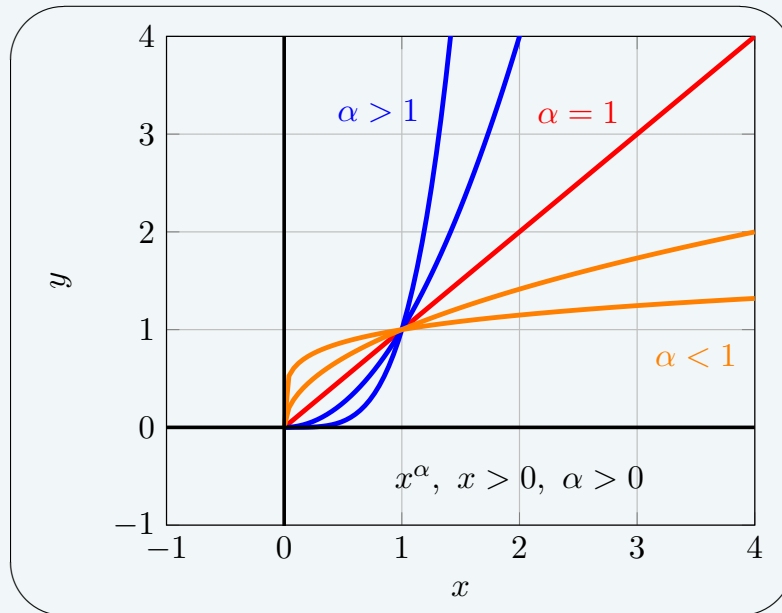
The power functions with real exponent α :

$$f(x) = x^\alpha$$

are convex on $(0, +\infty)$ if $\alpha > 1$ or $\alpha < 0$, and concave on $(0, +\infty)$ if $0 < \alpha < 1$, since:

$$f'(x) = \alpha x^{\alpha-1}; \quad f''(x) = \alpha(\alpha-1)x^{\alpha-2}$$

which has, for every $x > 0$, the sign of $\alpha(\alpha-1)$.



7.4 Convexity and tangent lines

- A useful geometric characterization of convexity involves the tangent lines to the graph of the function

Theorem 7.10

A function $f : (a, b) \rightarrow \mathbb{R}$, differentiable in (a, b) , is convex (concave) on (a, b) if and only if, for any choice of a point $x_0 \in (a, b)$, the graph of f lies on the whole of (a, b) above (below) the graph of its tangent line at $(x_0, f(x_0))$.

Proof. Omitted

□

Example 7.11: tangent lines and graphs of convex functions

Consider the function:

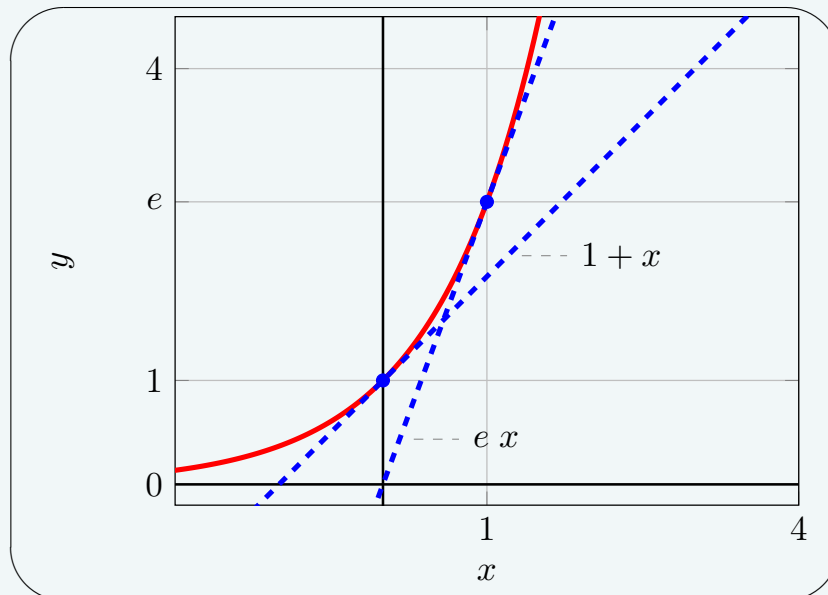
$$f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x \quad \text{is convex on all of } \mathbb{R}$$

The tangent line to the graph of f , for example at $x = 0$, that is, at the point $(0, 1)$, is:

$$y = 1 + x \quad \text{hence} \quad e^x \geq 1 + x, \quad x \in \mathbb{R}$$

The tangent line to the graph of f , for example at $x = 1$, that is, at the point $(1, e)$, is:

$$y = e + e(x - 1) \quad \text{hence} \quad e^x \geq ex, \quad x \in \mathbb{R}$$



A (differentiable) convex function lies above its tangent lines and, at the same time, below its chords (definition of convexity).

- This allows us to conclude that, given any two points on the graph of a convex function, the graph between those two points lies entirely in the **triangle** whose sides are the chord joining them and the tangent lines to the graph at the two points.

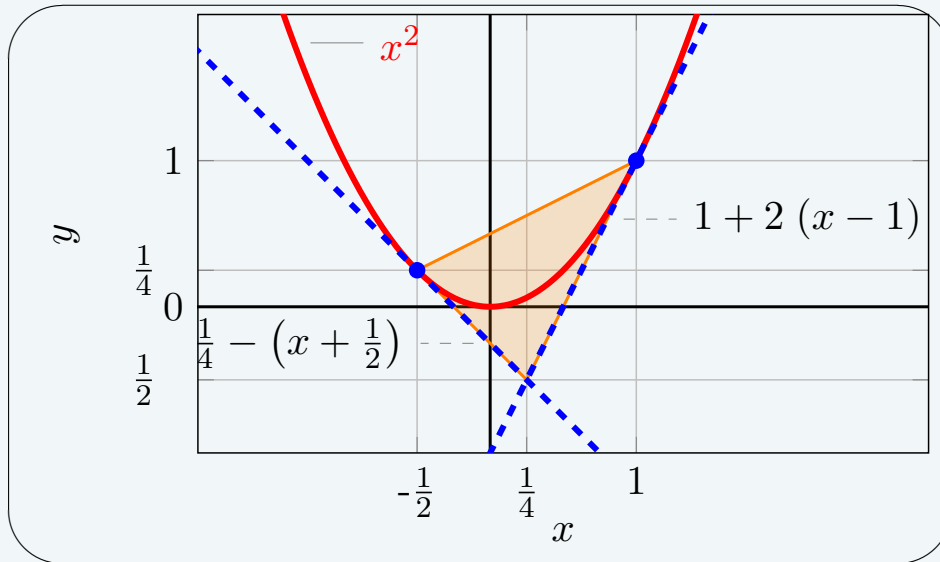
Example 7.12: tangent lines/chords and graphs of convex functions

Consider the function:

$$f(x) = x^2, \quad f'(x) = 2x, \quad f''(x) = 2 \quad \text{is convex on all of } \mathbb{R}$$

The tangent lines to the graph of f , for example at the points $x_1 = -0.5$ and $x_2 = 1$, are:

$$y = \frac{1}{4} - \left(x + \frac{1}{2}\right) \quad \text{and} \quad y = 1 + 2(x - 1)$$



7.5 Inflection points

- The direction of concavity of a function (that is, whether it is convex or concave) can change within its domain; this leads us to the concept of inflection point.

Definition 7.11: inflection point

Let $f : (a, b) \rightarrow \mathbb{R}$ be a function and let $x_0 \in (a, b)$ be a point of differentiability for f , or let $f'(x_0) = \pm\infty$. The point x_0 is called an **inflection point** of f if there exist a right neighborhood $(x_0, x_0 + h)$, $h > 0$, in which f is convex (concave) and a left neighborhood $(x_0 - h, x_0)$, $h > 0$, in which f is concave (convex).

- When crossing an inflection point, the second derivative of f (if it exists) changes sign. We then expect f'' to vanish at this point.

Theorem 7.12

Let x_0 be an inflection point of f ; if $f''(x_0)$ exists, then $f''(x_0) = 0$.

Proof. Note that, if we knew that f'' exists in a neighborhood of x_0 **and is continuous** at x_0 , then the thesis of the theorem would follow from the intermediate value theorem for continuous functions (applied to f''). The theorem can also be proved without these additional hypotheses (omitted). $\square$

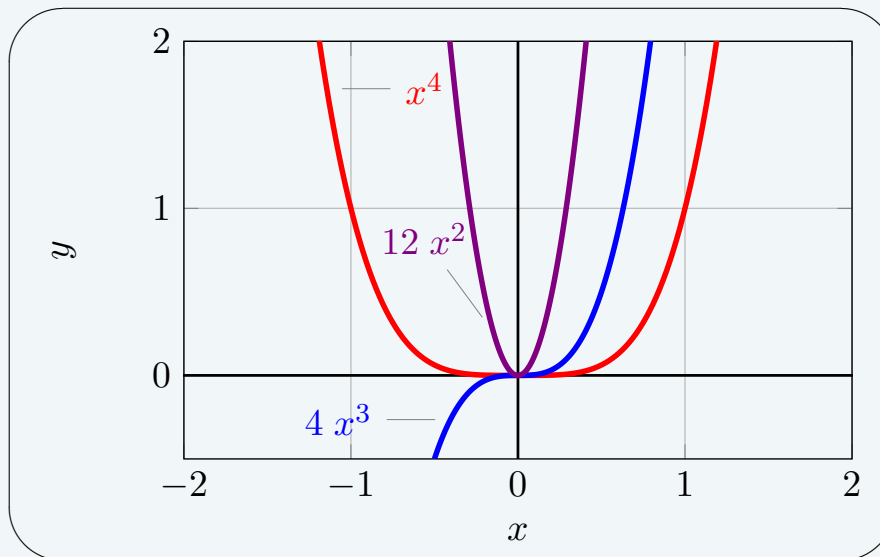
The converse of the implication stated by the theorem is not true: a point where the second derivative vanishes may not be an inflection point.

Example 7.13: points with zero second derivative that are not inflection points

Consider the function:

$$f(x) = x^4, \quad f'(x) = 4x^3, \quad f''(x) = 12x^2 \quad \text{is convex on all of } \mathbb{R}$$

Since $f'(x) > 0$ for $x > 0$, the function is increasing for $x > 0$, decreasing for $x < 0$ and has a minimum point at $x = 0$. We have $f''(0) = 0$, but the function is convex on all of $\mathbb{R}$, hence $x = 0$ is **not an inflection point**.



- The geometric meaning of inflection points is clarified by the following theorem.

Theorem 7.13

If $f : (a, b) \rightarrow \mathbb{R}$ is differentiable in (a, b) and $x_0 \in (a, b)$ is an inflection point, then the graph of $f(x)$ crosses its tangent line at $(x_0, f(x_0))$.

Proof. Let us draw the tangent line to the graph of $f(x)$ at the point with abscissa x_0 . If f is (for example) concave on (a, x_0) , since f is differentiable, the graph of f lies below the line on (a, x_0) ; on the other hand f is convex on (x_0, b) , so its graph lies above the line on (x_0, b) . Consequently, at x_0 the graph crosses the tangent line. $\square$

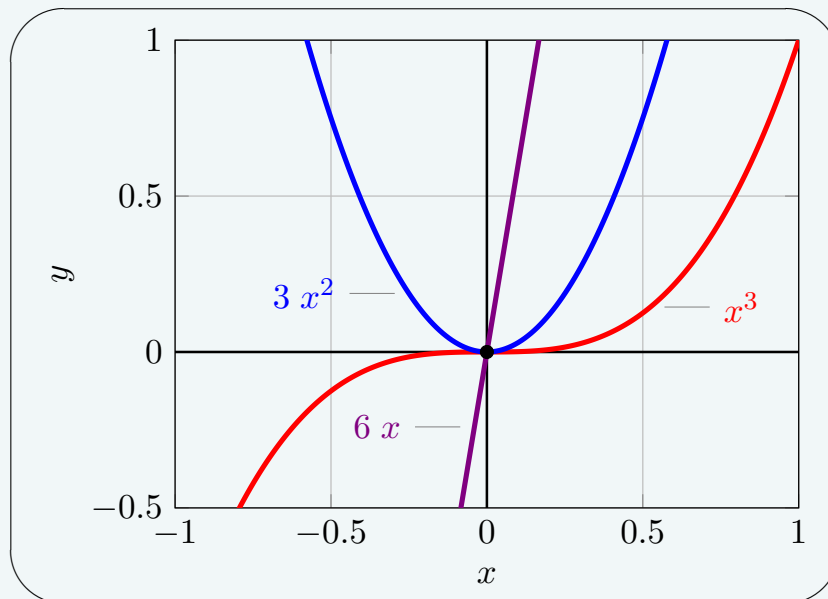
Example 7.14: inflection points (with horizontal tangent)

Consider the function:

$$f(x) = x^3, \quad f'(x) = 3x^2, \quad f''(x) = 6x$$

Since $f'(x) > 0$ for $x \neq 0$, the function is increasing on all of $\mathbb{R}$. It has a stationary point at $x = 0$, which, however, is not a maximum or minimum point, because the function is always increasing.

We have $f''(0) = 0$. Moreover, $f''(x) < 0$ for $x < 0$, hence it is concave for $x < 0$, and $f''(x) > 0$ for $x > 0$, hence it is convex for $x > 0$. Consequently, the point x_0 is **an inflection point with horizontal tangent**, and the graph of the function crosses its tangent line at $(0, 0)$, that is, the line $y = 0$.



Example 7.15: inflection points

Consider the function:

$$f(x) = e^{-x^2}, \quad f'(x) = -2x e^{-x^2}, \quad f''(x) = e^{-x^2} (4x^2 - 2)$$

We have $f'(x) > 0$ for $x < 0$, hence the function increases for $x \leq 0$; and $f'(x) < 0$ for $x > 0$, hence it decreases for $x \geq 0$, and moreover $f'(0) = 0$. Therefore it has a relative maximum point at $x = 0$.

We have:

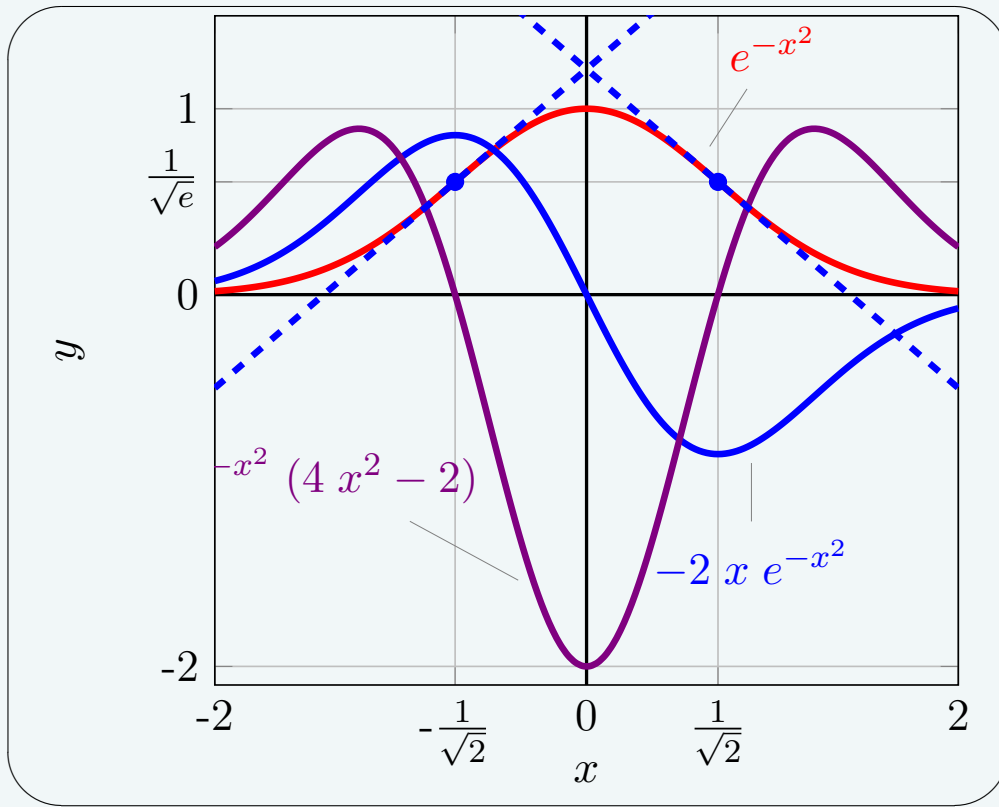
$$f''(x) = e^{-x^2} (4x^2 - 2) \geq 0 \text{ for } x^2 \geq \frac{1}{2} \text{ that is } x \in \left(-\infty, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, +\infty\right)$$

