



SAPIENZA
UNIVERSITÀ DI ROMA

Mathematical Analysis 1

Volume 4 — Limits

The fourth volume: limits, in two parts each numbered from 1 as in the notes. Limits of sequences: properties and limits, computing limits, Euler's number, comparisons and asymptotic estimates, hierarchies of infinities and the ratio test, recursive sequences. Limits of functions and continuity: limits and asymptotes, computing limits, polynomials and rational functions, continuous functions, comparison of infinities, standard limits, asymptotic expansions, the zeros theorem and bisection, Weierstrass and intermediate value theorems, monotone functions and invertibility. The exercises, with solutions, are in the volume Exercises 4.

Fabio Furini

fabiofurini.github.io

Department of Computer, Control and Management Engineering — Sapienza University of Rome

Online version and downloadable material:
fabiofurini.github.io/mathematical-analysis-1/
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Part I

Limits of sequences

Sequences and limits of sequences

1.1 Definition of sequence and properties

- Consider the set $\mathbb{N}$ of non-negative integers, ordered according to the natural order

$$\mathbb{N} : 0, 1, 2, 3, \dots, n, \dots$$

Definition 1.1: Sequence

A **sequence** is a relation that associates with each natural number $n \in \mathbb{N}$ (or from a certain natural number n_0 onward) a real number $a_n \in \mathbb{R}$.

- A sequence is therefore a **function**:

$$f : \mathbb{N} \rightarrow \mathbb{R}$$

$$f : n \mapsto a_n$$

or possibly

$$f : \{n \in \mathbb{N} : n \geq n_0\} \rightarrow \mathbb{R}$$

for some fixed integer $n_0 \in \mathbb{N}$.

The sequence associates with the **input variable** n the **output value** a_n .

- The fact that the domain of the function f is the set of natural numbers makes it possible to write the sequence by **listing its values**, in the order in which they follow one another as n increases:

$$a_0, a_1, a_2, \dots, a_n, \dots$$

- The dots after a_n indicate that we are not considering only the first n terms of the sequence (i.e., a **finite set** of numbers), but the whole sequence of *infinitely many terms* (i.e., an **infinite set** of numbers).

Example 1.1: Sequences

$n \in \mathbb{N}$	$n \mapsto n^2$	$0, 1, 4, 9, 16, \dots$
$n \in \mathbb{N}$	$n \mapsto (-1)^n$	$1, -1, 1, -1, 1, \dots$
$n \in \mathbb{N}$	$n \mapsto a \in \mathbb{R}$	$a, a, a, a, a, \dots$

The last sequence is called a **constant sequence**.

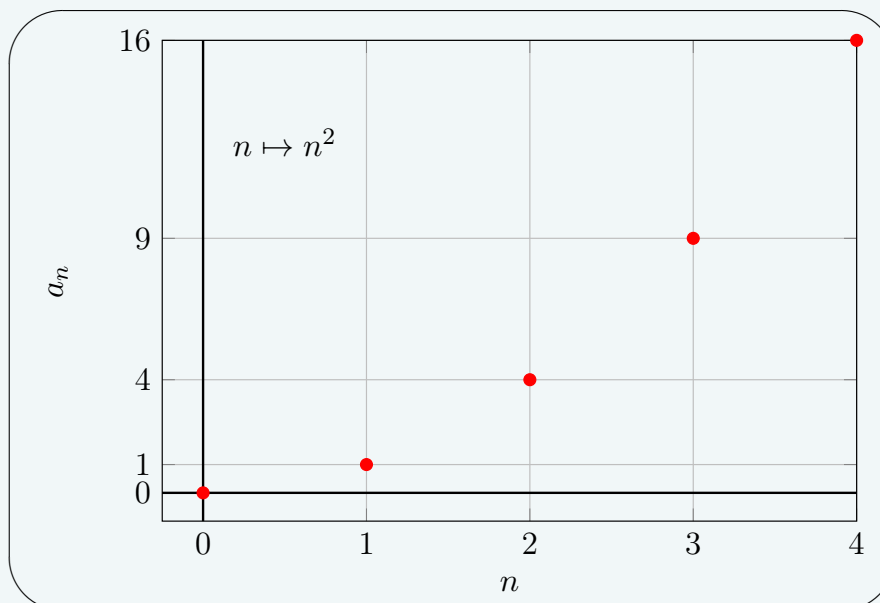
Example 1.2: Sequences

$n \in \mathbb{N}, n \geq 1$	$n \mapsto \frac{1}{n}$	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$
$n \in \mathbb{N}, n \geq 2$	$n \mapsto \frac{n+1}{n-1}$	$3, 2, \frac{5}{3}, \frac{6}{4}, \frac{7}{5}, \dots$

We can represent sequences graphically by the **points** of the Cartesian plane with coordinates (n, a_n) .

Example 1.3: Graph of a sequence

The graph of the sequence $n \mapsto n^2$ with $n \in \{0, 1, 2, 3, 4\}$ is:



To denote a sequence we will use the notation:

$$\{a_n\} \quad \text{or} \quad n \mapsto a_n$$

possibly specifying the set in which the input variable n varies (the whole set $\mathbb{N}$ or from a certain value $n_0 \in \mathbb{N}$ onward).

A sequence $\{a_n\}$ is **bounded** if there exist two numbers $m \in \mathbb{R}$ and $M \in \mathbb{R}$ such that:

$$m \leq a_n \leq M, \quad \forall n \in \mathbb{N}$$

It is **bounded below** if m exists. It is **bounded above** if M exists.

Example 1.4: Bounded sequences

- the sequence $\{(-1)^n\}$ is bounded
- the sequence $\{n^2\}$ is only bounded below
- the sequence $\{(-2)^n\}$ is not bounded (neither below nor above).

Definition 1.2: Property that holds eventually

We say that a sequence $\{a_n\}$ has (or acquires) a certain property **eventually** if there exists $\tilde{n} \in \mathbb{N}$ such that a_n satisfies that property for every $n \geq \tilde{n}$.

Definition 1.3: Positive and negative sequences

A sequence $\{a_n\}$ is called **non-negative** if:

$$a_n \geq 0, \quad \forall n$$

A sequence $\{a_n\}$ is called **positive** if:

$$a_n > 0, \quad \forall n$$

A sequence $\{a_n\}$ is called **non-positive** if:

$$a_n \leq 0, \quad \forall n$$

A sequence $\{a_n\}$ is called **negative** if:

$$a_n < 0, \quad \forall n$$

Definition 1.4: Eventually positive and eventually negative sequences

A sequence $\{a_n\}$ is called **eventually non-negative** if there exists $\tilde{n} \in \mathbb{N}$ such that:

$$a_n \geq 0, \forall n \geq \tilde{n}$$

A sequence $\{a_n\}$ is called **eventually positive** if there exists $\tilde{n} \in \mathbb{N}$ such that:

$$a_n > 0, \forall n \geq \tilde{n}$$

A sequence $\{a_n\}$ is called **eventually non-positive** if there exists $\tilde{n} \in \mathbb{N}$ such that:

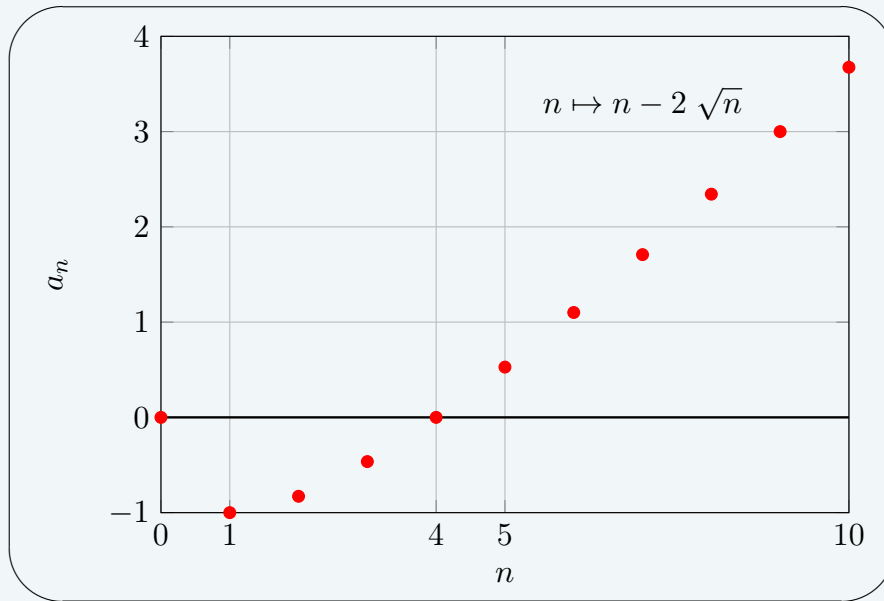
$$a_n \leq 0, \forall n \geq \tilde{n}$$

A sequence $\{a_n\}$ is called **eventually negative** if there exists $\tilde{n} \in \mathbb{N}$ such that:

$$a_n < 0, \forall n \geq \tilde{n}$$

Example 1.5: Properties that hold eventually

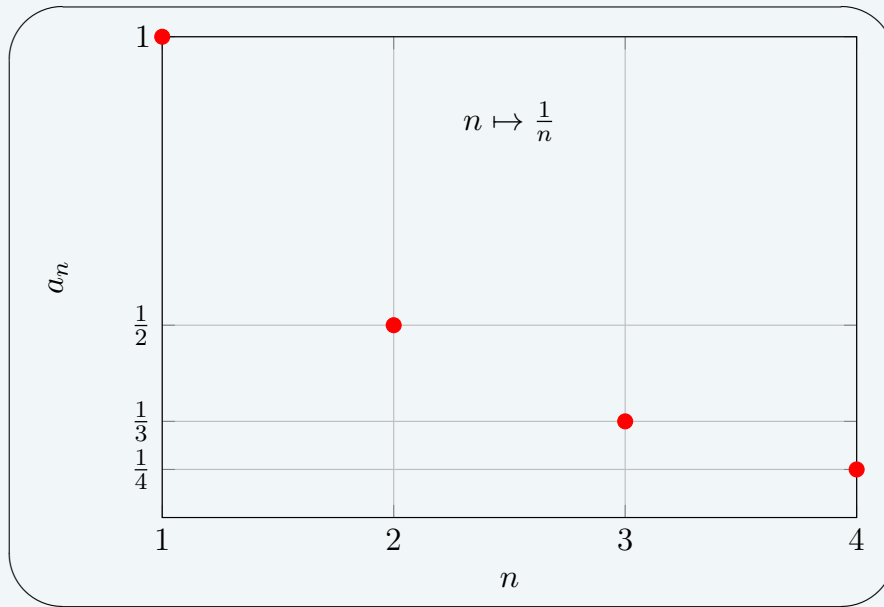
Consider the sequence $\{n - 2\sqrt{n}\}$. The graph of the sequence with $n \in \{0, 1, 2, \dots, 10\}$ is:



This sequence is eventually positive. With $n = 4$ we have $a_n = 0$, hence taking $\tilde{n} = 5$, we have $a_n > 0$ for $n \geq \tilde{n}$.

Example 1.6: Properties that hold eventually

Now consider the sequence $\{\frac{1}{n}\}$. The graph of the sequence with $n \in \{1, 2, \dots, 4\}$ is:



This sequence is eventually less than 10^{-100} . With $n = 10^{100}$ we have $a_n = 10^{-100}$, hence taking for example $\tilde{n} = 10^{100} + 1$ we have $a_n < 10^{-100}$ for $n \geq \tilde{n}$.

1.1.1 Convergent sequences and definition of the limit of a sequence

Definition 1.5: Convergent sequence

A sequence $\{a_n\}$ is called **convergent** if there exists a number $\ell \in \mathbb{R}$ such that:

$$|a_n - \ell| < \varepsilon, \quad \text{eventually} \quad (1.1)$$

for every $\varepsilon > 0$.

A sequence $\{a_n\}$ is therefore called **convergent** if for every $\varepsilon > 0$ (as small as we like) there exists a number $n(\varepsilon) \in \mathbb{N}$ such that:

$$|a_n - \ell| < \varepsilon \quad \text{for every} \quad n \geq n(\varepsilon)$$

The number $n(\varepsilon)$ depends (in general) on the value of ε . If the sequence $\{a_n\}$ is convergent, then the number $\ell \in \mathbb{R}$ is associated with it.

Definition 1.6: Limit of a sequence

The number $\ell \in \mathbb{R}$ appearing in inequality (1.1) is called the **limit of the sequence** $\{a_n\}$, and we write:

$$\lim_{n \rightarrow +\infty} a_n = \ell \quad \text{or, equivalently} \quad a_n \rightarrow \ell \quad \text{for} \quad n \rightarrow +\infty$$

- read, respectively:

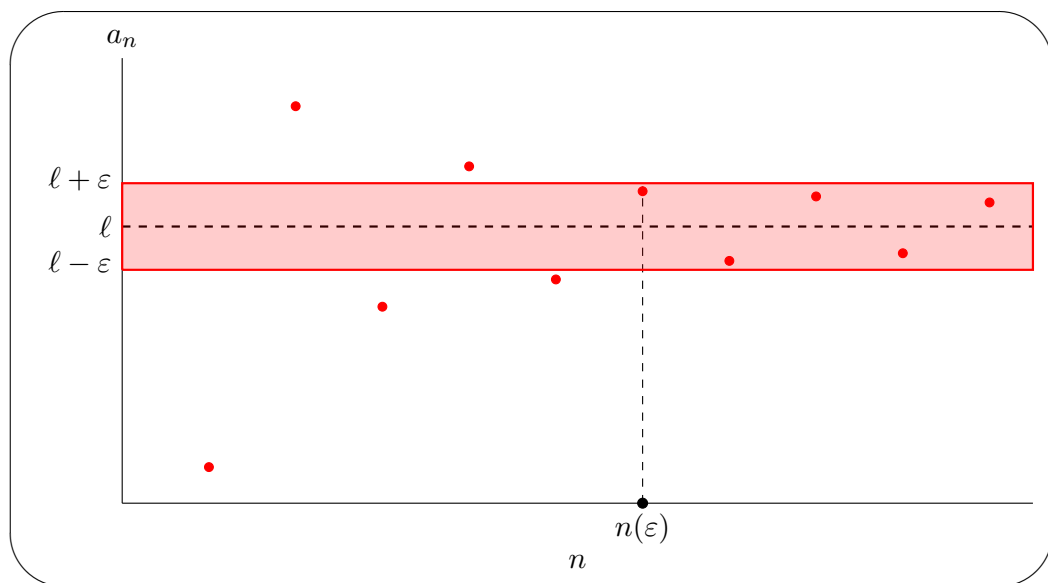
the limit of a_n , as n tends to infinity, is ℓ

a_n tends to ℓ as n tends to infinity

Inequality (1.1) corresponds to the following two:

$$\ell - \varepsilon < a_n < \ell + \varepsilon \quad (1.2)$$

- Representing the points of a sequence graphically, we have:



The convergence condition means that, having fixed a horizontal strip “as narrow as we like”:

$$(\ell - \varepsilon, \ell + \varepsilon)$$

from a certain value of n onward, called $n(\varepsilon)$, the points a_n of the sequence no longer leave this strip. In the previous graph, having fixed the width of the strip, the values a_n lie inside the strip for $n \geq n(\varepsilon)$.

Theorem 1.7: Uniqueness of the limit of a sequence

If a sequence $\{a_n\}$ converges to the limit $\ell \in \mathbb{R}$, then this limit is unique.

Proof. Suppose, by contradiction, that there exist two different limits, ℓ_1 and ℓ_2 , associated with the same sequence $\{a_n\}$. Then, eventually, for every $\varepsilon > 0$ we would have:

$$|\ell_1 - \ell_2| = |\ell_1 - a_n + a_n - \ell_2| \leq \underbrace{|\ell_1 - a_n|}_{=|a_n - \ell_1| < \varepsilon} + \underbrace{|a_n - \ell_2|}_{< \varepsilon} < 2\varepsilon \quad (1.3)$$

We used the triangle inequality. Since $\varepsilon > 0$ can be chosen arbitrarily small, (1.3) can be satisfied if and only if:

$$\ell_1 = \ell_2$$

Hence two different values ℓ_1 and ℓ_2 cannot exist and consequently the limit (if it exists) is unique. $\square$

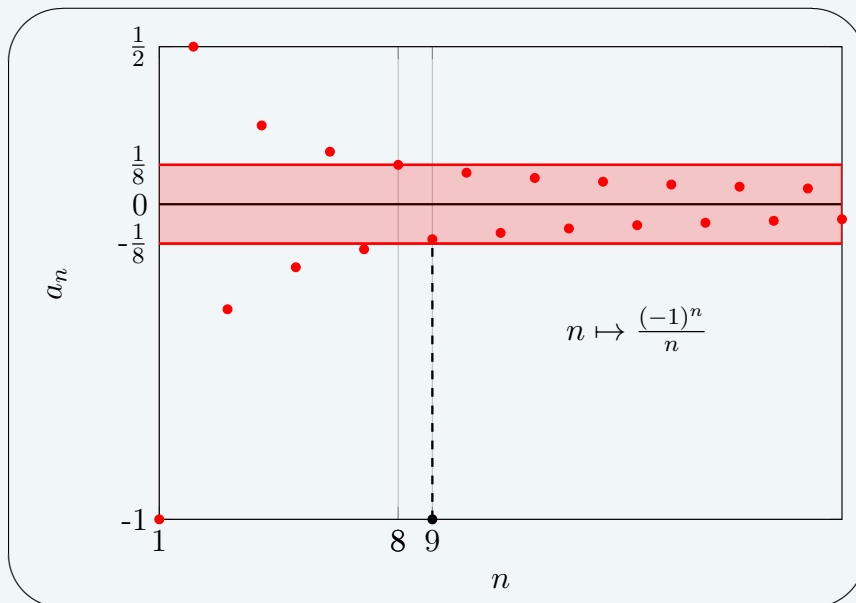
Example 1.7: Verifying the limit of a sequence

The graph of the sequence $n \mapsto \frac{(-1)^n}{n}$, for example starting from $n = 9$, lies within the horizontal strip:

$$\left(-\frac{1}{8}, \frac{1}{8}\right)$$

given by $\ell = 0$ and $\varepsilon = \frac{1}{8}$. With $n = 8$ we have $a_n = \frac{1}{8}$, with $n = 9$ we have $a_n = -\frac{1}{9}$, hence

$$|a_n| < \frac{1}{8} \quad \text{for every } n \geq 9$$



To prove that:

$$a_n \rightarrow 0 \quad \text{for } n \rightarrow +\infty$$

we must verify that for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$n \geq n(\varepsilon) \Rightarrow |a_n| < \varepsilon$$

The inequality is equivalent to

$$\left| \frac{(-1)^n}{n} \right| < \varepsilon \quad \text{which is satisfied for } n > \frac{1}{\varepsilon}$$

Having fixed $\varepsilon > 0$, it will suffice to choose the first integer

$$n(\varepsilon) > \frac{1}{\varepsilon}$$

to satisfy the condition required by the definition of limit.

Observation 1.8

Every convergent sequence is bounded.

Proof. Let $\{a_n\}$ be convergent to the limit $\ell \in \mathbb{R}$. Choosing $\varepsilon = 1$ in the definition of limit, there exists $n(1) \in \mathbb{N}$ such that

$$|a_n - \ell| < 1 \quad \text{for every } n \geq n(1)$$

By the triangle inequality we then have

$$|a_n| = |(a_n - \ell) + \ell| \leq |a_n - \ell| + |\ell| < 1 + |\ell| \quad \text{for every } n \geq n(1)$$

The remaining terms $a_0, a_1, \dots, a_{n(1)-1}$ are *finitely many*, hence we can set

$$M = \max \{ |a_0|, |a_1|, \dots, |a_{n(1)-1}|, 1 + |\ell| \} \in \mathbb{R}$$

and we obtain

$$-M \leq a_n \leq M \quad \text{for every } n \in \mathbb{N}$$

that is, the sequence is bounded. □

- By contraposition we also obtain: if a sequence is **not** bounded, then it is **not** convergent.
- The implication does not hold in the opposite direction: a bounded sequence is not necessarily convergent. A counterexample is the sequence $\{(-1)^n\}$, which is bounded ($-1 \leq (-1)^n \leq 1$ for every $n \in \mathbb{N}$) but is not convergent. Suppose indeed, for the sake of contradiction, that $(-1)^n \rightarrow \ell \in \mathbb{R}$: fixing $\varepsilon = \frac{1}{2}$, we should have $|(-1)^n - \ell| < \frac{1}{2}$ eventually, whereas

$$\text{if } \ell \geq 0, \text{ for every odd } n \text{ we have } |(-1)^n - \ell| = |-1 - \ell| = 1 + \ell \geq 1;$$

$$\text{if } \ell < 0, \text{ for every even } n \text{ we have } |(-1)^n - \ell| = |1 - \ell| = 1 + |\ell| > 1.$$

In both cases the inequality $|(-1)^n - \ell| < \frac{1}{2}$ is violated for infinitely many values of n (hence it does not hold eventually), and we have a contradiction. Therefore

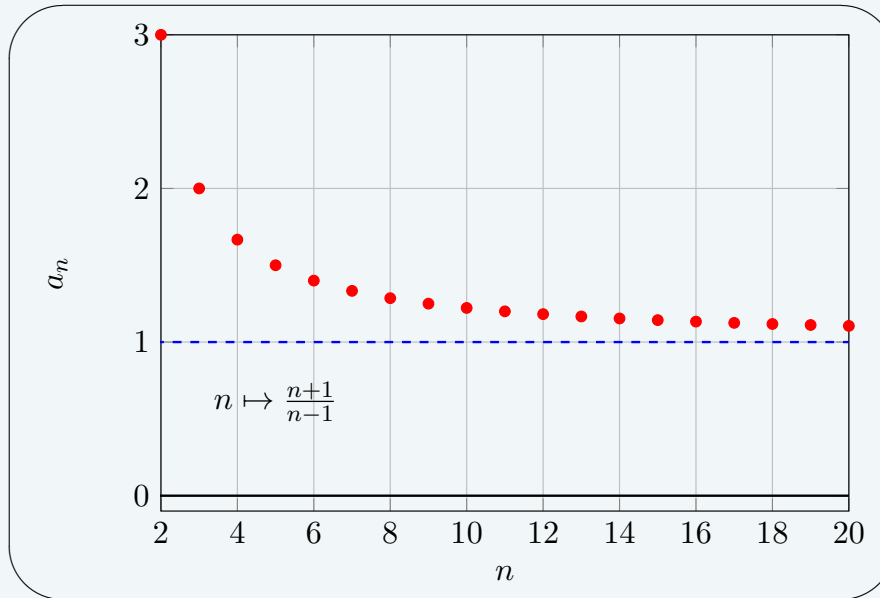
$$\{a_n\} \text{ bounded} \not\Rightarrow \{a_n\} \text{ convergent}$$

Example 1.8: Verifying the limit of a sequence

Consider the sequence:

$$n \mapsto \frac{n+1}{n-1} \quad \left(\frac{n+1}{n-1} = \frac{n-1+1+1}{n-1} = 1 + \frac{2}{n-1} \right)$$

This is the graph of the sequence with $n \in \{2, 3, \dots, 20\}$:



We see that the values a_n get closer to 1, hence we try to prove, using the definition of limit, that:

$$\lim_{n \rightarrow +\infty} \frac{n+1}{n-1} = 1$$

We must verify that for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that:

$$n \geq n(\varepsilon) \Rightarrow 1 - \varepsilon < \frac{n+1}{n-1} < 1 + \varepsilon$$

The left inequality is always satisfied (the numerator of the fraction is always larger than the denominator). We take the right one:

$$\frac{n+1}{n-1} - 1 < \varepsilon, \quad \frac{n+1 - (n-1)}{n-1} < \varepsilon, \quad \frac{2}{n-1} < \varepsilon, \quad n-1 > \frac{2}{\varepsilon}$$

$$\text{hence it is satisfied if } n > \frac{2+\varepsilon}{\varepsilon}$$

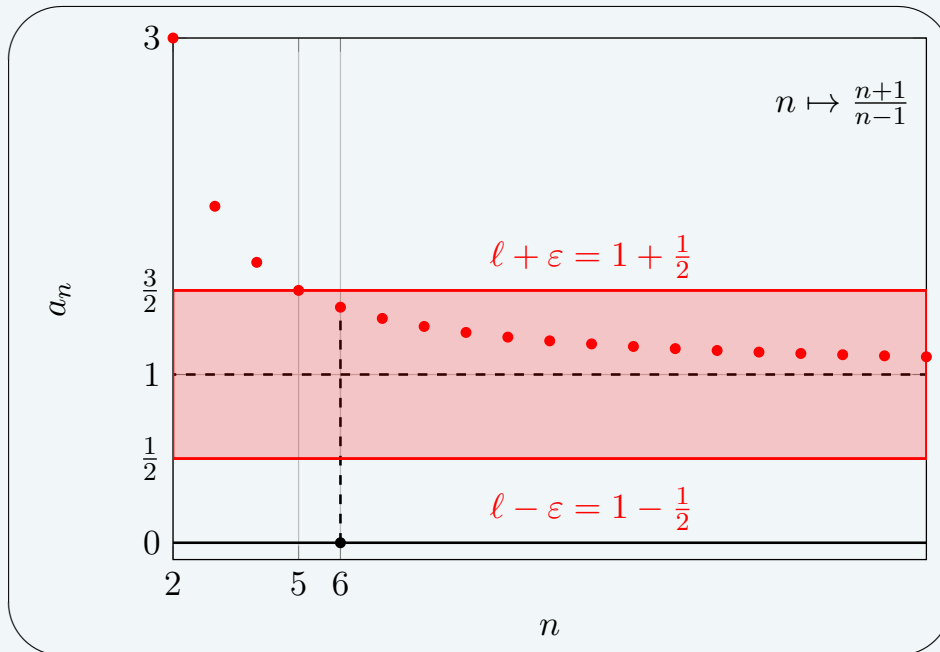
Having fixed $\varepsilon > 0$, it will suffice to choose the first integer

$$n(\varepsilon) > \frac{2+\varepsilon}{\varepsilon}$$

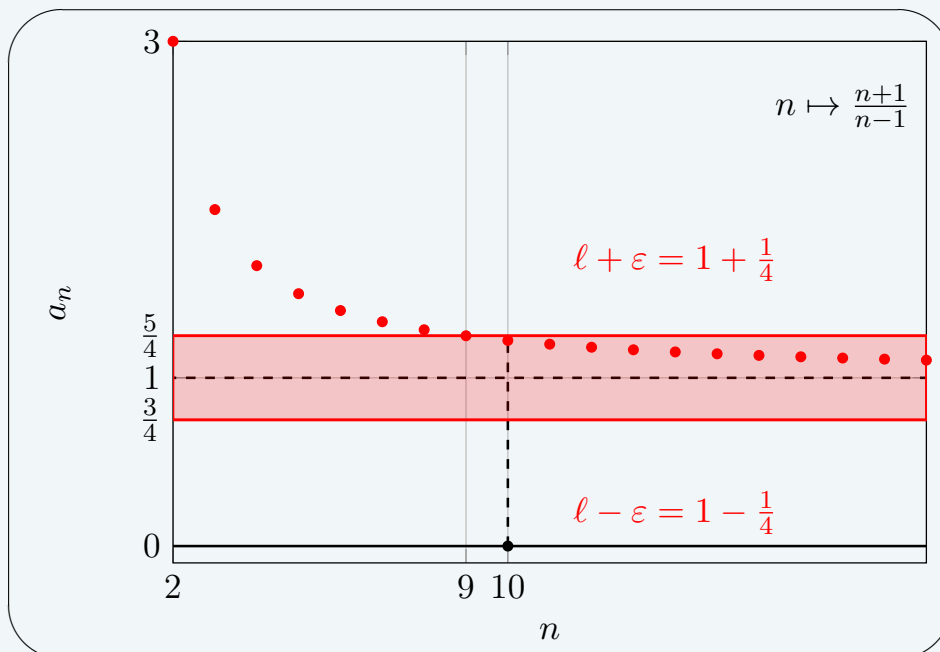
to satisfy the condition required by the definition of limit.

Example 1.9: Verifying the limit of a sequence

For the previous example, we proved that the limit equals 1. Let us now check what happens when we fix $\varepsilon = \frac{1}{2}$. In this case $n(\frac{1}{2}) > \frac{2+1/2}{1/2} = 5$.



If instead we fix $\varepsilon = \frac{1}{4}$, in this case we have $n(\frac{1}{4}) > \frac{2+1/4}{1/4} = 9$.

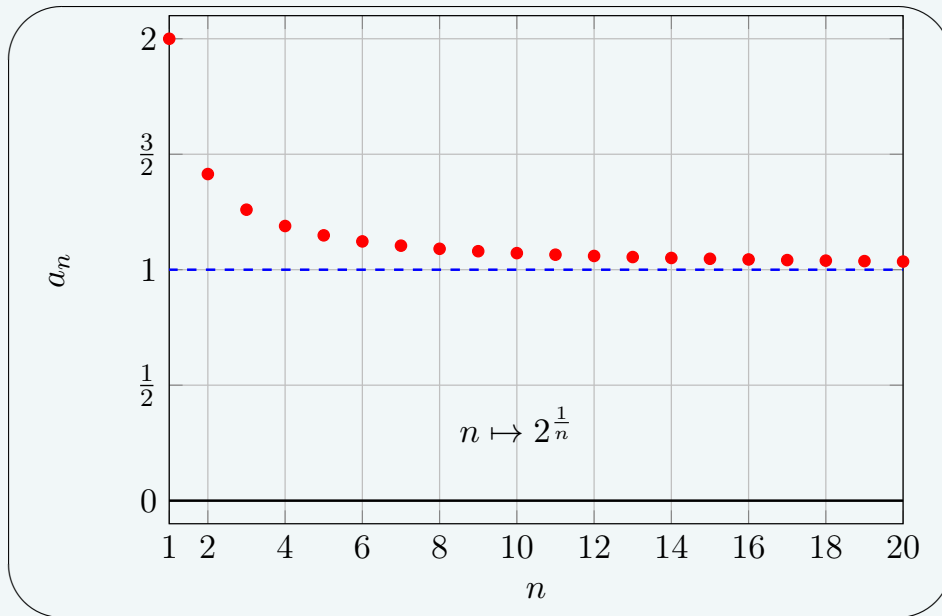


Example 1.10: Verifying the limit of a sequence

Consider the sequence:

$$n \mapsto 2^{\frac{1}{n}}$$

This is the graph of the sequence with $n \in \{1, 2, \dots, 20\}$:



We see that the values a_n get closer to 1, hence we try to prove, using the definition of limit, that:

$$\lim_{n \rightarrow +\infty} 2^{\frac{1}{n}} = 1$$

We must verify that for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that:

$$n \geq n(\varepsilon) \Rightarrow 1 - \varepsilon < 2^{\frac{1}{n}} < 1 + \varepsilon.$$

The left inequality is always satisfied (2 raised to a positive rational number), while for the right one, taking the logarithm to base 2, we obtain:

$$\frac{1}{n} < \log_2(1 + \varepsilon)$$

Hence it is satisfied if:

$$n > \frac{1}{\log_2(1 + \varepsilon)}$$

Having fixed $\varepsilon > 0$, it suffices to choose the first integer

$$n(\varepsilon) > \frac{1}{\log_2(1 + \varepsilon)}$$

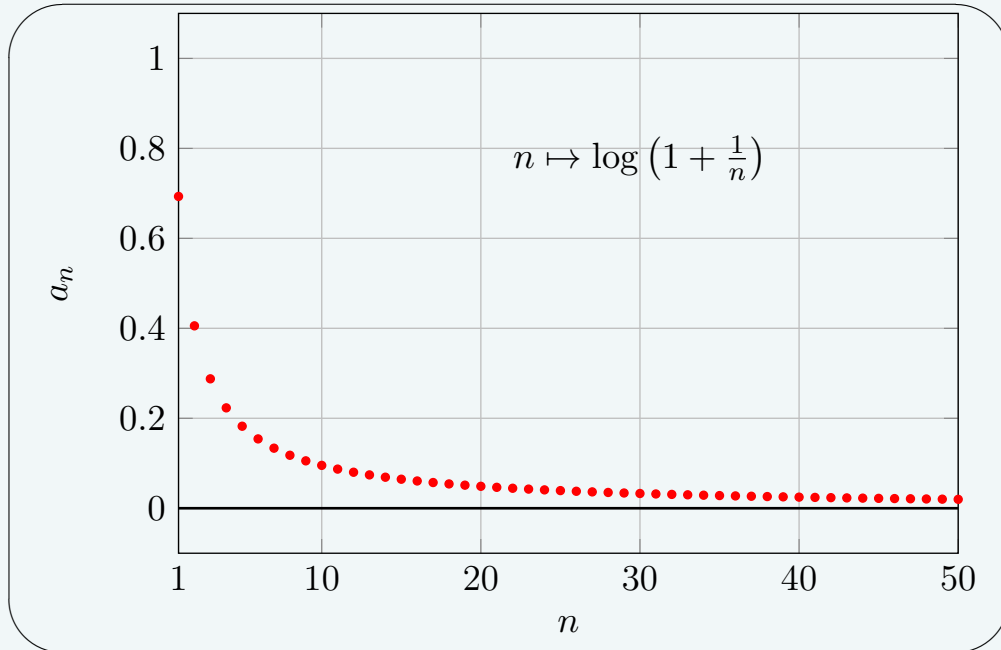
to satisfy the condition required by the definition of limit.