The function is convex for these values, and concave for $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$. Hence it has inflection points at $x = \pm \frac{1}{\sqrt{2}}$, with tangent line of slope:

$$f' \left(\pm \frac{1}{\sqrt{2}} \right) = \mp \sqrt{\frac{2}{e}} \text{ since } -2 \left(\pm \frac{1}{\sqrt{2}} \right) e^{-\left(\pm \frac{1}{\sqrt{2}}\right)^2} = \mp \sqrt{2} \cdot \frac{1}{\sqrt{e}}$$

The tangent lines are:

$$y = \frac{1}{\sqrt{e}} + \sqrt{\frac{2}{e}} \left(x + \frac{1}{\sqrt{2}} \right) \quad \text{and} \quad y = \frac{1}{\sqrt{e}} - \sqrt{\frac{2}{e}} \left(x - \frac{1}{\sqrt{2}} \right)$$

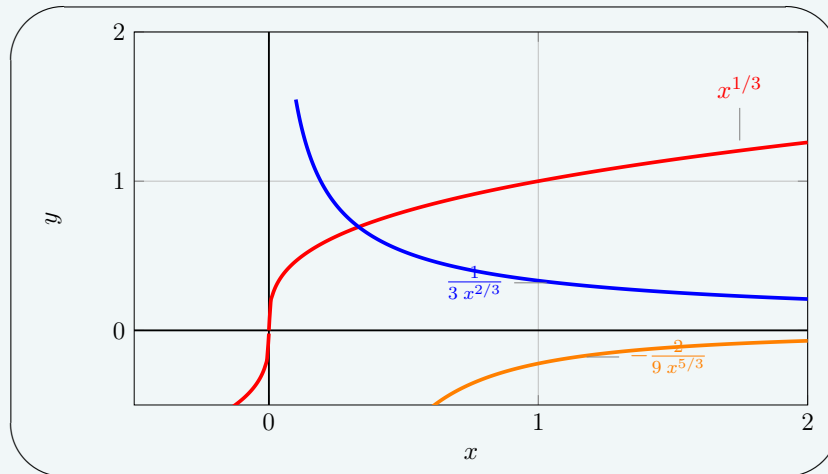


Example 7.16: inflection points (with vertical tangent)

Consider the function:

$$f(x) = x^{1/3}, \quad f'(x) = \frac{1}{3x^{2/3}}, \quad \text{for } x > 0, \quad f''(x) = -\frac{2}{9x^{5/3}}, \quad \text{for } x > 0$$

For $x_0 = 0$, we have seen that $f'(0) = +\infty$. In this case, however, $f''(0)$ does not exist and the function has an inflection point with vertical tangent.

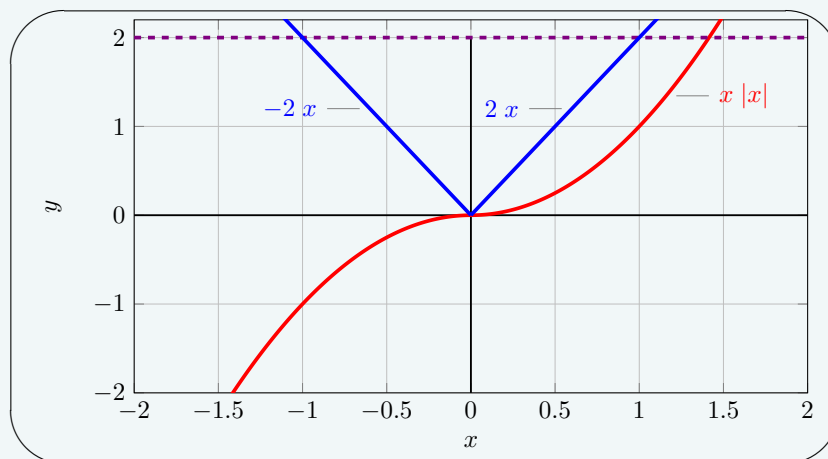

Example 7.17: inflection points (with horizontal tangent)

Consider the function:

$$f(x) = x|x|, \quad \text{with } x \neq 0, \quad f(x) = \begin{cases} x^2, & x > 0 \\ -x^2, & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x, & x > 0 \\ -2x, & x < 0 \end{cases} \quad f''(x) = \begin{cases} 2, & x > 0 \\ -2, & x < 0 \end{cases}$$

Hence, for $x > 0$ the function is convex and for $x < 0$ it is concave, and $f'(0) = 0$; therefore x_0 is an inflection point (but $f''(0)$ does not exist).



Differential calculus and approximations

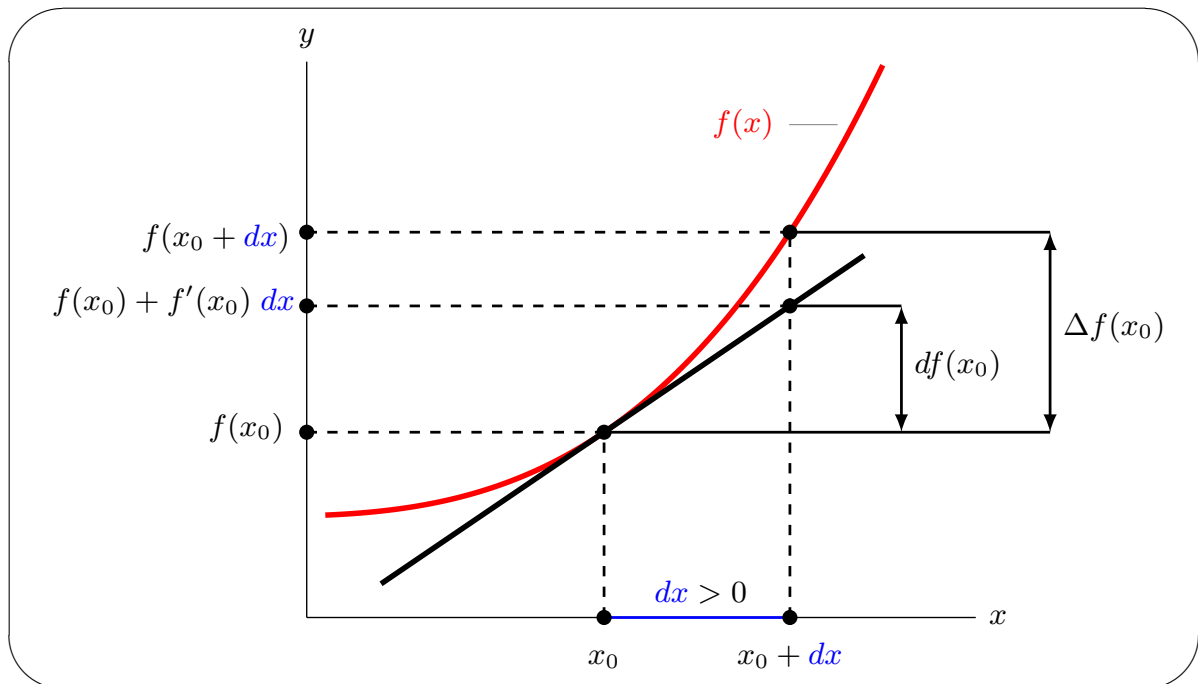
8.1 First-order asymptotic expansions

- A very frequent operation, both in mathematics and in its applications, is the linear approximation of a given differentiable function (an operation also called first-order asymptotic expansion).
- Let $f : (a, b) \rightarrow \mathbb{R}$ be a function differentiable at a point $x_0 \in (a, b)$. Consider the argument $x_0 + dx$ due to the increment/decrement dx with respect to x_0 . As a consequence, the values of f undergo the increment/decrement:

$$\Delta f(x_0) = f(x_0 + dx) - f(x_0)$$

which, in general, for fixed x_0 , is not proportional to dx , i.e., it is not linear with respect to dx . The increment/decrement evaluated along the tangent line is:

$$df(x_0) = f(x_0) + f'(x_0) dx - f(x_0) = f'(x_0) dx$$



- Given a function f differentiable at a point x_0 in the interior of its domain, the **linear approximation** consists in approximating the increment/decrement $\Delta f(x_0)$ of the values of the function due to an increment/decrement of the argument from x_0 to $x_0 + dx$, by replacing the function with its tangent line at the point $(x_0, f(x_0))$.
- The idea behind linear approximation therefore consists in estimating $\Delta f(x_0)$ with $df(x_0)$. To obtain good approximations, we consider increments that are very “small” in absolute value, that is, $|dx| \ll 1$.

Definition 8.1: differential of a function

Given a function $f : (a, b) \rightarrow \mathbb{R}$, differentiable at a point $x_0 \in (a, b)$, the **differential** of f at the point x_0 is the increment/decrement $df(x_0)$ of the values of the function from x_0 to $x_0 + dx$ evaluated along the tangent line:

$$df(x_0) = f'(x_0) dx$$

- Given two functions f and g , the rules for computing differentials are analogous to the differentiation rules:

$$d(f \pm g)(x_0) = df(x_0) \pm dg(x_0), \quad d(f \cdot g)(x_0) = g(x_0) df(x_0) + f(x_0) dg(x_0)$$

$$d\left(\frac{f}{g}\right)(x_0) = \frac{g(x_0) df(x_0) - f(x_0) dg(x_0)}{g^2(x_0)}$$

- In all types of approximation it is necessary to provide information (qualitative or quantitative) on the **error made**. What is the error made in the approximation? What is the value of $\Delta f(x_0) - df(x_0)$?

To answer, we observe that dx is equivalent to $x - x_0$, hence we have:

$$\Delta f(x_0) = f(x_0 + x - x_0) - f(x_0) = f(x) - f(x_0) \quad \text{and} \quad df(x_0) = f'(x_0) (x - x_0)$$

Moreover, from the definition of derivative we have:

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) &\Rightarrow \lim_{x \rightarrow x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right) = 0 \\ \Rightarrow \lim_{x \rightarrow x_0} \left(\frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} \right) &= 0 \end{aligned}$$

and from the definition of little- o we have:

$$\underbrace{f(x) - f(x_0)}_{=\Delta f(x_0)} - \underbrace{f'(x_0)(x - x_0)}_{=df(x_0)} = o(x - x_0) \quad \text{as } x \rightarrow x_0 \quad (8.1)$$

Hence the error $\Delta f(x_0) - df(x_0)$ tends to zero faster than $(x - x_0)$ as $x \rightarrow x_0$.

Definition 8.2: first-order asymptotic expansion

Given a function $f : (a, b) \rightarrow \mathbb{R}$, differentiable at a point $x_0 \in (a, b)$, the **first-order asymptotic expansion** (or **linear expansion**) of f at the point x_0 is the following expression:

$$f(x) = f(x_0) + f'(x_0) (x - x_0) + o(x - x_0) \quad \text{as } x \rightarrow x_0 \quad (8.2)$$

Expression (8.2) follows directly from (8.1), and in the case $x_0 = 0$ we have:

$$f(x) = f(0) + f'(0) x + o(x) \quad \text{as } x \rightarrow 0 \quad (8.3)$$

Example 8.1: first-order (linear) asymptotic expansions and approximation errors

We compute the first-order (linear) asymptotic expansion as $x \rightarrow 0$ of:

$$f(x) = \sqrt{1+x}$$

We have $x_0 = 0$ and moreover:

$$f'(x) = \frac{1}{2\sqrt{1+x}}, \quad f'(0) = \frac{1}{2} \quad \text{and} \quad f(0) = 1$$

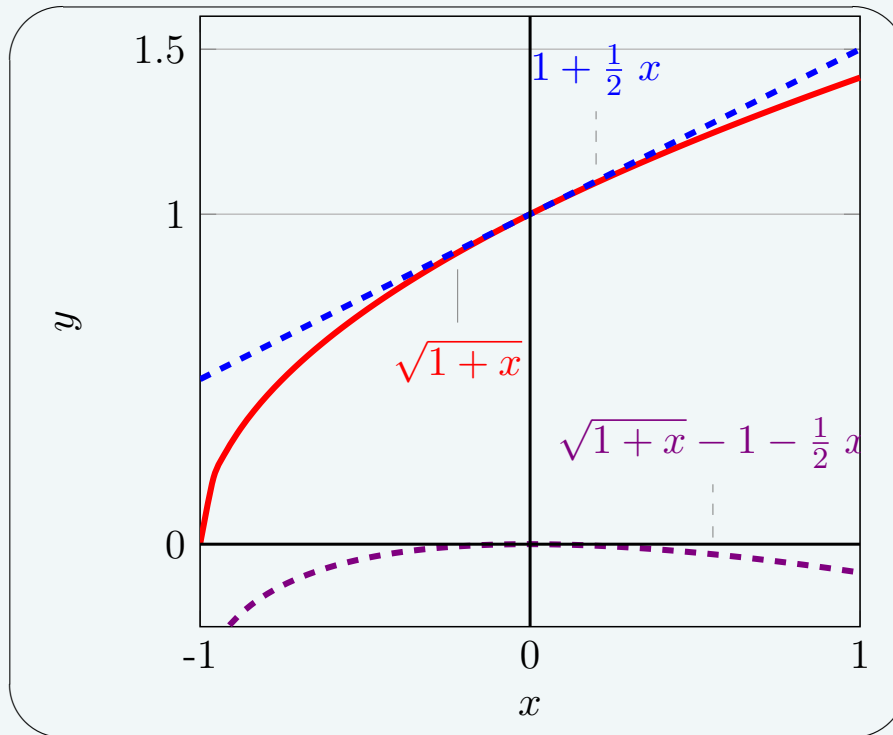
Hence we have the following first-order (linear) asymptotic expansion:

$$\sqrt{1+x} = 1 + \frac{1}{2} x + o(x) \quad \text{as } x \rightarrow 0$$

Moreover, the error made by the approximation is:

$$\underbrace{\sqrt{1+x} - 1}_{=\Delta f(0)=f(x)-f(0)} - \underbrace{\frac{1}{2} x}_{=df(0)=f'(0)x} = o(x) \quad \text{as } x \rightarrow 0$$

Graphically we have:



Example 8.2: first-order (linear) asymptotic expansions

We compute the first-order (linear) asymptotic expansion as $x \rightarrow 1$ of:

$$f(x) = \sqrt{1+x}$$

We have $x_0 = 1$ and moreover:

$$f'(x) = \frac{1}{2\sqrt{1+x}}, \quad f'(1) = \frac{1}{2\sqrt{2}} \quad \text{and} \quad f(1) = \sqrt{2}$$

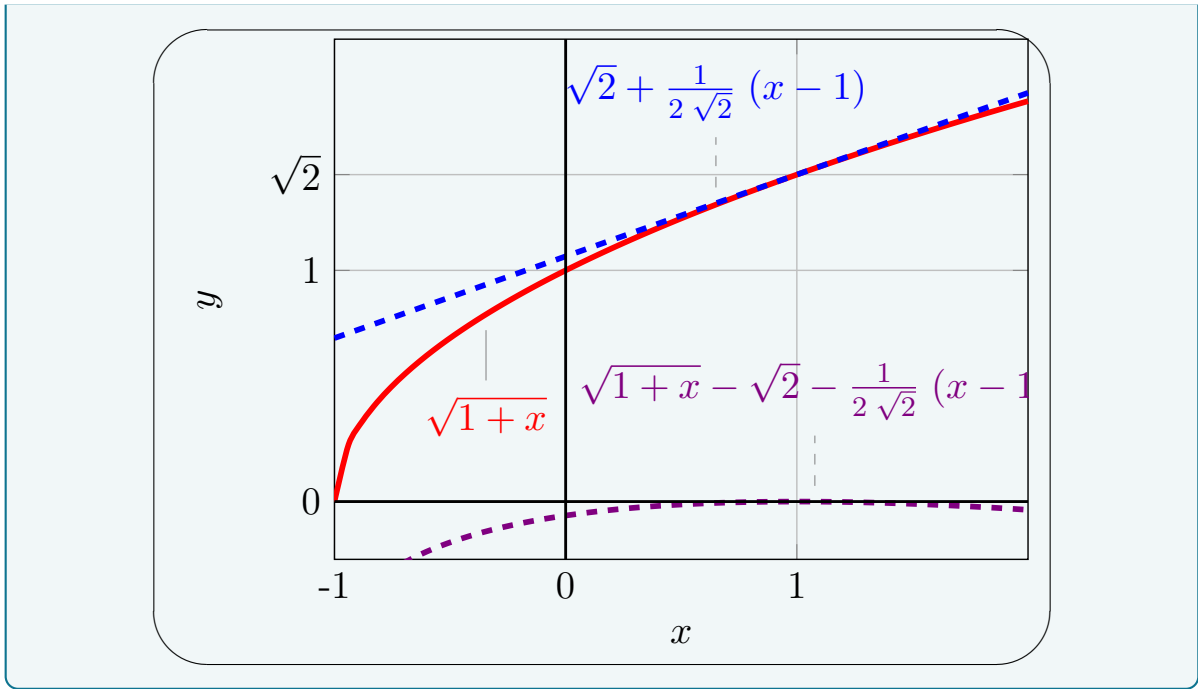
Hence we have the following first-order (linear) asymptotic expansion:

$$\sqrt{1+x} = \sqrt{2} + \frac{1}{2\sqrt{2}}(x-1) + o(x-1) \quad \text{as } x \rightarrow 1$$

Moreover, the error made by the approximation is:

$$\underbrace{\sqrt{1+x} - \sqrt{2}}_{=\Delta f(1)=f(x)-f(1)} - \underbrace{\frac{1}{2\sqrt{2}}(x-1)}_{=df(1)=f'(1)(x-1)} = o(x-1) \quad \text{as } x \rightarrow 1$$

Graphically we have:



8.2 Derivatives of polynomials

- A polynomial of degree n can be written as:

$$P_n(x) = \sum_{i=0}^n a_i \cdot x^i \quad \text{with } a_i \in \mathbb{R}, \text{ for } i \in \{0, 1, \dots, n\}, \text{ and } a_n \neq 0.$$

The value a_i is the **coefficient** of the monomial of degree i , with $i \in \{0, 1, \dots, n\}$, while x^i is the **literal part** of the monomial, a positive integer power of x . The value a_0 is the **constant term** of the polynomial.

The first and second derivative functions of a polynomial of degree $n \geq 2$ are:

$$P'_n(x) = \sum_{i=1}^n i a_i x^{i-1} \qquad P''_n(x) = \sum_{i=2}^n (i-1) \cdot i \cdot a_i x^{i-2}$$

$$P'_n(0) = a_1 \qquad P''_n(0) = 2 a_2$$

Example 8.3: first and second derivative of a polynomial

Consider the following polynomial of degree 5:

$$P_5(x) = 10 + 7x + 3x^2 - 6x^3 + 4x^4 - 2x^5$$

the first derivative function is:

$$\begin{aligned} P'_5(x) &= 1 \cdot 7 x^{1-1} + 2 \cdot 3 x^{2-1} + 3 \cdot (-6) x^{3-1} + 4 \cdot 4 x^{4-1} + 5 \cdot (-2) x^{5-1} \\ &= 7 + 6x - 18x^2 + 16x^3 - 10x^4 \end{aligned}$$

$$P'_5(0) = 7$$

the second derivative function is:

$$P_5''(x) = 1 \cdot 2 \cdot 3 x^{2-2} + 2 \cdot 3 \cdot (-6) x^{3-2} + 3 \cdot 4 \cdot 4 x^{4-2} + 4 \cdot 5 \cdot (-2) x^{5-2} = 6 - 36x + 48x^2 - 40x^3$$

$$P_5''(0) = 2 \cdot 3 = 6$$

The k -th derivative function of a polynomial of degree n with $k \in \{0, 1, \dots, n\}$ is:

$$P_n^{(k)}(x) = \sum_{i=k}^n \underbrace{(i-k+1) \cdots (i-2) \cdot (i-1) \cdot i}_{=\prod_{j=1}^k (i-j+1)} a_i x^{i-k}$$

$$P_n^{(k)}(0) = k! a_k$$

Example 8.4: third, fourth and fifth derivative of a polynomial

Consider the following polynomial of degree 5:

$$P_5(x) = 10 + 7x + 3x^2 - 6x^3 + 4x^4 - 2x^5$$

the third derivative function is:

$$P_5^{(3)}(x) = 1 \cdot 2 \cdot 3 \cdot (-6) x^{3-3} + 2 \cdot 3 \cdot 4 \cdot 4 x^{4-3} + 3 \cdot 4 \cdot 5 \cdot (-2) x^{5-3} = -36 + 96x - 120x^2$$

$$P_5^{(3)}(0) = 3! \cdot (-6) = 6 \cdot (-6) = -36$$

the fourth derivative function is:

$$P_5^{(4)}(x) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 4 x^{4-4} + 2 \cdot 3 \cdot 4 \cdot 5 \cdot (-2) x^{5-4} = 96 - 240x$$

$$P_5^{(4)}(0) = 4! \cdot 4 = 24 \cdot 4 = 96$$

the fifth derivative function is:

$$P_5^{(5)}(x) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot (-2) x^{5-5} = -240$$

$$P_5^{(5)}(0) = 5! \cdot (-2) = 120 \cdot (-2) = -240$$

- Given $x_0 \in \mathbb{R}$, equivalently, a polynomial of degree n can be written as:

$$P_n(x) = \sum_{i=0}^n a_i \cdot (x - x_0)^i \quad \text{with } a_i \in \mathbb{R}, \text{ for } i \in \{0, 1, \dots, n\}, \text{ and } a_n \neq 0.$$

The k -th derivative function, $k \in \{0, 1, \dots, n\}$, is in this case:

$$P_n^{(k)}(x) = \sum_{i=k}^n (i-k+1) \cdots (i-2) \cdot (i-1) \cdot i \cdot a_i \cdot (x - x_0)^{i-k}$$

$$P_n^{(k)}(x_0) = k! \cdot a_k$$

8.3 Taylor formula/expansion with Peano remainder

- Linear approximation (or linearization) approximates a function with its tangent line at a point $(x_0, f(x_0))$, that is, a first-degree polynomial whose derivative equals that of the function at the point x_0 and whose constant term is the value of the function at x_0 .

For simplicity, we start by reasoning with $x_0 = 0$.

- With first-degree polynomials we have:

$$P_1(x) = a_0 + a_1 x, \quad P'_1(x) = a_1, \quad \text{setting } a_0 = f(0) \text{ and } a_1 = f'(0)$$

from (8.2) we obtain the first-order asymptotic expansion:

$$f(x) = \underbrace{a_0}_{=f(0)} + \underbrace{a_1}_{=f'(0)} x + o(x)$$

and the error made by using the first-degree polynomial instead of the function:

$$\underbrace{f(x) - a_0}_{\Delta f(0)} - \underbrace{a_1 x}_{df(0)} = o(x)$$

tends to zero faster than any linear function.

- We now want to generalize the procedure of “**linear approximation**” to that of “**polynomial approximation**”.

We ask: given a function, differentiable as many times as necessary, does there exist a polynomial of degree $n \geq 2$ that, in a neighborhood of a fixed point, approximates the function “better” than its tangent line? **We proceed in two steps.**

First step

- We identify a candidate polynomial to approximate the function “well”, by looking for a polynomial whose derivatives up to order n are all equal to those of the function f at the point $x_0 = 0$. The polynomial must be of degree n in order to have its n -th derivative equal to $f^{(n)}(0)$.

Theorem 8.3: Maclaurin polynomial

Given a function $f : (a, b) \rightarrow \mathbb{R}$ differentiable $n-1$ times in (a, b) and n times at $0 \in (a, b)$, there exists one and only one polynomial $T_{n,f}$ of degree $\leq n$ with the property that:

$$T_{n,f}(0) = f(0), \quad T'_{n,f}(0) = f'(0), \quad \dots, \quad T^{(n)}_{n,f}(0) = f^{(n)}(0)$$

and this polynomial, called the **Maclaurin polynomial** of f of degree n , is:

$$T_{n,f}(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k \quad (\text{setting } f^{(0)} = f)$$

Proof. We prove that the Maclaurin polynomial of a function f satisfying the hypotheses of the theorem has the same derivatives at 0 as the function, up to order n . The k -th derivative at $x_0 = 0$ of the Maclaurin polynomial of degree n , with $k \in \{0, 1, \dots, n\}$, is:

$$T_{n,f}^{(k)}(0) = k! \cdot \frac{f^{(k)}(0)}{k!} = f^{(k)}(0) \quad \text{and with } k = 0 \text{ we have } T_{n,f}(0) = f(0)$$

Hence all the derivatives at 0 up to order n are equal to those of the function f (and it has the same value as the function at 0).

We now prove that this polynomial is unique. Let us now consider a generic polynomial of degree n :

$$P_n(x) = \sum_{i=0}^n a_i \cdot x^i \quad \text{with } a_i \in \mathbb{R}, \text{ for } i = 0, 1, \dots, n, \text{ and } a_n \neq 0.$$

Its k -th derivative at $x_0 = 0$, with $k \in \{0, 1, \dots, n\}$, is

$$P_n^{(k)}(0) = k! \cdot a_k$$

Hence, if the polynomial has the same k -th derivative as the function, we have:

$$k! \cdot a_k = f^{(k)}(0) \quad \text{which implies } a_k = \frac{f^{(k)}(0)}{k!}$$

that is, the coefficients of the Maclaurin polynomial. □

The expanded form of the Maclaurin polynomial of degree n of a function f is:

$$T_{n,f}(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \dots + \frac{1}{n!}f^{(n)}(0)x^n$$

- When the function we refer to is clear, for convenience we omit the subscript f of the Maclaurin polynomial and simply write $T_n(x)$.

Example 8.5: Computing the Maclaurin polynomial

We compute the Maclaurin polynomial of degree 3 of the function

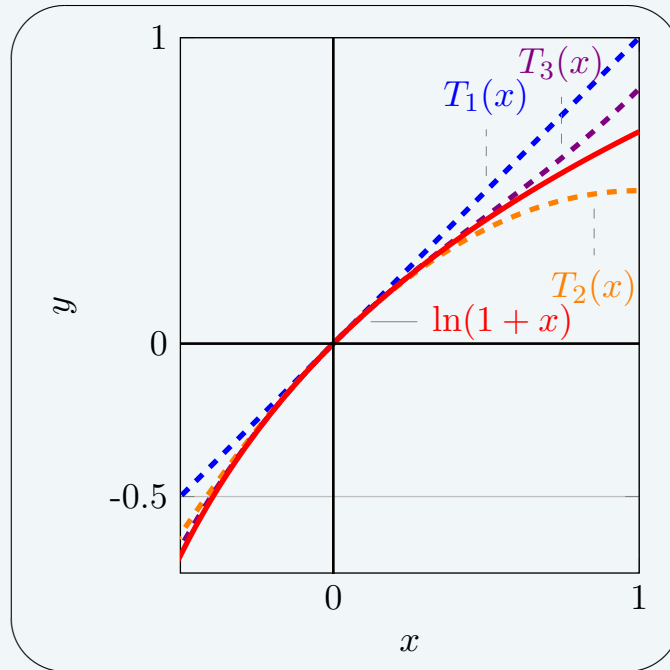
$$f(x) = \log(1+x)$$

We have $f(0) = 0$ and for the derivatives we have:

$$\begin{aligned} f^{(1)}(x) &= \frac{1}{1+x}, & f^{(1)}(0) &= 1, \\ f^{(2)}(x) &= -\frac{1}{(1+x)^2}, & f^{(2)}(0) &= -1, \\ f^{(3)}(x) &= \frac{2}{(1+x)^3}, & f^{(3)}(0) &= 2 \end{aligned}$$

Hence the Maclaurin polynomial of degree 3 of the function is:

$$T_3(x) = 0 + 1x + \frac{1}{2}(-1)x^2 + \frac{1}{6}2x^3 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$



The fourth derivative is:

$$f^{(4)}(x) = -\frac{6}{(1+x)^4}, \quad f^{(4)}(0) = -6$$

Generalizing to the n -th derivative, it follows that the Maclaurin polynomial of degree n of the function is:

$$T_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n-1} \frac{x^n}{n}$$

Example 8.6: Computing the Maclaurin polynomial

We compute the Maclaurin polynomial of degree 3 of the function

$$f(x) = (1+x)^\alpha$$

We have $f(0) = 1$ and for the derivatives we have:

$$f^{(1)}(x) = \alpha(1+x)^{\alpha-1}, \quad f^{(1)}(0) = \alpha, \quad f^{(2)}(x) = (\alpha-1)\alpha(1+x)^{\alpha-2}, \quad f^{(2)}(0) = (\alpha-1)\alpha$$

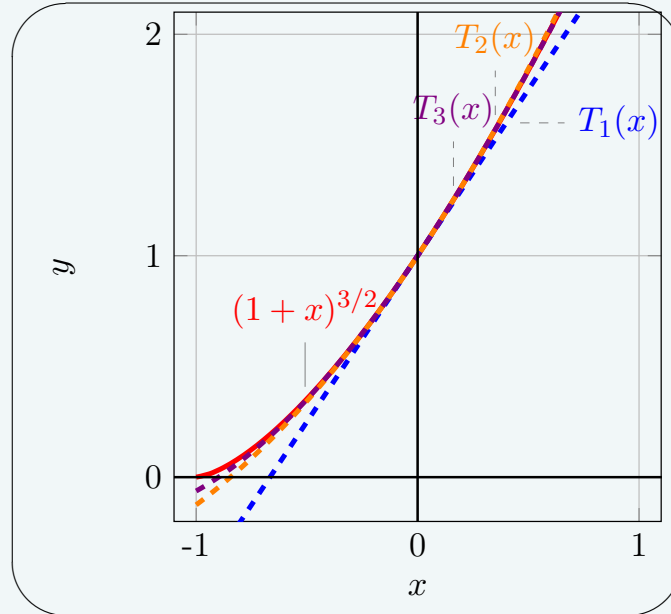
$$f^{(3)}(x) = (\alpha-2)(\alpha-1)\alpha(1+x)^{\alpha-3}, \quad f^{(3)}(0) = (\alpha-2)(\alpha-1)\alpha$$

Hence the Maclaurin polynomial of degree 3 of the function is:

$$T_3(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3$$

For example, with $\alpha = \frac{3}{2}$ we have:

$$T_3(x) = 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3$$



The fourth derivative is:

$$f^{(4)}(x) = (\alpha - 3)(\alpha - 2)(\alpha - 1)\alpha (1+x)^{\alpha-4}, \quad f^{(4)}(0) = (\alpha - 3)(\alpha - 2)(\alpha - 1)\alpha$$

Generalizing to the n -th derivative, it follows that the Maclaurin polynomial of degree n of the function is:

$$T_n(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$$

Second step

- We now prove that the Maclaurin polynomial approximates $f(x)$ “well” in a neighborhood of $x_0 = 0$. Precisely, the following theorem holds:

Theorem 8.4: Maclaurin formula of order n with Peano remainder

Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable $n - 1$ times in (a, b) and n times at $0 \in (a, b)$. Then

$$f(x) = T_{n,f}(x) + o(x^n) \quad \text{as } x \rightarrow 0$$

- The formula has the structure: **function to be approximated equals approximating polynomial plus approximation error**. The error is the term $o(x^n)$ and is called the Peano remainder. As $x \rightarrow 0$, the larger n is, the smaller the error.

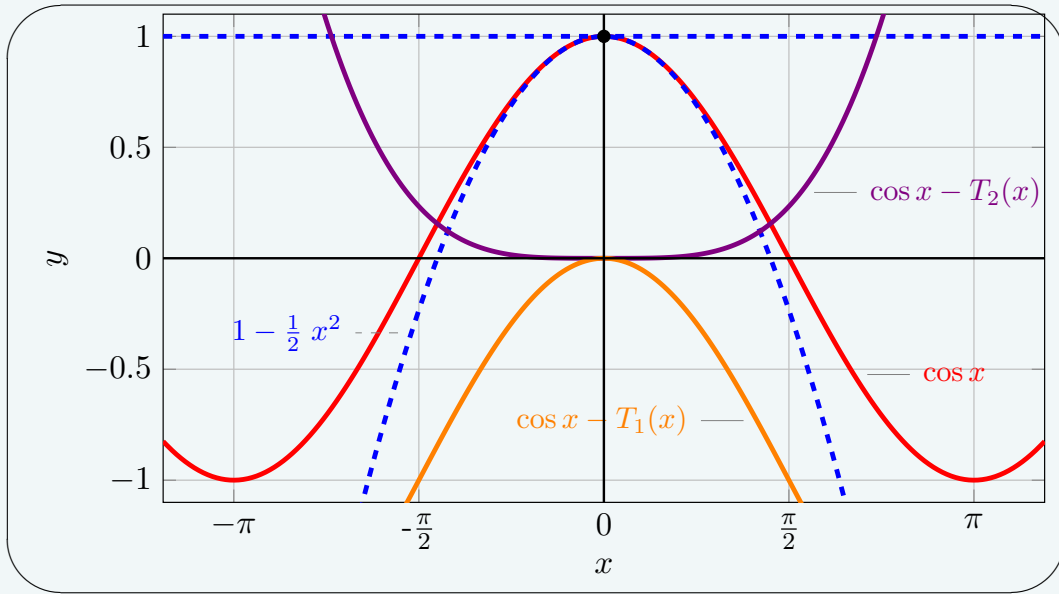
Example 8.7: errors of polynomial approximations

Consider $f(x) = \cos x$; we have:

$$f(0) = 1, \quad f^{(1)}(x) = -\sin x, \quad f^{(1)}(0) = 0, \quad f^{(2)} = -\cos x, \quad f^{(2)}(0) = -1$$

The Maclaurin polynomials of first and second degree and the approximation errors are:

$$T_1(x) = 1, \quad T_2(x) = 1 - \frac{1}{2}x^2, \quad \cos x - 1 = o(x) \quad \text{and} \quad \cos x - 1 + \frac{1}{2}x^2 = o(x^2)$$



The second-degree polynomial $T_2(x)$ approximates the function $\cos x$ near $x_0 = 0$ “better” than the first-degree polynomial $T_1(x)$. The difference between the function and $T_2(x)$ tends to zero faster than x^2 , while the difference with $T_1(x)$ tends to zero only faster than x .