Example 1.11: Limit of sequences

Consider the sequence:

$$n \mapsto \log \left(1 + \frac{1}{n} \right).$$

This is the graph of the sequence with $n \in \{1, 2, \dots, 50\}$:



We see that the values a_n get closer to 0, hence we try to prove, using the definition of limit, that:

$$\lim_{n \rightarrow +\infty} \log \left(1 + \frac{1}{n} \right) = 0$$

We must verify that for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that:

$$n \geq n(\varepsilon) \Rightarrow -\varepsilon < \log \left(1 + \frac{1}{n} \right) < \varepsilon$$

The left inequality is always satisfied (the logarithm to base e of a number greater than 1), while for the right one, exponentiating, we obtain:

$$\frac{1}{n} + 1 < e^\varepsilon \quad \text{hence it is satisfied if} \quad n > \frac{1}{e^\varepsilon - 1} \quad (e^\varepsilon - 1 > 0 \text{ with } \varepsilon > 0)$$

Having fixed $\varepsilon > 0$, it suffices to choose the first integer

$$n(\varepsilon) > \frac{1}{e^\varepsilon - 1}$$

to satisfy the condition required by the definition of limit.

1.1.2 Divergent sequences and irregular sequences

Definition 1.9: Sequence divergent to $+\infty$

A sequence $\{a_n\}$ is called **divergent** to $+\infty$ if for every $M > 0$ there exists a number $n(M) \in \mathbb{N}$ such that:

$$a_n > M \text{ for every } n \geq n(M)$$

Definition 1.10: Sequence divergent to $-\infty$

A sequence $\{a_n\}$ is called **divergent** to $-\infty$ if for every $M > 0$ there exists a number $n(M) \in \mathbb{N}$ such that:

$$a_n < -M \text{ for every } n \geq n(M)$$

The number $n(M)$ depends (in general) on the value of M .

- In the two cases we will say, respectively, that $+\infty$ and $-\infty$ are the limits of the sequence and we will write, respectively:

$$\lim_{n \rightarrow +\infty} a_n = +\infty \quad \text{or} \quad \lim_{n \rightarrow +\infty} a_n = -\infty$$

The values of a sequence divergent to $+\infty$ eventually exceed any fixed real number. The values of a sequence divergent to $-\infty$ eventually fall below any fixed real number.

The symbols $+\infty$ and $-\infty$ are not numbers.

- If we represent the real numbers on the **Euclidean line**, each number corresponds to a point and each point to a number.
- With the symbols $+\infty$ and $-\infty$ we agree to denote two “points”:

$+\infty$ lies to the *right* of every point of $\mathbb{R}$

$-\infty$ lies to the *left* of every point of $\mathbb{R}$

however, no number corresponds to these two points.

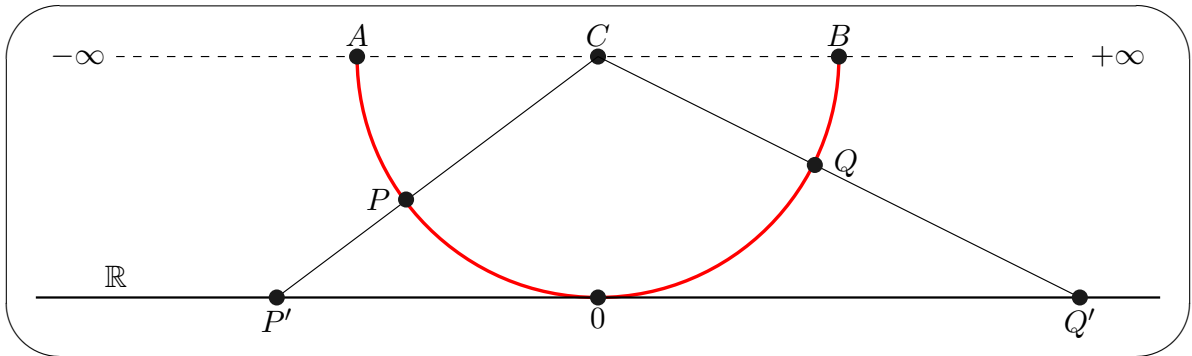
- On the symbols $+\infty$ and $-\infty$ the operations of sum and product with the properties stated in R_1 and R_2 are not defined, even though we will be able to perform these operations “partially” (as we will see later).

Definition 1.11: The set $\mathbb{R}^*$

The set of real numbers $\mathbb{R}$ with the addition of the two elements $+\infty$ and $-\infty$ will be denoted by:

$$\mathbb{R}^* = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$$

- We can represent the set $\mathbb{R}^*$ “visually” by putting the points of the **line** in one-to-one correspondence with those of a **semicircle** (projecting them from the center of the semicircle onto the line $\mathbb{R}$):



- No point on $\mathbb{R}$ corresponds to the points A and B ; we will say that $-\infty$ is the “counterpart” of the point A and $+\infty$ the “counterpart” of B .
- The limit operation becomes *fully meaningful* if set in $\mathbb{R}^*$ instead of $\mathbb{R}$, i.e., the limit of a sequence can be a real number ℓ or $+\infty$ or $-\infty$.

Sequences whose limit is a real number are **convergent**; those whose limit is $+\infty$ or $-\infty$ are **divergent**.

Example 1.12: Convergent and divergent sequences

- the canonical sequence $\{n\}$ is divergent to $+\infty$;
- the sequence $\{2^n\}$ is divergent to $+\infty$;
- the sequence $\{-2^n\}$ is divergent to $-\infty$;
- the sequence $\{2^{\frac{1}{n}}\}$ is convergent to 1.

Observation 1.12

$$\lim_{n \rightarrow +\infty} n^\alpha = \begin{cases} +\infty & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha < 0 \end{cases}$$

Proof. If $\alpha > 0$, we must prove that the sequence diverges to $+\infty$. We must therefore verify that for every $M > 0$ there exists $n(M) \in \mathbb{N}$ such that

$$n \geq n(M) \Rightarrow a_n > M$$

The inequality

$$n^\alpha > M \quad \text{is satisfied for } n > M^{\frac{1}{\alpha}}$$

Hence, having fixed $M > 0$, it will suffice to choose the first integer

$$n(M) > M^{\frac{1}{\alpha}}$$

to satisfy the required divergence condition.

Hence there is no $M > 0$ such that $n^\alpha \leq M$ for $n \in \mathbb{N}$, and the sequence is not bounded. Hence $n^\alpha \rightarrow +\infty$ for $n \rightarrow +\infty$.

If $\alpha < 0$, we must verify that for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$n \geq n(\varepsilon) \Rightarrow -\varepsilon < n^\alpha < \varepsilon$$

The left inequality is always satisfied. For the right inequality, since $n^\alpha = \frac{1}{n^{|\alpha|}}$, we obtain

$$\frac{1}{n^{|\alpha|}} < \varepsilon$$

which is satisfied if

$$n^{|\alpha|} > \frac{1}{\varepsilon} \quad \text{i.e., for } n > \frac{1}{\varepsilon^{1/|\alpha|}}$$

Having fixed $\varepsilon > 0$, it will suffice to choose the first integer

$$n(\varepsilon) > \frac{1}{\varepsilon^{1/|\alpha|}}$$

to satisfy the condition required by the definition of limit. □

- Finally, we observe that there are sequences that are neither convergent nor divergent

Definition 1.13: Irregular (indeterminate) sequence

A sequence that is neither convergent nor divergent is called **irregular** (oscillating) or **indeterminate**.

Example 1.13: Irregular sequences

- the sequence $\{(-1)^n\}$ is neither convergent nor divergent (but it is bounded)
- the sequence $\{(-2)^n\}$ is neither convergent nor divergent (and not even bounded).

Proof. We check that the sequence $\{(-1)^n\}$ is irregular, that is, that it is neither convergent nor divergent.

It is not convergent: we proved it in Remark 1.8, where $\{(-1)^n\}$ was used as a counterexample to the converse implication.

It is not divergent: for every $n \in \mathbb{N}$ we have $-1 \leq (-1)^n \leq 1$. Fixing $M = 1$, for every *odd* n we get $(-1)^n = -1 < M$, hence $(-1)^n \rightarrow +\infty$ is impossible; for every *even* n we get $(-1)^n = 1 > -M$, hence $(-1)^n \rightarrow -\infty$ is impossible.

The limit of $\{(-1)^n\}$ therefore does not exist. $\square$

- For **irregular** sequences, the limit operation is not defined, i.e., their **limit does not exist**.

Summarizing, the operation of computing the limit allows us to answer rigorously the question: **how do the numbers a_n behave as n becomes larger and larger?**

1.2 Unbounded sets and their suprema/infima

- It is convenient to adopt the convention introduced for limits also for the sup and the inf, extending the definition of these quantities as follows

Definition 1.14: Supremum and infimum sup and inf (unbounded sets)

If a set $E \subseteq \mathbb{R}$ is not bounded above (below), we will say that

$$\sup E = +\infty \quad (\inf E = -\infty)$$

- In this way, property R_4 of the real numbers can be stated as follows:

$R_4 \rightarrow$ every non-empty set $E \subseteq \mathbb{R}$ has a supremum and an infimum; $\sup E$ ($\inf E$) is a number if E is bounded above (below), otherwise it is $+\infty$ ($-\infty$).

1.3 Infinitesimal and infinite sequences

Definition 1.15: Infinitesimal sequence

A sequence $\{a_n\}$ tending to zero is called **infinitesimal**

Example 1.14: Infinitesimal sequences

- the sequence $\{\frac{1}{n}\}$ is infinitesimal
- the sequence $\{\frac{1}{n^2}\}$ is infinitesimal

Theorem 1.16: On infinitesimal sequences

A sequence $\{a_n\}$ is infinitesimal if and only if the sequence of absolute values $\{|a_n|\}$ is infinitesimal:

$$\lim_{n \rightarrow +\infty} a_n = 0 \iff \lim_{n \rightarrow +\infty} |a_n| = 0.$$

Proof. We prove the two implications.

($\Rightarrow$) If $\{a_n\}$ is infinitesimal, then $a_n \rightarrow 0$ and therefore, for every $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$\underbrace{|a_n - 0|}_{= |a_n - 0|} < \varepsilon \quad \text{for every } n \geq n(\varepsilon)$$

which is exactly the condition $|a_n| \rightarrow 0$.

($\Leftarrow$) If $|a_n| \rightarrow 0$, then for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$\underbrace{||a_n| - 0|}_{= |a_n| = |a_n - 0|} < \varepsilon \quad \text{for every } n \geq n(\varepsilon)$$

which is exactly the condition $a_n \rightarrow 0$, that is, $\{a_n\}$ is infinitesimal. □

- Beware: the theorem holds **only** for the limit 0. For instance $(-1)^n$ has $|(-1)^n| = 1 \rightarrow 1$, but the sequence $\{(-1)^n\}$ is not convergent.
- The concept of *infinitesimal* plays a central role and is also fundamental for having a correct and effective **intuitive picture** of the concepts of *infinitesimal calculus*.

an “**infinitesimal**” is not an “infinitely small number” (a meaningless concept) but a **variable quantity** (a sequence or, as we will see, a function) that **becomes indefinitely small**.

Definition 1.17: Infinite sequence

A sequence $\{a_n\}$ tending to $\pm\infty$ is called **infinite** (an infinity)

Example 1.15: Infinite sequences

- the sequence $\{n^2\}$ is infinite
- the sequence $\{n!\}$ is infinite
- Sometimes it is possible to specify whether a convergent sequence **approaches its limit from above or from below**

Definition 1.18: Limit from above

We say that the sequence $\{a_n\}$ tends to $\ell \in \mathbb{R}$ **from above** and we write

$$\lim_{n \rightarrow +\infty} a_n = \ell^+$$

if for every $\varepsilon > 0$ we have that

$$0 \leq a_n - \ell < \varepsilon, \quad \text{eventually.}$$

Definition 1.19: Limit from below

We say that the sequence $\{a_n\}$ tends to $\ell \in \mathbb{R}$ **from below** and we write

$$\lim_{n \rightarrow +\infty} a_n = \ell^-$$

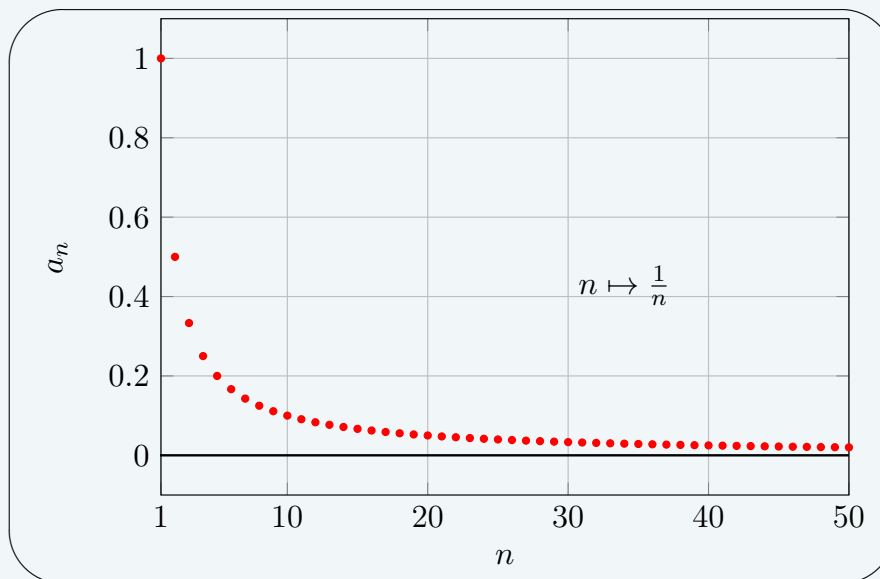
if for every $\varepsilon > 0$ we have that

$$0 \leq \ell - a_n < \varepsilon, \quad \text{eventually.}$$

- Saying that $a_n \rightarrow \ell^+$ for $n \rightarrow +\infty$ means stating that $a_n \rightarrow \ell$ and moreover $a_n \geq \ell$ eventually; hence a_n approaches ℓ *from above*, i.e., it approximates ℓ from above.
- Saying that $a_n \rightarrow \ell^-$ for $n \rightarrow +\infty$ means stating that $a_n \rightarrow \ell$ and moreover $a_n \leq \ell$ eventually; hence a_n approaches ℓ *from below*, i.e., it approximates ℓ from below.

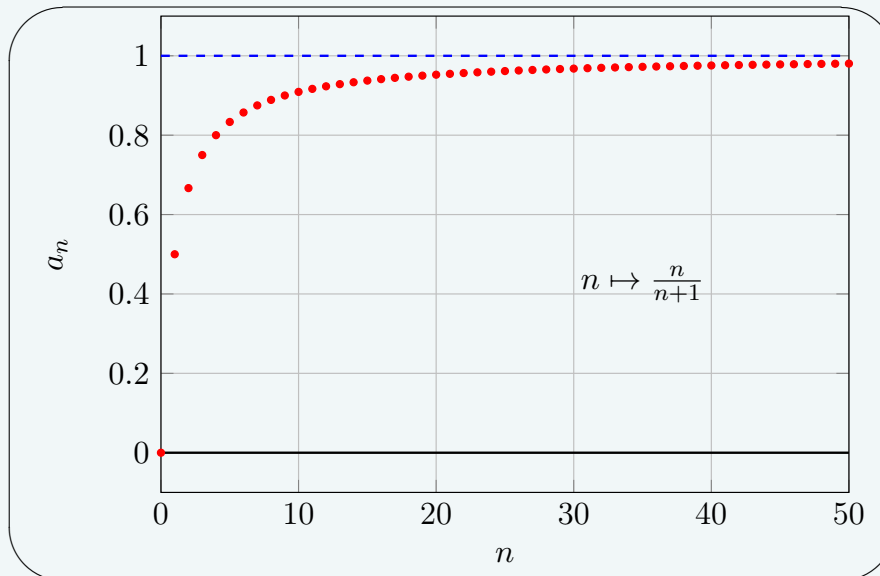
Example 1.16: Sequence with limit from above

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0^+$$



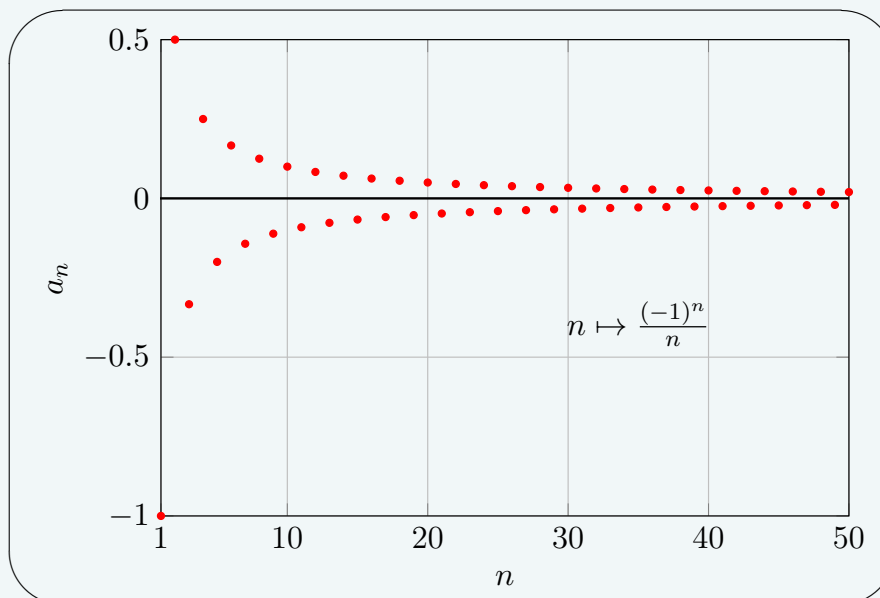
Example 1.17: Sequence with limit from below

$$\lim_{n \rightarrow +\infty} \frac{n}{n+1} = 1^- \quad \left(\frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1} \right)$$



Example 1.18: Sequence with a limit, but neither from below nor from above

$$\lim_{n \rightarrow +\infty} \frac{(-1)^n}{n} = 0$$



In this case we can state neither that $a_n \rightarrow 0^+$ nor that $a_n \rightarrow 0^-$ for $n \rightarrow +\infty$.

1.4 Monotone sequences

Definition 1.20: Monotone sequences

A sequence $\{a_n\}$ is called **non-decreasing** if:

$$a_n \leq a_{n+1}, \forall n$$

A sequence $\{a_n\}$ is called **increasing** if:

$$a_n < a_{n+1}, \forall n$$

A sequence $\{a_n\}$ is called **non-increasing** if:

$$a_n \geq a_{n+1}, \forall n$$

A sequence $\{a_n\}$ is called **decreasing** if:

$$a_n > a_{n+1}, \forall n$$

A sequence is **monotonic** if it is non-decreasing or non-increasing; it is **strictly monotonic** if it is increasing or decreasing.

Definition 1.21: Eventually monotone sequences

A sequence $\{a_n\}$ is called **eventually non-decreasing** if there exists $\tilde{n} \in \mathbb{N}$ such that:

$$a_n \leq a_{n+1}, \forall n \geq \tilde{n}$$

A sequence $\{a_n\}$ is called **eventually increasing** if there exists $\tilde{n} \in \mathbb{N}$ such that:

$$a_n < a_{n+1}, \forall n \geq \tilde{n}$$

A sequence $\{a_n\}$ is called **eventually non-increasing** if there exists $\tilde{n} \in \mathbb{N}$ such that:

$$a_n \geq a_{n+1}, \forall n \geq \tilde{n}$$

A sequence $\{a_n\}$ is called **eventually decreasing** if there exists $\tilde{n} \in \mathbb{N}$ such that:

$$a_n > a_{n+1}, \forall n \geq \tilde{n}$$

A sequence is **eventually monotonic** if it is eventually non-decreasing or eventually non-increasing; it is **eventually strictly monotonic** if it is eventually increasing or eventually decreasing.

Example 1.19: Increasing and decreasing sequences

- The sequence $\{n^2\}$ is increasing
 - The sequence $\{\frac{1}{n}\}$ is decreasing
 - The sequence $\{(-1)^n\}$ is not monotonic
 - every constant sequence is monotonic (it is both non-decreasing and non-increasing), but it is not strictly monotonic
- With regard to the limit operation, these sequences are of particular importance; indeed, they are **never irregular**, but are **convergent or divergent** depending on whether they are **bounded or not**.

Theorem 1.22: Monotone sequence theorem

- Let $\{a_n\}$ be a **non-decreasing sequence that is bounded above**. Then $\{a_n\}$ is convergent, and its limit equals

$$\sup\{a_n : n \in \mathbb{N}\}.$$

- Let $\{a_n\}$ be a **non-increasing sequence that is bounded below**. Then $\{a_n\}$ is convergent, and its limit equals

$$\inf\{a_n : n \in \mathbb{N}\}.$$

Proof. We consider the case of non-decreasing sequences that are bounded above.

Since the sequence is bounded above, the set of values taken by the sequence $\{a_n : n \in \mathbb{N}\}$ is bounded above.

By the *supremum property* enjoyed by $\mathbb{R}$, there therefore exists a finite supremum, which we denote by ℓ :

$$\ell = \sup\{a_n : n \in \mathbb{N}\} \quad \text{and} \quad \ell \in \mathbb{R}.$$

We now need to prove that

$$\lim_{n \rightarrow +\infty} a_n = \ell$$

and hence that for every $\varepsilon > 0$ we have

$$\ell - \varepsilon < a_n < \ell + \varepsilon, \quad \text{eventually.}$$

The second inequality is obvious. For every $n \in \mathbb{N}$ we have

$$a_n \leq \ell \quad (\text{and hence } a_n < \ell + \varepsilon)$$

since ℓ is the supremum of $\{a_n : n \in \mathbb{N}\}$ and hence an upper bound.

By definition of supremum, ℓ is the least of the upper bounds of the set $\{a_n : n \in \mathbb{N}\}$. Therefore, since

$$\ell - \varepsilon < \ell,$$

certainly $\ell - \varepsilon$ is not an upper bound of the set $\{a_n : n \in \mathbb{N}\}$. This means that there exists an $n(\varepsilon) \in \mathbb{N}$ for which

$$a_{n(\varepsilon)} > \ell - \varepsilon.$$

On the other hand, the sequence is non-decreasing, therefore for every $n \geq n(\varepsilon)$ we have $a_n \geq a_{n(\varepsilon)}$. We have thus proved that

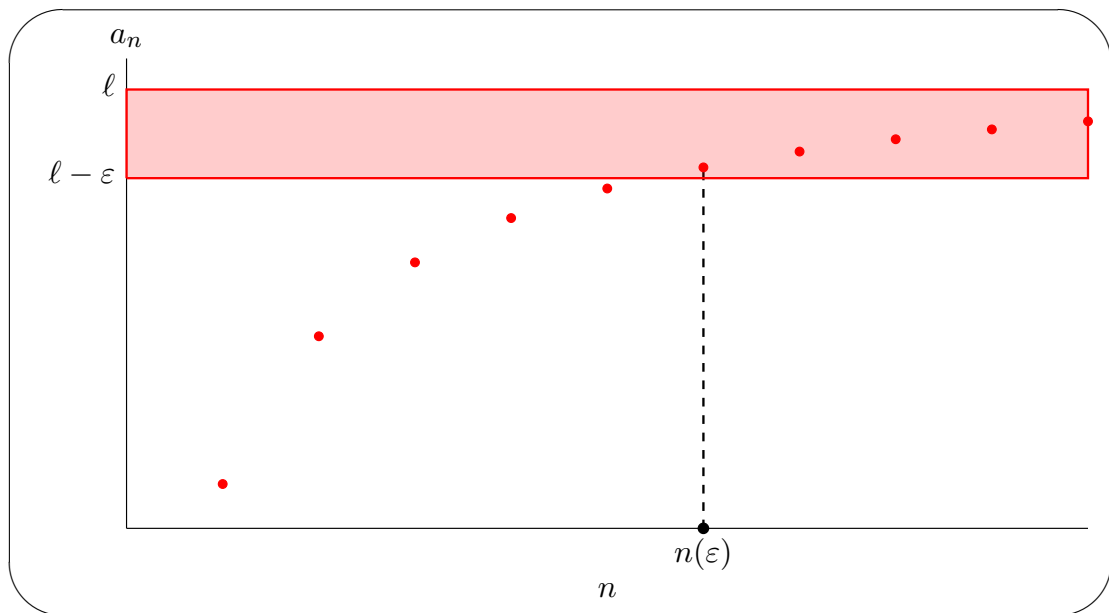
$$a_n \geq a_{n(\varepsilon)} > \ell - \varepsilon \quad \text{for every } n \geq n(\varepsilon)$$

Consequently, we have the claim:

$$\lim_{n \rightarrow +\infty} a_n = \ell$$

The case of non-increasing sequences that are bounded below is proved analogously. $\square$

- The idea of the proof is conveyed by the following figure:



- To express symbolically as well that the limit is the sup (or the inf) of a non-decreasing (or non-increasing) sequence, we use the notation

$$a_n \uparrow \ell \quad \text{or} \quad a_n \downarrow \ell$$

This in particular implies that $a_n \rightarrow \ell^-$ (respectively, ℓ^+), but it contains a further piece of information: the monotonicity of the sequence.

- This theorem is a consequence of the completeness axiom (axiom of continuity) R_4 of the real numbers and therefore holds if the setting we consider is $\mathbb{R}$. For example, it is not true that an increasing and bounded sequence of rational numbers always has a rational limit, i.e., in $\mathbb{Q}$.

Example 1.20: Increasing and bounded sequence in $\mathbb{Q}$

Let $\{a_n\}$ be the sequence defined as follows:

$$a_0 = 0, a_1 = 0.1, a_2 = 0.1011, a_3 = 0.10110111, a_4 = 0.1011011101111 \dots$$

At step n we append to the decimal number obtained at the previous step a digit zero followed by n digits equal to 1. The sequence $\{a_n\}$ is evidently increasing, and bounded above (for example, $a_n \leq 1$).

- In $\mathbb{R}$ the sequence converges to the number $\sup\{a_n : n \in \mathbb{N}\}$, which after the decimal point has an infinite, non-periodic decimal expansion (one digit 1, one digit 0, two digits 1, one digit 0, three digits 1, one digit 0, and so on forever)
- Hence the limit of the sequence is an irrational number. This example shows that in the set $\mathbb{Q}$ the monotone sequence theorem is false.

- The monotone sequence theorem can be completed with the next corollary, which considers bounded or unbounded sequences.

Corollary 1.23: Of the monotone sequence theorem

Let $\{a_n\}$ be a non-decreasing sequence. Then there exists

$$\lim_{n \rightarrow +\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}.$$

Proof. If $\{a_n\}$ is bounded above, the statement is contained in Theorem 1.22 (monotone sequence theorem).

If instead $\{a_n\}$ is unbounded above, this means that, having fixed $M > 0$, there exists an $n(M) \in \mathbb{N}$ such that

$$a_{n(M)} > M$$

On the other hand, the sequence is non-decreasing, therefore for every $n \geq n(M)$ we have

$$a_n \geq a_{n(M)} > M$$

We have thus proved that for every $M > 0$ we have $a_n > M$, eventually. This means that $a_n \rightarrow +\infty$ for $n \rightarrow +\infty$. □

Summarizing, we have:

- if $\{a_n\}$ is bounded above, then it converges (and its limit equals the supremum of its values, which in this case is a real number)
- if instead $\{a_n\}$ is unbounded above, then a_n tends to $+\infty$ (which in this case equals the supremum of its values).

A monotonic sequence either converges or diverges (it cannot be irregular).

1.5 Limits of geometric progressions

- Consider the **geometric progression** with common ratio $a \in \mathbb{R}$:

$$1, a, a^2, a^3, \dots, a^n, \dots$$

it is equivalent to the sequence $\{a^n\}$:

$$n \mapsto a^n$$

- If $a > 1$, the sequence is increasing and unbounded above.
- If $a = 1$, the sequence is constant.
- If $0 < a < 1$, the sequence is decreasing and tends to zero.
- If a is negative, the sequence is not monotonic.

Observation 1.24

$$\lim_{n \rightarrow +\infty} a^n = \begin{cases} +\infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \\ \text{does not exist} & \text{if } a \leq -1 \end{cases}$$

Proof. If $a = 0$, we have $a^n = 0$ for every $n \geq 1$: the sequence is eventually zero and therefore $a^n \rightarrow 0$.

If $0 < |a| < 1$, we must verify that for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$n \geq n(\varepsilon) \Rightarrow |a^n| < \varepsilon$$

Since $|a^n| = |a|^n$, we must verify that

$$n \geq n(\varepsilon) \Rightarrow |a|^n < \varepsilon$$

Taking the logarithm in base $|a|$, which is *decreasing* since $0 < |a| < 1$, the inequality is satisfied if

$$n > \log_{|a|} \varepsilon$$

Having fixed $\varepsilon > 0$, it will suffice to choose the first integer

$$n(\varepsilon) > \log_{|a|} \varepsilon$$

to satisfy the condition required by the definition of limit.

If $a > 1$, we must verify that for every $M > 0$ there exists $n(M) \in \mathbb{N}$ such that

$$n \geq n(M) \Rightarrow a_n > M$$

The inequality

$$a^n > M \quad \text{is satisfied for } n > \log_a M$$

Hence, having fixed $M > 0$, it will suffice to choose the first integer

$$n(M) > \log_a M$$

to satisfy the required divergence condition. □

Proof. If $a \leq -1$, we have $|a| \geq 1$ and therefore

$$a^n = (-1)^n |a|^n \quad \text{with} \quad |a|^n \geq 1, \quad \forall n \in \mathbb{N}$$

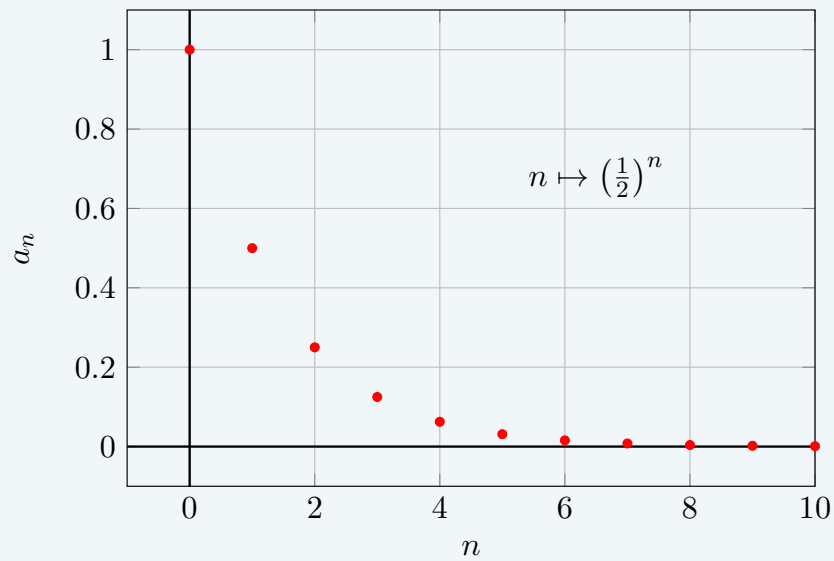
that is, $a^n \geq 1$ for n even and $a^n \leq -1$ for n odd. We show that the sequence is neither convergent nor divergent, that is, it is irregular.

It is not convergent: if, for the sake of contradiction, $a^n \rightarrow \ell \in \mathbb{R}$, then fixing $\varepsilon = \frac{1}{2}$ we should have $|a^n - \ell| < \frac{1}{2}$ eventually. Instead, if $\ell \geq 0$, for every *odd* n we have $|a^n - \ell| = |a|^n + \ell \geq 1$, while if $\ell < 0$, for every *even* n we have $|a^n - \ell| = |a|^n + |\ell| > 1$. In both cases the inequality is violated for infinitely many values of n .

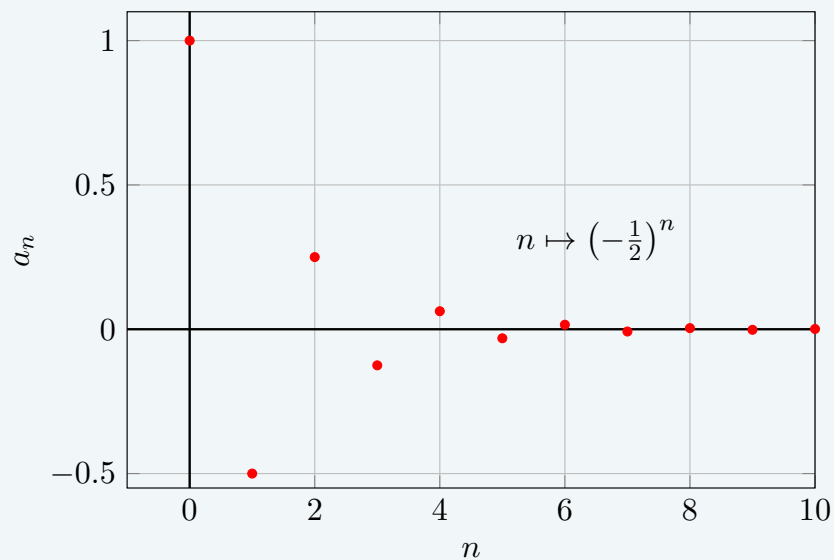
It is not divergent: fixing $M = 1$, for every *odd* n we have $a^n \leq -1 < M$, hence $a^n \rightarrow +\infty$ is impossible; for every *even* n we have $a^n \geq 1 > -M$, hence $a^n \rightarrow -\infty$ is impossible.

The limit therefore does not exist. □

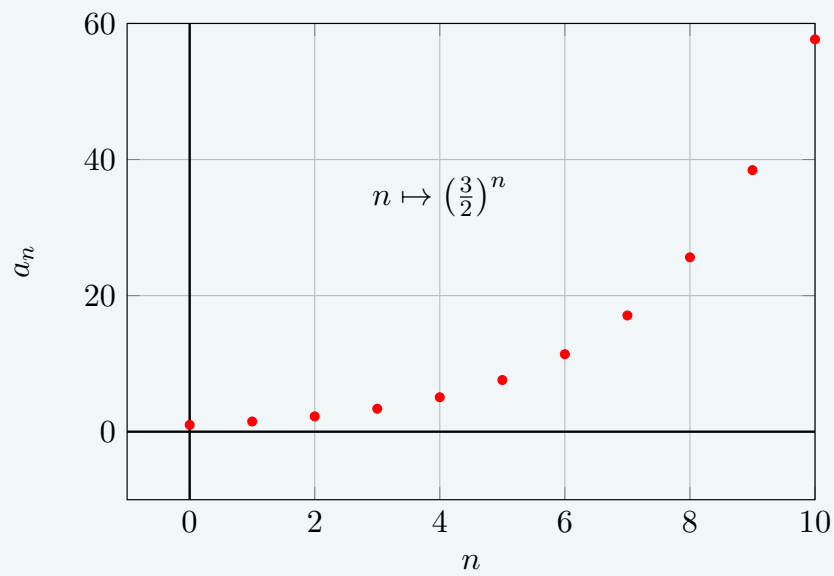
Example 1.21: Infinitesimal and decreasing geometric progression



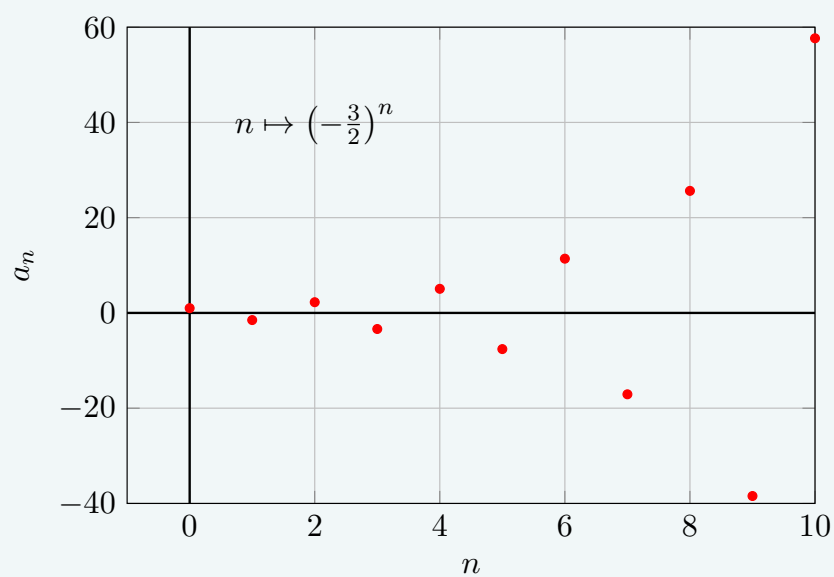
Example 1.22: Infinitesimal (but not monotonic) geometric progression



Example 1.23: Divergent geometric progression



Example 1.24: Geometric progression that is neither convergent nor divergent



Computing limits of sequences

2.1 Computing limits of sequences

- The proofs of the **basic theorems** on computing limits are based on the **definition of limit**, on the **use of inequalities**, and on the use of **properties that are eventually true**.
- In particular, these theorems illustrate the relation between the limit operation and the algebraic structures¹ and the order structures² present in $\mathbb{R}$.

We will simply write $a_n \rightarrow \ell$ to mean $a_n \rightarrow \ell$ for $n \rightarrow +\infty$

Properties of the limit operation with respect to algebraic operations.

Theorem 2.1: Algebra of limits, case of finite limits

Hypotheses:

$$1. a_n \rightarrow \ell_a \in \mathbb{R} \quad 2. b_n \rightarrow \ell_b \in \mathbb{R}.$$

Claim:

$$\begin{aligned}
 1. a_n \pm b_n &\rightarrow \ell_a \pm \ell_b & 2. \frac{a_n}{b_n} &\rightarrow \frac{\ell_a}{\ell_b} & (b_n, \ell_b \neq 0, \text{ eventually}) \\
 3. a_n b_n &\rightarrow \ell_a \ell_b & 4. a_n^{b_n} &\rightarrow \ell_a^{\ell_b} & (a_n, \ell_a > 0, \text{ eventually}).
 \end{aligned}$$

Proof. We prove that:

$$a_n \rightarrow \ell_a \in \mathbb{R}, b_n \rightarrow \ell_b \in \mathbb{R} \Rightarrow a_n + b_n \rightarrow \ell_a + \ell_b$$

We consider

$$|(a_n + b_n) - (\ell_a + \ell_b)| = |(a_n - \ell_a) + (b_n - \ell_b)| \leq |a_n - \ell_a| + |b_n - \ell_b| \quad (2.1)$$

by the triangle inequality. Since by hypothesis $a_n \rightarrow \ell_a$ and $b_n \rightarrow \ell_b$, we have that

$$|a_n - \ell_a| < \varepsilon_a \quad \text{and} \quad |b_n - \ell_b| < \varepsilon_b \quad (\text{eventually})$$

¹A set on which an operation is defined is called an algebraic structure

²A set is endowed with an order structure if an order relation is defined on it

for every $\varepsilon_a > 0$ and $\varepsilon_b > 0$. Bounding from above the terms on the right-hand side of (2.1) we conclude that

$$|(a_n + b_n) - (\ell_a + \ell_b)| < \underbrace{2 \max\{\varepsilon_a, \varepsilon_b\}}_{=\tilde{\varepsilon} \text{ and } >0}$$

Since $\tilde{\varepsilon}$ is arbitrary, the claim follows. $\square$

Proof. We prove that:

$$a_n \rightarrow \ell_a \in \mathbb{R}, b_n \rightarrow \ell_b \in \mathbb{R} \Rightarrow a_n - b_n \rightarrow \ell_a - \ell_b$$

For every $\varepsilon_a > 0$ and $\varepsilon_b > 0$, we have:

$$|(a_n - b_n) - (\ell_a - \ell_b)| = |(a_n - \ell_a) + (\ell_b - b_n)| \leq \underbrace{|a_n - \ell_a|}_{<\varepsilon_a} + \underbrace{|\ell_b - b_n|}_{=|b_n - \ell_b| < \varepsilon_b}$$

Hence

$$|(a_n - b_n) - (\ell_a - \ell_b)| < \underbrace{2 \max\{\varepsilon_a, \varepsilon_b\}}_{=\tilde{\varepsilon} \text{ and } >0}$$

Since $\tilde{\varepsilon}$ is arbitrary, the claim follows. $\square$

Proof. We prove that:

$$a_n \rightarrow \ell_a \in \mathbb{R}, b_n \rightarrow \ell_b \in \mathbb{R} \Rightarrow a_n b_n \rightarrow \ell_a \ell_b$$

We consider

$$\begin{aligned} |(a_n b_n) - (\ell_a \ell_b)| &= |a_n b_n - a_n \ell_b + a_n \ell_b - \ell_a \ell_b| \\ &= |a_n (b_n - \ell_b) + \ell_b (a_n - \ell_a)| \leq |a_n (b_n - \ell_b)| + |\ell_b (a_n - \ell_a)| \\ &= |a_n| |b_n - \ell_b| + |\ell_b| |a_n - \ell_a| \end{aligned}$$

by the triangle inequality and the properties of the absolute value. Hence

$$|(a_n b_n) - (\ell_a \ell_b)| \leq |a_n| |b_n - \ell_b| + |\ell_b| |a_n - \ell_a| \quad (2.2)$$

Since by hypothesis $a_n \rightarrow \ell_a$ and $b_n \rightarrow \ell_b$, for every $\varepsilon_a > 0$ and $\varepsilon_b > 0$ we have

$$|a_n - \ell_a| < \varepsilon_a \quad \text{and} \quad |b_n - \ell_b| < \varepsilon_b \quad (\text{eventually}).$$

Moreover, since

$$|a_n - \ell_a| \geq |a_n| - |\ell_a| \quad \text{we have} \quad |a_n| < |\ell_a| + \varepsilon_a, \quad \text{eventually.}$$

Therefore, bounding from above the terms on the right-hand side of (2.2), we conclude that

$$|(a_n b_n) - (\ell_a \ell_b)| < (|\ell_a| + \varepsilon_a) \varepsilon_b + |\ell_b| \varepsilon_a = \underbrace{|\ell_a| \varepsilon_b + |\ell_b| \varepsilon_a + \varepsilon_a \cdot \varepsilon_b}_{=\tilde{\varepsilon} \text{ and } >0}$$

Since $\tilde{\varepsilon}$ is arbitrary, the claim follows. $\square$

- Moreover, the **limit operation preserves the ordering**

Theorem 2.2: Sign-preservation, first form

Hypotheses:

$$1. a_n \rightarrow \ell_a \quad 2. \ell_a \leq 0.$$

Claim:

$$a_n \leq 0, \text{ eventually.}$$

Proof. We consider the case $\ell_a > 0$. By definition of limit we have that

$$|a_n - \ell_a| < \varepsilon, \text{ eventually,}$$

for every $\varepsilon > 0$, which we rewrite in the form:

$$\ell_a - \varepsilon < a_n < \ell_a + \varepsilon, \text{ eventually.}$$

Since $\ell_a > 0$, we can choose $\varepsilon > 0$ such that $\ell_a - \varepsilon > 0$; then the inequality

$$0 < \ell_a - \varepsilon < a_n$$

shows that $a_n > 0$, eventually.

The case $\ell_a < 0$ is proved analogously. □

Theorem 2.3: Permanence of sign (generalized form)

Hypotheses:

$$1. a_n \rightarrow \ell_a \in \mathbb{R} \quad 2. \lambda \in \mathbb{R} \quad 3. \ell_a > \lambda.$$

Claim:

$$a_n > \lambda, \text{ eventually.}$$

Proof. Since $\ell_a > \lambda$, the number $\varepsilon = \ell_a - \lambda$ is strictly positive. Applying the definition of limit with precisely this value of ε , we have, eventually:

$$\underbrace{\ell_a - (\ell_a - \lambda)}_{=\lambda} < a_n < \ell_a + (\ell_a - \lambda)$$

and therefore $a_n > \lambda$, eventually. □

- Analogously one proves that, if $a_n \rightarrow \ell_a \in \mathbb{R}$ and $\ell_a < \lambda$, then $a_n < \lambda$ eventually: it suffices to choose $\varepsilon = \lambda - \ell_a > 0$ and to obtain, eventually,

$$\ell_a - (\lambda - \ell_a) < a_n < \underbrace{\ell_a + (\lambda - \ell_a)}_{=\lambda}$$

- With $\lambda = 0$ we recover the permanence of sign theorem, 1st form.
- We can now prove the case of the **quotient** in Theorem 2.1 on the algebra of limits.

Proof. We first prove that:

$$b_n \rightarrow \ell_b \in \mathbb{R}, \ell_b \neq 0 \Rightarrow \frac{1}{b_n} \rightarrow \frac{1}{\ell_b}$$

The case of the quotient then follows from the case of the product, already proved, applied to the sequences $\{a_n\}$ and $\left\{\frac{1}{b_n}\right\}$:

$$\frac{a_n}{b_n} = a_n \frac{1}{b_n} \rightarrow \ell_a \frac{1}{\ell_b} = \frac{\ell_a}{\ell_b}$$

Suppose $\ell_b > 0$ (if $\ell_b < 0$ one argues in the same way on the sequence $\{-b_n\}$). By the permanence of sign theorem (generalized form), applied with $\lambda = \frac{\ell_b}{2} < \ell_b$, we have

$$b_n > \frac{\ell_b}{2} > 0, \quad \text{eventually}$$

in particular $b_n \neq 0$ eventually and the quotient $\frac{1}{b_n}$ is well defined. Moreover, for every $\varepsilon > 0$, we have $|b_n - \ell_b| < \varepsilon$ eventually, and therefore, eventually:

$$\begin{aligned} \left| \frac{1}{b_n} - \frac{1}{\ell_b} \right| &= \left| \frac{\ell_b - b_n}{b_n \ell_b} \right| = \frac{|b_n - \ell_b|}{|b_n| |\ell_b|} \\ &< \frac{2}{\ell_b |\ell_b|} |b_n - \ell_b| < \underbrace{\frac{2}{\ell_b^2} \varepsilon}_{= \tilde{\varepsilon} \text{ and } > 0} \end{aligned}$$

where we used $|b_n| > \frac{\ell_b}{2}$. Since $\tilde{\varepsilon}$ is arbitrary, the claim follows. $\square$

Theorem 2.4: Sign-preservation, second form (part I)

Hypotheses:

$$1. \quad a_n \rightarrow \ell_a \in \mathbb{R} \quad 2. \quad a_n \geq 0, \quad \text{eventually.}$$

Claim:

$$\ell_a \geq 0.$$

Proof. It follows from the previous theorem. Indeed, if by contradiction we had $\ell_a < 0$, from the previous theorem we would have $a_n < 0$ eventually, which is incompatible with the hypothesis that $a_n \geq 0$ eventually.

This case cannot occur, i.e., the opposite holds, which is the claim of the theorem. $\square$

Theorem 2.5: Sign-preservation, second form (part II)

Hypotheses:

1. $a_n \rightarrow \ell_a \in \mathbb{R}$ 2. $b_n \rightarrow \ell_b \in \mathbb{R}$ 3. $a_n \geq b_n$, eventually.

Claim:

$$\ell_a \geq \ell_b.$$

Proof. We consider the sequence $a_n - b_n$; by the theorem on the algebra of limits we have

$$a_n - b_n \rightarrow \ell_a - \ell_b.$$

Since by hypothesis

$$a_n \geq b_n \quad \text{and hence} \quad a_n - b_n \geq 0,$$

by the sign-preservation theorem, second form (part I), applied to the sequence $a_n - b_n$ we have

$$\ell_a - \ell_b \geq 0 \quad \text{and hence} \quad \ell_a \geq \ell_b.$$

□

- This theorem tells us that in an inequality between two sequences we can pass to the limit on both sides, keeping the “ $\leq$ ” or the “ $\geq$ ”.
- Note, instead, that in general strict inequalities “ $<$ ” and “ $>$ ” are not preserved when passing to the limit.

Example 2.1: Passing to the limit with strict inequalities

For example, even if the a_n are strictly positive, their limit ℓ_a is positive or zero, as shown by the simple example $\frac{1}{n} \rightarrow 0$.

Theorem 2.6: Comparison (squeeze) theorem

Hypotheses:

1. $a_n \rightarrow \ell, c_n \rightarrow \ell$ and $\ell \in \mathbb{R}$, 2. $a_n \leq b_n \leq c_n$, eventually.

Claim:

$$b_n \rightarrow \ell.$$

Proof. By definition of limit we have, eventually, that

$$\ell - \varepsilon_a < a_n < \ell + \varepsilon_a \quad \text{and} \quad \ell - \varepsilon_c < c_n < \ell + \varepsilon_c$$

for every $\varepsilon_a > 0$ and $\varepsilon_c > 0$. Hence:

$$\ell - \max\{\varepsilon_a, \varepsilon_c\} < a_n < \ell + \max\{\varepsilon_a, \varepsilon_c\} \quad \text{and} \quad \ell - \max\{\varepsilon_a, \varepsilon_c\} < c_n < \ell + \max\{\varepsilon_a, \varepsilon_c\}$$

From the hypotheses of the theorem we therefore have, eventually, that

$$\ell - \max\{\varepsilon_a, \varepsilon_c\} < a_n \leq b_n \leq c_n < \ell + \max\{\varepsilon_a, \varepsilon_c\}$$

But then, eventually, we have

$$\underbrace{\ell - \max\{\varepsilon_a, \varepsilon_c\}}_{=\tilde{\varepsilon} \text{ and } >0} < b_n < \underbrace{\ell + \max\{\varepsilon_a, \varepsilon_c\}}_{=\tilde{\varepsilon} \text{ and } >0}.$$

Since $\tilde{\varepsilon}$ is arbitrary, the claim follows. $\square$

Theorem 2.7: Comparison theorem for divergent sequences

Hypotheses:

$$1. \ a_n \rightarrow +\infty, \quad 2. \ a_n \leq b_n, \text{ eventually.}$$

Claim:

$$b_n \rightarrow +\infty.$$

Proof. Since $a_n \rightarrow +\infty$, for every $M > 0$ we have, eventually,

$$a_n > M$$

By hypothesis we also have $a_n \leq b_n$, eventually. Both properties hold eventually, hence, eventually, we have

$$b_n \geq a_n > M$$

Since $M > 0$ is arbitrary, we conclude that $b_n \rightarrow +\infty$. $\square$

- Analogously one proves that, if $a_n \rightarrow -\infty$ and $a_n \geq b_n$ eventually, then $b_n \rightarrow -\infty$.
- Frequently used special cases of the comparison theorem are expressed by the following corollaries, which are very useful when studying the product of an oscillating (but bounded) sequence and one that tends to zero

Corollary 2.8: Of the comparison theorem (part I)

Hypotheses:

$$1. \ c_n \rightarrow 0 \quad 2. \ |b_n| \leq c_n, \text{ eventually.}$$

Claim:

$$b_n \rightarrow 0.$$

Proof. We know that eventually we have $-c_n \leq b_n \leq c_n$. Obviously

$$\text{if } c_n \rightarrow 0 \text{ then } -c_n \rightarrow 0.$$

Hence by the comparison theorem (with $a_n = -c_n$ and $\ell = 0$) we have that $b_n \rightarrow 0$. $\square$

Corollary 2.9: Of the comparison theorem (part II)

Hypotheses:

1. $c_n \rightarrow 0$
2. $\{b_n\}$ is bounded (but not necessarily convergent).

Claim:

$$c_n b_n \rightarrow 0.$$

Proof. If $\{b_n\}$ is bounded, then $|b_n| \leq M$ for some $M > 0$ and for every $n \in \mathbb{N}$. We can therefore write

$$|b_n c_n| \leq M |c_n|.$$

Since

$$c_n \rightarrow 0 \text{ also } M |c_n| \rightarrow 0,$$

by Corollary 2.8 we conclude that $b_n c_n \rightarrow 0$. $\square$

The product of an infinitesimal sequence and a bounded one is infinitesimal.

Example 2.2: Application of the corollary

Consider the sequence given by a ratio of two expressions, each consisting of a sum of powers of n , such as:

$$n \mapsto \frac{n^{5/2} - 3n + 7}{n^3 + \sqrt{n} - 3n^2}$$

Factoring out the highest power both in the numerator and in the denominator we obtain:

$$\frac{n^{5/2} \left(1 - \frac{3}{n^{3/2}} + \frac{7}{n^{5/2}}\right)}{n^3 \left(1 + \frac{1}{n^{5/2}} - \frac{3}{n}\right)} = \frac{1}{\sqrt{n}} \frac{1 - \frac{3}{n^{3/2}} + \frac{7}{n^{5/2}}}{1 + \frac{1}{n^{5/2}} - \frac{3}{n}}$$

Now, by Theorem 2.1 on the algebra of limits and knowing that negative powers of n tend to zero, we can state that:

$$1 - \underbrace{\frac{3}{n^{3/2}}}_{\rightarrow 0} + \underbrace{\frac{7}{n^{5/2}}}_{\rightarrow 0} \rightarrow 1, \quad 1 + \underbrace{\frac{1}{n^{5/2}}}_{\rightarrow 0} - \underbrace{\frac{3}{n}}_{\rightarrow 0} \rightarrow 1 \quad \text{and} \quad \left(\frac{1 - \frac{3}{n^{3/2}} + \frac{7}{n^{5/2}}}{1 + \frac{1}{n^{5/2}} - \frac{3}{n}} \right) \rightarrow 1$$

hence the last sequence is convergent and consequently bounded. Now, by Corollary 2.9 and since

$$\frac{1}{\sqrt{n}} \rightarrow 0 \quad \text{we have} \quad \frac{n^{5/2} - 3n + 7}{n^3 + \sqrt{n} - 3n^2} \rightarrow 0$$

Example 2.3: Application of the corollary

The sequence

$$n \mapsto \frac{\sin n}{n}$$

is the product of two sequences

$$n \mapsto \frac{1}{n} \quad (\text{convergent, infinitesimal}) \quad \text{and} \quad n \mapsto \sin n \quad (\text{irregular}),$$

hence Theorem 2.1 on the algebra of limits cannot be applied (the second limit does not exist).

- However, Corollary 2.9 can be applied. The sequence $\{\frac{1}{n}\}$ is infinitesimal and, since $|\sin n| \leq 1$, the sequence $\{\sin n\}$ is bounded; therefore we have

$$\lim_{n \rightarrow +\infty} \frac{\sin n}{n} = 0$$

- The theorems on the algebra of limits seen so far work on pairs of sequences that are both convergent or at least bounded.

Sequences with limits $+\infty$ and $-\infty$

- Suppose, for example, that

$$a_n \rightarrow \ell_a \quad \text{and} \quad b_n \rightarrow +\infty$$

then it is easy (and intuitive) to see that

$$a_n + b_n \rightarrow +\infty$$

We will abbreviate this as follows:

$$\ell_a + \infty = +\infty$$

- Reasoning in a similar way we can summarize the rules for the limit of the sum (or difference) of two sequences, one or both of which are divergent.

Rules of partial arithmetization of the infinity symbol

Theorem 2.10: Partial arithmetization of the infinity symbol (addition)

Hypotheses:

$$1. a_n \rightarrow \ell_a \in \mathbb{R} \quad 2. b_n \rightarrow +\infty \quad 3. c_n \rightarrow +\infty.$$

Claim:

$$\begin{aligned} 1. a_n + b_n &\rightarrow \ell_a + \infty = +\infty & 2. a_n - b_n &\rightarrow \ell_a - \infty = -\infty \\ 3. b_n + c_n &\rightarrow +\infty + \infty = +\infty & 4. -b_n - c_n &\rightarrow -\infty - \infty = -\infty. \end{aligned}$$

Theorem 2.11: Partial arithmetization of the infinity symbol (product)

Hypotheses:

$$1. a_n \rightarrow \ell_a \in \mathbb{R} \quad 2. b_n \rightarrow 0^+ \text{ or } b_n \rightarrow 0^- \quad 3. c_n \rightarrow \infty.$$

Claim:

$$1. a_n c_n \rightarrow \ell_a \infty = \infty \quad (\ell_a \neq 0) \quad 2. \frac{a_n}{b_n} \rightarrow \frac{\ell_a}{0} = \infty \quad (\ell_a \neq 0) \quad 3. \frac{a_n}{c_n} \rightarrow \frac{\ell_a}{\infty} = 0.$$

- the **sign** of ∞ must be determined with the **usual rule of signs**;
- in claim **2.** it is understood, as usual, that $b_n \neq 0$ eventually, so that the quotient is well defined;
- hypothesis **2.**, namely that $\{b_n\}$ tends to zero *from above* or *from below* (and hence that it eventually has constant sign), is **necessary**: with $a_n = 1$ and $b_n = \frac{(-1)^n}{n} \rightarrow 0$ we have

$$\frac{a_n}{b_n} = (-1)^n n$$

which is an irregular sequence (it tends neither to $+\infty$ nor to $-\infty$).

Example 2.4: Rule of signs

We have:

- if $a_n \rightarrow \ell_a \in \mathbb{R}, \ell_a > 0$ and $b_n \rightarrow 0^+$ then $\frac{a_n}{b_n} \rightarrow +\infty$
- if $a_n \rightarrow \ell_a \in \mathbb{R}, \ell_a < 0$ and $b_n \rightarrow 0^+$ then $\frac{a_n}{b_n} \rightarrow -\infty$
- if $a_n \rightarrow \ell_a \in \mathbb{R}, \ell_a > 0$ and $b_n \rightarrow 0^-$ then $\frac{a_n}{b_n} \rightarrow -\infty$
- if $a_n \rightarrow \ell_a \in \mathbb{R}, \ell_a < 0$ and $b_n \rightarrow 0^-$ then $\frac{a_n}{b_n} \rightarrow +\infty$

Therefore, in order to apply the rules of partial arithmetization of the infinity symbol, it is necessary to determine whether b_n tends to zero from above or from below.

Proof. We prove that:

$$a_n \rightarrow \ell_a \in \mathbb{R}, \quad c_n \rightarrow +\infty \quad \Rightarrow \quad \frac{a_n}{c_n} \rightarrow 0.$$

For every $\varepsilon > 0$, since $a_n \rightarrow \ell_a$, eventually we have

$$\underbrace{|a_n - \ell_a|}_{\geq |a_n| - |\ell_a|} < \varepsilon \quad \text{hence} \quad |a_n| < |\ell_a| + \varepsilon.$$

Moreover, since $c_n \rightarrow +\infty$, eventually we have

$$c_n > \frac{1}{\varepsilon}$$

It follows that, eventually, we have

$$\frac{|a_n|}{|c_n|} < \frac{|\ell_a| + \varepsilon}{\frac{1}{\varepsilon}} \quad \text{and hence} \quad \left| \frac{a_n}{c_n} \right| < \varepsilon (|\ell_a| + \varepsilon) = \underbrace{|\ell_a| \varepsilon + \varepsilon^2}_{= \tilde{\varepsilon} \text{ and } > 0}$$

Since $\tilde{\varepsilon}$ is arbitrary, the claim follows. $\square$

The four missing operations:

$$+\infty - \infty, \quad 0 \cdot \infty, \quad \frac{0}{0} \quad \text{and} \quad \frac{\infty}{\infty}$$

are called **indeterminate forms**, since no rule can be established a priori to determine their result.

Example 2.5: Resolving indeterminate forms $+\infty - \infty$

Consider the sequence

$$n \mapsto \sqrt{n+1} - \sqrt{n-1}$$

We have the difference of two sequences:

$$n \mapsto \sqrt{n+1} \quad \text{with} \quad \lim_{n \rightarrow +\infty} \sqrt{n+1} = +\infty$$

$$n \mapsto \sqrt{n-1} \quad \text{with} \quad \lim_{n \rightarrow +\infty} \sqrt{n-1} = +\infty$$

Hence we fall into the indeterminate form $+\infty - \infty$.

Multiplying and dividing by $\sqrt{n+1} + \sqrt{n-1}$ we obtain

$$\frac{(\sqrt{n+1} - \sqrt{n-1})(\sqrt{n+1} + \sqrt{n-1})}{\sqrt{n+1} + \sqrt{n-1}} = \frac{(\sqrt{n+1})^2 - (\sqrt{n-1})^2}{\sqrt{n+1} + \sqrt{n-1}} = \frac{2}{\sqrt{n+1} + \sqrt{n-1}}$$

recalling that $a^2 - b^2 = (a - b)(a + b)$. Now, considering the sequence in the denominator, we have

$$\lim_{n \rightarrow +\infty} \sqrt{n+1} + \sqrt{n-1} = +\infty + \infty = +\infty$$

Using the rule

$$a_n \rightarrow \ell_a \in \mathbb{R}, \quad c_n \rightarrow \infty \quad \Rightarrow \quad \frac{a_n}{c_n} \rightarrow 0$$

of the theorem on the partial arithmetization of the infinity symbol (product), we have:

$$\lim_{n \rightarrow +\infty} \sqrt{n+1} - \sqrt{n-1} = \lim_{n \rightarrow +\infty} \frac{2}{\sqrt{n+1} + \sqrt{n-1}} = \frac{2}{+\infty} = 0$$

- Limits of sequences of the form:

$$\{a_n^{b_n}\}$$

can be handled by considering the sequence of their logarithms, taking for simplicity the base e .

Given a sequence $\{a_n^{b_n}\}$, we have that:

$$\text{if } b_n \log a_n \rightarrow \ell \text{ then } a_n^{b_n} \rightarrow e^\ell \quad (2.3)$$

$$\text{if } b_n \log a_n \rightarrow +\infty \text{ then } a_n^{b_n} \rightarrow +\infty \quad (2.4)$$

$$\text{if } b_n \log a_n \rightarrow 0 \text{ then } a_n^{b_n} \rightarrow 1 \quad (2.5)$$

$$\text{if } b_n \log a_n \rightarrow -\infty \text{ then } a_n^{b_n} \rightarrow 0 \quad (2.6)$$

If the sequence $\{b_n \log a_n\}$ is indeterminate (has no limit), then $\{a_n^{b_n}\}$ is indeterminate as well.

Example 2.6: Computing limits by taking logarithms

$$\lim_{n \rightarrow +\infty} (3n)^{(-3n^2+7)}$$

taking logarithms we have

$$\underbrace{(-3n^2+7)}_{b_n} \log \underbrace{3n}_{a_n}$$

since

$$\lim_{n \rightarrow +\infty} (-3n^2+7) \log(3n) = -\infty \cdot +\infty = -\infty$$

then

$$\lim_{n \rightarrow +\infty} (3n)^{(-3n^2+7)} = 0$$

Example 2.7: Computing limits by taking logarithms (alternative method)

$$\lim_{n \rightarrow +\infty} (3n)^{(-3n^2+7)} = \lim_{n \rightarrow +\infty} e^{\log((3n)^{(-3n^2+7)})} = \lim_{n \rightarrow +\infty} e^{(-3n^2+7) \log(3n)}$$

since

$$\lim_{n \rightarrow +\infty} (-3n^2 + 7) \log(3n) = -\infty \cdot +\infty = -\infty$$

then

$$\lim_{n \rightarrow +\infty} (3n)^{(-3n^2+7)} = e^{-\infty} = 0$$

We also have the following **indeterminate forms**:

$$1^\infty, \quad 0^0, \quad \infty^0$$

Taking logarithms, they correspond to the indeterminate form

$$0 \cdot \infty$$

since:

$$\log(1^\infty) = \infty \log 1 = \infty \cdot 0$$

$$\log(0^0) = 0 \log 0 = 0 \cdot -\infty$$

$$\log(+\infty^0) = 0 \log(+\infty) = 0 \cdot +\infty$$

In the powers $\{a_n^{b_n}\}$ the base is always (eventually) positive, hence in the form ∞^0 the base is $+\infty$.