Proof. We prove the theorem in the case $n = 2$, i.e.:

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + o(x^2) \quad \text{as } x \rightarrow 0$$

We therefore need to prove that:

$$f(x) - \left(f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 \right) = o(x^2) \quad \text{as } x \rightarrow 0$$

i.e. (by definition of little- o) that:

$$\lim_{x \rightarrow 0} \frac{f(x) - \left(f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 \right)}{x^2} = 0$$

This limit gives an indeterminate form $[0/0]$; we now apply L'Hôpital's rule and obtain:

$$\lim_{x \rightarrow 0} \frac{f(x) - \left(f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 \right)}{x^2} = \lim_{x \rightarrow 0} \frac{f'(x) - (f'(0) + f''(0)x)}{2x}$$

which is still in the indeterminate form $[0/0]$. We apply L'Hôpital's rule again and obtain:

$$\lim_{x \rightarrow 0} \frac{f'(x) - (f'(0) + f''(0)x)}{2x} = \lim_{x \rightarrow 0} \frac{f''(x) - f''(0)}{2} = 0$$

the limit equals 0 if we add the hypothesis that $f''(x)$ is continuous at $x = 0$.

Let us see how to proceed also without requiring the continuity of f'' at 0, and prove directly that:

$$\frac{1}{2} \lim_{x \rightarrow 0} \frac{f'(x) - (f'(0) + f''(0)x)}{x} = 0$$

By hypothesis, f is twice differentiable at 0, hence f' is differentiable; if we apply the linear approximation (equation (8.3)) to the function $f'(x)$ we obtain:

$$f'(x) = f'(0) + f''(0)x + o(x) \quad \text{as } x \rightarrow 0$$

hence it follows that (definition of little- o):

$$\frac{f'(x) - (f'(0) + f''(0)x)}{x} \rightarrow 0 \quad \text{as } x \rightarrow 0$$

□

Proof. The general case (any n) can be proved by induction on n . We have already proved that the thesis holds for $n = 2$.

Suppose now that we know that for any function differentiable $n - 1$ times at $x = 0$ the Maclaurin formula holds, and let us prove that

$$f(x) = T_{n,f}(x) + o(x^n) \quad \text{as } x \rightarrow 0 \quad \text{i.e.} \quad \lim_{x \rightarrow 0} \frac{f(x) - T_{n,f}(x)}{x^n} = 0 \quad (8.4)$$

Applying L'Hôpital's rule, we have:

$$\lim_{x \rightarrow 0} \frac{f'(x) - T'_{n,f}(x)}{n x^{n-1}} \quad (8.5)$$

We now observe that the derivative of the polynomial $T_{n,f}(x)$ of f is nothing but the polynomial $T_{n-1,f'}(x)$ of the function f' :

$$T'_{n,f}(x) = T_{n-1,f'}(x)$$

as can be verified directly from the definition of $T_{n,f}$. On the other hand, by the inductive hypothesis applied to the function f' , we know that

$$f'(x) = T_{n-1,f'}(x) + o(x^{n-1}) \quad \text{as } x \rightarrow 0 \quad \text{i.e.} \quad \lim_{x \rightarrow 0} \frac{f'(x) - T_{n-1,f'}(x)}{x^{n-1}} = 0$$

hence the limit (8.5) is zero and, by L'Hôpital's rule, the limit (8.4) is also zero, as we wanted to prove. □

Example 8.8: computing Maclaurin polynomials

To better understand the equality:

$$T'_{n,f}(x) = T_{n-1,f'}(x)$$

consider a function f ; its Maclaurin polynomial, for example of degree 3, is:

$$T_{3,f}(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3$$

and the derivative of the polynomial $T_{3,f}(x)$ is:

$$T'_{3,f}(x) = f'(0) + f''(0)x + \frac{1}{2}f'''(0)x^2$$

Let us now consider its derivative function $f'(x)$ and compute its Maclaurin polynomial of degree 2:

$$T_{2,f'}(x) = f'(0) + f''(0)x + \frac{1}{2}f'''(0)x^2$$

that is, we obtain the same polynomial; the reasoning easily extends to degree n .

- The derivative $T'_{n,f}(x)$ of the Maclaurin polynomial $T_{n,f}(x)$ of f is the Maclaurin polynomial $T_{n-1,f'}(x)$ of degree $n-1$ of the derivative function f' , that is:

$$T'_{n,f}(x) = \sum_{k=1}^n k \frac{f^{(k)}(0)}{k!} x^{k-1} = \sum_{k=1}^n \frac{f^{(k)}(0)}{(k-1)!} x^{k-1} = \sum_{k=0}^{n-1} \frac{f^{(k+1)}(0)}{(k)!} x^k = T_{n-1,f'}(x)$$

Example 8.9: Maclaurin formula or expansion of the sine

Consider

$$f(x) = \sin x$$

$f(0) = 0$ and moreover we have:

$$\begin{aligned} f^{(1)}(x) &= \cos x, & f^{(2)}(x) &= -\sin x, & f^{(3)}(x) &= -\cos x, \\ f^{(4)}(x) &= \sin x = f(x), & f^{(5)}(x) &= \cos x = f^{(1)}(x) \dots \end{aligned}$$

It follows that the derivatives of order $2k$ (even) vanish at $x = 0$, while those of order $2k+1$ (odd) are alternately $+1$ and -1 at $x = 0$. The Maclaurin polynomials of the sine function therefore contain only terms of odd degree:

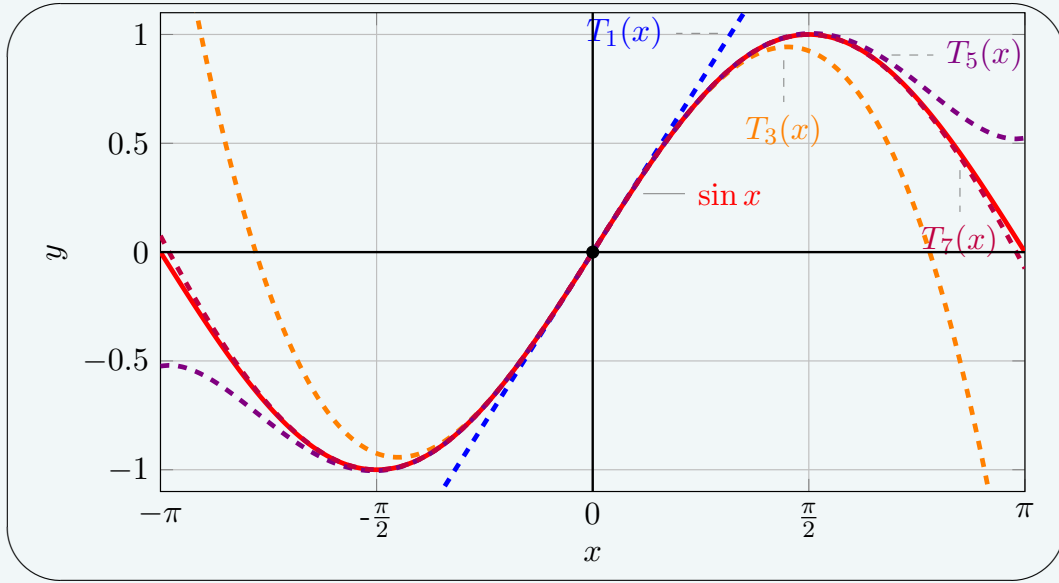
$$\sin x = T_1(x) + o(x) = x + o(x) \quad \text{as } x \rightarrow 0$$

$$\sin x = T_3(x) + o(x^3) = x - \frac{x^3}{6} + o(x^3) \quad \text{as } x \rightarrow 0$$

$$\sin x = T_5(x) + o(x^5) = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5) \quad \text{as } x \rightarrow 0$$

$$\sin x = T_7(x) + o(x^7) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + o(x^7) \quad \text{as } x \rightarrow 0$$

Graphically, the Maclaurin polynomials of the function $f(x) = \sin x$ are:



Example 8.10: Maclaurin formula or expansion of the cosine

Consider

$$f(x) = \cos x$$

$f(0) = 1$ and moreover we have:

$$\begin{aligned} f^{(1)}(x) &= -\sin x, & f^{(2)}(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\ f^{(4)}(x) &= \cos x = f(x), & f^{(5)}(x) &= -\sin x = f^{(1)}(x) \dots \end{aligned}$$

It follows that the derivatives of order $2k + 1$ (odd) vanish at $x = 0$, while those of order $2k$ (even) are alternately -1 and $+1$ at $x = 0$. The Maclaurin polynomials of the cosine function therefore contain only terms of even degree:

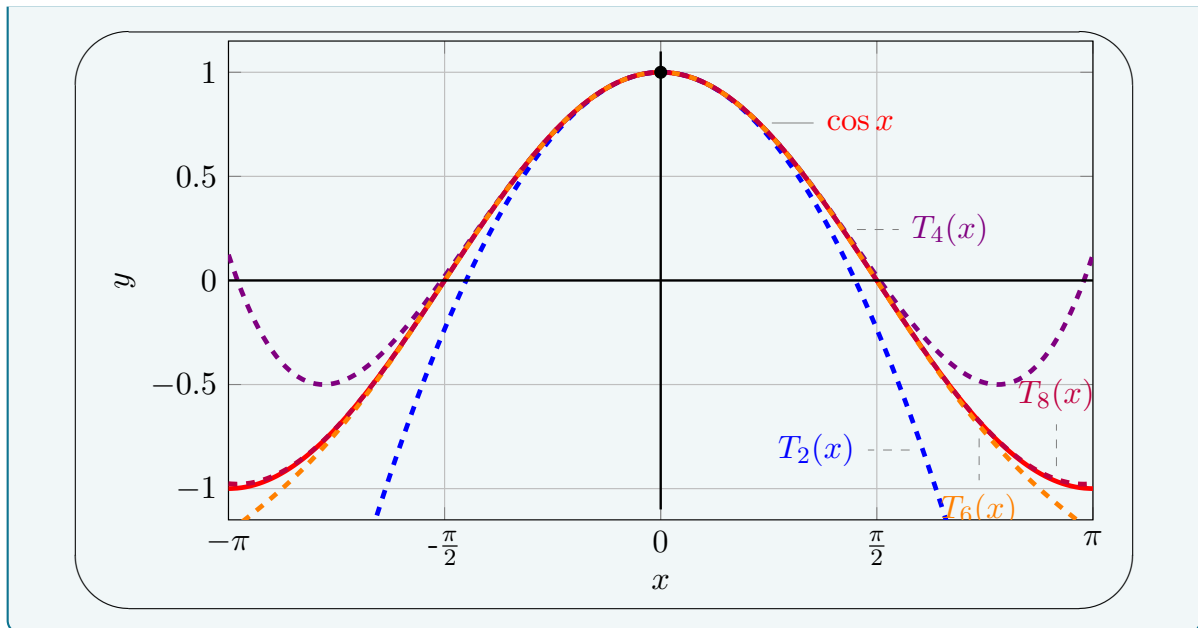
$$\cos x = T_2(x) + o(x^2) = 1 - \frac{1}{2} x^2 + o(x^2) \quad \text{as } x \rightarrow 0$$

$$\cos x = T_4(x) + o(x^4) = 1 - \frac{1}{2} x^2 + \frac{x^4}{24} + o(x^4) \quad \text{as } x \rightarrow 0$$

$$\cos x = T_6(x) + o(x^6) = 1 - \frac{1}{2} x^2 + \frac{x^4}{24} - \frac{x^6}{720} + o(x^6) \quad \text{as } x \rightarrow 0$$

$$\cos x = T_8(x) + o(x^8) = 1 - \frac{1}{2} x^2 + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} + o(x^8) \quad \text{as } x \rightarrow 0$$

Graphically, the Maclaurin polynomials of the function $f(x) = \cos x$ are:



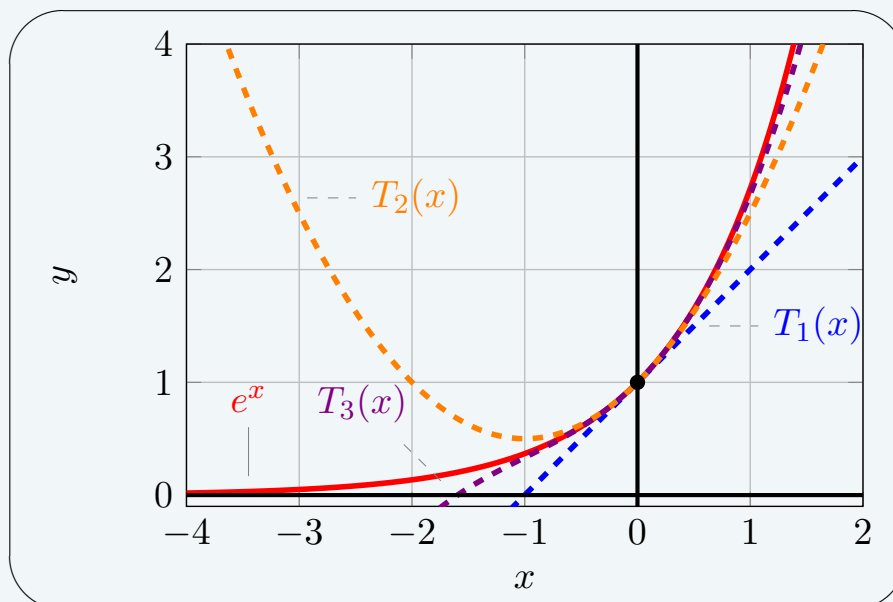
Example 8.11: Maclaurin formula or expansion

Consider:

$$f(x) = e^x \quad \text{we have} \quad f^{(n)}(x) = e^x \quad \text{and} \quad f^{(n)}(0) = 1, \quad \forall n \in \mathbb{N}, n \geq 1$$

Hence:

$$e^x = T_n(x) + o(x^n) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n) \quad \text{as } x \rightarrow 0$$



Example 8.12: Computing limits with the Maclaurin expansion

We want to compute:

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \left[\frac{0}{0} \right]$$

Expanding the function to first order, $\cos x = 1 + o(x)$, we have:

$$\lim_{x \rightarrow 0} \frac{1 - (1 + o(x))}{x^2} = \lim_{x \rightarrow 0} \frac{o(x)}{x^2}$$

The symbol $o(x)$ as $x \rightarrow 0$ denotes the set of functions that, divided by x , tend to 0 as $x \rightarrow 0$. Hence, for example, $5x^2 = o(x)$ but also $x^3 = o(x)$, and the value of the limit cannot yet be determined. Expanding the function to second order, $\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$, we have:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{1 - (1 - \frac{1}{2}x^2 + o(x^2))}{x^2} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2 + o(x^2)}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2 + o(x^2)}{x^2} = \lim_{x \rightarrow 0} \frac{1}{2} + \underbrace{\frac{o(x^2)}{x^2}}_{\rightarrow 0} = \frac{1}{2} \end{aligned}$$

The symbol $o(x^2)$ as $x \rightarrow 0$ denotes the set of functions that, divided by x^2 , tend to 0 as $x \rightarrow 0$; hence the second term tends to 0 as $x \rightarrow 0$. Equivalently, $\frac{o(x^2)}{x^2} = o(1)$, that is, a generic function that tends to 0 as $x \rightarrow 0$.