Limits of the form

$$0^{+\infty} \quad \text{and} \quad 0^{-\infty}$$

are not indeterminate forms. We have:

$$0^{+\infty} = 0 \quad \text{and} \quad 0^{-\infty} = +\infty.$$

since, taking logarithms, we have

$$\log(0^{+\infty}) = +\infty \cdot \log 0 = +\infty \cdot -\infty = -\infty \quad \text{and} \quad e^{-\infty} = 0$$

$$\log(0^{-\infty}) = -\infty \cdot \log 0 = -\infty \cdot -\infty = +\infty \quad \text{and} \quad e^{+\infty} = +\infty$$

Example 2.8: Limits of the forms $0^{+\infty}$ and $0^{-\infty}$

- Example of the case $0^{+\infty} = 0$:

$$\lim_{n \rightarrow +\infty} \left(\frac{1}{n}\right)^{\log n} = 0^{+\infty} = 0.$$

Indeed we have

$$\left(\frac{1}{n}\right)^{\log n} = e^{\log\left(\frac{1}{n}\right)^{\log n}} = e^{\log n \log \frac{1}{n}}$$

hence

$$\lim_{n \rightarrow +\infty} \left(\frac{1}{n}\right)^{\log n} = \lim_{n \rightarrow +\infty} e^{\overbrace{\log n}^{\rightarrow +\infty} \overbrace{\log \frac{1}{n}}^{\rightarrow -\infty}} = e^{-\infty} = 0.$$

- Example of the case $0^{-\infty} = +\infty$:

$$\lim_{n \rightarrow +\infty} \left(\frac{1}{n}\right)^{-\log n} = 0^{-\infty} = +\infty.$$

Indeed we have

$$\left(\frac{1}{n}\right)^{-\log n} = e^{\log\left(\frac{1}{n}\right)^{-\log n}} = e^{-\log n \log \frac{1}{n}}$$

hence

$$\lim_{n \rightarrow +\infty} \left(\frac{1}{n}\right)^{-\log n} = \lim_{n \rightarrow +\infty} e^{\overbrace{-\log n}^{\rightarrow -\infty} \overbrace{\log \frac{1}{n}}^{\rightarrow -\infty}} = e^{+\infty} = +\infty.$$

Euler's number

3.1 Euler's number e (Napier's constant)

Theorem 3.1

The sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n \quad \text{with } n \geq 1$$

is convergent.

Proof. We will prove that the sequence $\{a_n\}$ is non-decreasing ($a_n \geq a_{n-1}, \forall n \in \mathbb{N}, n \geq 2$) and bounded ($m \leq a_n \leq M, \forall n \in \mathbb{N}, n \geq 1$); hence it is convergent by the monotone sequence theorem. The proof is split into three parts: (1) the monotonicity of $\{a_n\}$ and its lower bound; (2) the monotonicity and the upper bound of an auxiliary sequence $\{b_n\}$; (3) the synthesis of the two results.

Part (1). To prove that $\{a_n\}$ is non-decreasing, we study, for $n \geq 2$, the ratio:

$$\begin{aligned} \frac{a_n}{a_{n-1}} &= \frac{\left(1 + \frac{1}{n}\right)^n}{\left(1 + \frac{1}{n-1}\right)^{n-1}} = \frac{\left(\frac{n+1}{n}\right)^n}{\left(\frac{n}{n-1}\right)^{n-1}} \\ &= \left(\frac{n+1}{n}\right)^n \frac{1}{\left(\frac{n}{n-1}\right)^{n-1}} = \left(\frac{n^2-1}{n^2}\right)^n \frac{1}{\left(\frac{n-1}{n}\right)} \\ &= \frac{\left(1 - \frac{1}{n^2}\right)^n}{1 - \frac{1}{n}} \geq \frac{1 - n \cdot \frac{1}{n^2}}{1 - \frac{1}{n}} = 1 \end{aligned}$$

where for the “ $\geq$ ” we applied Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx \quad \text{with } x = -\frac{1}{n^2} \geq -1 \quad \text{and } n \geq 2$$

Hence we have

$$\frac{a_n}{a_{n-1}} \geq 1$$

i.e., $a_n \geq a_{n-1}$ and the sequence is non-decreasing.

To prove that $\{a_n\}$ is bounded *from below*, we observe that, since $a_1 = 2$ and the sequence is non-decreasing, it follows that $a_n \geq 2, \forall n \geq 1$. $\square$

Proof. Part (2). Consider the auxiliary sequence

$$b_n = \left(1 + \frac{1}{n}\right)^{n+1} \quad \text{with } n \geq 1$$

To prove that $\{b_n\}$ is decreasing, we study, for $n \geq 2$, the ratio:

$$\begin{aligned} \frac{b_n}{b_{n-1}} &= \frac{\left(1 + \frac{1}{n}\right)^{n+1}}{\left(1 + \frac{1}{n-1}\right)^n} = \frac{\left(\frac{n+1}{n}\right)^{n+1}}{\left(\frac{n}{n-1}\right)^n} \\ &= \left(\frac{\frac{n+1}{n}}{\frac{n}{n-1}}\right)^n \left(\frac{n+1}{n}\right) = \frac{1}{\left(\frac{n^2}{n^2-1}\right)^n} \left(\frac{n+1}{n}\right) \\ &= \frac{1}{\left(1 + \frac{1}{n^2-1}\right)^n} \left(\frac{n+1}{n}\right) \leq \frac{1}{\left(1 + \frac{n}{n^2-1}\right)} \left(\frac{n+1}{n}\right) \\ &< \frac{1}{\left(1 + \frac{1}{n}\right)} \left(1 + \frac{1}{n}\right) = 1 \end{aligned}$$

where for the “ $\leq$ ” we applied Bernoulli’s inequality:

$$(1+x)^n \geq 1 + nx \quad \text{with } x = \frac{1}{n^2-1} \geq -1 \quad \text{and } n \geq 2$$

and for the “ $<$ ” we applied the inequality

$$\frac{n}{n^2-1} > \frac{1}{n} \quad \text{since } n^2 > n^2-1 \quad \text{for } n \geq 2$$

Hence we have proved that

$$\frac{b_n}{b_{n-1}} < 1 \quad \text{hence } b_n < b_{n-1}$$

and the sequence $\{b_n\}$ is decreasing.

Since $b_1 = 4$ and the sequence is decreasing, we therefore obtain

$$b_n \leq b_1 = 4, \quad \forall n \geq 1$$

that is, $\{b_n\}$ is bounded from above. $\square$

Proof. Part (3). The two sequences are related by

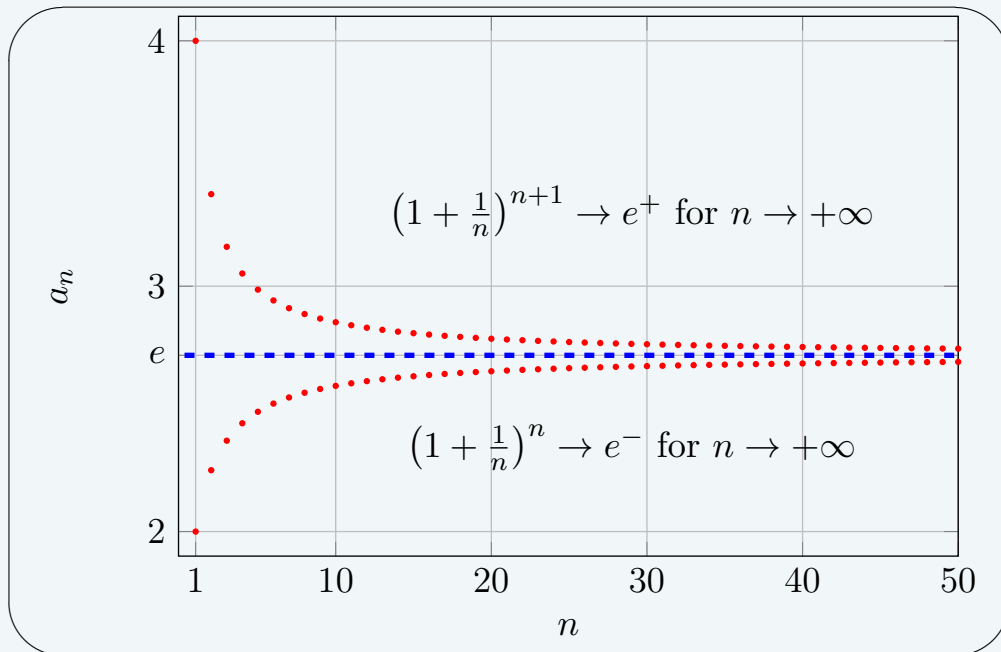
$$b_n = a_n \underbrace{\left(1 + \frac{1}{n}\right)}_{>1, \forall n \geq 1} \text{ therefore } b_n > a_n, \forall n \in \mathbb{N}, n \geq 1$$

and consequently

$$2 \leq a_n < b_n \leq 4, \forall n \geq 1$$

The sequence $\{a_n\}$ is therefore non-decreasing and bounded and, by the monotone sequence theorem, it is convergent. $\square$

Example 3.1: Graphs of the sequences $\{a_n\}$ and $\{b_n\}$



- The limit of the sequence a_n just studied is an **irrational** number that is very important in mathematics. This limit is denoted by the letter e (**Euler's number**, or Napier's constant) and its decimal representation begins as follows:

$$2.7182818284\dots$$

By definition, we have:

$$e = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n \quad (3.1)$$

- This number is very often used as the base of logarithms, which, when this base is used, are called natural or Napierian logarithms (after the mathematician **John Napier**) and are denoted simply by the symbol $\log$ (or $\ln$) without indicating the base.

- Consequently, we have:

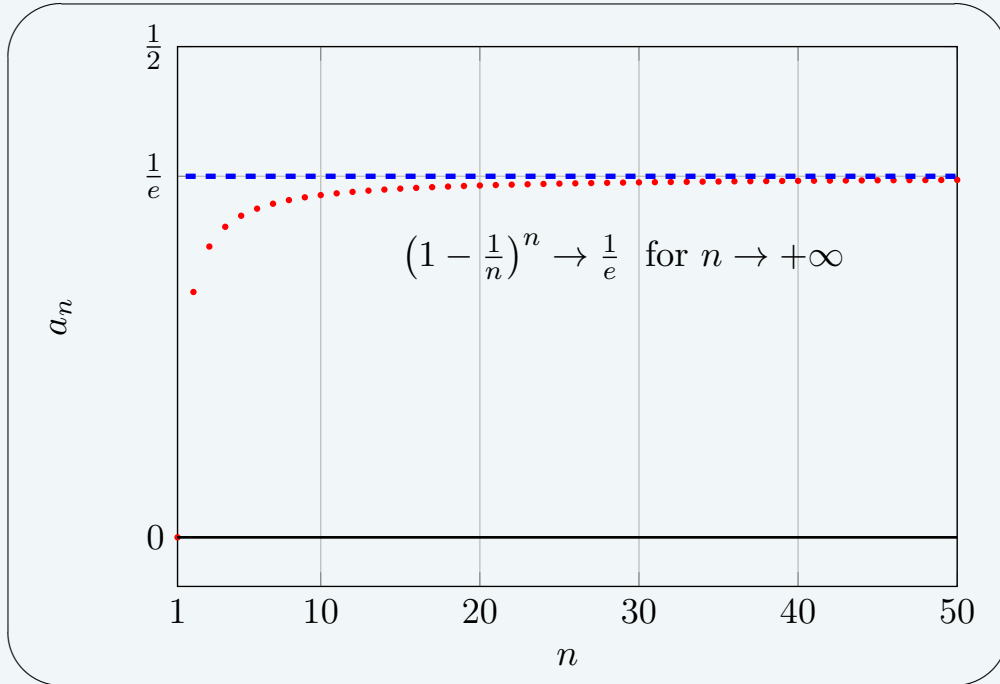
$$\lim_{n \rightarrow +\infty} n \log \left(1 + \frac{1}{n} \right) = \lim_{n \rightarrow +\infty} \log \left(1 + \frac{1}{n} \right)^n = \log e = 1$$

Example 3.2: Computing limits with the sequence tending to e

$$\lim_{n \rightarrow +\infty} \left(1 - \frac{1}{n} \right)^n = \frac{1}{e}$$

Since:

$$\left(\frac{n-1}{n} \right)^n = \frac{1}{\left(\frac{n}{n-1} \right)^n} = \frac{1}{\left(1 + \frac{1}{n-1} \right)^n} = \frac{1}{\underbrace{\left(1 + \frac{1}{n-1} \right)^{n-1}}_{\rightarrow e} \cdot \underbrace{\left(1 + \frac{1}{n-1} \right)}_{\rightarrow 1}}$$


Theorem 3.2

Let $\{c_n\}$ be any divergent sequence (to $+\infty$ or $-\infty$); then

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{c_n} \right)^{c_n} = e, \quad \lim_{n \rightarrow +\infty} \left(1 - \frac{1}{c_n} \right)^{c_n} = \frac{1}{e} \quad (3.2)$$

Proof. Recall that, given $x \in \mathbb{R}$, its integer part, denoted by $\lfloor x \rfloor$, is the largest integer not exceeding x .

If $c_n \rightarrow +\infty$, since $\lfloor c_n \rfloor > c_n - 1$, by comparison we also have $\lfloor c_n \rfloor \rightarrow +\infty$. Hence, by (3.1) and by the definition of limit we have

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{\lfloor c_n \rfloor + 1}\right)^{\lfloor c_n \rfloor + 1} = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{\lfloor c_n \rfloor}\right)^{\lfloor c_n \rfloor} = e$$

Using:

$$\lfloor c_n \rfloor \leq c_n < \lfloor c_n \rfloor + 1$$

we obtain

$$\left(1 + \frac{1}{c_n}\right)^{c_n} < \left(1 + \frac{1}{\lfloor c_n \rfloor}\right)^{\lfloor c_n \rfloor + 1} = \underbrace{\left(1 + \frac{1}{\lfloor c_n \rfloor}\right)^{\lfloor c_n \rfloor}}_{\rightarrow e} \cdot \underbrace{\left(1 + \frac{1}{\lfloor c_n \rfloor}\right)}_{\rightarrow 1}$$

and also

$$\left(1 + \frac{1}{c_n}\right)^{c_n} > \left(1 + \frac{1}{\lfloor c_n \rfloor + 1}\right)^{\lfloor c_n \rfloor} = \underbrace{\left(1 + \frac{1}{\lfloor c_n \rfloor + 1}\right)^{\lfloor c_n \rfloor + 1}}_{\rightarrow e} \cdot \underbrace{\left(1 + \frac{1}{\lfloor c_n \rfloor + 1}\right)^{-1}}_{\rightarrow 1}$$

hence the first limit of the theorem follows from the comparison theorem.

If instead $c_n \rightarrow -\infty$, then the sequence $d_n = -c_n \rightarrow +\infty$, and we have

$$\left(1 + \frac{1}{c_n}\right)^{c_n} = \left(1 - \frac{1}{d_n}\right)^{-d_n} = \left(\frac{d_n}{d_n - 1}\right)^{d_n} = \left(1 + \frac{1}{d_n - 1}\right)^{d_n - 1} \cdot \left(1 + \frac{1}{d_n - 1}\right)$$

Since $d_n - 1 \rightarrow +\infty$, the claim follows from the previous case.

The second limit of the theorem is proved analogously.

□

This theorem is useful for computing limits involving the indeterminate form 1^∞

Example 3.3: Computing limits with the sequence tending to e

We compute

$$\lim_{n \rightarrow +\infty} \left(\frac{n}{3+n}\right)^{5n+1} = [1^\infty]$$

since:

$$\lim_{n \rightarrow +\infty} \frac{n}{3+n} = \lim_{n \rightarrow +\infty} 1 - \frac{3}{3+n} = 1, \quad \lim_{n \rightarrow +\infty} 5n+1 = +\infty$$

We rewrite the sequence as follows:

$$\begin{aligned} \left(\frac{n}{3+n} \right)^{5n+1} &= \left(\frac{3+n}{n} \right)^{-(5n+1)} = \frac{1}{\left(1 + \frac{3}{n} \right)^{5n+1}} \\ &= \frac{1}{\left(\left(1 + \frac{3}{n} \right)^{\frac{n}{3}} \right)^{\frac{3(5n+1)}{n}}} = \frac{1}{\left(\left(1 + \frac{1}{\frac{n}{3}} \right)^{\frac{n}{3}} \right)^{\frac{3(5n+1)}{n}}} \end{aligned}$$

and we consider the denominator

$$\underbrace{\left(\left(1 + \frac{1}{\frac{n}{3}} \right)^{\frac{n}{3}} \right)}_{\rightarrow e}^{\frac{3(5n+1)}{n}} \rightarrow e^{15} \quad \text{since} \quad \frac{n}{3} \rightarrow +\infty \text{ for } n \rightarrow +\infty$$

Summarizing, we have:

$$\lim_{n \rightarrow +\infty} \left(\frac{n}{3+n} \right)^{5n+1} = \frac{1}{e^{15}}$$

Alternative method:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left(\frac{n}{3+n} \right)^{5n+1} &= \lim_{n \rightarrow +\infty} \left(1 - \frac{3}{3+n} \right)^{5n+1} \\ &= \lim_{n \rightarrow +\infty} \left(\underbrace{\left(1 - \frac{1}{\frac{3+n}{3}} \right)^{\frac{3+n}{3}}}_{\rightarrow \frac{1}{e}} \right)^{\frac{3(5n+1)}{3+n}} = \frac{1}{e^{15}} \end{aligned}$$

since

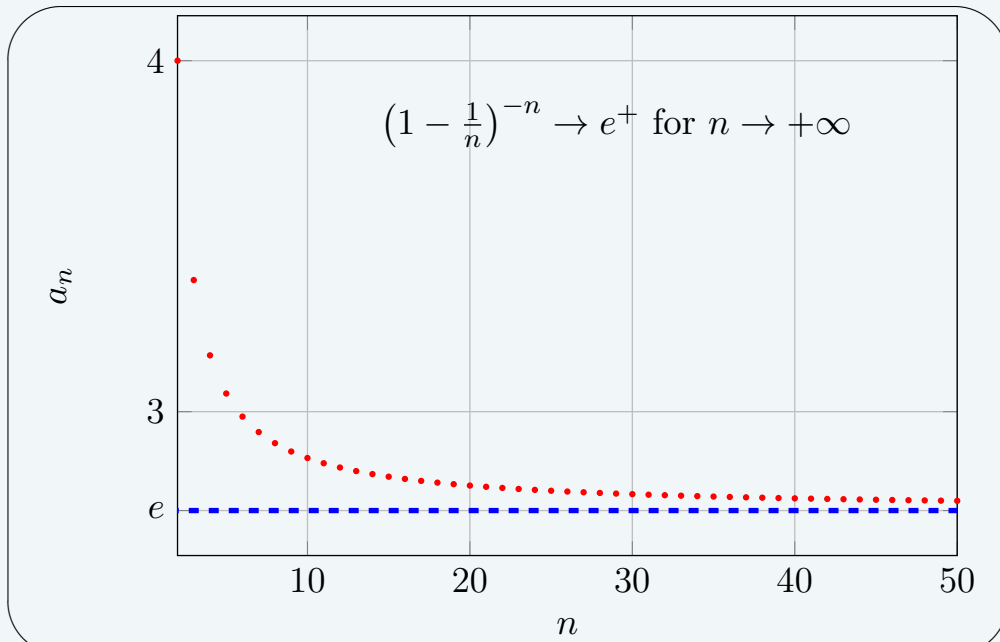
$$\frac{3+n}{3} \rightarrow +\infty \text{ for } n \rightarrow +\infty$$

Example 3.4: Computing limits with the sequence tending to e

$$\lim_{n \rightarrow +\infty} \left(1 - \frac{1}{n}\right)^{-n} = e$$

Since:

$$\left(\frac{n-1}{n}\right)^{-n} = \left(\frac{n}{n-1}\right)^n = \left(1 + \frac{1}{n-1}\right)^n = \underbrace{\left(1 + \frac{1}{n-1}\right)^{n-1}}_{\rightarrow e} \cdot \underbrace{\left(1 + \frac{1}{n-1}\right)}_{\rightarrow 1}$$


Example 3.5: Computing limits with the sequence tending to e

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{\alpha}{n}\right)^n = e^\alpha \quad \text{with } \alpha \in \mathbb{R}$$

If $\alpha = 0$ the result is immediate, since the sequence is constant:

$$\left(1 + \frac{0}{n}\right)^n = 1^n = 1 = e^0$$

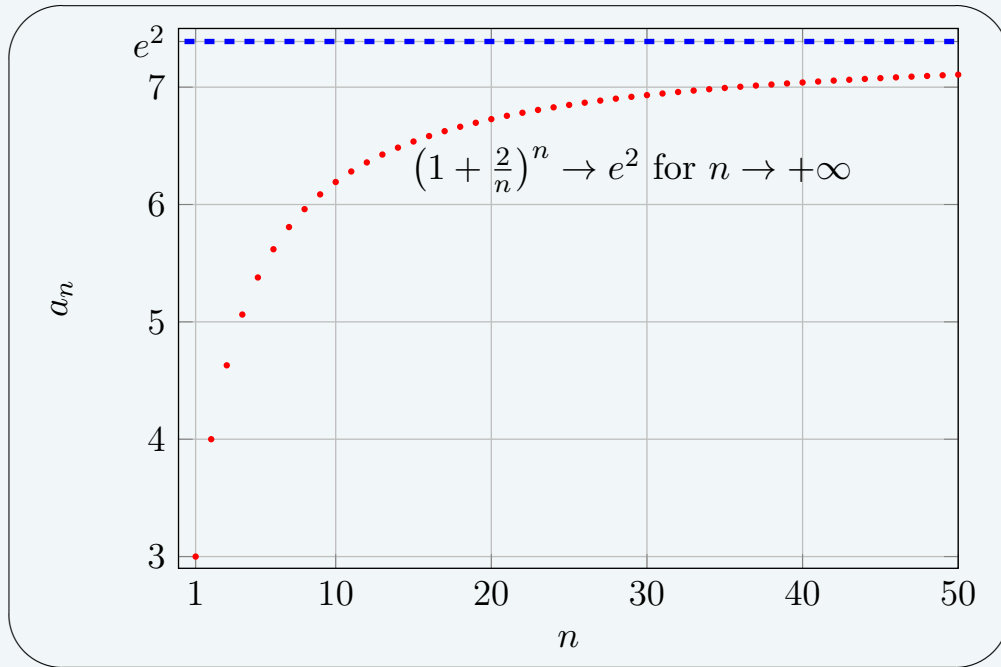
If instead $\alpha \neq 0$ (so that we can divide by α), we have:

$$\left(1 + \frac{\alpha}{n}\right)^n = \left(1 + \frac{1}{\frac{n}{\alpha}}\right)^n = \left(\underbrace{\left(1 + \frac{1}{\frac{n}{\alpha}}\right)^{\frac{n}{\alpha}}}_{\rightarrow e}\right)^\alpha$$

where we used the previous theorem with $c_n = \frac{n}{\alpha}$, a sequence divergent to $+\infty$ if $\alpha > 0$ and to $-\infty$ if $\alpha < 0$.

For example, with $\alpha = 2$ we have:

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{2}{n}\right)^n = e^2$$



3.2 Euler's number in finance

- Suppose we own a capital of unit value and an annual interest rate t . If the interest is paid annually, after one year the capital owned will be

$$1 + t \cdot 1 = 1 + t$$

- If instead the interest is paid monthly, we will have

- after the first month, a capital equal to

$$1 + \frac{t}{12} \cdot 1 = \underbrace{1 + \frac{t}{12}}_{\alpha}$$

- after the second month, a capital equal to

$$\underbrace{1 + \frac{t}{12}}_{\alpha} + \frac{t}{12} \underbrace{\left(1 + \frac{t}{12}\right)}_{\alpha} = \underbrace{\left(1 + \frac{t}{12}\right)^2}_{\beta}$$

- after the third month, a capital equal to

$$\underbrace{\left(1 + \frac{t}{12}\right)^2}_{\beta} + \frac{t}{12} \underbrace{\left(1 + \frac{t}{12}\right)^2}_{\beta} = \left(1 + \frac{t}{12}\right)^3$$

– at the end of the year we will have a capital equal to

$$\left(1 + \frac{t}{12}\right)^{12}$$

- If the interest is computed every n -th of a year, at the end we will have a capital equal to

$$\left(1 + \frac{t}{n}\right)^n$$

- For $t = 1$ (a 100% return) we obtain exactly the sequence that defines e

Hence, even if the interest is paid an infinite number of times per year, the capital does not grow to infinity but tends to e , since:

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Comparisons and asymptotic estimates

4.1 Comparisons and asymptotic estimates

- We have seen that a sequence tending to 0 is an **infinitesimal**; a sequence that diverges (to $+\infty$, to $-\infty$) is called an **infinity** (infinite quantity).
- When two sequences are both infinitesimals or both infinities, it is useful to be able to compare them, to understand which of the two tends “**more rapidly**” to 0 or to infinity.

Example 4.1: Infinities

Examples of infinities are the following sequences:

$$\{\log n\}, \quad \{\sqrt{n}\}, \quad \{n^2\}, \quad \{2^n\}$$

Example 4.2: Infinitesimals

Examples of infinitesimals are the following sequences:

$$\left\{\frac{1}{\log n}\right\}, \quad \left\{\frac{1}{\sqrt{n}}\right\}, \quad \left\{\frac{1}{n^2}\right\}, \quad \left\{\frac{1}{2^n}\right\}$$

- Let $\{a_n\}$ and $\{b_n\}$ be **two infinities**; considering the limit of their ratio we have 4 cases:

$$\lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = \begin{cases} 0 & \text{case 1): } \{a_n\} \text{ is an infinity of } \mathbf{lower\ order} \text{ than } \{b_n\} \\ l \in \mathbb{R}, l \neq 0 & \text{case 2): } \{a_n\} \text{ and } \{b_n\} \text{ are infinities of the } \mathbf{same\ order} \\ \pm\infty & \text{case 3): } \{a_n\} \text{ is an infinity of } \mathbf{higher\ order} \text{ than } \{b_n\} \\ \text{nonexistent} & \text{case 4): } \{a_n\} \text{ and } \{b_n\} \text{ are not comparable} \end{cases}$$

- Let $\{a_n\}$ and $\{b_n\}$ be **two infinitesimals** (with b_n eventually different from zero); considering the limit of their ratio we have 4 cases:

$$\lim_{n \rightarrow +\infty} \frac{a_n}{b_n} = \begin{cases} 0 & \text{case 1): } \{a_n\} \text{ is an infinitesimal of } \mathbf{higher} \text{ order} \\ & \text{than } \{b_n\} \\ l \in \mathbb{R}, l \neq 0 & \text{case 2): } \{a_n\} \text{ and } \{b_n\} \text{ are infinitesimals} \\ & \text{of the } \mathbf{same} \text{ order} \\ \pm\infty & \text{case 3): } \{a_n\} \text{ is an infinitesimal of } \mathbf{lower} \text{ order} \\ & \text{than } \{b_n\} \\ \text{nonexistent} & \text{case 4): } \{a_n\} \text{ and } \{b_n\} \text{ are not comparable} \end{cases}$$

- The case

$$\frac{a_n}{b_n} \rightarrow 1$$

is particularly important: in this case we say that the two sequences $\{a_n\}$ and $\{b_n\}$ are **asymptotic** (asymptotically equivalent).

- To indicate this, we write

$$a_n \sim b_n$$

(read: a_n is asymptotic to b_n)

- The asymptotic symbol is very useful in computing limits, thanks to the following **properties**:

Proposition 4.1: Asymptotic behavior

1. If $a_n \sim b_n$, the two sequences have the same behavior:

- either they converge to the same limit,
- or they both diverge to $\pm\infty$,
- or neither of them has a limit.

2. We can write chains of asymptotic relations, i.e.:

$$\text{if } a_n \sim b_n \sim \dots \sim c_n \quad \text{then} \quad a_n \sim c_n$$

3. An expression made of a product or quotient of several factors can be estimated factor by factor:

$$\text{if } a_n \sim a'_n, b_n \sim b'_n, c_n \sim c'_n \quad \text{then} \quad \frac{a_n b_n}{c_n} \sim \frac{a'_n b'_n}{c'_n}$$

- **Warning:** the same does not hold for sums or for exponentials.

Proof. 1. We prove the first statement

if $a_n \sim b_n$ then $\{a_n\}$ and $\{b_n\}$ have the same behavior

- If $a_n \rightarrow l \in \mathbb{R}$, since

$$b_n = \frac{b_n}{a_n} \cdot a_n \quad \text{and} \quad \frac{b_n}{a_n} \rightarrow 1 \quad (\text{by definition of asymptotic}),$$

then by the theorem on the algebra of limits we have

$$b_n \rightarrow 1 \cdot l = l.$$

- With the same steps, the theorem on the partial arithmetization of the infinity symbol allows us to conclude that

$$\text{if } a_n \rightarrow \pm\infty \quad \text{and} \quad a_n \sim b_n \quad \text{then} \quad b_n \rightarrow \pm\infty.$$

Observing that the asymptotic relation is symmetric, what we have just proved also shows that if $\{b_n\}$ converges (diverges), then $\{a_n\}$ converges (diverges) as well.

- We conclude that if $\{a_n\}$ is irregular, then $\{b_n\}$ is irregular as well, because if, by contradiction, it were not, then by what we have just proved $\{a_n\}$ would also be convergent or divergent.

2. We prove the transitivity of the asymptotic relation:

$$\text{if } a_n \sim b_n \sim c_n \quad \text{then} \quad a_n \sim c_n.$$

The hypotheses mean that

$$\frac{a_n}{b_n} \rightarrow 1 \quad \text{and} \quad \frac{b_n}{c_n} \rightarrow 1$$

Then by the theorem on the algebra of limits we have

$$\frac{a_n}{c_n} = \frac{a_n}{b_n} \cdot \frac{b_n}{c_n} \rightarrow 1.$$

3. The third property is proved analogously. □

Example 4.3: Asymptotic sequences by applying the definition

We prove that:

$$\log n \sim \log(n+1)$$

Factoring out n in the second sequence we obtain

$$\log(n+1) = \log\left(n \left(1 + \frac{1}{n}\right)\right) = \log n + \log\left(1 + \frac{1}{n}\right)$$

Hence

$$\lim_{n \rightarrow +\infty} \frac{\log n}{\log(n+1)} = \lim_{n \rightarrow +\infty} \frac{\log n}{\log n + \underbrace{\log\left(1 + \frac{1}{n}\right)}_{\rightarrow 0}} = 1$$

- A typical way to show that

$$a_n \sim b_n$$

consists in writing

$$a_n = b_n c_n \quad \text{with} \quad c_n \rightarrow 1.$$

That is, decomposing $\{a_n\}$ into the product of a sequence $\{b_n\}$ and a sequence $\{c_n\}$ that tends to 1.

Example 4.4: Asymptotic sequences via the decomposition method

Example:

$$\underbrace{2n^2 + 3n + 1}_{a_n} = \underbrace{2n^2}_{b_n} \underbrace{\left(1 + \frac{3}{2n} + \frac{1}{2n^2}\right)}_{c_n} \sim 2n^2 \quad \text{hence} \quad a_n \sim b_n$$

since

$$\underbrace{\left(1 + \frac{3}{2n} + \frac{1}{2n^2}\right)}_{c_n} \rightarrow 1$$

Example 4.5: Computing limits with asymptotic estimates

We compute the limit

$$\lim_{n \rightarrow +\infty} \frac{2n^3 + 4n + 1}{5(n+1)^3} = \left[\frac{\infty}{\infty} \right] \quad (\text{indeterminate form})$$

Proceeding as before we can write

$$\begin{aligned} \underbrace{2n^3 + 4n + 1}_{a_n} &= \underbrace{2n^3}_{a'_n} \underbrace{\left(1 + \frac{2}{n^2} + \frac{1}{2n^3}\right)}_{\rightarrow 1} \sim 2n^3 \\ 5(n+1)^3 &= 5 \underbrace{(n^3 + 3n^2 + 3n + 1)}_{c_n} = \underbrace{5n^3}_{c'_n} \underbrace{\left(1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}\right)}_{\rightarrow 1} \sim 5n^3 \end{aligned}$$

Using point 3 of Proposition (4.1) on asymptotic behavior we can write:

$$\text{if } a_n \sim a'_n, c_n \sim c'_n \quad \text{then} \quad \frac{a_n}{c_n} \sim \frac{a'_n}{c'_n}$$

and obtain

$$\frac{2n^3 + 4n + 1}{5(n+1)^3} \sim \frac{2n^3}{5n^3}$$

that is, the two sequences have the same behavior. Hence

$$\lim_{n \rightarrow +\infty} \frac{2n^3 + 4n + 1}{5(n+1)^3} = \lim_{n \rightarrow +\infty} \frac{2n^3}{5n^3} = \frac{2}{5}$$

- Another way to show that

$$a_n \sim b_n$$

is to use the **substitution principle**.

- For the logarithm the substitution principle reads as follows: if

$$a_n \sim b_n \quad \text{and} \quad \lim_{n \rightarrow +\infty} b_n = +\infty \quad \text{then} \quad \log a_n \sim \log b_n$$

- Indeed, since $a_n \sim b_n$, we have $\frac{a_n}{b_n} \rightarrow 1$ and therefore $\log \frac{a_n}{b_n} \rightarrow \log 1 = 0$. We can then write:

$$\frac{\log a_n}{\log b_n} = \frac{\log \left(b_n \frac{a_n}{b_n} \right)}{\log b_n} = \frac{\log b_n + \log \frac{a_n}{b_n}}{\log b_n} = 1 + \frac{\overbrace{\log \frac{a_n}{b_n}}^{\rightarrow 0}}{\underbrace{\log b_n}_{\rightarrow +\infty}} \rightarrow 1$$

- **Warning:** it is not enough that two sequences both diverge to $+\infty$ for them to be asymptotic. For example $\log n$ and $\log e^n = n$ both diverge to $+\infty$, but

$$\lim_{n \rightarrow +\infty} \frac{\log n}{n} = 0 \neq 1$$

hence $\log n$ and $\log e^n$ are not asymptotic.

Example 4.6: Computing limits with the substitution principle and asymptotic estimates

We compute the limit

$$\lim_{n \rightarrow +\infty} \log_3(n^2 + 4n + 1)$$

using the substitution principle and asymptotic estimates. With the decomposition method we have

$$\underbrace{n^2 + 4n + 1}_{a_n} = \underbrace{n^2}_{b_n} \underbrace{\left(1 + \frac{4}{n} + \frac{1}{n^2}\right)}_{\rightarrow 1} \sim n^2 \quad \text{and} \quad \lim_{n \rightarrow +\infty} n^2 = +\infty$$

then, by the substitution principle,

$$\log_3(n^2 + 4n + 1) \sim \log_3(n^2) = 2 \log_3 n$$

and hence

$$\lim_{n \rightarrow +\infty} \log_3(n^2 + 4n + 1) = \lim_{n \rightarrow +\infty} 2 \log_3 n = +\infty$$

The fact that the asymptotic relation satisfies the 3 properties:

1. *Reflexive*: $a_n \sim a_n$
2. *Symmetric*: if $a_n \sim b_n$ then $b_n \sim a_n$
3. *Transitive*: if $a_n \sim b_n$ and $b_n \sim c_n$ then $a_n \sim c_n$

means that “**asymptotic**” is an **equivalence relation**.

Hierarchies of infinities and the ratio test

5.1 Hierarchies of infinities for sequences, part 1 and part 2

Theorem 5.1: Hierarchy of infinities (part I)

$$\lim_{n \rightarrow +\infty} \frac{\log_a n}{n^\alpha} = 0 \quad (5.1)$$

for every $a > 1$ and $\alpha > 0$.

Proof. We begin by establishing a useful inequality between any positive real number and its logarithm.

For $x \in \mathbb{R}$, $x > 0$, let k be the integer part of x , i.e., the integer k for which we have

$$k \leq x < k + 1, \quad \text{moreover we have } 2^x \geq 2^k = (1 + 1)^k \geq 1 + k > x,$$

the first inequality follows from the monotonicity of the exponential function, the second from the binomial expansion (or from Bernoulli's inequality). Taking logarithms to base $a > 1$, we obtain

$$\log_a x < x \log_a 2 \quad (\text{for every } x \in \mathbb{R}, x > 0).$$

We now apply this inequality to the number $x = n^{\alpha/2}$, and we have

$$\frac{\alpha}{2} \log_a n < n^{\alpha/2} \log_a 2 \quad \text{and} \quad \frac{\log_a n}{n^{\alpha/2}} < \frac{2}{\alpha} \log_a 2,$$

hence

$$\frac{\log_a n}{n^\alpha} = \frac{\log_a n}{n^{\alpha/2}} \frac{1}{n^{\alpha/2}} \leq \underbrace{\frac{2}{\alpha} \log_a 2}_{\text{constant}} \frac{1}{n^{\alpha/2}} \quad \text{and} \quad \frac{1}{n^{\alpha/2}} \rightarrow 0.$$

By the corollary of the comparison theorem, the claim follows. □

Theorem 5.2: Hierarchy of infinities (part II)

$$\lim_{n \rightarrow +\infty} \frac{n^\alpha}{a^n} = 0 \quad (5.2)$$

for every $a > 1$ and $\alpha > 0$.

Proof. We use the hierarchy of infinities theorem (part I), replacing the integer n with the integer 2^n :

$$0 = \lim_{n \rightarrow +\infty} \frac{\log_a(2^n)}{(2^n)^\alpha} = \lim_{n \rightarrow +\infty} \frac{n \overbrace{\log_a 2}^{\text{constant}}}{(2^\alpha)^n} \text{ hence } \frac{n}{(2^\alpha)^n} \rightarrow 0.$$

If now $a > 1$ is fixed, choosing $\alpha > 0$ such that $2^\alpha = a$ we obtain that $\frac{n}{a^n} \rightarrow 0$, that is, the second relation in the special case where n is raised to the exponent 1. The general case follows from the identity:

$$\frac{n^\alpha}{a^n} = \left(\frac{n}{a^{n/\alpha}} \right)^\alpha = \left(\frac{n}{(a^{1/\alpha})^n} \right)^\alpha$$

Indeed, by the previous result $\frac{n}{(a^{1/\alpha})^n} \rightarrow 0$ (the base $a^{1/\alpha}$ is still a number > 1), hence it follows that

$$\left(\frac{n}{(a^{1/\alpha})^n} \right)^\alpha \rightarrow 0 \text{ and from this the claim.}$$

□

- These limits describe the “speed” at which logarithms (with base > 1), powers (with exponent > 0) and exponentials (with base > 1) go to infinity. Logarithms with base > 1 go more slowly than any power with exponent > 0 ; powers with exponent > 0 go more slowly than any exponential with base > 1 .
- Exponentials with base > 1 are infinities of higher order than powers with exponent > 0 and than logarithms with base > 1 . Powers with exponent > 0 are infinities of higher order than logarithms with base > 1 .

Example 5.1: Computing limits using the hierarchy of infinities

We compute the limit

$$\lim_{n \rightarrow +\infty} \sqrt[n]{n} = [\infty^0] \quad (\text{indeterminate form})$$

Now we write

$$\sqrt[n]{n} = n^{\frac{1}{n}} = e^{\log n^{1/n}} = e^{\frac{\log n}{n}}$$

and studying the sequence in the exponent

$$a_n = \frac{\log n}{n} \quad \text{we have} \quad a_n \rightarrow 0$$

thanks to the hierarchy of infinities theorem. Hence we have:

$$\lim_{n \rightarrow +\infty} \sqrt[n]{n} = \lim_{n \rightarrow +\infty} e^{\frac{\log n}{n}} = 1$$

Example 5.2: Computing limits using the hierarchy of infinities

We compute the limit

$$\lim_{n \rightarrow +\infty} \frac{2^n + n}{2^{n+1}} = \left[\frac{\infty}{\infty} \right] \quad (\text{indeterminate form})$$

We can write:

$$\underbrace{2^n + n}_{a_n} = \underbrace{2^n}_{a'_n} \underbrace{\left(1 + \frac{n}{2^n}\right)}_{\rightarrow 1} \sim 2^n$$

since, thanks to the hierarchy of infinities theorem, we have

$$\frac{n}{2^n} \rightarrow 0,$$

i.e., 2^n is an infinity of higher order than n . Hence we can write:

$$\frac{2^n + n}{2^{n+1}} \sim \frac{2^n}{2^{n+1}}$$

Now, since

$$\frac{2^n}{2^{n+1}} = \frac{2^n}{2^n \cdot 2} = \frac{1}{2}$$

then

$$\lim_{n \rightarrow +\infty} \frac{2^n + n}{2^{n+1}} = \frac{1}{2}$$

5.2 The ratio test theorem

Theorem 5.3: Ratio test

Let $\{a_n\}$ be a positive sequence (i.e., $a_n > 0$ for every n).

$$\text{If there exists } \lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = l \quad \text{and} \quad \begin{cases} l < 1, & \text{then } a_n \rightarrow 0 \\ l > 1 \text{ (or } l = +\infty), & \text{then } a_n \rightarrow +\infty \end{cases}$$

- The previous theorem reduces the study of the limit of a positive sequence $\{a_n\}$ to the

computation of the limit of another sequence, the sequence (of ratios)

$$n \mapsto \frac{a_{n+1}}{a_n}$$

In some cases the latter is easier to study than the original one, as we will see in the examples.

Note that in the case $l = 1$ the theorem does not allow us to conclude anything.

Proof.

1. Suppose that:

$$\frac{a_{n+1}}{a_n} \rightarrow l < 1$$

Then, for every $\varepsilon > 0$, we have, for $n \geq n(\varepsilon)$,

$$\frac{a_{n+1}}{a_n} < l + \varepsilon$$

We can then write the chain of inequalities:

$$a_{n(\varepsilon)+1} < (l + \varepsilon) a_{n(\varepsilon)}, \quad a_{n(\varepsilon)+2} < (l + \varepsilon) \underbrace{a_{n(\varepsilon)+1}}_{< (l+\varepsilon) a_{n(\varepsilon)}} < (l + \varepsilon)^2 a_{n(\varepsilon)}, \quad \dots$$

hence

$$a_{n(\varepsilon)+k} < (l + \varepsilon)^k a_{n(\varepsilon)}$$

Choosing ε small enough to have $l + \varepsilon < 1$, we have

$$(l + \varepsilon)^k \rightarrow 0 \quad \text{for } k \rightarrow +\infty$$

On the other hand, $n(\varepsilon)$ is fixed and consequently $a_{n(\varepsilon)}$ is fixed as well; hence for k large enough the right-hand side (and therefore the left-hand side) is as small as we like. This proves the first claim, that is: $a_n \rightarrow 0$.

2. Suppose that:

$$\frac{a_{n+1}}{a_n} \rightarrow l > 1$$

Then, for every $\varepsilon > 0$, we have, for $n \geq n(\varepsilon)$,

$$\frac{a_{n+1}}{a_n} > l - \varepsilon$$

We choose ε small enough to have $l - \varepsilon > 1$; with steps similar to before we can write

$$a_{n(\varepsilon)+k} > (l - \varepsilon)^k a_{n(\varepsilon)} \quad \text{and} \quad (l - \varepsilon)^k \rightarrow +\infty \quad \text{for } k \rightarrow +\infty.$$

On the other hand, $n(\varepsilon)$ is fixed and consequently $a_{n(\varepsilon)}$ is fixed as well; hence for k large enough the right-hand side (and therefore the left-hand side) is as large as we like. This proves the second claim, that is: $a_n \rightarrow +\infty$.

□

Example 5.3: Using the ratio test theorem

Let us try to compute, with the ratio test, the limit

$$\lim_{n \rightarrow +\infty} \frac{\log n}{n} = \left[\frac{\infty}{\infty} \right] \quad (\text{indeterminate form})$$

we have

$$\frac{a_{n+1}}{a_n} = \frac{\log(n+1)}{n+1} \cdot \frac{n}{\log n} = \frac{n}{n+1} \cdot \frac{\log(n+1)}{\log n}$$

we have

$$\frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \underbrace{\frac{1}{n+1}}_{\rightarrow 0} \quad \text{hence} \quad \lim_{n \rightarrow +\infty} \frac{n}{n+1} = 1$$

moreover we have

$$\frac{\log(n+1)}{\log n} \sim \frac{\log n}{\log n} = 1 \quad \text{hence} \quad \lim_{n \rightarrow +\infty} \frac{\log(n+1)}{\log n} = 1$$

where $\log(n+1) \sim \log(n)$ by the substitution principle, since $n+1 \sim n$ and $\lim_{n \rightarrow +\infty} n = +\infty$. Now, using the theorem on the algebra of limits, we have

$$\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = 1 \cdot 1 = 1$$

hence Theorem (5.3) (ratio test) does not allow us, in this case, to conclude anything.

5.3 Hierarchies of infinities for sequences, part 3 and part 4

Theorem 5.4: Hierarchy of infinities (part III)

$$\lim_{n \rightarrow +\infty} \frac{a^n}{n!} = 0 \quad (5.3)$$

for every $a > 0$.

- hence exponentials with base > 0 go more slowly than the factorial

Proof. We apply the ratio test to the sequence

$$b_n = \frac{a^n}{n!} \quad \text{we have} \quad \frac{b_{n+1}}{b_n} = \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n} = \frac{a}{n+1} \quad \text{and} \quad \frac{a}{n+1} \rightarrow 0$$

Using the ratio test theorem, we obtain the claim. $\square$

Theorem 5.5: Hierarchy of infinities (part IV)

$$\lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0 \quad (5.4)$$

- hence the factorial goes more slowly than n^n

Proof. We apply the ratio test to the sequence

$$b_n = \frac{n!}{n^n}$$

we have

$$\begin{aligned} \frac{b_{n+1}}{b_n} &= \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1) \cdot n!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \\ &= \frac{(n+1) \cdot n^n}{(n+1)^n \cdot (n+1)} = \left(\frac{n}{n+1} \right)^n = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \end{aligned}$$

and since

$$\left(1 + \frac{1}{n}\right)^n \rightarrow e$$

then

$$\frac{b_{n+1}}{b_n} \rightarrow \frac{1}{e} < 1$$

Using the ratio test theorem, we obtain the claim. □

Recursively defined sequences

6.1 Recursively defined sequences

- We begin with the following problem:

The bank lends us a_0 euros, with a management fee of b euros per year, at a fixed annual interest rate r ($r \in (0, 1)$). How much will we have to pay back after n years?

- After one year, the amount to be paid back will be

$$a_1 = a_0 + r a_0 + b = (1 + r) a_0 + b$$

after two years

$$a_2 = a_1 + r a_1 + b = (1 + r) a_1 + b$$

after three years

$$a_3 = a_2 + r a_2 + b = (1 + r) a_2 + b$$

and so on... Each year the amount is computed using the amount of the previous year. We therefore have the sequence:

$$\underbrace{a_0}_{\text{given value}} \quad \text{given,} \quad \underbrace{a_{n+1} = (1 + r) a_n + b}_{\text{recursive relation}} \quad \text{with } n \in \mathbb{N}$$

This is a very simple case of a **recursively defined sequence**, in which each term is defined starting from the previous one (or the previous ones).

- In this case it is easy to convince ourselves that the sequence can be rewritten in **closed form** (i.e., explicitly).

After one year:

$$a_1 = (1 + r) a_0 + b$$

after two years:

$$\begin{aligned} a_2 &= (1 + r) a_1 + b = (1 + r) ((1 + r) a_0 + b) + b \\ &= (1 + r)^2 a_0 + b (1 + (1 + r)) \end{aligned}$$

after three years:

$$\begin{aligned} a_3 &= (1+r) a_2 + b = (1+r) \left((1+r)^2 a_0 + b (1 + (1+r)) \right) + b \\ &= (1+r)^3 a_0 + b (1 + (1+r) + (1+r)^2) \end{aligned}$$

and so on ... hence

$$a_n = s^n a_0 + b (1 + s + s^2 + \dots + s^{n-1}), \text{ where } s = (1+r) > 1$$

We can therefore easily compute the limit as follows:

$$\lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow +\infty} s^n a_0 + b (1 + s + s^2 + \dots + s^{n-1}) = +\infty$$

However, it is not always possible to make explicit a sequence given in iterative form. In that case, we are left with the **problem of understanding its behavior and its limit**.

Example 6.1: Behavior of a recursively defined sequence

Given $b > 0$, we determine the behavior of the sequence defined by

$$a_0 = b, \quad a_{n+1} = \frac{a_n}{1 + a_n} \text{ with } n \in \mathbb{N}$$

All the terms of the sequence are positive, hence the denominator of the fraction appearing in the definition never vanishes.

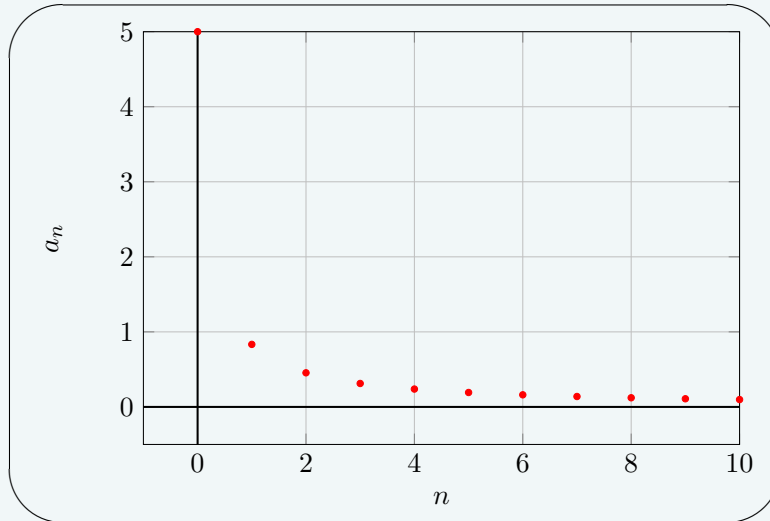
To verify this we proceed by induction:

$$a_0 = b > 0, \text{ and if } a_n > 0 \text{ then } a_{n+1} = \frac{a_n}{1 + a_n} > 0 \text{ (quotient of two positive numbers)}$$

Consequently, all the terms of the sequence after the initial one are less than 1, since $a_n < 1 + a_n$.

For example, with $b = 5$, we have:

$$a_0 = 5, \quad a_1 = \frac{5}{6}, \quad a_2 = \frac{\frac{5}{6}}{1 + \frac{5}{6}} = \frac{5}{11}, \quad a_3 = \frac{\frac{5}{11}}{1 + \frac{5}{11}} = \frac{5}{16} \dots$$



Example 6.2: Behavior of a recursively defined sequence

Let us try to understand whether or not the sequence is monotone. We know that $a_n > 0, \forall n \in \mathbb{N}$; then the inequality $a_n \leq a_{n+1}$ becomes

$$a_n \leq a_{n+1} \iff a_n \leq \frac{a_n}{1 + a_n} \iff 1 + a_n \leq 1 \iff a_n \leq 0$$

which we know to be false; hence the opposite inequality holds, i.e., the sequence is decreasing ($a_n > a_{n+1}, \forall n \in \mathbb{N}$).

Since it is also bounded below (by zero), a_n converges to a finite non-negative limit, which we call $\ell \geq 0$.

We pass to the limit in the recursive relation.

$$\underbrace{\lim_{n \rightarrow +\infty} a_{n+1}}_{=\ell} = \underbrace{\lim_{n \rightarrow +\infty} \frac{a_n}{1 + a_n}}_{=\frac{\ell}{1+\ell}} \quad \text{hence} \quad \ell = \frac{\ell}{1 + \ell} \quad \text{and consequently} \quad \ell = 0$$

since

$$\ell - \frac{\ell}{1 + \ell} = 0, \quad \frac{\ell(1 + \ell) - \ell}{1 + \ell} = 0, \quad \frac{\ell^2}{1 + \ell} = 0 \iff \ell = 0$$

Example 6.3: Closed formula of the recurrence

Given $b > 0$, we look for the closed formula of the sequence:

$$a_0 = b, \quad a_{n+1} = \frac{a_n}{1 + a_n} \quad \text{for } n \in \mathbb{N}$$

With $n = 1$, we have

$$a_1 = \frac{a_0}{1 + a_0}$$

With $n = 2$, we have

$$a_2 = \frac{a_1}{1 + a_1} = \frac{\frac{a_0}{1+a_0}}{1 + \frac{a_0}{1+a_0}} = \frac{\frac{a_0}{1+a_0}}{\frac{1+a_0}{1+a_0}} = \frac{a_0}{1+a_0} \frac{1+a_0}{1+2a_0} = \frac{a_0}{1+2a_0}$$

With $n = 3$, we have

$$a_3 = \frac{a_2}{1 + a_2} = \frac{\frac{a_0}{1+2a_0}}{1 + \frac{a_0}{1+2a_0}} = \frac{\frac{a_0}{1+2a_0}}{\frac{1+3a_0}{1+2a_0}} = \frac{a_0}{1+2a_0} \frac{1+2a_0}{1+3a_0} = \frac{a_0}{1+3a_0}$$

and so on ... hence

$$a_n = \frac{a_0}{1 + n a_0} \quad \text{with } n \in \mathbb{N} \quad \text{and we have } \lim_{n \rightarrow +\infty} \frac{a_0}{1 + n a_0} = 0$$

The previous example illustrates a good two-step strategy for studying the behavior of recursively defined sequences:

1. we try to understand whether the sequence satisfies some monotonicity property;
2. if so, we examine the possible limits, taking into account the recursive relation.

6.2 Heron's algorithm

- Given $b > 0$ and $c > 0$, we study the following recursively defined sequence:

$$a_0 = b, \quad a_{n+1} = \frac{1}{2} \left(a_n + \frac{c}{a_n} \right) \quad \text{with } n \in \mathbb{N}$$

It is evident that the sequence consists of positive numbers. To verify this we proceed by induction:

$$a_0 = b > 0, \quad \text{and if } a_n > 0 \quad \text{then } a_{n+1} = \frac{1}{2} \left(a_n + \frac{c}{a_n} \right) = \frac{1}{2} \left(\frac{a_n^2 + c}{a_n} \right) > 0$$

since we have the quotient of two positive numbers.

Let us try to understand whether or not the sequence is monotone. We know that $a_n > 0, \forall n \in \mathbb{N}$; the inequality $a_{n+1} < a_n$ becomes

$$a_{n+1} < a_n \iff \frac{1}{2} \left(a_n + \frac{c}{a_n} \right) < a_n \iff c < a_n^2 \iff a_n > \sqrt{c}$$

since

$$\frac{1}{2} \left(a_n + \frac{c}{a_n} \right) < a_n, \quad \frac{a_n^2 + c}{a_n} < 2 a_n, \quad a_n^2 + c < 2 a_n^2, \quad c < a_n^2$$

hence the sequence is decreasing if and only if:

$$a_n > \sqrt{c}$$

We now need to understand whether or not this condition is true. We have

$$a_{n+1} = \frac{1}{2} \left(a_n + \frac{c}{a_n} \right) > \sqrt{c} \iff (a_n - \sqrt{c})^2 > 0 \iff a_n \neq \sqrt{c}$$

since

$$\frac{1}{2} \left(a_n + \frac{c}{a_n} \right) - \sqrt{c} > 0, \quad \frac{a_n^2 + c}{2 a_n} - \sqrt{c} > 0, \quad \frac{a_n^2 + c - 2 a_n \sqrt{c}}{2 a_n} > 0, \quad \frac{(a_n - \sqrt{c})^2}{2 a_n} > 0$$

hence $a_n > \sqrt{c}$ for every $n \geq 1$ if $\underbrace{b}_{=a_0} \neq \sqrt{c}$, whereas, if $b = \sqrt{c}$, the recursive relation

gives $a_1 = \frac{1}{2} \left(\sqrt{c} + \frac{c}{\sqrt{c}} \right) = \sqrt{c}$ and, by induction, $a_n = \sqrt{c}$ for every $n \geq 1$. In either case we therefore have

$$a_n \geq \sqrt{c}, \quad \forall b > 0 \text{ and } n \geq 1$$

1. If $b > \sqrt{c}$ the sequence is decreasing and is **bounded from below** by $\sqrt{c}$, since $a_0 = b > \sqrt{c}$ and $a_n > \sqrt{c}$ for every $n \geq 1$. By the monotone sequence theorem it therefore has a finite limit

$$\ell = \inf \{a_n : n \in \mathbb{N}\} \geq \sqrt{c} > 0$$

in particular $\ell > 0$ and we may divide by ℓ .

To identify it, we pass to the limit in the recursive relation, obtaining

$$\underbrace{\lim_{n \rightarrow +\infty} a_{n+1}}_{=\ell} = \underbrace{\lim_{n \rightarrow +\infty} \frac{1}{2} \left(a_n + \frac{c}{a_n} \right)}_{=\frac{1}{2}(\ell + \frac{c}{\ell})}$$

$$\text{hence } \ell = \frac{1}{2} \left(\ell + \frac{c}{\ell} \right) \text{ and consequently } \ell = \sqrt{c}$$

since

$$\ell - \frac{1}{2} \left(\ell + \frac{c}{\ell} \right) = 0, \quad \ell - \frac{1}{2} \left(\frac{\ell^2 + c}{\ell} \right) = 0, \quad \left(\frac{2\ell^2 - \ell^2 - c}{2\ell} \right) = 0, \quad \left(\frac{\ell^2 - c}{2\ell} \right) = 0$$

that is $\ell^2 = c$ and, since $\ell > 0$, we obtain $\ell = \sqrt{c}$ (and not $\ell = -\sqrt{c}$).

Therefore, if $b > \sqrt{c}$, the whole sequence converges to $\sqrt{c}$ while decreasing.

2. If instead $0 < b < \sqrt{c}$, then by what we have seen before we will have $a_1 > \sqrt{c}$, and from that point on the sequence will start decreasing, again tending to $\sqrt{c}$.

3. Finally, if $b = \sqrt{c}$, the sequence remains constant, since

$$a_1 = \frac{1}{2} \left(\sqrt{c} + \frac{c}{\sqrt{c}} \right) = \frac{1}{2} \frac{\sqrt{c}\sqrt{c} + c}{\sqrt{c}} = \frac{1}{2} \frac{2c}{\sqrt{c}} = \frac{\sqrt{c}\sqrt{c}}{\sqrt{c}} = \sqrt{c}$$

and all the other terms a_n , with $n > 1$, will consequently have the same value $\sqrt{c}$.

- Moreover, $\forall b > 0$ and $n \geq 1$ we have:

$$\frac{c}{a_n} \leq \sqrt{c} \quad \text{since} \quad a_n \geq \sqrt{c}, \quad c = \sqrt{c} \cdot \underbrace{\sqrt{c}}_{\leq a_n} \quad \text{and} \quad c \leq \sqrt{c} \cdot a_n$$

Summarizing, $\forall b > 0$ and $n \geq 1$ we have:

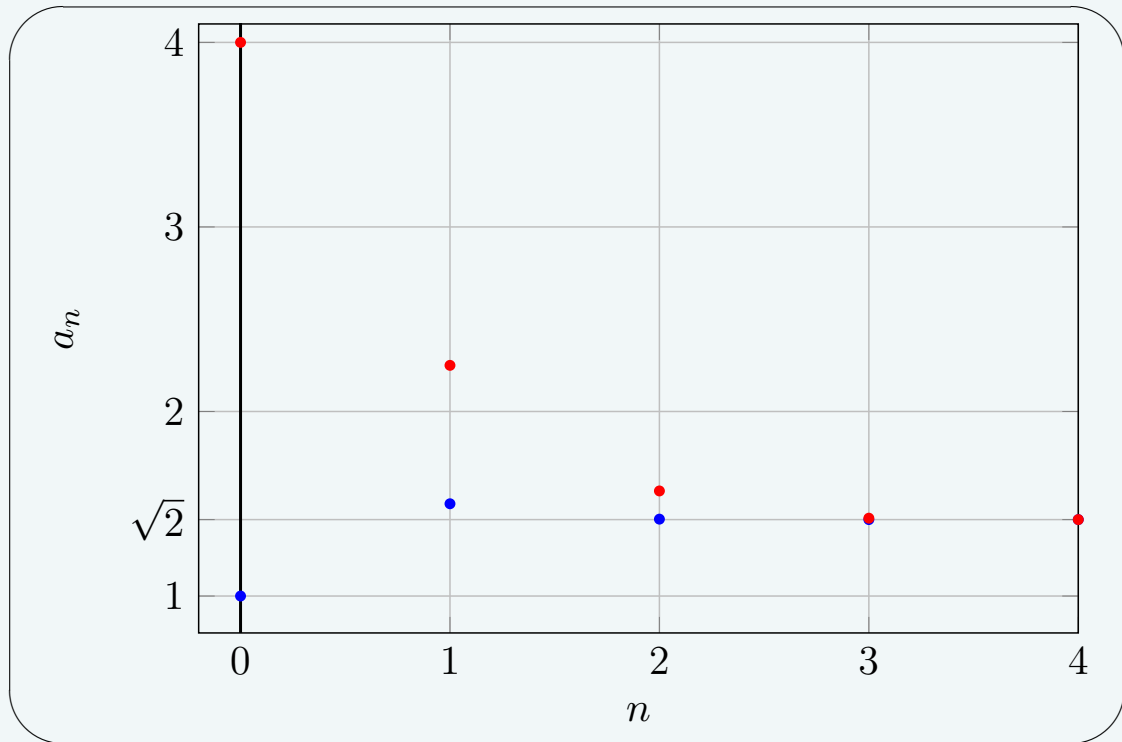
$$\frac{c}{a_n} \leq \sqrt{c} \leq a_n, \quad a_n \rightarrow \sqrt{c} \quad \text{and also} \quad \frac{c}{a_n} \rightarrow \sqrt{c} \quad \text{since} \quad c = \sqrt{c}\sqrt{c}$$

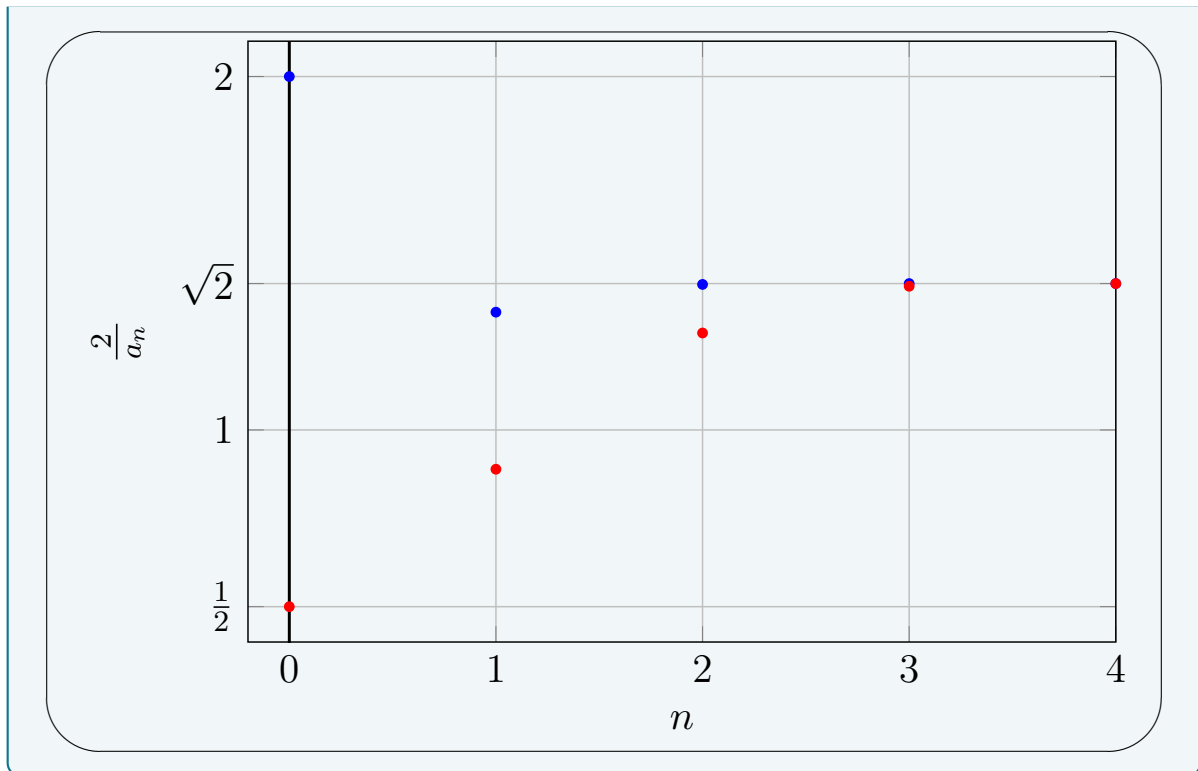
The inequalities are strict for every $b \neq \sqrt{c}$ and hold as equalities for $b = \sqrt{c}$.

For every choice of $b > 0$ the sequence a_n tends to $\sqrt{c}$ and can therefore be used to approximate the square root.

Example 6.4: Behavior of the sequence

For example, with $c = 2$ and $b = 4 > \sqrt{2}$ (red points) or $b = 1 < \sqrt{2}$ (blue points), we have:





The recursive computation of the values of the sequence just seen is called Heron's **algorithm**, originally proposed to compute **the side of a square of area c** .

- The steps of the algorithm are:
 1. Having fixed the base equal to a_0 (the initial value b), we construct the associated rectangle of area c . Its height will be $\frac{c}{a_0}$, since the area of the rectangle is fixed equal to that of the square, that is:

$$c = \left(a_0 \cdot \frac{c}{a_0} \right)$$

2. Since the side of the square we are looking for lies between a_0 and $\frac{c}{a_0}$, we take as the new approximation a_1 the average of a_0 and $\frac{c}{a_0}$:

$$a_1 = \frac{1}{2} \left(a_0 + \frac{c}{a_0} \right)$$

which becomes the new base of the rectangle. Then we take a_1 in place of a_0 and repeat, and so on.