Example 8.13: Computing limits with the Maclaurin expansion

We want to compute:

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \left[\frac{0}{0} \right]$$

Expanding the functions to first order we have:

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \lim_{x \rightarrow 0} \frac{1 + x + o(x) - (1 - x + o(x)) - 2x}{x - (x + o(x))} = \lim_{x \rightarrow 0} \frac{o(x)}{o(x)}$$

The symbol $o(x)$ as $x \rightarrow 0$ denotes the set of functions that, divided by x , tend to 0 as $x \rightarrow 0$. Hence, for example, $5x^2 = o(x)$ but also $x^3 = o(x)$, and the value of the limit cannot yet be determined. Expanding the functions to second order we have:

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} = \lim_{x \rightarrow 0} \frac{1 + x + \frac{x^2}{2} + o(x^2) - (1 - x + \frac{x^2}{2} + o(x^2)) - 2x}{x - (x + o(x^2))} = \lim_{x \rightarrow 0} \frac{o(x^2)}{o(x^2)}$$

The limit still cannot be solved; expanding the functions to third order we have:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} &= \lim_{x \rightarrow 0} \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) - (1 - x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3)) - 2x}{x - (x - \frac{x^3}{6} + o(x^3))} \\ &= \lim_{x \rightarrow 0} \frac{\frac{x^3}{3} + o(x^3)}{\frac{x^3}{6} + o(x^3)} = \lim_{x \rightarrow 0} \frac{x^3 \left(\frac{1}{3} + \frac{o(x^3)}{x^3} \right)}{x^3 \left(\frac{1}{6} + \frac{o(x^3)}{x^3} \right)} = \lim_{x \rightarrow 0} \frac{\frac{1}{3} + o(1)}{\frac{1}{6} + o(1)} = 2 \end{aligned}$$

Example 8.14: Computing the Maclaurin polynomial

Compute the Maclaurin polynomial of degree 4 of the function

$$f(x) = \log(1 + \sin x)$$

The Maclaurin polynomials of degree 4 of the functions $\sin x$ and $\log(1 + x)$ are:

$$\sin x = x - \frac{x^3}{6}, \quad \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

Hence, substituting, the Maclaurin polynomial of the function $f(x)$ is:

$$T_4(x) = \left(x - \frac{x^3}{6}\right) - \frac{\left(x - \frac{x^3}{6}\right)^2}{2} + \frac{\left(x - \frac{x^3}{6}\right)^3}{3} - \frac{\left(x - \frac{x^3}{6}\right)^4}{4}$$

We have $\left(x - \frac{x^3}{6}\right)^2 = x^2 - \frac{2x^4}{6} + \frac{x^6}{36}$ and the last term has degree greater than 4. Since we are looking only for terms up to degree 4, we discard the last summand. Applying the same reasoning to the other numerators we have:

$$T_4(x) = x - \frac{x^3}{6} - \frac{x^2 - \frac{2x^4}{6}}{2} + \frac{x^3}{3} - \frac{x^4}{4} = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}$$

Maclaurin formulas or expansions of some elementary functions, with the Peano remainder, as $x \rightarrow 0$:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + o(x^n)$$

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n-1} \frac{x^n}{n} + o(x^n)$$

$$(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)}{2} x^2 + \dots + \frac{\alpha(\alpha - 1) \dots (\alpha - n + 1)}{n!} x^n + o(x^n), \quad \forall \alpha \in \mathbb{R}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n})$$

$$\operatorname{Sh} x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$$

$$\operatorname{Ch} x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + o(x^{2n})$$

The expansions clearly also hold if we replace x with any function $\varepsilon(x)$ that tends to 0, that is, an infinitesimal (it does not matter what x tends to in this case).

- All of this can be generalized to a point $x_0 \neq 0$. The proofs are analogous to the previous

ones and are left as an exercise.

Theorem 8.5: Taylor polynomial

Given a function $f : (a, b) \rightarrow \mathbb{R}$ differentiable $n - 1$ times in (a, b) and n times at $x_0 \in (a, b)$, there exists one and only one polynomial T_{n,f,x_0} of degree $\leq n$ with the property that:

$$T_{n,f,x_0}(x_0) = f(x_0), \quad T'_{n,f,x_0}(x_0) = f'(x_0), \quad \dots, \quad T^{(n)}_{n,f,x_0}(x_0) = f^{(n)}(x_0)$$

and this polynomial, called the **Taylor polynomial** of f of degree n , is:

$$T_{n,f,x_0}(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \quad (\text{setting } f^{(0)} = f)$$

The expanded form of the Taylor polynomial of degree n of a function f at x_0 is:

$$\begin{aligned} T_{n,f,x_0}(x) = & f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \dots \\ & + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n \end{aligned}$$

- When the function we refer to is clear, for convenience we omit the subscript f of the Taylor polynomial and simply write $T_{n,x_0}(x)$.

Theorem 8.6: Taylor formula of order n with Peano remainder

Let $f : (a, b) \rightarrow \mathbb{R}$ be differentiable $n - 1$ times in (a, b) and n times at $x_0 \in (a, b)$. Then

$$f(x) = T_{n,f,x_0}(x) + o((x - x_0)^n) \quad \text{as } x \rightarrow x_0$$

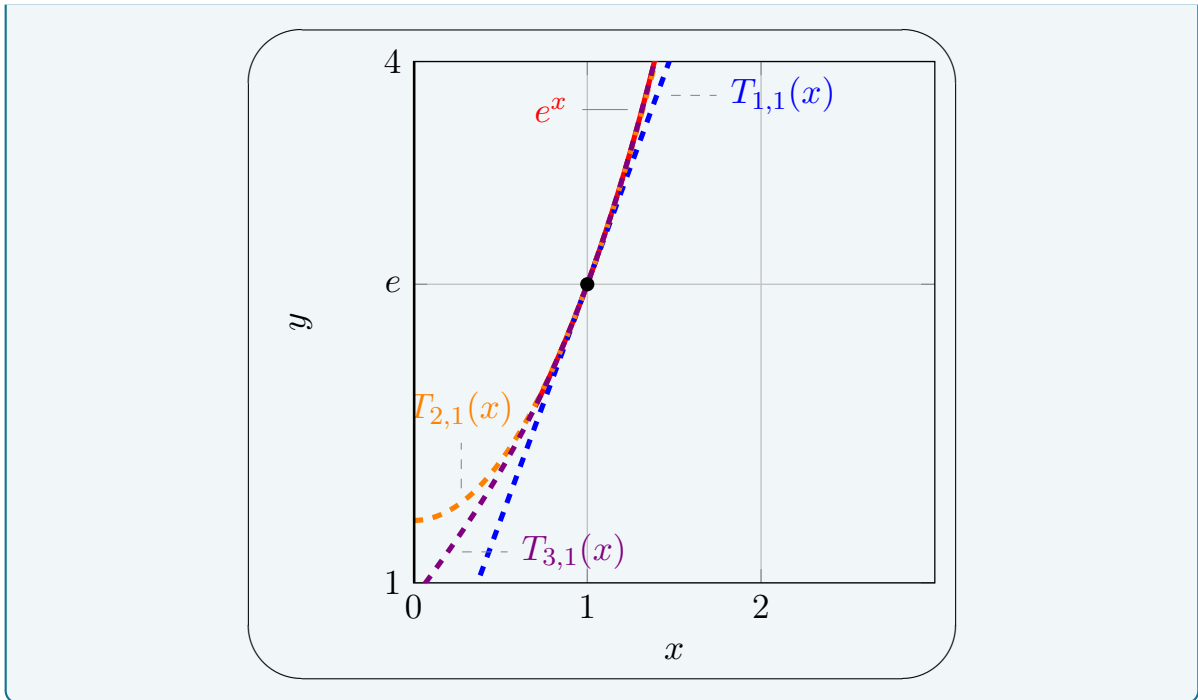
Example 8.15: Taylor formula or expansion

Consider $x_0 = 1$ and:

$$f(x) = e^x \quad \text{we have} \quad f^{(n)}(x) = e^x \quad \text{and} \quad f^{(n)}(1) = e, \quad \forall n \in \mathbb{N}, n \geq 1$$

Hence:

$$\begin{aligned} e^x &= T_{n,1}(x) + o((x - 1)^n) \\ &= e + e(x - 1) + e \frac{(x - 1)^2}{2} + e \frac{(x - 1)^3}{3!} \\ &\quad + \dots + e \frac{(x - 1)^n}{n!} + o((x - 1)^n) \quad \text{as } x \rightarrow 1 \end{aligned}$$



8.4 Taylor formula/expansion with Lagrange remainder

- In the Taylor-Maclaurin formula with Peano remainder, the information we have on the error made in approximating f with its Taylor-Maclaurin polynomial is of a **dynamic type**: as the increment $(x - x_0)$ tends to zero, we know that the remainder tends to zero faster than $(x - x_0)^n$.

For a fixed value of the increment $(x - x_0)$ the Taylor-Maclaurin formula with Peano remainder says nothing about the size of the error made.

- In various problems of approximate computation, instead, it is essential to estimate the error made in approximating a function with its Taylor polynomial when the increment $(x - x_0)$ has a fixed value, or a value that does not exceed a fixed threshold.
- This type of problem is addressed by the next result, which gives an alternative way of quantifying the approximation error made.

Theorem 8.7: Taylor formula of order n with Lagrange remainder

Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable $n + 1$ times in $[a, b]$ and let $x_0 \in [a, b]$. Then there exists a point c between x_0 and x such that:

$$f(x) = T_{n,f,x_0}(x) + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1} \quad (8.6)$$

- The formula has the structure: **function to be approximated equals approximating polynomial plus approximation error**. The error is the term $\frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$ and is called the Lagrange remainder.

- For $n = 0$, the theorem coincides with Lagrange's theorem: setting $x_0 = a$ and $b = x$ we have:

$$\exists c \in [a, b] : f(b) = f(a) + f'(c)(b - a) \quad \text{that is} \quad \frac{f(b) - f(a)}{b - a} = f'(c)$$

which is Lagrange's theorem. Warning: the interval (a, b) becomes $[a, b]$.

The point c depends on x_0 , x and n , and lies between x_0 and x . If we manage to prove that

$$|f^{(n+1)}(t)| \leq M, \quad \forall t \in [x_0, x] \quad (8.7)$$

then the Taylor formula with Lagrange remainder says that

$$|f(x) - T_{n,x_0}(x)| \leq \frac{M}{(n+1)!} |x - x_0|^{n+1} \quad (8.8)$$

Example 8.16: error estimates with the Lagrange remainder

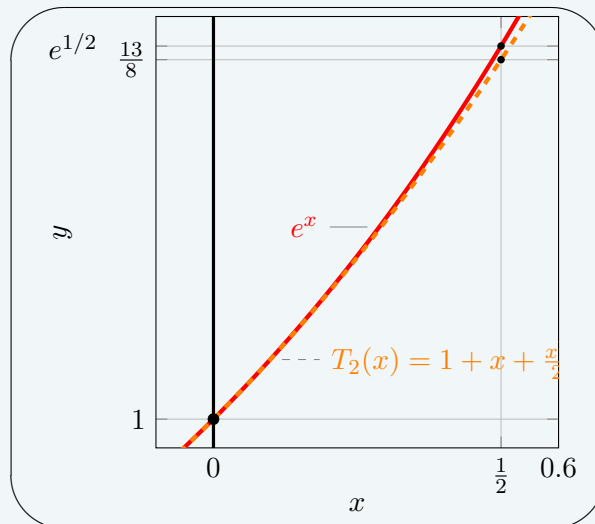
Consider $f(x) = e^x$; we know that the second- and third-order Maclaurin polynomials of e^x are:

$$T_2(x) = 1 + x + \frac{1}{2}x^2, \quad T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

Now, if we wanted to use the polynomial to compute an approximate value of $e^{1/2}$, i.e., if we planned to approximate

$$e^{1/2} \quad \text{with} \quad T_2\left(\frac{1}{2}\right) = \frac{13}{8} \quad \text{or with} \quad T_3\left(\frac{1}{2}\right) = \frac{79}{48}$$

how can we estimate a priori (i.e., without already knowing the true value of $e^{1/2}$) how large our error is?



We have $x_0 = 0$, $x = \frac{1}{2}$. In formula (8.6), expanding up to the third degree ($n = 3$), we

obtain:

$$e^{1/2} = \underbrace{T_3\left(\frac{1}{2}\right)}_{=\frac{79}{48}} + \frac{e^c}{4!} \left(\frac{1}{2}\right)^4 \quad \text{with } c \in \left[0, \frac{1}{2}\right]$$

The value c is unknown, hence the value e^c is also unknown; we therefore need to use (8.7) to bound e^c from above. We have:

$$f^{(4)}(t) = e^t \quad \text{consequently} \quad |e^t| \leq \underbrace{3^{1/2}}_{=M}, \quad \forall t \in \left[0, \frac{1}{2}\right], \quad \text{since } e < 3$$

Hence from (8.8) we obtain:

$$\left| e^{1/2} - T_3\left(\frac{1}{2}\right) \right| \leq \frac{3^{1/2}}{2^4 \cdot 4!}$$

This means that by approximating $e^{1/2}$ with the value $\frac{79}{48}$ we make an error not greater than $\sqrt{3}/384$, which is approximately 0.0045.

Proof. We prove the theorem in the case $n = 1$. Setting for convenience $x_0 = a$, $x = b$, the statement becomes: if $f : [a, b] \rightarrow \mathbb{R}$ is twice differentiable in $[a, b]$, then there exists a point $c \in [a, b]$ such that:

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(c)(b-a)^2$$

We set

$$f(b) - (f(a) + f'(a)(b-a)) = k(b-a)^2$$

and we try to determine the form of k .

Let us therefore consider the function:

$$g(x) = f(b) - f(x) - f'(x)(b-x) - k(b-x)^2$$

and let us apply to it the mean value (Lagrange's) theorem. Since $g(b) = g(a) = 0$ (the second equality follows from the definition of k), we find that there exists $c \in (a, b)$ such that

$$0 = \frac{g(b) - g(a)}{b-a} = g'(c)$$

But we have:

$$g'(x) = -f'(x) - f''(x)(b-x) + f'(x) + 2k(b-x) = (b-x) \cdot (2k - f''(x))$$

and hence $g'(c) = 0$ implies, since $c \neq b$:

$$k = \frac{1}{2} f''(c)$$

The procedure just seen can be extended to any n .

We now define:

$$g(x) = f(b) - \sum_{j=0}^n \frac{f^{(j)}(x) (b-x)^j}{j!} - k (b-x)^{n+1}$$

with k defined implicitly by the identity

$$f(b) - T_{n,a}(b) = k (b-a)^{n+1} \quad (8.9)$$

□

Proof. We now proceed as follows:

1. We verify that $g(a) = g(b) = 0$ since

$$g(b) = f(b) - f(b) = 0, \quad g(a) = f(b) - T_{n,a}(b) - k (b-a)^{n+1} = 0 \text{ by def. of } k$$

2. We apply Lagrange's theorem to g on $[a, b]$, and we show that the statement "there exists $c \in (a, b)$ such that $g'(c) = 0$ " is exactly the theorem.

We start by computing g' :

$$\begin{aligned} g'(x) &= - \sum_{j=0}^n \frac{f^{(j+1)}(x) (b-x)^j}{j!} + \sum_{j=1}^n \frac{f^{(j)}(x) (b-x)^{j-1}}{(j-1)!} + k (n+1) (b-x)^n \\ &= - \sum_{j=0}^n \frac{f^{(j+1)}(x) (b-x)^j}{j!} + \sum_{j=0}^{n-1} \frac{f^{(j+1)}(x) (b-x)^j}{j!} + k (n+1) (b-x)^n \\ &= - \frac{f^{(n+1)}(x) (b-x)^n}{n!} + k (n+1) (b-x)^n \end{aligned}$$

Then $g'(c) = 0$ means:

$$- \frac{f^{(n+1)}(c) (b-c)^n}{n!} + k (n+1) (b-c)^n = 0$$

that is

$$k = \frac{f^{(n+1)}(c)}{(n+1)!}$$

which, substituted into (8.9), gives the thesis.