For example, let us try to compute the side of a square of area $c = 2$; in other words, we try to estimate the value $\sqrt{2}$.

At the **first iteration** (with $n=0$) we consider, for example, a rectangle with base $a_0 = b = 4$ (the first upper **approximation** of $\sqrt{2}$); consequently, the height is $\frac{c}{a_0} = \frac{1}{2}$ (the first lower approx. of $\sqrt{2}$). Hence: $\frac{1}{2} < \sqrt{2} < 4$.

We compute the value a_1 :

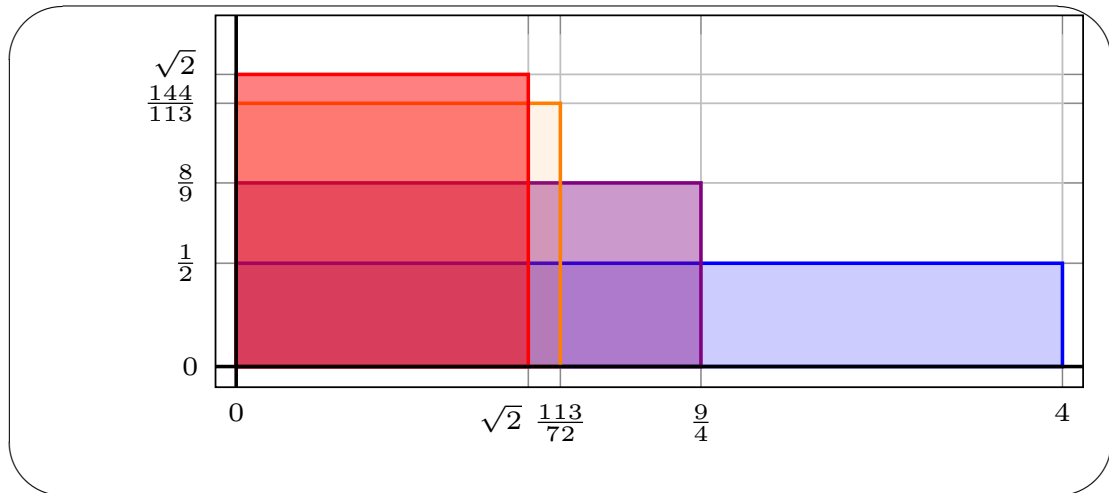
$$a_1 = \frac{1}{2} \left(4 + \frac{1}{2} \right) = \frac{9}{4}$$

At the **second iteration** (with $n=1$) we have a rectangle with base $a_1 = \frac{9}{4}$ (the second upper approx. of $\sqrt{2}$); consequently, the height is $\frac{c}{a_1} = \frac{2}{9/4} = \frac{8}{9}$ (the second lower approx. of $\sqrt{2}$). Hence: $\frac{8}{9} < \sqrt{2} < \frac{9}{4}$.

We compute the value a_2 :

$$a_2 = \frac{1}{2} \left(\frac{9}{4} + \frac{8}{9} \right) = \frac{113}{72}$$

At the **third iteration** (with $n=2$) we have a rectangle with base $a_2 = \frac{113}{72}$ (the third upper approx. of $\sqrt{2}$); consequently, the height is $\frac{c}{a_2} = \frac{2}{113/72} = \frac{144}{113}$ (the third lower approx. of $\sqrt{2}$). Hence: $\frac{144}{113} < \sqrt{2} < \frac{113}{72}$.



We compute the value a_3 :

$$a_3 = \frac{1}{2} \left(\frac{113}{72} + \frac{144}{113} \right) = \frac{23,137}{16,272}$$

At the **fourth iteration** (with $n=3$) we have a rectangle with base $a_3 = \frac{23,137}{16,272}$ (the fourth upper approx. of $\sqrt{2}$); consequently, the height is $\frac{c}{a_3} = \frac{2}{23,137/16,272} = \frac{32,544}{23,137}$ (the fourth lower approx. of $\sqrt{2}$). Hence: $\frac{32,544}{23,137} < \sqrt{2} < \frac{23,137}{16,272}$.

Already at the fourth iteration we have a fairly good estimate of $\sqrt{2}$:

$$\frac{23137}{16272} = 1.421890363\dots \quad \text{and} \quad \sqrt{2} = 1.414213562\dots$$

that is, an estimate that is correct up to the first decimal digit.

Finally, we compute the value a_4 :

$$a_4 = \frac{1}{2} \left(\frac{23,137}{16,272} + \frac{32,544}{23,137} \right) = \frac{1,064,876,737}{752,970,528}$$

With the new estimate we have:

$$\frac{1,064,876,737}{752,970,528} = 1.414234285\dots \quad \text{and} \quad \sqrt{2} = 1.414213562\dots$$

which is correct up to the fourth decimal digit, i.e., with just one additional iteration we have fixed 3 more decimal digits.

6.2.1 Error estimates for Heron's algorithm

- The **relative error** $\tilde{\varepsilon}_n$ at iteration n is:

$$\tilde{\varepsilon}_n = \frac{|a_n - \sqrt{c}|}{\sqrt{c}}$$

we have $\sqrt{c} \leq a_n, \forall b > 0$ and $n \geq 1$, hence we can simply consider:

$$\tilde{\varepsilon}_n = \frac{a_n - \sqrt{c}}{\sqrt{c}}$$

The value of $\sqrt{c}$ is unknown, hence we need a way to analyze the values of the relative errors that is independent of $\sqrt{c}$.

We first consider the **absolute error**:

$$\varepsilon_n = a_n - \sqrt{c}$$

we know that

$$\frac{c}{a_n} \leq \sqrt{c} \leq a_n, \quad \forall b > 0, n \geq 1$$

hence we can obtain an estimate of the absolute error ε_n made at iteration n as follows:

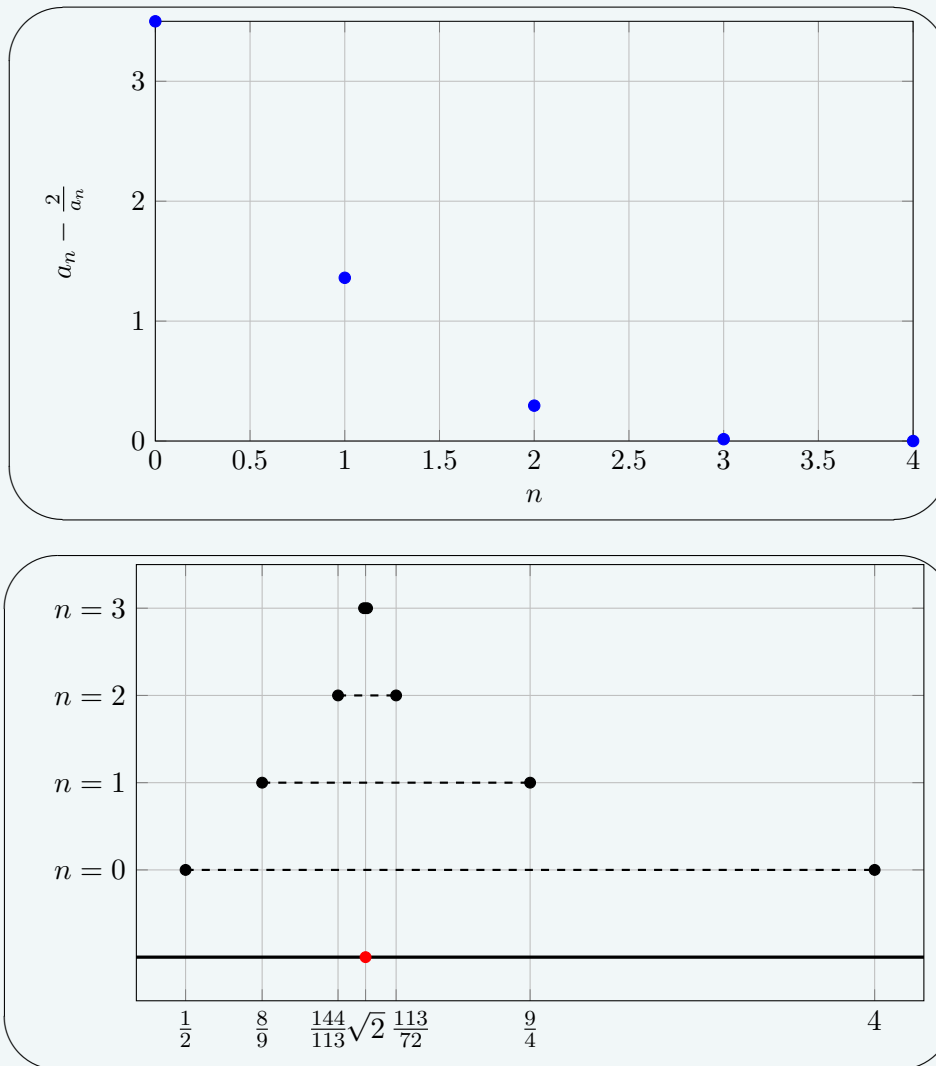
$$\varepsilon_n = a_n - \sqrt{c} \leq a_n - \frac{c}{a_n} \quad \text{and} \quad \varepsilon_n \rightarrow 0 \quad \text{for} \quad n \rightarrow +\infty \quad \text{since} \quad a_n \rightarrow \sqrt{c}$$

Example 6.5: Error estimates for Heron's algorithm ($\sqrt{2} = 1.414213562\dots$)

We go back to the previous example with $c = 2$ and $b = 4$, and we have:

iter.	$\frac{2}{a_n}$	a_n	estimate of $\sqrt{2}$	ε_n
$n = 0$	$\frac{1}{2}$	4	4	$< \frac{7}{2} = 3.5$
$n = 1$	$\frac{8}{9}$	$\frac{9}{4}$	2.25	$< \frac{49}{36} = 1.361\dots$
$n = 2$	$\frac{144}{113}$	$\frac{113}{72}$	1.569444444...	$< \frac{2,401}{8,136} = 0.295\dots$
$n = 3$	$\frac{32,544}{23,137}$	$\frac{23,137}{16,272}$	1.421890363...	$< \frac{5,764,801}{376,485,264} = 0.0153\dots$
$n = 4$	$\frac{1,505,941,056}{1,064,876,737}$	$\frac{1,064,876,737}{752,970,528}$	1.414234285...	$< \frac{33,232,930,569,601}{801,820,798,913,807,136} = 0.000041\dots$

Example 6.6: Errors and intervals considered by Heron's algorithm



We now want to understand how “fast” the error decreases by comparing the relative errors of two consecutive iterations. This estimate gives us information about the quality of the algorithm, i.e., we want to determine the “**rate of convergence**” of the algorithm.

We begin by observing that the value $\sqrt{c}$ can be written as follows:

$$\sqrt{c} = \frac{1}{2} \left(\sqrt{c} + \frac{c}{\sqrt{c}} \right) \quad \text{since} \quad \frac{1}{2} \left(\sqrt{c} + \frac{c}{\sqrt{c}} \right) = \frac{1}{2} \frac{\sqrt{c}\sqrt{c} + c}{\sqrt{c}} = \frac{1}{2} \frac{2c}{\sqrt{c}} = \frac{\sqrt{c}\sqrt{c}}{\sqrt{c}} = \sqrt{c}$$

We now establish the relation between ε_{n+1} and ε_n ; we have:

$$\begin{aligned} \varepsilon_{n+1} &= a_{n+1} - \sqrt{c} = \frac{1}{2} \left(a_n + \frac{c}{a_n} \right) - \frac{1}{2} \left(\sqrt{c} + \frac{c}{\sqrt{c}} \right) \\ &= \frac{1}{2} \left(a_n - \sqrt{c} + \frac{c}{a_n} - \frac{c}{\sqrt{c}} \right) = \frac{1}{2} \left(\varepsilon_n + \frac{c}{a_n} - \frac{c}{\sqrt{c}} \right) \\ &= \frac{1}{2} \left(\varepsilon_n + \frac{c(\sqrt{c} - a_n)}{\sqrt{c} a_n} \right) = \frac{1}{2} \left(\varepsilon_n - \frac{c \varepsilon_n}{\sqrt{c} a_n} \right) = \frac{1}{2} \left(\varepsilon_n - \frac{\sqrt{c} \varepsilon_n}{a_n} \right) \\ &= \frac{1}{2} \varepsilon_n \left(1 - \frac{\sqrt{c}}{a_n} \right) = \frac{1}{2} \varepsilon_n \left(\frac{a_n - \sqrt{c}}{a_n} \right) \\ &= \frac{1}{2} \frac{\varepsilon_n^2}{a_n} \end{aligned}$$

We now look for the relation between $\tilde{\varepsilon}_{n+1}$ and $\tilde{\varepsilon}_n$, and we substitute $\varepsilon_n = \sqrt{c} \tilde{\varepsilon}_n$ into the previous formula:

$$\sqrt{c} \tilde{\varepsilon}_{n+1} = \frac{1}{2} \frac{(\sqrt{c} \tilde{\varepsilon}_n)^2}{a_n} = \frac{1}{2} \frac{\sqrt{c} \sqrt{c} \tilde{\varepsilon}_n^2}{a_n} \quad \text{hence} \quad \tilde{\varepsilon}_{n+1} = \frac{1}{2} \frac{\sqrt{c} \tilde{\varepsilon}_n^2}{a_n}$$

We have:

$$\tilde{\varepsilon}_n = \frac{\varepsilon_n}{\sqrt{c}} = \frac{a_n - \sqrt{c}}{\sqrt{c}} = \frac{a_n}{\sqrt{c}} - 1 \quad \text{hence} \quad \frac{\sqrt{c}}{a_n} = \frac{1}{\tilde{\varepsilon}_n + 1}$$

and hence we obtain:

$$\tilde{\varepsilon}_{n+1} = \frac{1}{2} \frac{\sqrt{c} \tilde{\varepsilon}_n^2}{a_n} = \frac{1}{2} \frac{\tilde{\varepsilon}_n^2}{1 + \tilde{\varepsilon}_n}$$

We have obtained the relation between $\tilde{\varepsilon}_{n+1}$ and $\tilde{\varepsilon}_n$, independent of both a_n and $\sqrt{c}$.

From the formula just obtained we can deduce the following simplified and more intuitive one:

$$0 \leq \tilde{\varepsilon}_{n+1} \leq \frac{1}{2} \min \{ \tilde{\varepsilon}_n, \tilde{\varepsilon}_n^2 \}$$

with strict inequality when $\tilde{\varepsilon}_n > 0$ (if $\tilde{\varepsilon}_n = 0$ then $\tilde{\varepsilon}_{n+1} = 0$ and equality holds). Hence:

$$\text{if } 0 < \tilde{\varepsilon}_n < 1, \quad \text{we have} \quad \tilde{\varepsilon}_{n+1} < \frac{1}{2} \tilde{\varepsilon}_n^2$$

For example, if the relative error at a certain iteration equals 10^{-3} , at the next step it will be less than $5 \cdot 10^{-7}$. In other words, the number of zeros after the decimal point (which equals the number of correctly estimated digits after the decimal point) doubles at each iteration.

The relative error of Heron's algorithm at each step is proportional to the square of the relative error at the previous step. **The rate of convergence is quadratic.**

Part II

Limits of functions and continuity

Limits of functions, asymptotes and continuity

1.1 Sequential definition of limit

The limit operation can be extended from sequences to real functions of a real variable.

In this way we can describe the behavior of the function when the independent variable **moves close to a given point** or **becomes very large** (in absolute value).

- Consider an **interval** I , a **point** $c \in I$ and a **function** f defined in I , except possibly at the point c .
- The **interval** I can be **bounded** or **unbounded**, **closed** or **open**.
- The **point** c can be **interior** to the interval or **one of its endpoints** (possibly $+\infty$ or $-\infty$).

Definition 1.1: Limit (sequential definition)

With $\ell, c \in \mathbb{R}^*$, we write

$$f(x) \rightarrow \ell \text{ as } x \rightarrow c$$

if for every sequence $\{x_n\}$ of points of I such that $x_n \rightarrow c$ as $n \rightarrow +\infty$ and $x_n \neq c, \forall n$, we have

$$f(x_n) \rightarrow \ell \text{ as } n \rightarrow +\infty$$

In these cases we can also write:

$$\lim_{x \rightarrow c} f(x) = \ell$$

which is read: the limit of $f(x)$, as x tends to c , is ℓ .

In other words, if **for any sequence** $\{x_n\}$ **of inputs** the **sequence of outputs** $\{f(x_n)\}$ tends to the limit ℓ (finite or infinite), then we say that the limit of $f(x)$, as x tends to c , is ℓ .

The sequential definition of limit reduces the concept of limit of a function to that of limit of a sequence.

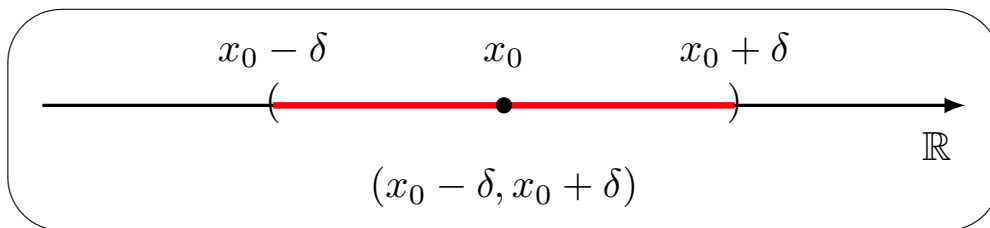
It is not the only possible definition. Another definition is the *topological definition of limit*. The two definitions are perfectly equivalent.

1.1.1 Neighborhoods and topological definition of limit

Definition 1.2: Centered neighborhood of a point

Given $\delta > 0$ and $x_0 \in \mathbb{R}$, the **neighborhood** centered at x_0 associated with δ is the open interval:

$$(x_0 - \delta, x_0 + \delta) = \{x \in \mathbb{R} : x_0 - \delta < x < x_0 + \delta\}$$



Saying that “ x moves in a neighborhood of x_0 ” means stating that

$$x \in (x_0 - \delta, x_0 + \delta), \text{ where we think of } 0 < \delta \ll 1.$$

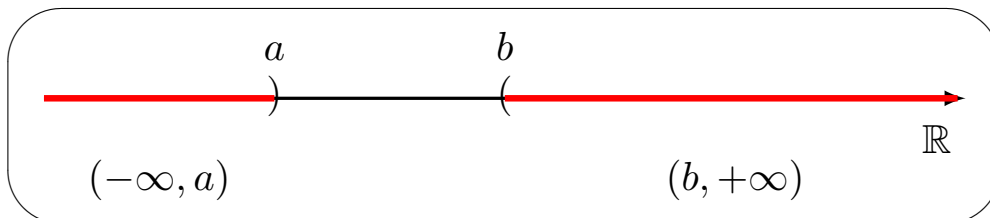
Definition 1.3: Neighborhood of $\pm\infty$

Given $a, b \in \mathbb{R}$, the neighborhood of $-\infty$ associated with a is the interval:

$$(-\infty, a) = \{x \in \mathbb{R} : x < a\}$$

and the neighborhood of $+\infty$ associated with b is the interval:

$$(b, +\infty) = \{x \in \mathbb{R} : x > b\}$$



- We now introduce the expression “eventually as $x \rightarrow c$ ”, analogously to what we did in the case of sequences

Definition 1.4: Property eventually satisfied by functions

We say that a function f **eventually** has a certain property as $x \rightarrow c \in \mathbb{R}^*$ if there exists a neighborhood U of c such that the property holds for $f(x)$ for every $x \in U, x \neq c$.

- We now give the topological definition of limit which, as already mentioned, is equivalent to the sequential one seen above, but turns out to be useful.

Definition 1.5: Limit (topological definition)

Let $c, \ell \in \mathbb{R}^*$, and let f be a function defined at least eventually as $x \rightarrow c$. We write

$$f(x) \rightarrow \ell \quad \text{as} \quad x \rightarrow c$$

if for every neighborhood U_ℓ of ℓ there exists a neighborhood U_c of c such that

$$\forall x \in U_c, x \neq c, \quad \text{we have} \quad f(x) \in U_\ell$$

- This definition can be specialized to the specific cases:

Finite limit at a finite point: $f(x) \rightarrow \ell \in \mathbb{R}$ as $x \rightarrow c \in \mathbb{R}$

$$\forall \varepsilon > 0, \exists \delta > 0 : \quad \forall x \neq c, |x - c| < \delta \implies |f(x) - \ell| < \varepsilon$$

Infinite limit at a finite point: $f(x) \rightarrow \pm\infty$ as $x \rightarrow c \in \mathbb{R}$

$$\forall M > 0, \exists \delta > 0 : \quad \forall x \neq c, |x - c| < \delta \implies |f(x)| > M$$

Finite limit at infinity: $f(x) \rightarrow \ell \in \mathbb{R}$ as $x \rightarrow \pm\infty$

$$\forall \varepsilon > 0, \exists b > 0 : \quad \forall |x| > b \implies |f(x) - \ell| < \varepsilon$$

Infinite limit at infinity: $f(x) \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$

$$\forall M > 0, \exists b > 0 : \quad \forall |x| > b \implies |f(x)| > M$$

The fact that we have already developed the basics of *limit calculus for sequences* will make it very convenient to use the sequential definition of limit to prove the theorems on limits of functions starting from those seen for sequences.

Theorem 1.6: Uniqueness of the limit of functions

Given a function f , if $f(x) \rightarrow \ell$ as $x \rightarrow c$, then this limit ℓ is unique.

Proof. If there were two limits ℓ_1 and ℓ_2 , different from each other, taking any sequence $\{x_n\}$ such that $x_n \rightarrow c$ and $x_n \neq c, \forall n$, we would have:

$$f(x_n) \rightarrow \ell_1 \quad \text{and} \quad f(x_n) \rightarrow \ell_2 \quad \text{as} \quad n \rightarrow +\infty$$

Hence the sequence $\{f(x_n)\}$ would have two distinct limits: a contradiction. $\square$

Definition 1.7: Infinitesimal function

A function f such that $f(x) \rightarrow 0$ as $x \rightarrow c$ is called **infinitesimal** as $x \rightarrow c$.

Definition 1.8: Infinite function

A function f such that $f(x) \rightarrow \pm\infty$ as $x \rightarrow c$ is called **infinite** as $x \rightarrow c$.

- When we write:

$$f(x) \rightarrow \ell \quad \text{as} \quad x \rightarrow c \quad \text{or} \quad \lim_{x \rightarrow c} f(x) = \ell$$

we have

$$\text{a} \quad \begin{cases} \text{finite limit} \\ \text{infinite limit} \end{cases} \quad \text{if} \quad \begin{cases} \ell \in \mathbb{R} \\ \ell = \pm\infty \end{cases}$$

we have

$$\text{a limit} \quad \begin{cases} \text{at a finite point} \\ \text{at infinity} \end{cases} \quad \text{if} \quad \begin{cases} c \in \mathbb{R} \\ c = \pm\infty \end{cases}$$

1.1.2 Finite limit at infinity

$$f(x) \rightarrow \ell \quad \text{as} \quad x \rightarrow c \quad \text{with} \quad \ell \in \mathbb{R} \quad \text{and} \quad c = \pm\infty$$

Example 1.1: Finite limit at infinity (sequential definition of limit)

Let us prove that:

$$e^x \rightarrow 0 \quad \text{as} \quad x \rightarrow -\infty$$

By the definition, we have to prove that for every sequence $\{x_n\}$ such that $x_n \rightarrow -\infty$ as $n \rightarrow +\infty$, we have:

$$e^{x_n} \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty$$

By the definition of limit of a sequence, this means proving that, for every $\varepsilon > 0$, we have:

$$|e^{x_n}| < \varepsilon, \quad \text{eventually, that is} \quad e^{x_n} < \varepsilon, \quad \text{eventually,}$$

since the exponential is always positive. The last inequality is equivalent to:

$$x_n < \log \varepsilon$$

If $\varepsilon > 0$ is a small number (< 1), $\log \varepsilon$ is a negative number (and large in absolute value). We set

$$M(\varepsilon) = -\log \varepsilon > 0$$

We therefore have to prove that, for every $M(\varepsilon) > 0$, we have:

$$x_n < -M(\varepsilon), \text{ eventually.}$$

But this is exactly what holds by hypothesis, because $x_n \rightarrow -\infty$. Hence $e^x \rightarrow 0$ as $x \rightarrow -\infty$.

1.1.2.1 Horizontal asymptote

Definition 1.9: Horizontal asymptote

We say that f has a **horizontal asymptote** with equation $y = \ell \in \mathbb{R}$ as $x \rightarrow +\infty$ ($x \rightarrow -\infty$) if:

$$f(x) \rightarrow \ell \in \mathbb{R} \quad \text{as } x \rightarrow +\infty \quad (x \rightarrow -\infty)$$

- Every situation of *finite limit at infinity*, therefore, corresponds graphically to the **presence of a horizontal asymptote**, that is, a horizontal line which the graph of the function gets closer and closer to.

1.1.2.2 Limit from above or from below

- When a function has a finite limit at infinity, it is **sometimes** possible to specify whether this limit is **from above** (ℓ^+) or **from below** (ℓ^-). Graphically, this means that the graph of the function approaches the level $y = \ell$ from **above** or from **below**.

Definition 1.10: Limit from above or from below

With $\ell, c \in \mathbb{R}^*$, we write

$$f(x) \rightarrow \ell^+ \quad (\ell^-) \quad \text{as } x \rightarrow c$$

if for every sequence $\{x_n\}$ of points of I such that $x_n \rightarrow c$ as $n \rightarrow +\infty$ and $x_n \neq c, \forall n$, we have

$$f(x_n) \rightarrow \ell^+ \quad (\ell^-) \quad \text{as } n \rightarrow +\infty$$

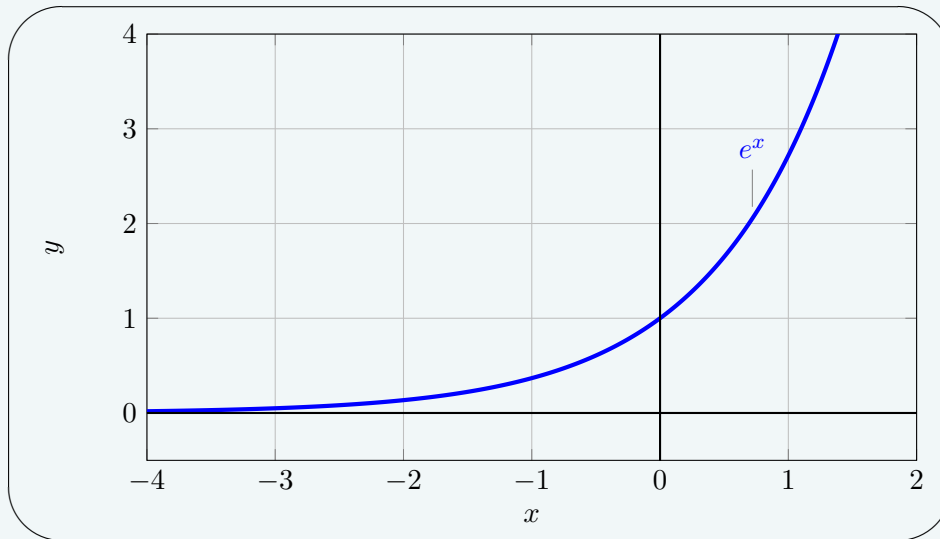
- Stating that $f(x) \rightarrow \ell^+$ means that $f(x) \rightarrow \ell$ and moreover $f(x) \geq \ell$, eventually.
- Stating that $f(x) \rightarrow \ell^-$ means that $f(x) \rightarrow \ell$ and moreover $f(x) \leq \ell$, eventually.
- In these cases we say that $f(x)$ tends to ℓ from above (from below) as x tends to c .

Example 1.2: Limits from above/below

For example:

$$\lim_{x \rightarrow -\infty} e^x = 0^+$$

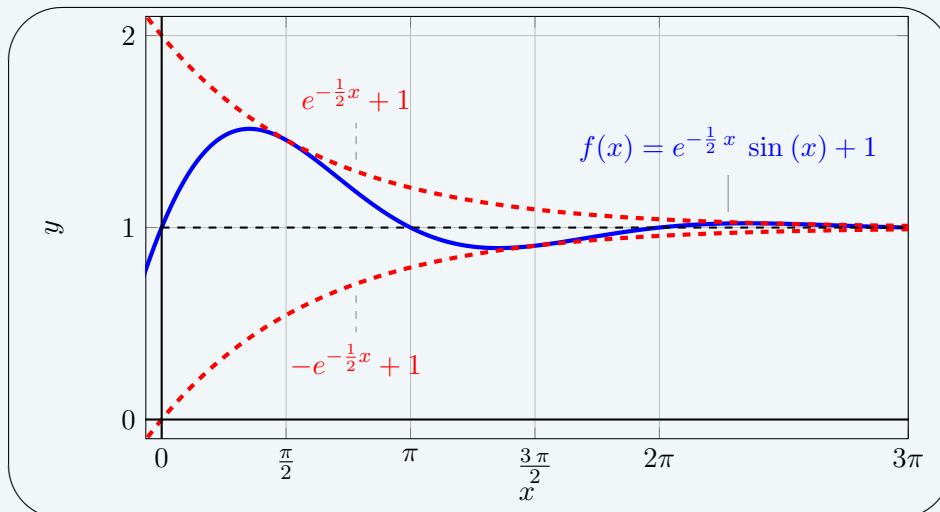
the notation 0^+ means that the function tends to 0 from above, that is, the values of $f(x)$ tend to zero while remaining non-negative.



Example 1.3: Limit from above/below

Note that not every finite limit is necessarily from above or from below, as shown for example by:

$$\lim_{x \rightarrow +\infty} e^{-\frac{1}{2}x} \sin(x) + 1 = 1$$



The function $f(x)$ tends to 1 as $x \rightarrow +\infty$, but we can state neither $f(x) \rightarrow 1^+$ nor $f(x) \rightarrow 1^-$.

1.1.3 Infinite limit at infinity

$$f(x) \rightarrow \ell \text{ as } x \rightarrow c \text{ with } \ell = \pm\infty \text{ and } c = \pm\infty$$

Example 1.4: Infinite limit at infinity (sequential definition of limit)

Let us prove that:

$$\log_{1/2} x \rightarrow -\infty \text{ as } x \rightarrow +\infty$$

By the definition, we have to prove that for every sequence $\{x_n\}$ such that $x_n \rightarrow +\infty$ as $n \rightarrow +\infty$, we have:

$$\log_{1/2} x_n \rightarrow -\infty \text{ as } n \rightarrow +\infty$$

This means proving that for every $M > 0$ we have:

$$\log_{1/2} x_n < -M, \text{ eventually, that is } x_n > \left(\frac{1}{2}\right)^{-M} = 2^M, \text{ eventually}$$

But by hypothesis $x_n \rightarrow +\infty$, so, having fixed the positive quantity 2^M , we certainly have $x_n > 2^M$, eventually. Hence $\log_{1/2} x \rightarrow -\infty$ as $x \rightarrow +\infty$.

1.1.3.1 Oblique asymptote

- When a function has an infinite limit at infinity, it may happen (but it does not always happen) that there exists an oblique line which the graph of the function gets closer and closer to.

Definition 1.11: Oblique asymptote

We say that f has an **oblique asymptote** with equation $y = mx + q$ ($q, m \in \mathbb{R}, m \neq 0$) as $x \rightarrow +\infty$ ($-\infty$) if:

$$(f(x) - (mx + q)) \rightarrow 0 \text{ as } x \rightarrow +\infty \text{ } (-\infty)$$

Example 1.5: Oblique asymptote

Consider

$$f(x) = 2x + 1 + e^x$$

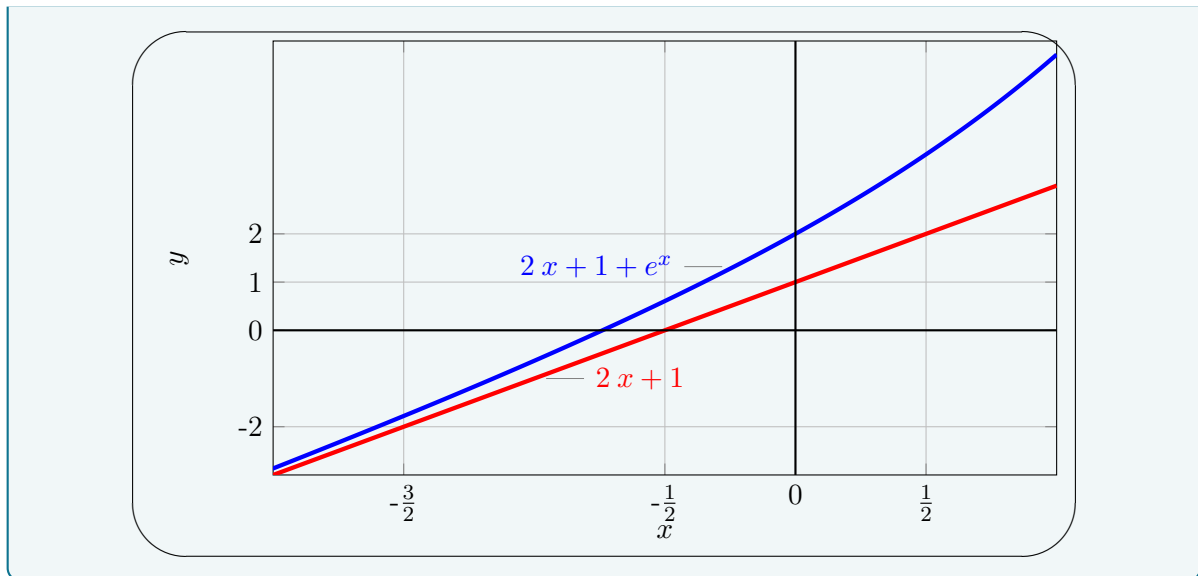
and we want to test whether or not $2x + 1$ is an oblique asymptote as $x \rightarrow -\infty$. We have

$$\lim_{x \rightarrow -\infty} (f(x) - (2x + 1)) = \lim_{x \rightarrow -\infty} e^x = 0$$

Therefore, by the definition of oblique asymptote, the line

$$y = 2x + 1$$

is an oblique asymptote of f as $x \rightarrow -\infty$.



- In less elementary cases, instead of having to “guess” the oblique asymptote, it is useful to have an operational criterion to find it.

Proposition 1.12: Existence of an oblique asymptote

A function f has an oblique asymptote with equation $y = mx + q$ as $x \rightarrow +\infty$ if and only if the following two conditions hold:

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = m \in \mathbb{R}, m \neq 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} (f(x) - mx) = q \in \mathbb{R}$$

An analogous criterion holds for $x \rightarrow -\infty$.

- Note that the first condition requires $f(x)$ to have the same order of infinity as $y = x$ as $x \rightarrow +\infty$, since:

$$\lim_{x \rightarrow -\infty} x = -\infty \quad \text{and} \quad \lim_{x \rightarrow +\infty} x = +\infty$$

Moreover, if $m > 0$ we have:

$$\lim_{x \rightarrow -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow +\infty} f(x) = +\infty$$

and if $m < 0$ we have:

$$\lim_{x \rightarrow -\infty} f(x) = +\infty \quad \text{and} \quad \lim_{x \rightarrow +\infty} f(x) = -\infty$$

The oblique asymptote $y = mx + q$ is a (non-horizontal) line that approximates the behavior of a function that diverges as $x \rightarrow \pm\infty$.

Proof. The first condition rules out the possibility that the asymptote is horizontal and requires $f(x)$ to have the same order of infinity as $y = x$. From the second condition:

$$\text{if } \lim_{x \rightarrow +\infty} (f(x) - mx) = q \quad \text{then} \quad \lim_{x \rightarrow +\infty} (f(x) - (mx + q)) = 0$$

Analogous reasoning for $x \rightarrow -\infty$. □

Example 1.6: Oblique asymptote

Consider

$$f(x) = 3x + \sqrt{x}$$

and compute

$$\lim_{x \rightarrow +\infty} \frac{3x + \sqrt{x}}{x} = \lim_{x \rightarrow +\infty} \left(3 + \frac{1}{\sqrt{x}} \right) = 3, \quad \lim_{x \rightarrow +\infty} (3x + \sqrt{x} - 3x) = \lim_{x \rightarrow +\infty} \sqrt{x} = +\infty$$

since the second limit is infinite, the second condition of the proposition does not hold and consequently the function has no oblique asymptote as $x \rightarrow +\infty$.

1.1.4 Infinite limit at a finite point

$$f(x) \rightarrow \ell \text{ as } x \rightarrow c \quad \text{with} \quad \ell = \pm\infty \quad \text{and} \quad c \in \mathbb{R}$$

Example 1.7: Infinite limit at a finite point (sequential definition of limit)

Let us prove that:

$$\frac{1}{x^2} \rightarrow +\infty \quad \text{as} \quad x \rightarrow 0$$

By the definition, we have to prove that for every sequence $\{x_n\}$ such that $x_n \rightarrow 0$ as $n \rightarrow +\infty$ and $x_n \neq 0, \forall n$, we have:

$$\frac{1}{x_n^2} \rightarrow +\infty \quad \text{as} \quad n \rightarrow +\infty$$

Note that the condition $x_n \neq 0, \forall n$, contained in the definition of limit, was irrelevant in the previous examples where we had $x \rightarrow \pm\infty$, but now it clearly becomes relevant.

We therefore have to prove that for every $M > 0$ we have:

$$\frac{1}{x_n^2} > M, \quad \text{eventually, that is} \quad |x_n| < \frac{1}{\sqrt{M}}, \quad \text{eventually}$$

But by hypothesis $x_n \rightarrow 0$, so, having fixed the positive quantity $\frac{1}{\sqrt{M}}$, we certainly have $|x_n| < \frac{1}{\sqrt{M}}$ eventually. Hence $1/x^2 \rightarrow +\infty$ as $x \rightarrow 0$.

1.1.4.1 Right and left limits

- Sometimes a function behaves differently, as far as its limit is concerned, depending on whether x approaches c from the **right** or from the **left**.

Definition 1.13: Right and left limits

With $\ell \in \mathbb{R}^*$ and $c \in \mathbb{R}$, we write

$$f(x) \rightarrow \ell \quad \text{as } x \rightarrow c^+ \quad (c^-)$$

if for every sequence $\{x_n\}$ of points of I such that $x_n \rightarrow c^+ \quad (c^-)$ as $n \rightarrow +\infty$ and $x_n \neq c^+ \quad (c^-)$, $\forall n$, we have

$$f(x_n) \rightarrow \ell \quad \text{as } n \rightarrow +\infty$$

The limit $\lim_{x \rightarrow c} f(x)$ exists if and only if the right limit and the left limit exist and are both equal to ℓ . That is, if and only if:

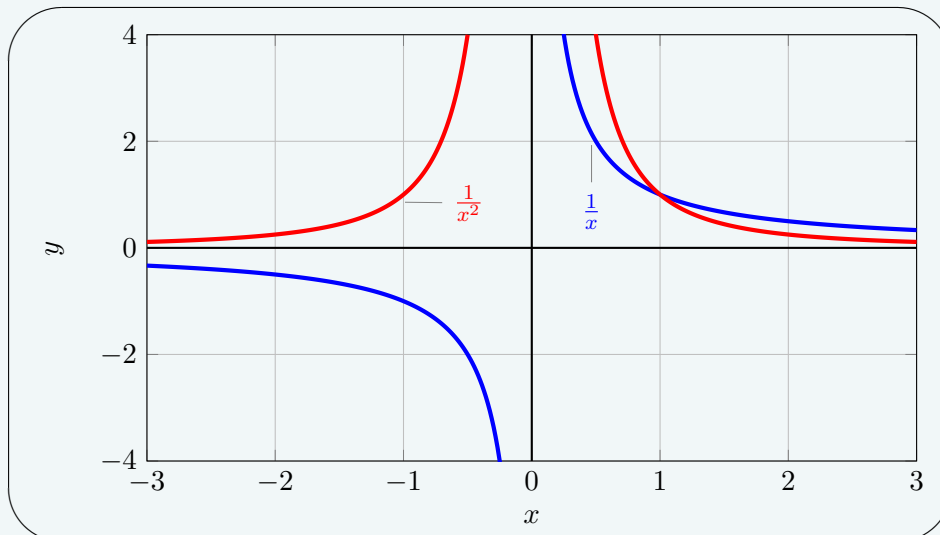
$$\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) = \ell$$

However, it may happen that the right limit and the left limit exist but are different from each other, or that only one of the two exists. In these cases:

$$\lim_{x \rightarrow c} f(x) \quad \text{does not exist}$$

Example 1.8: Right and left limits

Consider the functions $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$:



$$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty, \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty \quad \text{while} \quad \lim_{x \rightarrow 0} \frac{1}{x} \quad \text{does not exist}$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x^2} = +\infty, \quad \lim_{x \rightarrow 0^-} \frac{1}{x^2} = +\infty \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty$$

If f is defined in (a, b) , the limit as $x \rightarrow a$ (respectively $x \rightarrow b$) is automatically a right limit (respectively a left limit).

1.1.4.2 Vertical asymptote

Definition 1.14: Vertical asymptote

We say that f has a **vertical asymptote** with equation $x = c \in \mathbb{R}$ as $x \rightarrow c$ (c^+ or c^-) if:

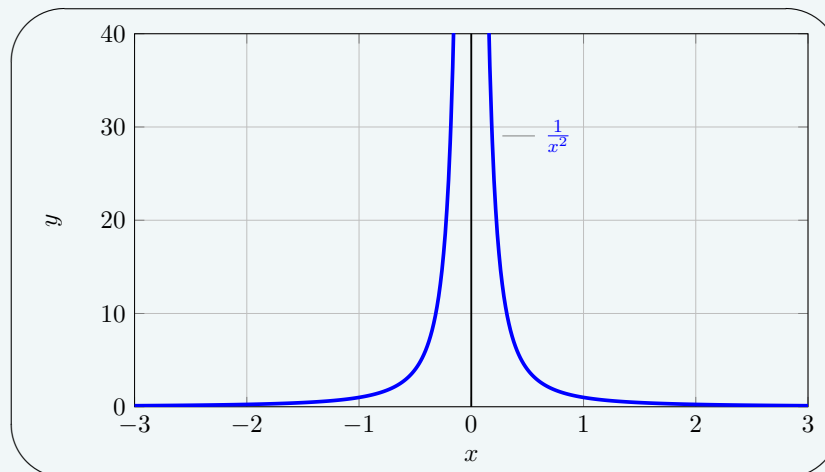
$$f(x) \rightarrow \pm\infty \quad \text{as } x \rightarrow c \quad (c^+ \text{ or } c^-)$$

- Every situation of **infinite limit at a finite point**, therefore, corresponds graphically to the presence of a **vertical asymptote**, that is, a **vertical line** which the graph of the function gets closer and closer to.

Example 1.9: Vertical asymptote

- $x = 0$ is a vertical asymptote of

$$\frac{1}{x^2} \quad \text{as } x \rightarrow 0$$



- $x = 0$ is a vertical asymptote of

$$\frac{1}{x} \quad \text{as } x \rightarrow 0^+ \text{ and as } x \rightarrow 0^-$$

- $x = 0$ is a vertical asymptote of

$$\log x \quad \text{as } x \rightarrow 0^+$$

1.1.5 Finite limit at a finite point

$$f(x) \rightarrow \ell \text{ as } x \rightarrow c \text{ with } \ell \in \mathbb{R} \text{ and } c \in \mathbb{R}$$

Example 1.10: Finite limit at a finite point (sequential definition of limit)

Consider the function $f(x) = \sin x$ and let us prove that:

$$\sin x \rightarrow 0 \text{ as } x \rightarrow 0$$

By the definition, we have to prove that for every sequence $\{x_n\}$ such that $x_n \rightarrow 0$ as $n \rightarrow +\infty$ and $x_n \neq 0, \forall n$, we have:

$$\sin x_n \rightarrow 0 \text{ as } n \rightarrow +\infty$$

We start from the following elementary inequality:

$$|\sin x| \leq |x|, \quad \forall x \in \mathbb{R} \quad \text{hence} \quad |\sin x_n| \leq |x_n|$$

By the comparison theorem:

$$\text{if } x_n \rightarrow 0 \text{ as } n \rightarrow +\infty \text{ then also } \sin x_n \rightarrow 0 \text{ as } n \rightarrow +\infty$$

Hence $\sin x \rightarrow 0$ as $x \rightarrow 0$ and we have:

$$\lim_{x \rightarrow 0} \sin x = f(0)$$

Example 1.11: Finite limit at a finite point (sequential definition of limit)

Consider the function:

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and let us prove that:

$$f(x) \rightarrow 1 \text{ as } x \rightarrow 0$$

By the definition, we have to prove that for every sequence $\{x_n\}$ such that $x_n \rightarrow 0$ as $n \rightarrow +\infty$ and $x_n \neq 0, \forall n$, we have:

$$f(x_n) \rightarrow 1 \text{ as } n \rightarrow +\infty$$

We have

$$f(x_n) = 1, \quad \forall n, \quad \text{hence } f(x_n) \rightarrow 1 \text{ as } n \rightarrow +\infty$$

Consequently $f(x) \rightarrow 1$ as $x \rightarrow 0$. In this case, therefore (unlike the previous example), we have:

$$\lim_{x \rightarrow 0} f(x) \neq f(0)$$

- Think about the two examples just seen. In both cases the limit at a finite point of a certain function exists and is finite.
- In the first case, this limit coincides with the value of the function at the point considered; in the second case it does not.
- We therefore introduce the concept of continuity to distinguish these two cases.

1.1.6 Continuity

Definition 1.15: Continuity

If $f : I \rightarrow \mathbb{R}$ and $c \in I$, we say that f is **continuous** at c if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

We say that f is continuous in I if it is continuous at each point of I .

- The continuity property on an interval has a simple geometric interpretation:

“the graph of a function continuous on an interval can be drawn, on that interval, without lifting the pen from the paper”

- A function that is **not continuous** at a point c is called **discontinuous** at c .

Example 1.12: Continuity

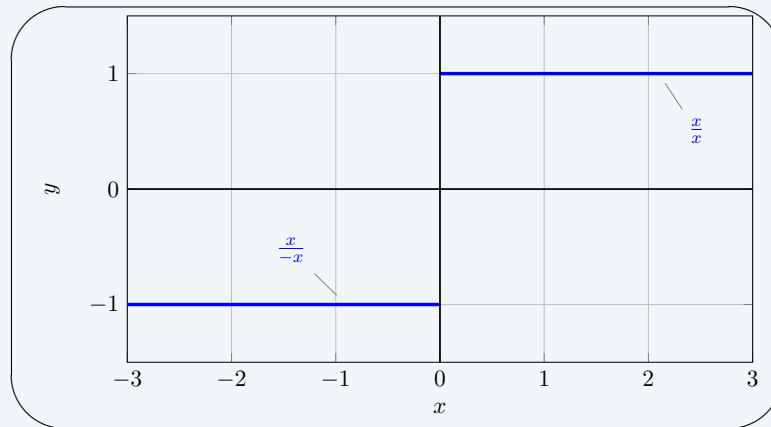
The function:

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is discontinuous at 0. The function $\sin x$ is continuous at 0.

Example 1.13: Discontinuity

Consider the function: $f(x) = x/|x|$



In this case it is not possible to compute $f(0)$ and the function $f(x)$ is discontinuous at 0 since:

$$\lim_{x \rightarrow 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} f(x) = -1$$

Definition 1.16: Jump discontinuity point

We say that c is a **jump discontinuity point** of f when the right and left limits at c exist and are finite, but are different from each other. The jump is given by the difference of the limits:

$$\text{jump at } c = \lim_{x \rightarrow c^+} f(x) - \lim_{x \rightarrow c^-} f(x)$$

Example 1.14: Jump discontinuity

The function

$$f(x) = \frac{x}{|x|}$$

has a jump discontinuity point at $x = 0$, with jump equal to 2.

If one of the two limits, the right limit or the left limit (as $x \rightarrow c$), coincides with $f(c)$, we say that f is **continuous from the right or from the left**, respectively.

- Functions with jump discontinuities are well suited to model phenomena that undergo sudden changes.
- Continuous functions, on the other hand, owe their importance to the fact that

$$\lim_{x \rightarrow c} f(x) = f(c)$$

can be interpreted by saying that “if x is close to c ” then “ $f(x)$ is close to $f(c)$ ”, that is, small variations of x imply small variations of $f(x)$.

1.1.7 Non-existence of the limit

- The limit of a function may also not exist

$$\lim_{x \rightarrow c} f(x) \text{ does not exist}$$

Example 1.15: Non-existence of the limit (sequential definition of limit)

Let us prove that:

$$\lim_{x \rightarrow +\infty} \sin x \text{ does not exist}$$

To prove it, it is enough to find two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \rightarrow +\infty$ and $y_n \rightarrow +\infty$ as $n \rightarrow +\infty$, with $\{\sin x_n\}$ and $\{\sin y_n\}$ tending to two different limits.

We take:

$$x_n = n\pi \quad \text{and} \quad y_n = \frac{\pi}{2} + 2n\pi, \quad \forall n$$

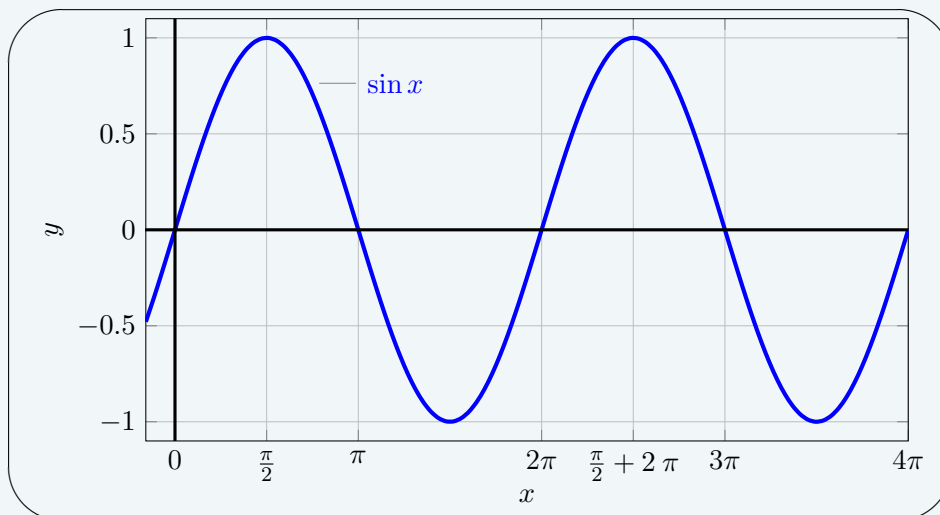
consequently we have:

$$\{n\pi\} \rightarrow +\infty \quad \text{and} \quad \left\{\frac{\pi}{2} + 2n\pi\right\} \rightarrow +\infty \quad \text{as} \quad n \rightarrow +\infty$$

For these sequences we have

$$\sin x_n = 0 \quad \text{and} \quad \sin y_n = 1, \quad \forall n$$

and consequently the two sequences tend to two different limits. Hence the definition of limit is not satisfied and the limit does not exist.



Example 1.16: Non-existence of the limit (sequential definition of limit)

Let us prove that:

$$\lim_{x \rightarrow 0} \sin \frac{1}{x} \text{ does not exist}$$

To prove it, it is enough to find two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \rightarrow 0$, $y_n \rightarrow 0$ as $n \rightarrow +\infty$ and $x_n, y_n \neq 0, \forall n$, with $\left\{\sin \frac{1}{x_n}\right\}$ and $\left\{\sin \frac{1}{y_n}\right\}$ tending to two different limits.

We take:

$$x_n = \frac{1}{n\pi} \quad \text{and} \quad y_n = \frac{1}{\frac{\pi}{2} + 2n\pi}, \quad \forall n > 1$$

consequently we have:

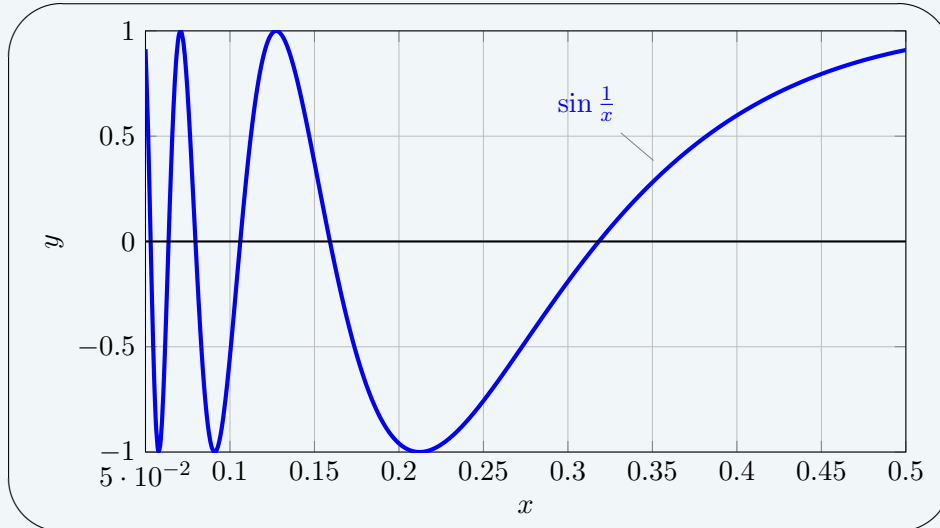
$$\left\{\frac{1}{n\pi}\right\} \rightarrow 0 \quad \text{and} \quad \left\{\frac{1}{\frac{\pi}{2} + 2n\pi}\right\} \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty$$

For these sequences we have

$$\sin \frac{1}{x_n} = 0 \quad \text{and} \quad \sin \frac{1}{y_n} = 1, \quad \forall n > 1$$

and consequently the two sequences tend to two different limits. Hence the definition of limit is not satisfied and the limit does not exist.

The function has infinitely many oscillations in a finite space, so it is not possible to draw its graph on an interval $(0, b]$ with $b \in \mathbb{R}_+$.



Computing limits of functions

2.1 Computing limits of functions

- We state the theorems on limits of functions that follow immediately from the corresponding theorems on limits of sequences and from the sequential definition of limit

2.1.1 Algebra of limits theorem

Theorem 2.1: Algebra of limits, case of finite limits

Hypotheses as $x \rightarrow c$:

$$1. f(x) \rightarrow \ell_1 \in \mathbb{R}, \quad 2. g(x) \rightarrow \ell_2 \in \mathbb{R}.$$

Thesis as $x \rightarrow c$:

$$1. f(x) \pm g(x) \rightarrow \ell_1 \pm \ell_2, \quad 2. f(x) g(x) \rightarrow \ell_1 \ell_2,$$

$$3. \frac{f(x)}{g(x)} \rightarrow \frac{\ell_1}{\ell_2} \quad (\ell_2 \neq 0, g(x) \neq 0, \text{ eventually as } x \rightarrow c).$$

Proof. Let $\{x_n\}$ be any sequence such that

$$x_n \neq c, \forall n, \quad \text{and} \quad x_n \rightarrow c \quad \text{as} \quad n \rightarrow +\infty.$$

By hypothesis we have

$$f(x_n) \rightarrow \ell_1 \quad \text{and} \quad g(x_n) \rightarrow \ell_2 \quad (\ell_1, \ell_2 \in \mathbb{R}).$$

From the theorem on the algebra of limits for sequences we therefore conclude that

$$f(x_n) \pm g(x_n) \rightarrow \ell_1 \pm \ell_2,$$

and hence

$$f(x) \pm g(x) \rightarrow \ell_1 \pm \ell_2.$$

Claims 2 and 3 of the theorem are proved in a perfectly analogous way. □

2.1.2 Sign-preservation theorems

In the following statements ℓ and c will be points of $\mathbb{R}^*$ unless otherwise stated.

Theorem 2.2: Sign preservation, 1st form

Hypotheses:

$$1. f(x) \rightarrow \ell \text{ as } x \rightarrow c, \quad 2. \ell \leq 0.$$

Thesis:

$$f(x) \leq 0 \text{ eventually as } x \rightarrow c.$$

Proof. Let $\{x_n\}$ be any sequence such that

$$x_n \neq c, \forall n, \text{ and } x_n \rightarrow c \text{ as } n \rightarrow +\infty$$

By the hypothesis we have

$$f(x_n) \rightarrow \ell > 0$$

hence, by the sign-preservation theorem for sequences, applied to the sequence $\{f(x_n)\}$, we conclude that

$$f(x_n) > 0, \text{ eventually.}$$

Since this holds for every sequence such that $x_n \rightarrow c$, we conclude that

$$f(x) > 0, \text{ eventually, as } x \rightarrow c$$

□

Theorem 2.3: Sign preservation for functions, 2nd form

Hypotheses:

$$1. f(x) \rightarrow \ell \in \mathbb{R} \text{ as } x \rightarrow c, \\ 2. f(x) \geq 0 \text{ eventually as } x \rightarrow c.$$

Thesis:

$$\ell \geq 0.$$

Proof. It follows from the corresponding theorem for sequences. □

- For functions we also have the following sign-preservation theorem.

Theorem 2.4: Sign preservation for continuous functions*Hypotheses:*

1. f is continuous at $c \in \mathbb{R}$,
2. $f(c) > 0$.

Thesis:

$$f(x) > 0 \text{ eventually as } x \rightarrow c.$$

Proof. If f is continuous at c , then

$$f(c) = \lim_{x \rightarrow c} f(x)$$

hence the hypothesis $f(c) > 0$ means that

$$f(c) = \lim_{x \rightarrow c} f(x) > 0$$

and this, by the sign-preservation theorem (1st form), implies that

$$f(x) > 0 \text{ eventually as } x \rightarrow c.$$

□

2.1.3 Comparison theorem**Theorem 2.5: Comparison theorem***Hypotheses:*

1. $f(x) \rightarrow \ell$ and $g(x) \rightarrow \ell$ as $x \rightarrow c$,
2. $f(x) \leq h(x) \leq g(x)$ eventually as $x \rightarrow c$.

Thesis:

$$h(x) \rightarrow \ell \text{ as } x \rightarrow c.$$

Proof. Let $\{x_n\}$ be any sequence such that

$$x_n \neq c, \forall n, \text{ and } x_n \rightarrow c \text{ as } n \rightarrow +\infty$$

We want to prove that

$$h(x_n) \rightarrow \ell \text{ as } n \rightarrow +\infty$$

By hypothesis we know that:

$$f(x_n) \leq h(x_n) \leq g(x_n), \text{ eventually,}$$

$$f(x_n) \rightarrow \ell \text{ and } g(x_n) \rightarrow \ell, \text{ as } x_n \rightarrow c$$

Hence, by the comparison theorem for sequences applied to:

$$\{f(x_n)\}, \{h(x_n)\} \text{ and } \{g(x_n)\}$$

we conclude that

$$h(x_n) \rightarrow \ell \text{ as } n \rightarrow +\infty$$

□

Corollary 2.6: Of the comparison theorem (part I)

Hypotheses:

1. $g(x) \rightarrow 0$ as $x \rightarrow c$,
2. $|h(x)| \leq g(x)$ eventually as $x \rightarrow c$.

Thesis:

$$h(x) \rightarrow 0 \text{ as } x \rightarrow c.$$

Proof. It follows from the corresponding corollary for sequences. □

Example 2.1: Corollary of the comparison theorem

Let us prove that:

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

We have

$$x \rightarrow 0 \text{ as } x \rightarrow 0 \text{ and } \lim_{x \rightarrow 0} \sin \frac{1}{x} \text{ does not exist}$$

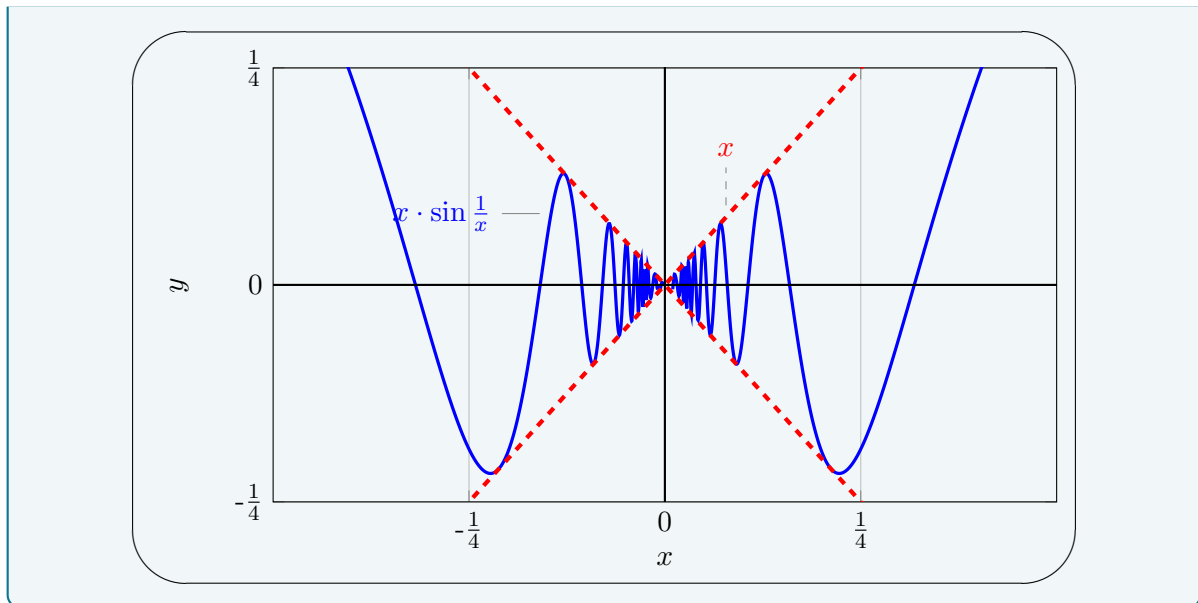
hence we cannot apply Theorem (2.1) on the algebra of limits for functions.

We have

$$\left| \sin \frac{1}{x} \right| \leq 1 \text{ hence } \left| x \sin \frac{1}{x} \right| \leq |x| \text{ and } |x| \rightarrow 0 \text{ as } x \rightarrow 0.$$

Hence, by Corollary (2.6) of the comparison theorem for functions, we have:

$$x \sin \frac{1}{x} \rightarrow 0 \text{ as } x \rightarrow 0.$$


Corollary 2.7: Of the comparison theorem (part II)

Hypotheses:

1. $f(x) \rightarrow 0$ as $x \rightarrow c$,
2. $g(x)$ is bounded, eventually, as $x \rightarrow c$.

Thesis:

$$f(x) g(x) \rightarrow 0 \text{ as } x \rightarrow c.$$

Proof. It follows from the corresponding corollary for sequences. □

Example 2.2: Corollary of the comparison theorem

Let us prove that:

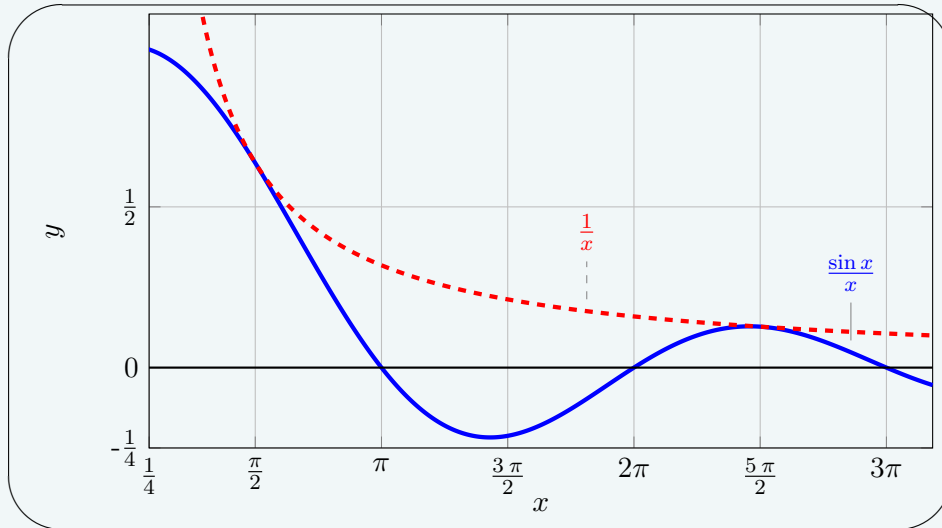
$$\lim_{x \rightarrow +\infty} \frac{x + \sin x}{2x + \cos x} = \frac{1}{2}$$

Factoring, we obtain

$$\frac{x + \sin x}{2x + \cos x} = \frac{x \left(1 + \frac{\sin x}{x}\right)}{2x \left(1 + \frac{\cos x}{2x}\right)} = \frac{1}{2} \left(\frac{1 + \frac{\sin x}{x}}{1 + \frac{\cos x}{2x}} \right)$$

By Corollary (2.7) of the comparison theorem for functions, we have:

$$\frac{\sin x}{x} \rightarrow 0 \text{ and } \frac{\cos x}{2x} \rightarrow 0 \text{ as } x \rightarrow +\infty.$$



Then, by Theorem (2.1) on the algebra of limits, we therefore have:

$$\frac{x + \sin x}{2x + \cos x} \rightarrow \frac{1}{2} \text{ as } x \rightarrow +\infty.$$

2.1.4 Theorems of partial arithmetization of the infinity symbol

Theorem 2.8: Partial arithmetization of the infinity symbol (addition)

Hypotheses as $x \rightarrow c$:

$$1. f(x) \rightarrow \ell \in \mathbb{R}, \quad 2. g(x) \rightarrow +\infty, \quad 3. h(x) \rightarrow +\infty.$$

Thesis as $x \rightarrow c$:

$$\begin{aligned} 1. f(x) + g(x) &\rightarrow \ell + \infty = +\infty, & 2. f(x) - g(x) &\rightarrow \ell - \infty = -\infty, \\ 3. g(x) + h(x) &\rightarrow +\infty + \infty = +\infty, & 4. -g(x) - h(x) &\rightarrow -\infty - \infty = -\infty. \end{aligned}$$

Proof. It follows from the corresponding theorem for sequences. □

Theorem 2.9: Partial arithmetization of the infinity symbol (product)

Hypotheses as $x \rightarrow c$:

$$1. f(x) \rightarrow \ell \in \mathbb{R}, \quad 2. g(x) \rightarrow 0, \quad 3. h(x) \rightarrow \infty.$$

Thesis as $x \rightarrow c$:

$$1. f(x) h(x) \rightarrow \ell \cdot \infty = \infty \quad (\ell \neq 0), \quad 2. \frac{f(x)}{g(x)} \rightarrow \frac{\ell}{0} = \infty \quad (\ell \neq 0),$$

$$3. \frac{f(x)}{h(x)} \rightarrow \frac{\ell}{\infty} = 0.$$

Proof. It follows from the corresponding theorem for sequences. □

- As for sequences, the **sign** of ∞ must be determined with the **rule of signs**.