□

8.4.1 Relations with convexity

- Consider formula (8.6) with $n = 1$:

$$f(x) = f(x_0) + f'(x_0) (x - x_0) + \frac{1}{2} f''(c) (x - x_0)^2$$

and suppose that at every point of (a, b) we have $f''(x) \geq 0$, that is, f is convex on (a, b) . Then we have

$$\frac{1}{2} f''(c) (x - x_0)^2 \geq 0$$

and therefore we can write, for every pair of points x_0 and $x \in (a, b)$,

$$f(x) \geq f(x_0) + f'(x_0) (x - x_0) \quad (8.10)$$

Geometrically, this means that the graph of $f(x)$ lies on the whole of (a, b) above the graph of its tangent line at x_0 (and this holds for every choice of the point $x_0 \in (a, b)$).

We have therefore proved that if a (twice differentiable) function is convex, it is also “convex with respect to tangents”

- If instead f is concave, i.e., $f''(x) \leq 0$ on the whole of (a, b) , we have

$$f(x) \leq f(x_0) + f'(x_0) (x - x_0) \quad (8.11)$$

and the graph of $f(x)$ lies on the whole of (a, b) below the graph of its tangent line at x_0 .

We have therefore proved that if a (twice differentiable) function is concave, it is also “concave with respect to tangents”

Curve sketching

9.1 Graphs of real functions of a real variable

- The infinitesimal and differential calculus developed so far allows us to tackle completely the problem of drawing the graph of a function f , that is, to carry out **curve sketching** (function analysis).

The steps of curve sketching:

1. Determine the (maximal) **domain** of f .
2. Compute the **limits at the boundary** of the domain. Determine any **horizontal/vertical asymptotes** and **points of discontinuity**. Study the **sign of the function** and the **points where it vanishes** (possible only for relatively simple functions).
3. If the function tends to ∞ as $x \rightarrow \infty$, determine any **oblique asymptotes**.
4. Compute the **derivative function** f' at the points where it exists. Study the points where f is continuous but not differentiable and determine their nature (**corner points, inflection points with vertical tangent** or **cusps**). At corner points or at the endpoints of the domain it is useful to compute the **right or left derivatives**, which determine the **slope of the graph at those points**.
5. Study the sign of f' (on its domain) to obtain information on the **monotonicity** of f and on its **relative maximum and minimum points**. Then determine the **absolute maximum and minimum points**.

Additional information:

1. It can be useful to determine an **asymptotic estimate at infinity** that tells us whether the function tends to ∞ in a *superlinear* or *sublinear* way or in a *linear* way (in which case it may have an oblique asymptote). The asymptotic estimate at infinity normally also gives information on the **concavity** of f at infinity.
2. Compute the **second derivative function** f'' and study its sign, to deduce information on the **concavity** and the **inflection points** of f .

3. Determine any symmetry of f (**even or odd functions**) and restrict the study to $x \geq 0$.
4. Determine any periodicity of f (**periodic functions**) and restrict the study to one period.

9.2 Examples

Example 9.1: Function analysis and graph

Let us study the function and draw its graph:

$$f(x) = e^{-|x|} \sqrt{x^2 - 5x + 6}$$

We expect a corner point at $x = 0$ due to the presence of $|x|$, and points with vertical tangent where the radicand vanishes.

Step 1. The (maximal) domain of f is:

$$x^2 - 5x + 6 \geq 0, \quad (x-3)(x-2) \geq 0 \quad \text{that is} \quad x \in (-\infty, 2] \cup [3, +\infty)$$

Step 2. The limits at the boundary are:

$$\lim_{x \rightarrow +\infty} e^{-|x|} \sqrt{x^2 - 5x + 6} = \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2 - 5x + 6}}{e^x} = 0^+$$

$$\lim_{x \rightarrow -\infty} e^{-|x|} \sqrt{x^2 - 5x + 6} = \lim_{x \rightarrow -\infty} \sqrt{e^{2x}(x^2 - 5x + 6)} = \lim_{y \rightarrow +\infty} \sqrt{\frac{y^2 + 5y + 6}{e^{2y}}} = 0^+$$

$$\lim_{x \rightarrow 3^+} e^{-|x|} \sqrt{x^2 - 5x + 6} = \lim_{x \rightarrow 3^+} e^{-x} \sqrt{(x-3)(x-2)} = 0^+$$

$$\lim_{x \rightarrow 2^-} e^{-|x|} \sqrt{x^2 - 5x + 6} = \lim_{x \rightarrow 2^-} e^{-x} \sqrt{(x-3)(x-2)} = 0^+$$

Hence the line $y = 0$ is a horizontal asymptote as $x \rightarrow \pm\infty$ and there are no vertical asymptotes. In the domain there are no points of discontinuity. We have $f(x) \geq 0$ for every x in the domain and $f(x) = 0$ for $x = 2$ and $x = 3$.

Step 3. The function has no oblique asymptotes.

Step 4. We compute the derivative function for $x \neq 0$:

$$\begin{aligned} f'(x) &= e^{-|x|} \left(-\operatorname{sgn}(x) \sqrt{x^2 - 5x + 6} + \frac{2x - 5}{2\sqrt{x^2 - 5x + 6}} \right) \\ &= \frac{e^{-|x|}}{2\sqrt{(x-3)(x-2)}} \left(-2\operatorname{sgn}(x)(x^2 - 5x + 6) + 2x - 5 \right) \end{aligned}$$

Or, equivalently:

$$f'(x) = \begin{cases} e^{-x} \cdot \frac{-2x^2 + 12x - 17}{2\sqrt{(x-3)(x-2)}} & \text{if } x > 0 \\ e^x \cdot \frac{2x^2 - 8x + 7}{2\sqrt{(x-3)(x-2)}} & \text{if } x < 0 \end{cases}$$

Example 9.2: Function analysis and graph

We compute the right and left derivatives at $x = 0$:

$$f'_+(0) = \lim_{x \rightarrow 0^+} \frac{e^{-x} (-2x^2 + 12x - 17)}{2\sqrt{(x-3)(x-2)}} = -\frac{17}{2\sqrt{6}}$$

$$f'_-(0) = \lim_{x \rightarrow 0^-} \frac{e^x (2x^2 - 8x + 7)}{2\sqrt{(x-3)(x-2)}} = \frac{7}{2\sqrt{6}}$$

hence $x = 0$ is a corner point, that is, $f'(0)$ does not exist. We compute the right-hand limit at $x = 3$ and the left-hand limit at $x = 2$ of the derivative function:

$$f'_+(3) = \lim_{x \rightarrow 3^+} \frac{e^{-x} \overbrace{(-2x^2 + 12x - 17)}^{\rightarrow 1}}{2\sqrt{(x-3)(x-2)}} = +\infty$$

$$f'_-(2) = \lim_{x \rightarrow 2^-} \frac{e^{-x} \overbrace{(-2x^2 + 12x - 17)}^{\rightarrow -1}}{2\sqrt{(x-3)(x-2)}} = -\infty$$

Hence the function has points with vertical tangent at $x = 2$ and $x = 3$.

Step 5. We study the sign of the derivative function:

- For $x > 0$ we have:

$$f'(x) \geq 0 \quad \text{if} \quad 2x^2 - 12x + 17 \leq 0, \quad \text{that is} \quad x \in \left[\underbrace{\frac{6 - \sqrt{2}}{2}}_{\approx 2.3}, \underbrace{\frac{6 + \sqrt{2}}{2}}_{\approx 3.7} \right]$$

hence:

$$f \text{ is decreasing for } x \in [0, 2] \cup \left[\frac{6 + \sqrt{2}}{2}, +\infty \right],$$

$$f \text{ is increasing for } x \in \left[3, \frac{6 + \sqrt{2}}{2} \right]$$

We have $f'\left(\frac{6+\sqrt{2}}{2}\right) = 0$, hence $x = \frac{6+\sqrt{2}}{2}$ is a relative maximum point. The relative maximum is $f\left(\frac{6+\sqrt{2}}{2}\right) \approx 0.027$

- For $x < 0$ we have:

$$f'(x) \geq 0 \quad \text{if} \quad 2x^2 - 8x + 7 \geq 0, \quad \text{that is, if}$$

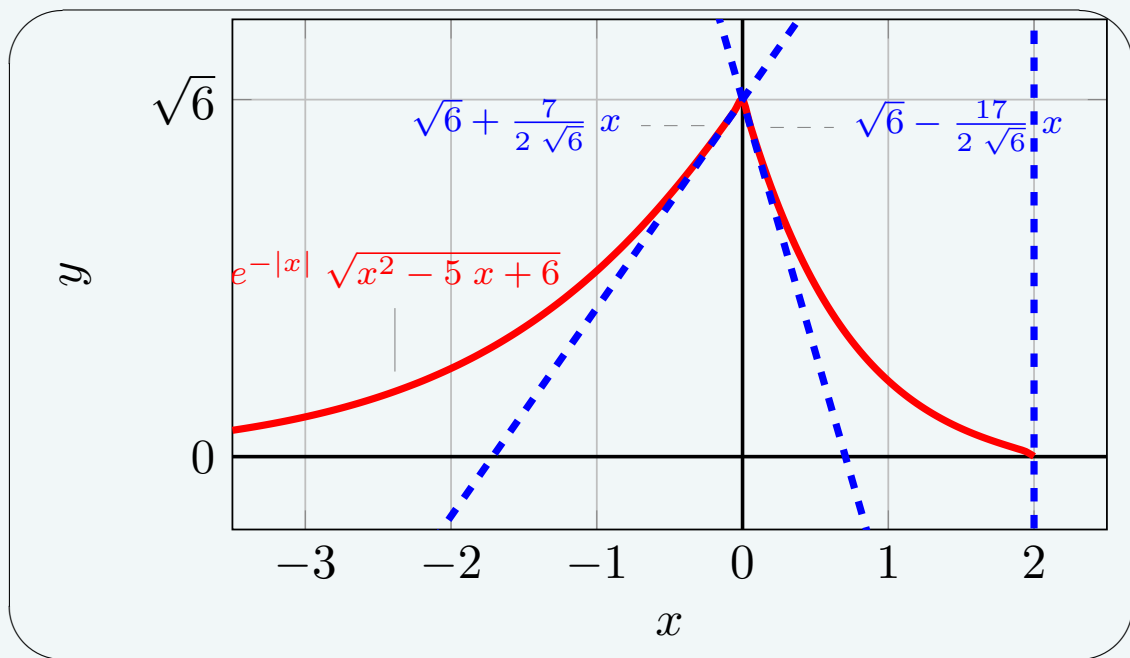
$$x \in \left[-\infty, \frac{4 - \sqrt{2}}{2}\right] \cup \left[\frac{4 + \sqrt{2}}{2}, +\infty\right]$$

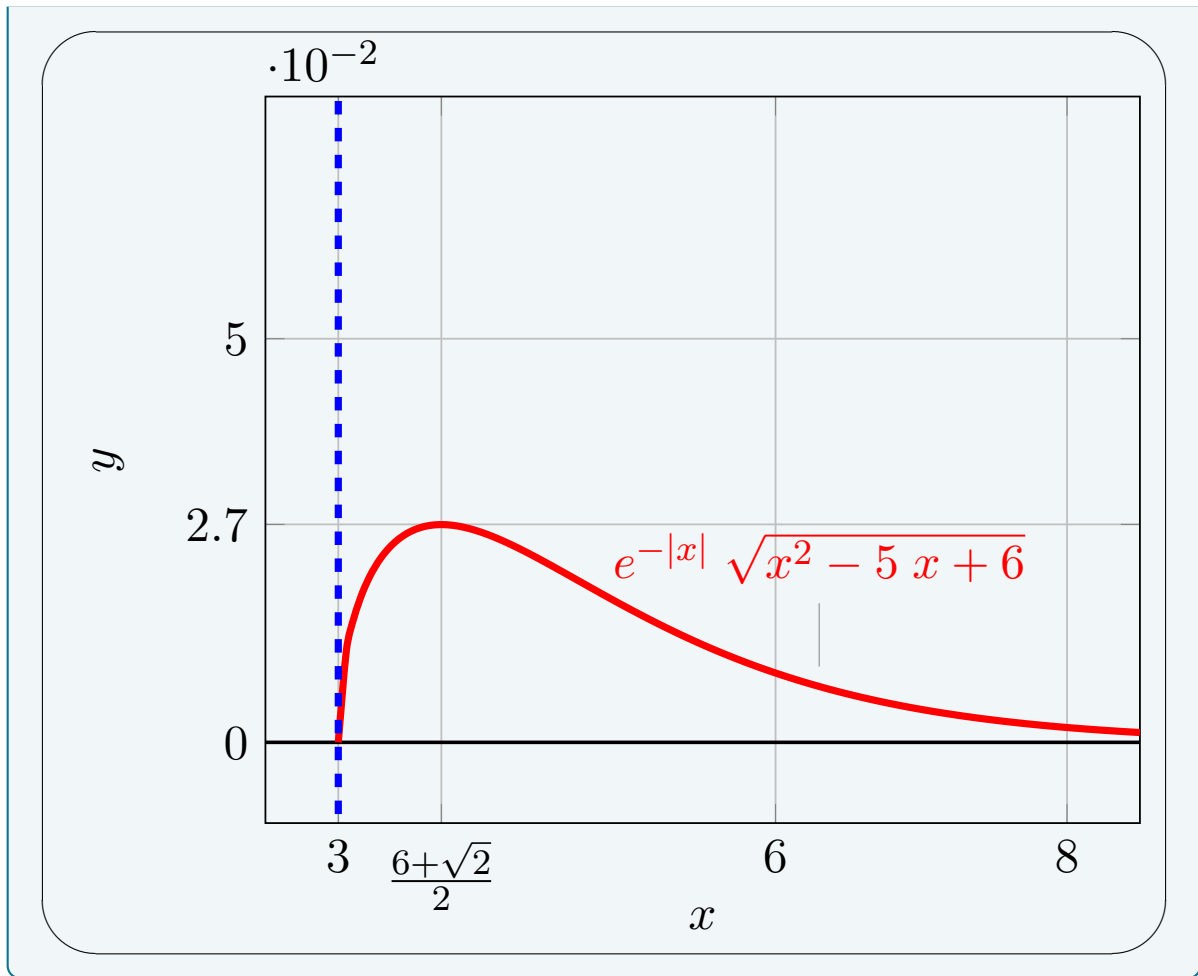
hence $f(x)$ is increasing for $x < 0$.

We can deduce that $x = 0$ (where f is not differentiable) is an absolute maximum point; the absolute maximum is $\sqrt{6} \approx 2.44$.

There are also two inflection points, one in $(0, 2)$ and one in $(3, +\infty)$, which can be obtained by studying the sign of the second derivative.

Example 9.3: Function analysis and graph





Example 9.4: Function analysis and graph

Let us study the function and draw its graph:

$$f(x) = x \cdot \exp\left(\frac{x+2}{x-1}\right)$$

Step 1. The (maximal) domain of f is:

$$x \in \mathbb{R} \setminus \{1\}$$

Step 2. The limits at the boundary are:

$$\lim_{x \rightarrow +\infty} x \cdot \exp\left(\frac{x+2}{x-1}\right) = +\infty$$

$$\lim_{x \rightarrow -\infty} x \cdot \exp\left(\frac{x+2}{x-1}\right) = -\infty$$

$$\frac{x+2}{x-1} \rightarrow +\infty \text{ as } x \rightarrow 1^+, \quad \text{hence } \lim_{x \rightarrow 1^+} x \exp\left(\frac{x+2}{x-1}\right) = +\infty$$

$$\frac{x+2}{x-1} \rightarrow -\infty \text{ as } x \rightarrow 1^-, \quad \text{hence } \lim_{x \rightarrow 1^-} x \exp\left(\frac{x+2}{x-1}\right) = 0$$

Hence there are no horizontal asymptotes and $x = 1$ is a point of discontinuity. Moreover, $x = 1$ is a vertical asymptote as $x \rightarrow 1^+$. We have $f(x) \geq 0$ for $x \geq 0$ and $f(x) = 0$ for $x = 0$.

Step 3. We compute an asymptotic estimate; we have:

$$\frac{x+2}{x-1} \rightarrow 1 \quad \text{as } x \rightarrow +\infty, \quad \frac{x+2}{x-1} \rightarrow 1 \quad \text{as } x \rightarrow -\infty$$

hence

$$x \cdot \exp\left(\frac{x+2}{x-1}\right) \sim x \cdot e \quad \text{as } x \rightarrow \pm\infty \quad \text{therefore } f \text{ has linear growth}$$

We therefore check for the presence of oblique asymptotes. We try to compute the following limit:

$$\lim_{x \rightarrow \pm\infty} \left(x \cdot \exp\left(\frac{x+2}{x-1}\right) - x e \right) = \lim_{x \rightarrow \pm\infty} x \cdot e \cdot \left(\exp\left(\frac{x+2}{x-1} - 1\right) - 1 \right)$$

We have

$$\frac{x+2}{x-1} - 1 = \frac{3}{x-1} \rightarrow 0 \quad \text{as } x \rightarrow \pm\infty$$

therefore

$$\begin{aligned} \exp\left(\frac{x+2}{x-1} - 1\right) - 1 &\sim \frac{3}{x-1} \quad \text{as } x \rightarrow \pm\infty; \\ x \cdot e \cdot \left(\exp\left(\frac{x+2}{x-1} - 1\right) - 1 \right) &\sim \frac{3 \cdot x \cdot e}{x-1} \quad \text{as } x \rightarrow \pm\infty \end{aligned}$$

consequently:

$$\lim_{x \rightarrow \pm\infty} \left(x \cdot \exp\left(\frac{x+2}{x-1}\right) - x e \right) = \lim_{x \rightarrow \pm\infty} \frac{3 \cdot x \cdot e}{x-1} = 3 \cdot e$$

Hence the function has the oblique asymptote $y = e x + 3 e$ as $x \rightarrow \pm\infty$.

Example 9.5: Function analysis and graph

Step 4. We compute the derivative function for $x \neq 1$:

$$\begin{aligned} f'(x) &= \exp\left(\frac{x+2}{x-1}\right) \cdot \left(1 + x \cdot \frac{(x-1) - (x+2)}{(x-1)^2} \right) = \exp\left(\frac{x+2}{x-1}\right) \cdot \frac{(x-1)^2 - 3x}{(x-1)^2} \\ &= \exp\left(\frac{x+2}{x-1}\right) \cdot \frac{x^2 - 5x + 1}{(x-1)^2} \end{aligned}$$

For every $x \neq 1$, f' is defined. We compute the left-hand limit at $x = 1$:

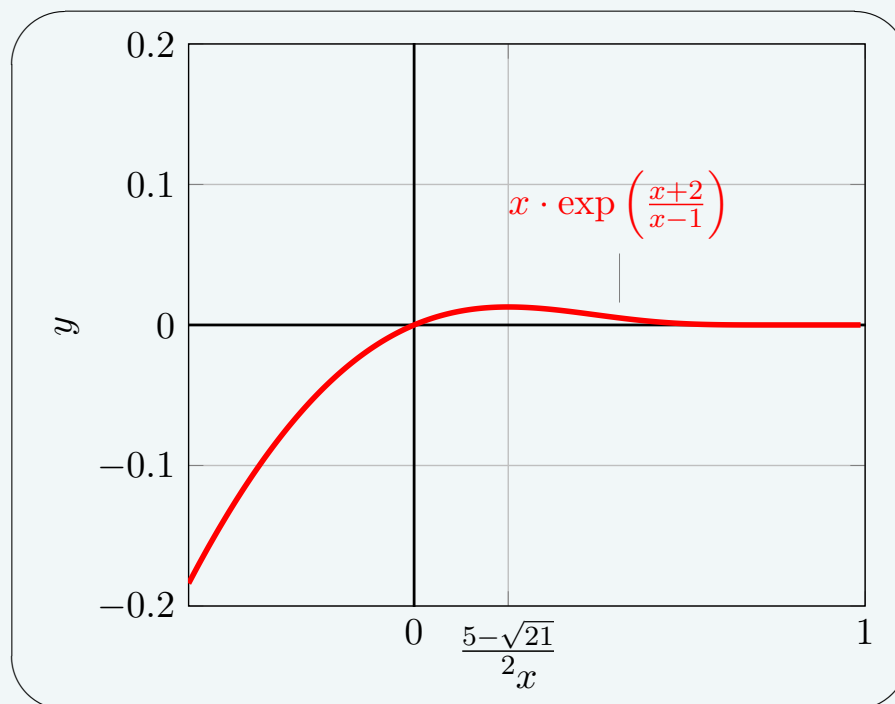
$$f'_-(1) = \lim_{x \rightarrow 1^-} \exp\left(\frac{x+2}{x-1}\right) \cdot \frac{x^2 - 5x + 1}{(x-1)^2} = 0$$

the exponential goes to zero faster than $(x - 1)^2$; hence the graph reaches $x = 1$ with a horizontal tangent, from the left.

Step 5. We study the sign of the derivative function:

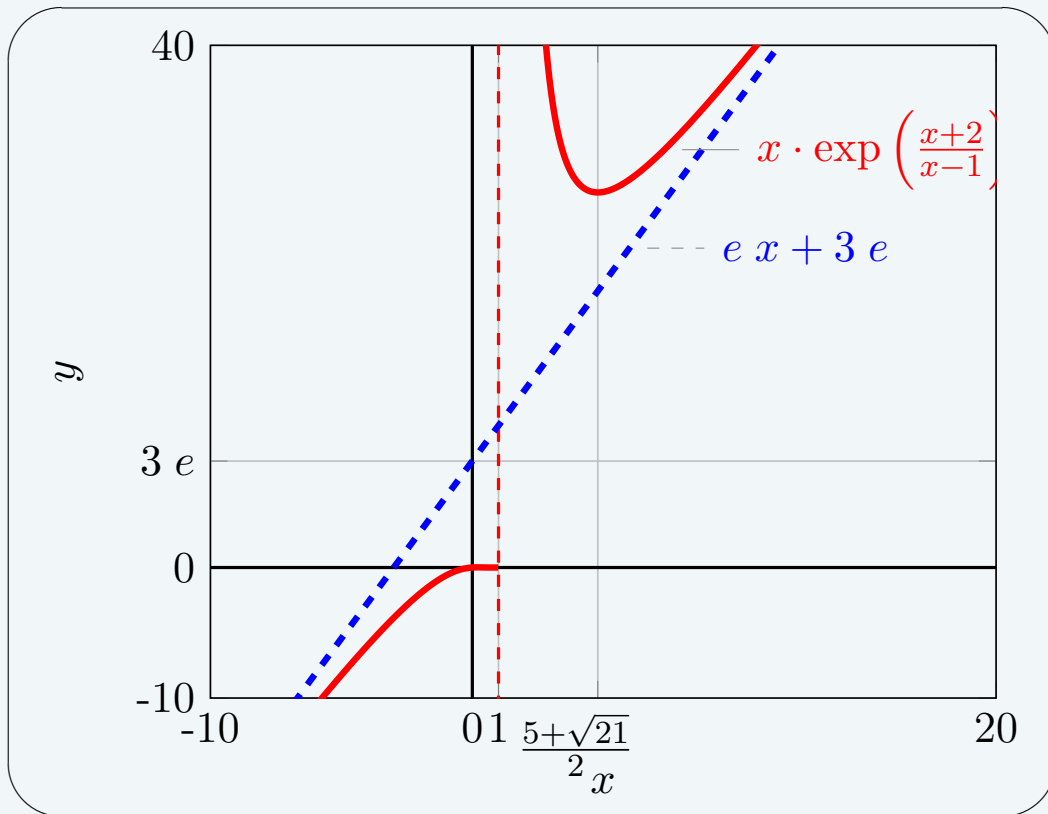
$$f'(x) \geq 0 \quad \text{for} \quad x^2 - 5x + 1 \geq 0 \quad \text{that is} \quad x \in \left[-\infty, \frac{5 - \sqrt{21}}{2}\right] \cup \left[\frac{5 + \sqrt{21}}{2}, +\infty\right]$$

hence $x = \frac{5 + \sqrt{21}}{2}$ is a relative minimum point and $x = \frac{5 - \sqrt{21}}{2}$ is a relative maximum point.



There is an inflection point in $\left(\frac{5 - \sqrt{21}}{2}, 1\right)$, which can be obtained by studying the sign of the second derivative.

Example 9.6: Function analysis and graph



We have $f'_-(1) = 0$ since:

$$f'_-(1) = \lim_{x \rightarrow 1^-} \exp\left(\frac{x+2}{x-1}\right) \cdot \frac{x^2 - 5x + 1}{(x-1)^2} = \lim_{x \rightarrow 1^-} \frac{\exp\left(\frac{x+2}{x-1}\right)}{(x-1)^2} \cdot \underbrace{(x^2 - 5x + 1)}_{\rightarrow -3 \text{ as } x \rightarrow 1^-}$$

with the change of variable $y = \frac{x+2}{x-1}$: if $x \rightarrow 1^-$ then $y \rightarrow -\infty$; moreover:

$$y = \frac{x+2}{x-1}, \quad y = \frac{x-1+3}{x-1}, \quad y = 1 + \frac{3}{x-1} \quad \text{and} \quad x = 1 + \frac{3}{y-1}$$

hence we have:

$$\lim_{x \rightarrow 1^-} \frac{\exp\left(\frac{x+2}{x-1}\right)}{(x-1)^2} = \lim_{y \rightarrow -\infty} \frac{e^y}{\left(\frac{3}{y-1}\right)^2} = \frac{1}{9} \lim_{y \rightarrow -\infty} e^y \cdot \underbrace{(y-1)^2}_{\sim y^2 \text{ as } y \rightarrow -\infty}$$

with a second change of variable $z = -y$: if $y \rightarrow -\infty$ then $z \rightarrow +\infty$ and we have

$$\frac{1}{9} \lim_{y \rightarrow -\infty} e^y \cdot y^2 = \frac{1}{9} \lim_{z \rightarrow +\infty} e^{-z} \cdot (-z)^2 = \frac{1}{9} \lim_{z \rightarrow +\infty} \frac{z^2}{e^z} = 0$$

by the theorem on the hierarchy of infinities.

Newton's method

10.1 Newton's method

- Suppose we want to solve the equation

$$f(x) = 0$$

which is equivalent to looking for the intersections of the graph of f with the x -axis, or the zeros of the function. Suppose moreover: i) that the solution $x = c$ exists, is unique and lies inside the interval $[a, b]$; ii) that the function is differentiable in $[a, b]$; iii) that $f'(x) \leq 0$ (decreasing function) and $f''(x) \geq 0$ (convex function) for every $x \in [a, b]$.

- Let us then start from $x_0 = a$ and linearize the equation $f(x) = 0$ by replacing f with the tangent line to its graph at the point $(x_0, f(x_0))$. This line has equation:

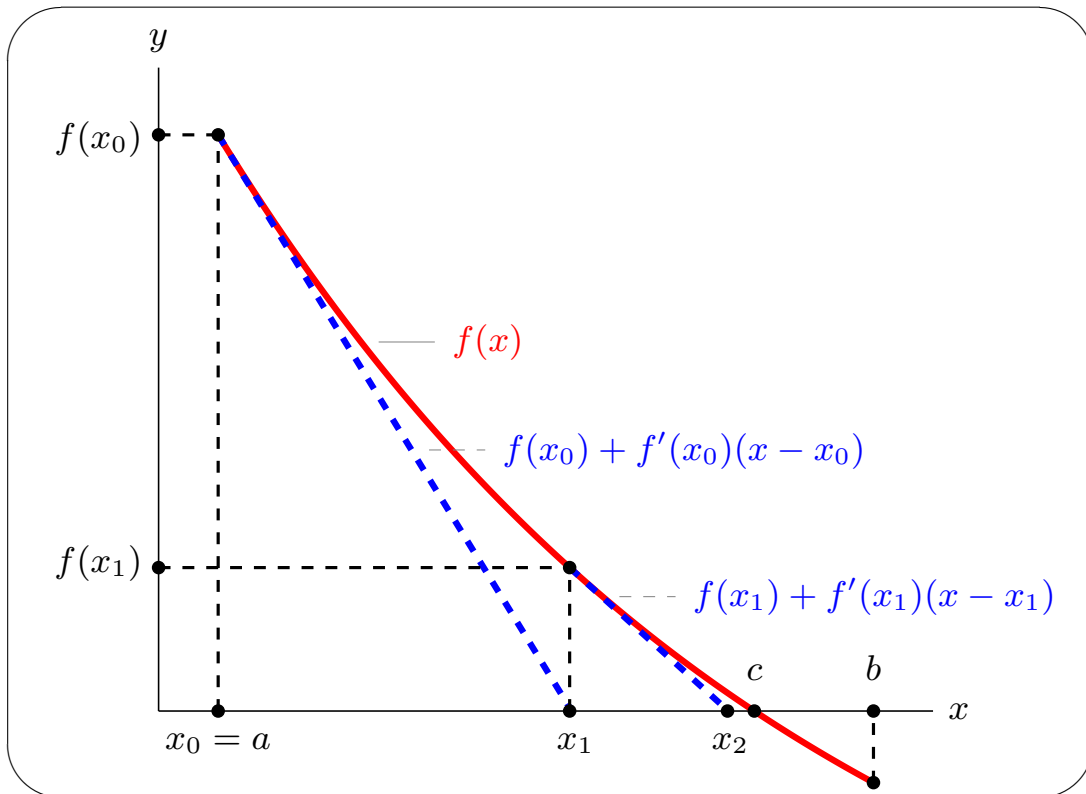
$$y = f(x_0) + f'(x_0)(x - x_0)$$

Instead of solving $f(x) = 0$, we solve:

$$f(x_0) + f'(x_0)(x - x_0) = 0$$

calling the solution x_1 we obtain:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad \text{the first approximation of } c$$



- Let us now start from x_1 and linearize as before with the tangent line at the new point $(x_1, f(x_1))$. This line has equation:

$$y = f(x_1) + f'(x_1)(x - x_1)$$

As before, we solve:

$$f(x_1) + f'(x_1)(x - x_1) = 0$$

obtaining

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \quad \text{the second approximation of } c$$

Proceeding with x_2 as with x_1 and continuing, we obtain the following recursively defined sequence:

$$x_0 = a, \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{with } n \in \mathbb{N}$$

In this way, the term x_n can be constructed starting from x_0 with n iterations of the same algorithm. The method is therefore very well suited to automatic computation.

- The main idea of this method, called **Newton's method**, is therefore to construct a sequence that, **under certain hypotheses**, converges to c .

Theorem 10.1: Newton's method

Let $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable in $[a, b]$. If the following three hypotheses hold:

1. $f(a) \cdot f(b) < 0$
2. $f'(x)$ and $f''(x)$ have constant sign in $[a, b]$
3. $f(a) \cdot f''(a) > 0$

then there exists one and only one point $c \in (a, b)$ such that $f(c) = 0$, and the sequence

$$x_0 = a, \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

converges to c from below. If, instead of hypothesis 3, the following hypothesis holds

- 3'. $f(b) \cdot f''(b) > 0$

then the sequence

$$x_0 = b, \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

converges to c from above.

- Note that, under hypotheses 1. and 2., either 3. or 3'. is certainly satisfied. The function f'' has the same sign at a and b by 2., and the function f has opposite signs at a and b by 1. Therefore either 3. or 3'. holds.
- This means that the method is applicable whenever hypotheses 1. and 2. are satisfied: we only need to choose appropriately whether to set $x_0 = a$ or $x_0 = b$.
- Condition 2, apparently restrictive, will in general be satisfied provided we choose an interval $[a, b]$ small enough.

Proof. Since f is continuous, being differentiable in $[a, b]$, and 1 holds, by the intermediate zero theorem there exists at least one $c \in (a, b)$ such that $f(c) = 0$.

Moreover, since $f'(x)$ has constant sign in (a, b) , f is strictly monotone, hence this point c is unique (a strictly monotone function cannot vanish at two distinct points).

This proves the existence and uniqueness of the point c at which f vanishes. We will now prove that the sequence is monotone. From this it will follow that the sequence is convergent, by the theorem on monotone sequences.

We start by showing that if the sequence x_n converges, then $x_n \rightarrow c$. If $x_n \rightarrow \ell \in [a, b]$, passing to the limit in the equality

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{we get} \quad \ell = \ell - \frac{f(\ell)}{f'(\ell)}$$

(we used the fact that f is continuous, and f' is also continuous, since f' is differentiable, because f'' exists by hypothesis). From the last equality it follows that $f(\ell) = 0$, so that

$\ell = c$ (the only point where the function vanishes).

Let us therefore prove that x_n is monotone, under hypotheses 1, 2, 3. Without loss of generality, we assume $f(a) < 0$ and, by 1, consequently $f(b) > 0$. Since, by 2, $f'(x)$ has constant sign in $[a, b]$, we must have $f'(x) > 0$ on all of $[a, b]$ (if the other inequality held, f would be decreasing, and we could not have $f(a) < 0 < f(b)$). Since $f(a) < 0$ and 3 holds, $f''(a) < 0$; hence, since by 2 $f''(x)$ has constant sign, $f''(x) < 0$ on all of $[a, b]$.

We are therefore proving the case of an increasing and concave function. The other cases are proved analogously. $\square$

Proof. Consider the function:

$$g(x) = x - \frac{f(x)}{f'(x)}, \quad \text{we have } x_{n+1} = g(x_n) \quad \text{and} \quad g(c) = c \quad \text{since } f(c) = 0$$

We have:

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2} > 0 \iff f(x) < 0$$

because $f''(x) < 0$ on all of $[a, b]$. Hence $g'(x) > 0$ for $x \in [a, c]$, i.e., on $[a, c]$ the function g is strictly increasing.

Keeping these facts in mind, we now prove that:

$$x_n < x_{n+1} < c, \quad \forall n \in \mathbb{N} \quad (10.1)$$

For $n = 0$ we have

$$x_1 = a - \frac{f(a)}{f'(a)} > a$$

because $f(a) < 0$ and $f'(a) > 0$, hence $x_1 > x_0$. Moreover $a < c$ (and $x_0 = a$), and since g is increasing, this implies:

$$g(a) < g(c) \quad \text{that is} \quad x_1 < c$$

Hence we have proved that

$$x_0 < x_1 < c$$

Applying now g to the previous inequalities (g is strictly increasing on $[a, c]$, and the three points lie in this interval, hence g preserves the inequalities), we have

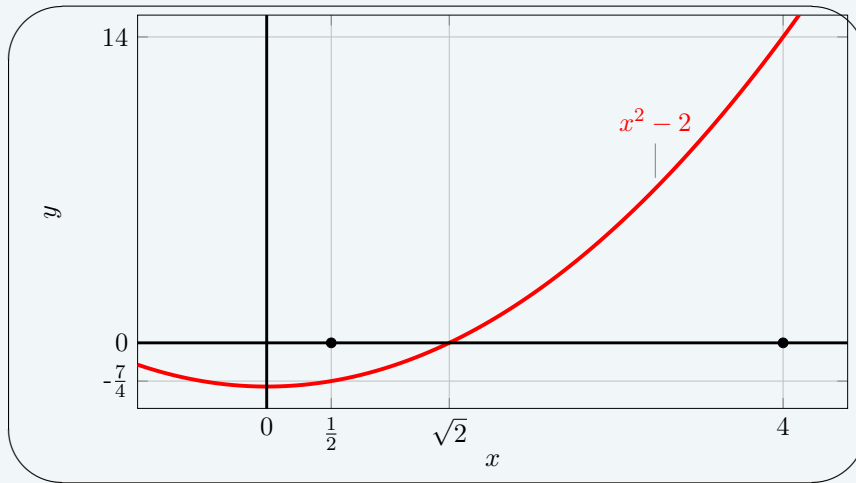
$$g(x_0) < g(x_1) < g(c) \quad \text{that is} \quad x_1 < x_2 < c$$

Applying g again, we find $x_2 < x_3 < c$, and so on. Therefore (10.1) is true, and in particular x_n is monotone. This concludes the proof. $\square$

Example 10.1: Newton's method

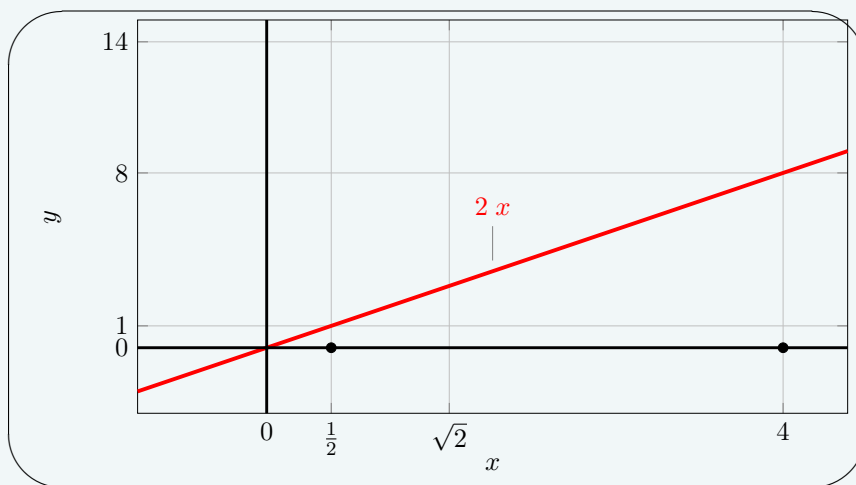
Consider the function:

$$f(x) = x^2 - 2, \quad f'(x) = 2x, \quad f''(x) = 2$$



We have:

$$f\left(\frac{1}{2}\right) = -\frac{7}{4} < 0 \quad \text{and} \quad f(4) = 14 > 0$$



Consider the interval $[\frac{1}{2}, 4]$; we have:

$$f'(x) > 0 \quad \text{and} \quad f''(x) > 0, \quad \forall x \in \left[\frac{1}{2}, 4\right]$$

Example 10.2: Newton's method

Hence on the interval $[\frac{1}{2}, 4]$ the function $f(x) = x^2 - 2$ satisfies the hypotheses of the theorem.

We are in the case of a function that is strictly increasing on $[\frac{1}{2}, 4]$, since

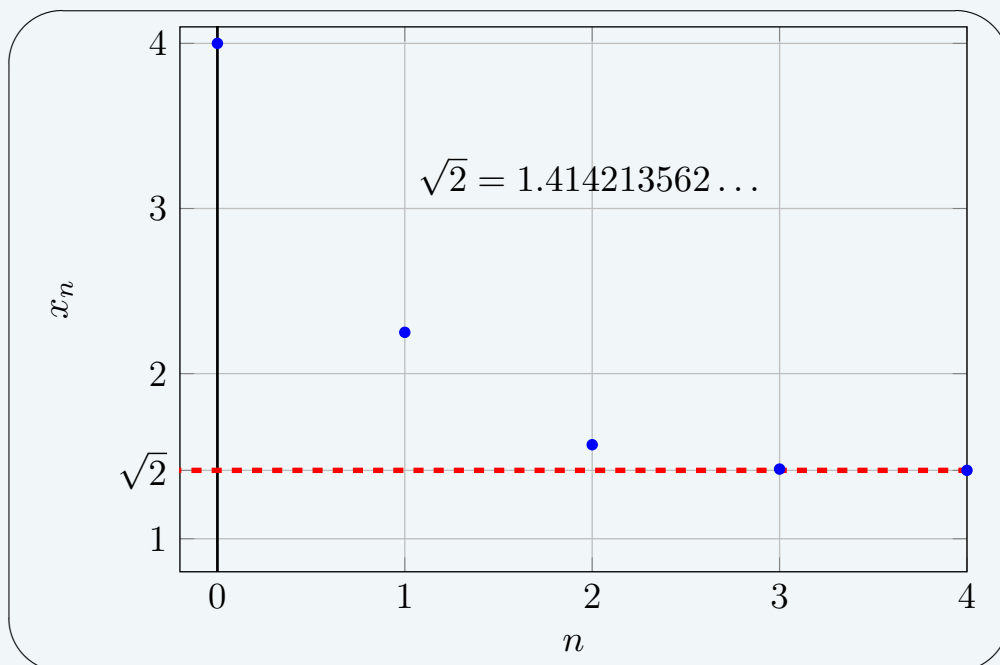
$f'(x) > 0, \forall x \in [\frac{1}{2}, 4]$, and convex on $[\frac{1}{2}, 4]$, since $f''(x) > 0, \forall x \in [\frac{1}{2}, 4]$. The endpoints of the chosen interval are $a = \frac{1}{2}$ and $b = 4$.

Hypothesis 3' holds, that is, $f(4) \cdot f''(4) > 0$, and the sequence becomes:

$$x_0 = 4, \quad x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{1}{2} \left(x_n + \frac{2}{x_n} \right) \quad \text{with } n \in \mathbb{N}$$

Computing the first values of the sequence we have:

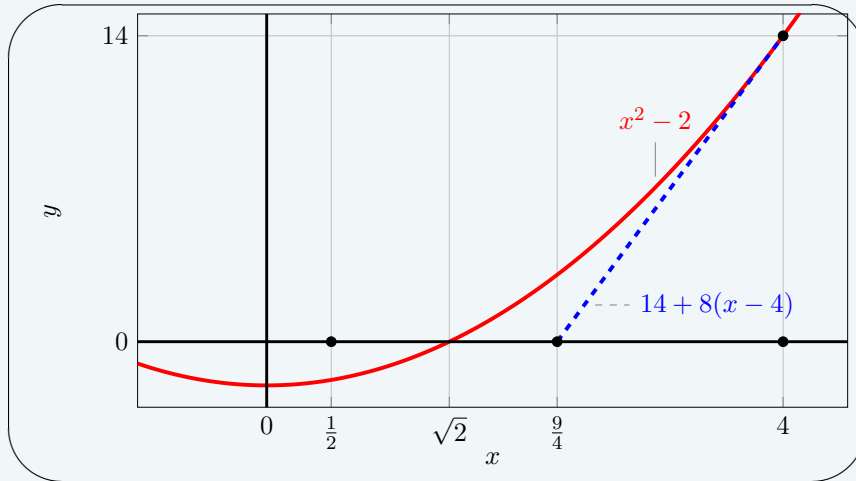
$$\begin{aligned} x_0 &= 4, \quad x_1 = \frac{9}{4} = 2.25, \quad x_2 = \frac{113}{72} = 1.569444444 \dots \\ x_3 &= \frac{23,137}{16,272} = 1.421890363 \dots, \quad x_4 = \frac{1,064,876,737}{752,970,528} = 1.414234285 \dots \end{aligned}$$



Example 10.3: Newton's method

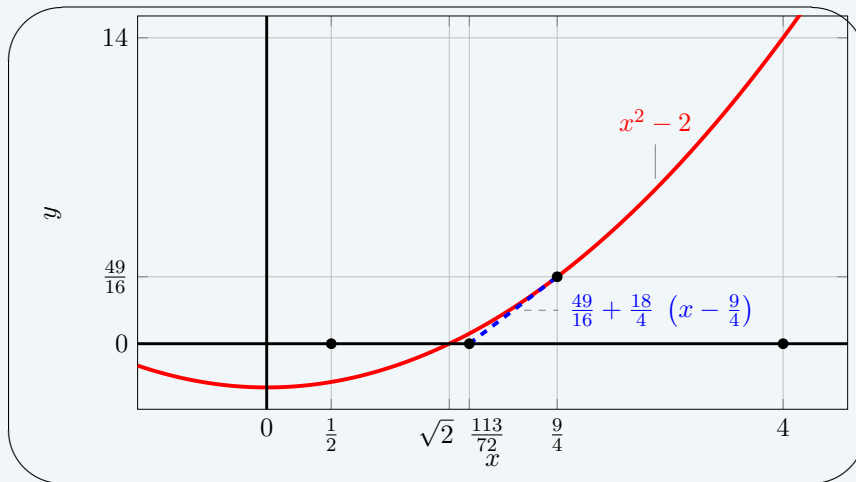
At the first iteration the tangent line is:

$$y = 14 + 8(x - 4) \quad \text{and} \quad x_1 = \frac{9}{4}$$



At the second iteration the tangent line is:

$$y = \frac{49}{16} + \frac{18}{4} \left(x - \frac{9}{4} \right) \quad \text{and} \quad x_2 = \frac{113}{72}$$



- Newton's method can be used, for example, to compute the k -th root of a number $c \in \mathbb{R}_+$, by looking for the zeros of the following function:

$$f(x) = x^k - c \quad \text{that is} \quad x = \sqrt[k]{c}$$

Given a value $b \geq \sqrt[k]{c}$, we have the recursively defined sequence:

$$x_0 = b, \quad x_{n+1} = x_n - \frac{x_n^k - c}{k x_n^{k-1}} \quad \text{with } n \in \mathbb{N}$$

expanding we obtain

$$x_n - \frac{x_n^k - c}{k x_n^{k-1}} = \frac{k x_n^k - x_n^k + c}{k x_n^{k-1}} = \frac{1}{k} \left((k-1)x_n + \frac{c}{x_n^{k-1}} \right)$$

and hence the sequence becomes:

$$x_0 = b, \quad x_{n+1} = \frac{1}{k} \left((k-1)x_n + \frac{c}{x_n^{k-1}} \right) \quad \text{with } n \in \mathbb{N}$$

With $k = 2$ we obtain the sequence of Heron's algorithm:

$$x_0 = b, \quad x_{n+1} = \frac{1}{2} \left(x_n + \frac{c}{x_n} \right) \quad \text{with } n \in \mathbb{N}$$

10.2 Error estimates for Newton's method

- The **absolute error** of Newton's method is:

$$\varepsilon_n = |x_n - c|$$

where $c \in (a, b)$ is the zero of the function, that is, $f(c) = 0$.

The value of c is unknown, hence we need a way to analyze the values of the absolute errors that is independent of c . We also want to understand how “fast” the error decreases by comparing the errors of two consecutive iterations, that is, we want to determine the “**rate of convergence**” of the algorithm.

- Using the theorem on the Taylor formula with Lagrange remainder we can write:

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{1}{2} f''(\mu)(x - x_n)^2$$

for some μ between x_n and x . In particular, choosing $x = c$ we have:

$$0 = f(c) = f(x_n) + f'(x_n)(c - x_n) + \frac{1}{2} f''(\mu)(c - x_n)^2 \quad (10.2)$$

Recalling that x_{n+1} was obtained as the solution of the equation:

$$f(x_n) + f'(x_n)(x - x_n) = 0$$

we have

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n) \quad (10.3)$$

Subtracting (10.3) from (10.2), we obtain:

$$0 = f'(x_n)(c - x_{n+1}) + \frac{1}{2} f''(\mu)(c - x_n)^2$$

which in turn implies:

$$x_{n+1} - c = \frac{1}{2} \frac{f''(\mu)}{f'(x_n)} (x_n - c)^2 \quad \text{hence} \quad |x_{n+1} - c| = \frac{1}{2} \frac{|f''(\mu)|}{|f'(x_n)|} |x_n - c|^2 \quad (10.4)$$

Now, if x_n is close to c (that is, $x_n \approx c$), then μ , which lies between x_n and c , is also close to c ; hence in these cases:

$$f''(\mu) \approx f''(c) \quad \text{and} \quad f'(x_n) \approx f'(c)$$

then, substituting, we obtain:

$$|x_{n+1} - c| \approx \frac{1}{2} \frac{|f''(c)|}{|f'(c)|} |x_n - c|^2 \quad \text{hence} \quad \varepsilon_{n+1} \approx \frac{1}{2} \frac{|f''(c)|}{|f'(c)|} \varepsilon_n^2 \quad (10.5)$$

The absolute error of Newton's method at each step is proportional to the square of the absolute error at the previous step. **The rate of convergence is quadratic.**

- However, the error is also proportional to the constant:

$$\frac{|f''(c)|}{2|f'(c)|}$$

hence, if $|f'(c)|$ is very small, or zero, or $|f''(c)|$ is very large, the convergence may be very slow or may even fail to occur.

- Let us go back to equation (10.4), and determine $L, M > 0$ such that:

$$|f'(x_n)| \geq L \quad \text{and} \quad |f''(\mu)| \leq M$$

then we can write:

$$|x_{n+1} - c| \leq \frac{1}{2} \frac{M}{L} |x_n - c|^2 \quad \text{hence} \quad \varepsilon_{n+1} \leq \frac{1}{2} \frac{M}{L} \varepsilon_n^2$$

We have obtained the relation between ε_{n+1} and ε_n , which is independent of both x_n and c (but depends on M and L).

Example 10.4: Error estimates for Newton's method ($\sqrt{2} = 1.414213562\dots$)

Let us go back to the previous example:

$$f(x) = x^2 - 2, \quad f'(x) = 2x \quad \text{and} \quad f''(x) = 2$$

Hence we can choose $M = 2$ and, since $x_n \geq 1, \forall n \in \mathbb{N}$, we can choose $L = 2$ (in this case $\frac{M}{L} = \frac{2}{2} = 1$).

We know that $\sqrt{2} > 1$, hence with $n = 2$ we have $\varepsilon_2 \leq \left| \frac{113}{72} - 1 \right| \approx 0.56944$, and we have:

iter.	x_n	estimate of $\sqrt{2}$	ε_n
$n = 3$	$\frac{23,137}{16,272}$	1.421890363...	$\leq \frac{1}{2} \varepsilon_2^2 \approx 0.16213\dots$
$n = 4$	$\frac{1,064,876,737}{752,970,528}$	1.414234285...	$\leq \frac{1}{2} \varepsilon_3^2 \approx 0.01314\dots$