Example 2.3: Partial arithmetization of the infinity symbol

$$\lim_{x \rightarrow -\infty} \left(\frac{1}{x} - 2 \right) x^3 = +\infty$$

Since $\left(\frac{1}{x} - 2 \right) \rightarrow -2$ and $x^3 \rightarrow -\infty$ as $x \rightarrow -\infty$.

2.2 Change of variable theorem for limits

Theorem 2.10: Change of variable in the limit

Let f and g be two functions for which the composition $f \circ g$ is defined, at least eventually as $x \rightarrow x_0 \in \mathbb{R}^*$. If:

1. $\lim_{x \rightarrow x_0} g(x) = t_0 \in \mathbb{R}^*$, 2. $\lim_{t \rightarrow t_0} f(t) = \ell \in \mathbb{R}^*$
3. $g(x) \neq t_0$, eventually, as $x \rightarrow x_0$

Then:

$$\lim_{x \rightarrow x_0} f(g(x)) = \lim_{t \rightarrow t_0} f(t).$$

Hypothesis 3. is not necessary when f is continuous at t_0 , or when $t_0 = \pm\infty$.

Proof. Let $\{x_n\}$ be any sequence such that

$$x_n \neq x_0, \forall n, \text{ and } x_n \rightarrow x_0 \text{ as } n \rightarrow +\infty.$$

We have

$$g(x_n) \rightarrow t_0 \text{ as } n \rightarrow +\infty \text{ (by hypothesis 1)}$$

and

$$g(x_n) \neq t_0, \text{ eventually (by hypothesis 3).}$$

Therefore

$$f(g(x_n)) \rightarrow \ell \text{ (by hypothesis 2).}$$

If $t_0 = \pm\infty$ the condition $g(x) \neq \pm\infty$ is obviously satisfied, while if f is continuous at t_0 , $\ell = f(t_0)$, so if $g(x_n) = t_0$ for some n we would have

$$f(g(x_n)) = f(t_0) = \ell$$

and hence the convergence

$$f(g(x_n)) \rightarrow \ell \text{ as } n \rightarrow +\infty$$

would be guaranteed anyway. □

Example 2.4: Computing a limit with the change of variable theorem

Let us compute the limit

$$\lim_{x \rightarrow +\infty} \log \left(\frac{2x^3 + 4x + 1}{5(x+1)^3} \right)$$

The functions f and g are:

$$g(x) = \left(\frac{2x^3 + 4x + 1}{5(x+1)^3} \right) \quad \text{and} \quad f(t) = \log t.$$

We have $t = g(x)$ and

$$(f \circ g)(x) = f(g(x)) = \log \left(\frac{2x^3 + 4x + 1}{5(x+1)^3} \right) \quad \text{and} \quad x_0 = +\infty.$$

We compute

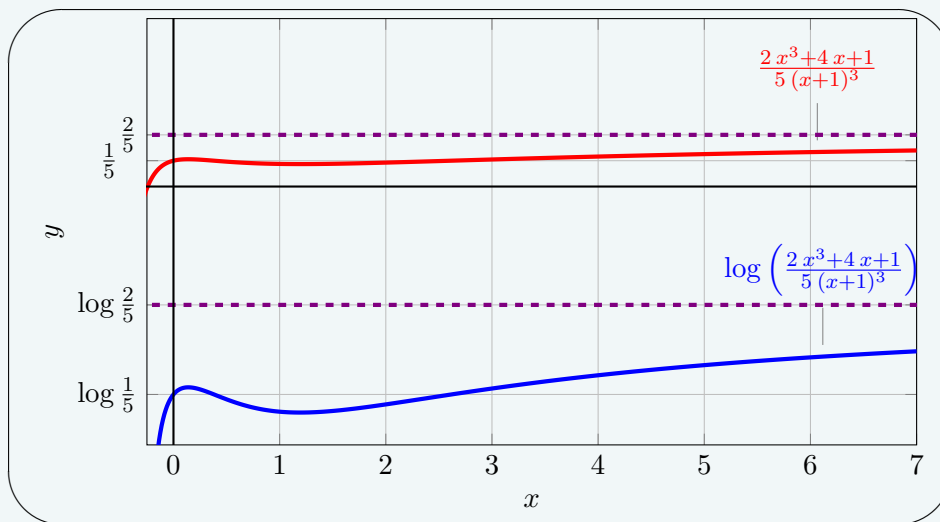
$$\lim_{x \rightarrow +\infty} g(x) = \frac{2}{5} \quad \text{and consequently} \quad t_0 = \frac{2}{5}.$$

Now we compute

$$\lim_{t \rightarrow 2/5} \log t = \log \frac{2}{5} \quad (\text{as we will see, the logarithm is continuous in } \mathbb{R}_+)$$

Hence

$$\lim_{x \rightarrow +\infty} f(g(x)) = \log \frac{2}{5}$$



Moreover:

$$\lim_{x \rightarrow 0} f(g(x)) = \log \frac{1}{5}$$

using the theorem we have:

$$\lim_{x \rightarrow 0} g(x) = \frac{1}{5} \quad \text{and} \quad \lim_{t \rightarrow 1/5} \log t = \log \frac{1}{5}$$

since

$$\lim_{x \rightarrow 0} 2x^3 + 4x + 1 = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} 5(x+1)^3 = 5$$

Example 2.5: Computing a limit with the change of variable theorem

Let us compute the limits

$$\lim_{x \rightarrow 0^+} e^{\frac{1}{x}} \quad \text{and} \quad \lim_{x \rightarrow 0^-} e^{\frac{1}{x}}.$$

The functions f and g are:

$$g(x) = \frac{1}{x} \quad \text{and} \quad f(t) = e^t.$$

We have $t = g(x)$ and

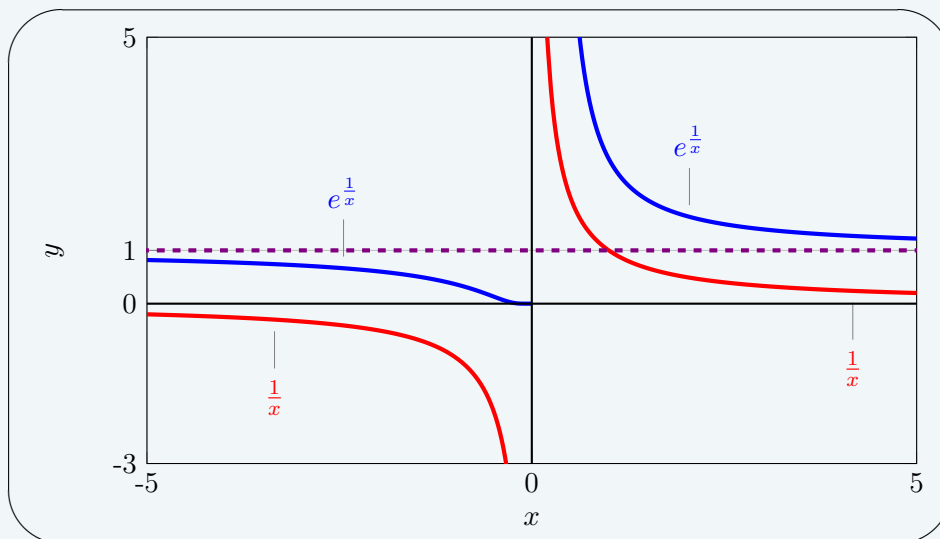
$$(f \circ g)(x) = f(g(x)) = e^{\frac{1}{x}} \quad \text{moreover} \quad x_0 = 0^+ \quad \text{and} \quad x_0 = 0^-.$$

1. $\frac{1}{x} \rightarrow +\infty$ as $x \rightarrow 0^+$ and hence $t_0 = +\infty$. Now we compute:

$$\lim_{t \rightarrow +\infty} e^t = +\infty \quad \text{hence} \quad \lim_{x \rightarrow 0^+} e^{\frac{1}{x}} = +\infty.$$

2. $\frac{1}{x} \rightarrow -\infty$ as $x \rightarrow 0^-$ and hence $t_0 = -\infty$. Now we compute:

$$\lim_{t \rightarrow -\infty} e^t = 0 \quad \text{hence} \quad \lim_{x \rightarrow 0^-} e^{\frac{1}{x}} = 0.$$



Limits of polynomials and rational functions

3.1 Limits of polynomials at $\pm\infty$

- A polynomial of (maximum) degree n can be written as:

$$P_n(x) = \sum_{i=0}^n a_i x^i \quad \text{with } a_i \in \mathbb{R}, \text{ for } i \in \{0, 1, \dots, n\}, \text{ and } a_n \neq 0.$$

The value a_i is the **coefficient of the monomial** i , with $i = 0, 1, \dots, n$, while x^i is the **literal part** of the monomial. In this notation, without loss of generality, the monomials are ordered by increasing values of the exponents.¹

- Factoring out $a_n x^n$, that is, the last monomial, which is the one of maximum degree, we have:

$$P_n(x) = a_n x^n \left(\overbrace{\frac{a_0}{a_n x^n} + \frac{a_1}{a_n x^{n-1}} + \dots + \frac{a_{n-1}}{a_n x}}^{\rightarrow 0 \text{ as } x \rightarrow \pm\infty} + 1 \right)$$

$$\lim_{x \rightarrow \pm\infty} P_n(x) = \lim_{x \rightarrow \pm\infty} a_n x^n$$

To compute the limit of a polynomial as $x \rightarrow \pm\infty$, it is enough to compute the limit of the monomial of maximum degree.

Example 3.1: Limits of polynomials as $x \rightarrow \pm\infty$

For example:

$$\lim_{x \rightarrow +\infty} (8 - x^2 + 3x^3) = \lim_{x \rightarrow +\infty} 3x^3 \left(\underbrace{\frac{8}{3x^3} - \frac{1}{3x}}_{\rightarrow 0 \text{ as } x \rightarrow +\infty} + 1 \right) = \lim_{x \rightarrow +\infty} 3x^3 = +\infty$$

¹If a monomial of degree i is missing we have $a_i = 0$; a_0 is the constant term since $x^0 = 1$.

$$\lim_{x \rightarrow -\infty} (120 + 4x - 2x^3) = \lim_{x \rightarrow -\infty} -2x^3 \left(\underbrace{\frac{120}{-2x^3} + \frac{4}{-2x^2} + 1}_{\rightarrow 0 \text{ as } x \rightarrow -\infty} \right) = \lim_{x \rightarrow -\infty} -2x^3 = +\infty$$

3.2 Limits of rational functions at $\pm\infty$

- Let f be a rational function (ratio of polynomials):

$$f(x) = \frac{P_n(x)}{P_m(x)}$$

where $P_n(x)$ and $P_m(x)$ are polynomials of degree n and m , respectively:

$$P_n(x) = \sum_{i=0}^n a_i x^i \quad \text{with } a_i \in \mathbb{R}, \text{ for } i = 0, 1, \dots, n, \quad \text{and } a_n \neq 0.$$

$$P_m(x) = \sum_{i=0}^m b_i x^i \quad \text{with } b_i \in \mathbb{R}, \text{ for } i = 0, 1, \dots, m, \quad \text{and } b_m \neq 0.$$

- Factoring out $a_n x^n$ in the numerator and $b_m x^m$ in the denominator, that is, the monomials of maximum degree, we have:

$$\frac{P_n(x)}{P_m(x)} = \frac{a_n x^n}{b_m x^m} \frac{\left\{ \underbrace{\frac{a_0}{a_n x^n} + \frac{a_1}{a_n x^{n-1}} + \dots + \frac{a_{n-1}}{a_n x} + 1}_{\rightarrow 0 \text{ as } x \rightarrow \pm\infty} \right\}}{\left\{ \underbrace{\frac{b_0}{b_m x^m} + \frac{b_1}{b_m x^{m-1}} + \dots + \frac{b_{m-1}}{b_m x} + 1}_{\rightarrow 0 \text{ as } x \rightarrow \pm\infty} \right\}}.$$

$$\lim_{x \rightarrow \pm\infty} \frac{P_n(x)}{P_m(x)} = \lim_{x \rightarrow \pm\infty} \frac{a_n x^n}{b_m x^m} = \lim_{x \rightarrow \pm\infty} \frac{a_n}{b_m} x^{n-m}$$

To compute the limit of a ratio of polynomials as $x \rightarrow \pm\infty$, it is enough to compute the limit of the ratio of the monomials of maximum degree.

Example 3.2: Limits of rational functions as $x \rightarrow \pm\infty$

For example:

$$\lim_{x \rightarrow -\infty} \frac{1 + 3x^3}{2 + x^2} = \lim_{x \rightarrow -\infty} \frac{3x^3}{x^2} = \lim_{x \rightarrow -\infty} 3x = -\infty$$

$$\lim_{x \rightarrow +\infty} \frac{1 + 3x^3}{1 - 10x + x^4} = \lim_{x \rightarrow +\infty} \frac{3x^3}{x^4} = \lim_{x \rightarrow +\infty} 3 \frac{1}{x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{7x^2 + 6x^5}{-4 + 7x^5} = \lim_{x \rightarrow -\infty} \frac{6x^5}{7x^5} = \lim_{x \rightarrow -\infty} \frac{6}{7} = \frac{6}{7}$$

To compute the limit of rational functions (without the constant term) as $x \rightarrow 0$ we must factor out, in the numerator and in the denominator, the powers of minimum degree.

Example 3.3: Limits of rational functions as $x \rightarrow 0$

For example:

$$\lim_{x \rightarrow 0} \frac{-2x^2 + 3x^3 - 6x^5}{x + x^2 - x^7} = \lim_{x \rightarrow 0} \frac{-2x^2 \left(1 - \overbrace{\frac{3}{2}x + 3x^3}^{\rightarrow 0 \text{ as } x \rightarrow 0} \right)}{x \left(1 + \underbrace{x - x^6}_{\rightarrow 0 \text{ as } x \rightarrow 0} \right)} = \lim_{x \rightarrow 0} -2x = 0$$

3.3 Limits of quotients of sums of powers with rational exponents

- The same rules hold for quotients of sums of powers with rational exponents:

as $x \rightarrow \infty$ we must factor out the power with the maximum exponent

as $x \rightarrow 0^+$ we must factor out the power with the minimum exponent

Raising to a power can also be defined with a negative base if the exponent is a rational number (fraction) with an odd denominator. Hence in this case we can compute the limit as $x \rightarrow 0$ (otherwise only as $x \rightarrow 0^+$).

Example 3.4: Limits of quotients of sums of powers with rational exponents

For example:

$$\lim_{x \rightarrow 0} \frac{x^{2/3}}{3x^{1/3} + x + x^2}$$

Factoring out in the denominator the power with the minimum exponent, we have:

$$\begin{aligned} \frac{x^{2/3}}{3x^{1/3} + x + x^2} &= \frac{x^{2/3}}{3x^{1/3} \left(1 + \frac{x}{3x^{1/3}} + \frac{x^2}{3x^{1/3}} \right)} \\ &= \frac{x^{2/3}}{3x^{1/3} \left(1 + \frac{x^{2/3}}{3} + \frac{x^{5/3}}{3} \right)} \\ &= \frac{1}{3} x^{1/3} \frac{1}{\left(1 + \frac{x^{2/3}}{3} + \frac{x^{5/3}}{3} \right)} \end{aligned}$$

hence

$$\lim_{x \rightarrow 0} \frac{x^{2/3}}{3x^{1/3} + x + x^2} = \frac{1}{3} \lim_{x \rightarrow 0} x^{1/3} \frac{1}{\left(1 + \underbrace{\frac{x^{2/3}}{3} + \frac{x^{5/3}}{3}}_{\rightarrow 0 \text{ as } x \rightarrow 0}\right)} = 0$$

Another option to compute limits as $x \rightarrow 0$ is to make a change of variable:

$$x = \frac{1}{y}, \text{ if } x \rightarrow 0^+ \text{ then } y \rightarrow +\infty, \text{ if } x \rightarrow 0^- \text{ then } y \rightarrow -\infty$$

and hence we reduce to the case of limits at $\pm\infty$.

Example 3.5: Computing the limit of rational functions as $x \rightarrow 0^+$ with a change of variable

We want to compute the following limit:

$$\lim_{x \rightarrow 0} \frac{x^{2/3}}{3x^{1/3} + x + x^2}, \quad f(x) = \frac{x^{2/3}}{3x^{1/3} + x + x^2}$$

Making the change of variable $x = \frac{1}{y}$, we therefore have to compute the following two limits:

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{y \rightarrow +\infty} \frac{\left(\frac{1}{y}\right)^{2/3}}{3\left(\frac{1}{y}\right)^{1/3} + \frac{1}{y} + \left(\frac{1}{y}\right)^2} \quad \text{and} \\ \lim_{x \rightarrow 0^-} f(x) &= \lim_{y \rightarrow -\infty} \frac{\left(\frac{1}{y}\right)^{2/3}}{3\left(\frac{1}{y}\right)^{1/3} + \frac{1}{y} + \left(\frac{1}{y}\right)^2} \end{aligned}$$

We have:

$$\frac{\left(\frac{1}{y}\right)^{2/3}}{3\left(\frac{1}{y}\right)^{1/3} + \frac{1}{y} + \left(\frac{1}{y}\right)^2} = \frac{y^{-2/3}}{3y^{-1/3} + y^{-1} + y^{-2}}$$

Factoring out in the denominator the power with the maximum exponent, we have:

$$\frac{y^{-2/3}}{3y^{-1/3} + y^{-1} + y^{-2}} = \frac{y^{-2/3}}{3y^{-1/3} \left(1 + \frac{y^{-1}}{3y^{-1/3}} + \frac{y^{-2}}{3y^{-1/3}}\right)} = \frac{1}{3} \frac{1}{y^{1/3}} \frac{1}{\left(1 + \frac{1}{3y^{2/3}} + \frac{1}{3y^{5/3}}\right)}$$

Hence:

$$\frac{1}{3} \lim_{y \rightarrow -\infty} \frac{1}{y^{1/3}} \frac{1}{\left(1 + \frac{1}{3y^{2/3}} + \frac{1}{3y^{5/3}}\right)} = \frac{1}{3} \lim_{y \rightarrow +\infty} \frac{1}{y^{1/3}} \frac{1}{\left(1 + \frac{1}{3y^{2/3}} + \frac{1}{3y^{5/3}}\right)} = 0$$

since, as $y \rightarrow \pm\infty$, we have

$$\frac{1}{y^{1/3}} \rightarrow 0 \quad \text{and} \quad \left(1 + \frac{1}{3y^{2/3}} + \frac{1}{3y^{5/3}}\right) \rightarrow 1 \quad \text{since} \quad \frac{1}{3y^{2/3}} \rightarrow 0, \quad \frac{1}{3y^{5/3}} \rightarrow 0.$$

Since the right limit and the left limit of $f(x)$ as $x \rightarrow 0$ exist and are equal to 0, then:

$$\lim_{x \rightarrow 0} f(x) = 0$$

Continuous functions

4.1 Algebra of continuous functions theorem

Theorem 4.1: Algebra of continuous functions

Let f and g be two functions defined at least in a neighborhood of $x_0 \in \mathbb{R}$ and continuous at x_0 . Then:

1. $f(x) \pm g(x)$ is continuous at x_0 ; 2. $f(x) \cdot g(x)$ is continuous at x_0 ;

3. $\frac{f(x)}{g(x)}$ is continuous at x_0 provided that $g(x_0) \neq 0$.

Proof. Let us prove, for example, claim 3., the others being analogous. By hypothesis we know that f and g are continuous at x_0 , that is:

$$f(x) \rightarrow f(x_0) \quad \text{and} \quad g(x) \rightarrow g(x_0) \quad \text{as} \quad x \rightarrow x_0.$$

Moreover $g(x_0) \neq 0$ and hence, by the sign-preservation theorem for continuous functions, $g(x) \neq 0$ eventually as $x \rightarrow x_0$.

Then, by the theorem on the algebra of limits, we conclude that

$$\frac{f(x)}{g(x)} \rightarrow \frac{f(x_0)}{g(x_0)} \quad \text{as} \quad x \rightarrow x_0$$

that is, $f(x)/g(x)$ is continuous at x_0 . □

4.2 Continuity of elementary functions theorem

Theorem 4.2: Continuity of elementary functions

The following elementary functions are continuous at all points of their domain:

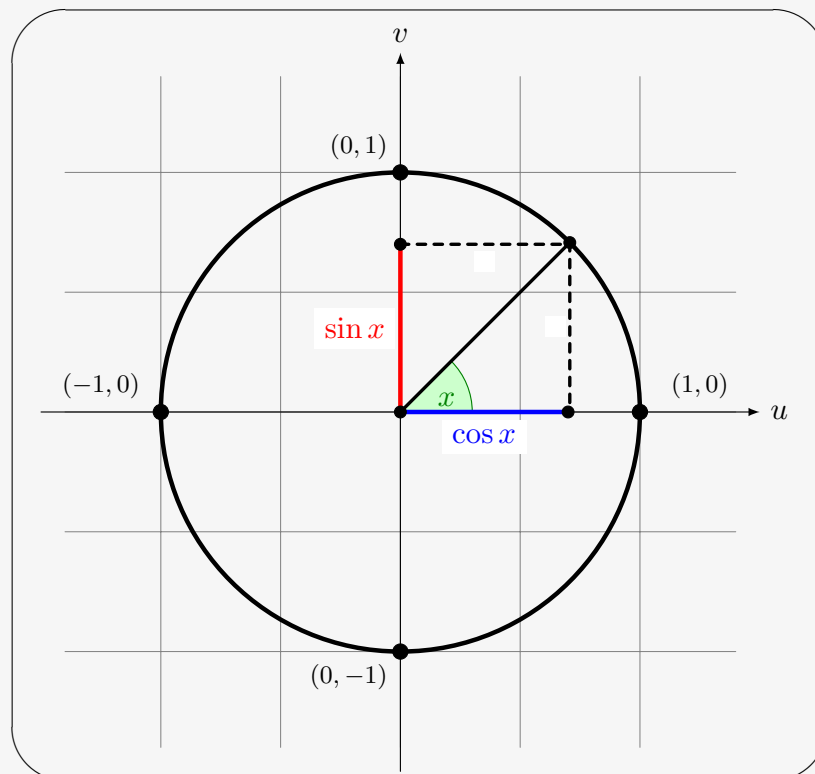
1. Powers with integer, rational or real exponent;
2. Exponential functions;
3. Logarithmic functions;
4. Elementary trigonometric functions ($\sin x$, $\cos x$)

Proof. For example, let us prove the continuity on the whole of $\mathbb{R}$ of the functions $\sin x$ and $\cos x$.

- We have seen that $\sin x$ is continuous at $x = 0$; let us show that $\cos x$ is also continuous at $x = 0$.
- The unit (trigonometric) circle shows that, if x is an angle in the first quadrant,

$$\sin x + \cos x \geq 1$$

since 1 is the hypotenuse of a right triangle with legs $\sin x$, $\cos x$.



- It follows that

$$0 \leq 1 - \cos x \leq \sin x \quad \text{for } x \in \left[0, \frac{\pi}{2}\right], \text{ and hence}$$

$$0 \leq 1 - \cos x \leq |\sin x| \quad \text{for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

- Then, by the comparison theorem,

$$1 - \cos x \rightarrow 0 \text{ as } x \rightarrow 0 \text{ hence } \cos x \rightarrow 1 \text{ as } x \rightarrow 0$$

and therefore $\cos x$ is continuous at 0.

□

Proof. • To now prove the continuity of $\sin x$ at a generic point $x_0 \in \mathbb{R}$ we consider the chain of relations:

$$\begin{aligned} |\sin(x_0 + h) - \sin x_0| &= |\sin x_0 \cos h + \sin h \cos x_0 - \sin x_0| \\ &= |\sin x_0 (\cos h - 1) + \cos x_0 \sin h| \\ &\leq |\sin x_0 (\cos h - 1)| + |\cos x_0 \sin h| \\ &= |\sin x_0| |\cos h - 1| + |\cos x_0| |\sin h| \end{aligned}$$

Now $|\sin h|$ and $|\cos h - 1|$ tend to zero as $h \rightarrow 0$, by what we have just proved, while $|\sin x_0|$ and $|\cos x_0|$ are constants, so

$$\sin(x_0 + h) - \sin x_0 \rightarrow 0 \text{ as } h \rightarrow 0$$

that is

$$\sin(x_0 + h) \rightarrow \sin x_0 \text{ as } h \rightarrow 0$$

and $\sin x$ is continuous at x_0 . An analogous argument shows the continuity of $\cos x$.

□

- Having proved that $\sin x$, $\cos x$ are continuous on the whole of $\mathbb{R}$, we deduce that the functions $\tan x$, $\cot x$ are continuous on their domain (by the theorem on the algebra of continuous functions)
- Power functions with integer exponent are continuous (on the whole of $\mathbb{R}$), since $f(x) = x$ is obviously continuous, and $f(x) = x^n$ is the product of n continuous functions.
- Polynomials are continuous functions, since they are obtained by adding functions of the type $c x^n$, which are continuous by the previous point.
- Rational functions (i.e., quotients of polynomials) are continuous functions, except at

the points where the denominator vanishes (the denominator is a polynomial, hence it vanishes at a finite number of points)

4.3 Continuity of composite functions theorem

Theorem 4.3: Continuity of the composite function

Let:

- g be a function defined at least in a neighborhood of x_0 and continuous at x_0 ,
- f be a function defined at least in a neighborhood of $t_0 = g(x_0)$ and continuous at t_0 ,

then $f \circ g$ is defined at least in a neighborhood of x_0 and is continuous at x_0 .

Proof. Since g is continuous at x_0 ,

$$\lim_{x \rightarrow x_0} g(x) = g(x_0) = t_0$$

then, by the change of variable theorem for limits, we have:

$$\lim_{x \rightarrow x_0} f(g(x)) = \lim_{t \rightarrow t_0} f(t)$$

and since f is continuous at t_0 , we have

$$\lim_{t \rightarrow t_0} f(t) = f(t_0)$$

and the thesis is proved. □

- It follows that all the functions that can be obtained from elementary functions by sums, products, quotients and compositions are continuous on their domain. **Hence, by combining functions in this way, we still obtain continuous functions.**

To summarize:

- sums, products and quotients of continuous functions give continuous functions (where the denominator does not vanish);
 - the elementary functions of mathematical analysis are continuous on their domain;
 - the composition of continuous functions gives a continuous function.
- It is therefore possible to know a priori that a function is continuous on its domain, without applying the definition of continuity case by case.

For continuous functions, if x_0 is a point of the domain, the limit as $x \rightarrow x_0$ is computed simply by substituting x_0 into the analytic expression of the function, that is:

$$f(x) \rightarrow f(x_0) \quad \text{as} \quad x \rightarrow x_0$$

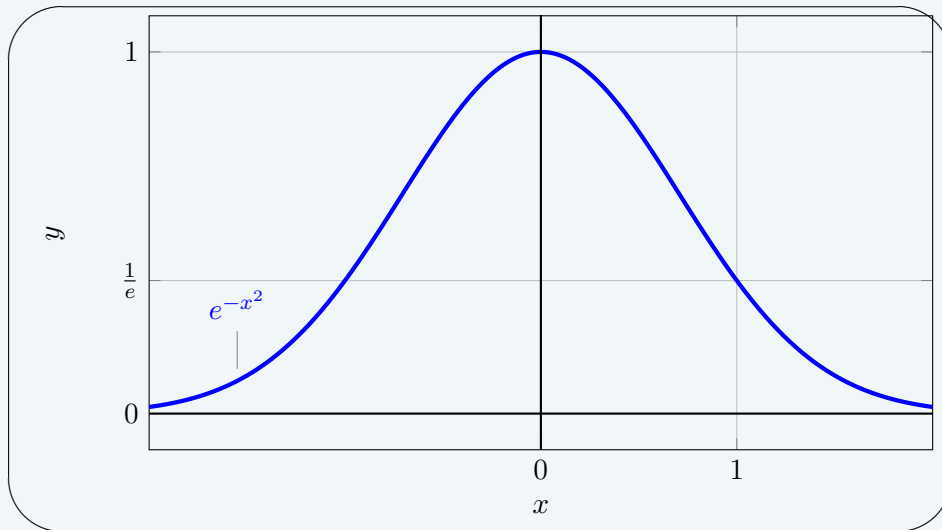
Example 4.1: Continuous functions and finite limits at a finite point

The following function is continuous for $x \in \mathbb{R}$:

$$f(x) = e^{-x^2}$$

hence:

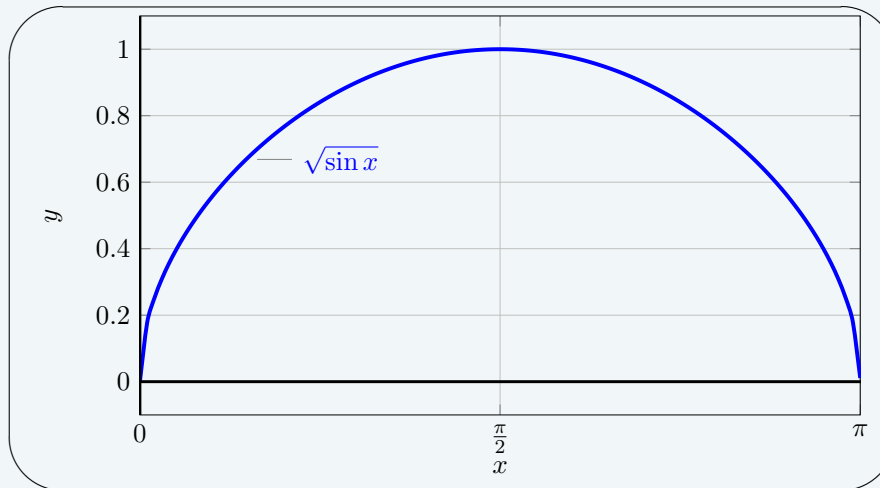
$$\lim_{x \rightarrow 1} e^{-x^2} = e^{-1} = \frac{1}{e}$$



Example 4.2: Continuous functions and finite limits at a finite point

The following function is continuous for $x \in \mathbb{R}$ with $2k\pi \leq x \leq 2k\pi + \pi, \forall k \in \mathbb{Z}$:

$$f(x) = \sqrt{\sin x} \quad \text{hence} \quad \lim_{x \rightarrow \pi^-} \sqrt{\sin x} = \sqrt{\sin \pi} = 0$$


Example 4.3: Continuous functions and finite limits at a finite point

The following function is continuous for $x \in \mathbb{R}$ with $x \neq (2k+1)\frac{\pi}{2}, \forall k \in \mathbb{Z}$:

$$f(x) = \log_a (1 + (\tan x)^2), \quad \forall a > 0, a \neq 1$$

hence:

$$\lim_{x \rightarrow 0} \log_a (1 + (\tan x)^2) = \log_a (1 + (\tan 0)^2) = \log_a 1 = 0$$

For example, with $a = e$ we have:



Comparison of infinities

5.1 Comparison of infinities for functions

Theorem 5.1: Comparison of infinities

Given $\alpha, \beta, \lambda > 0$, $a, b > 1$, we have

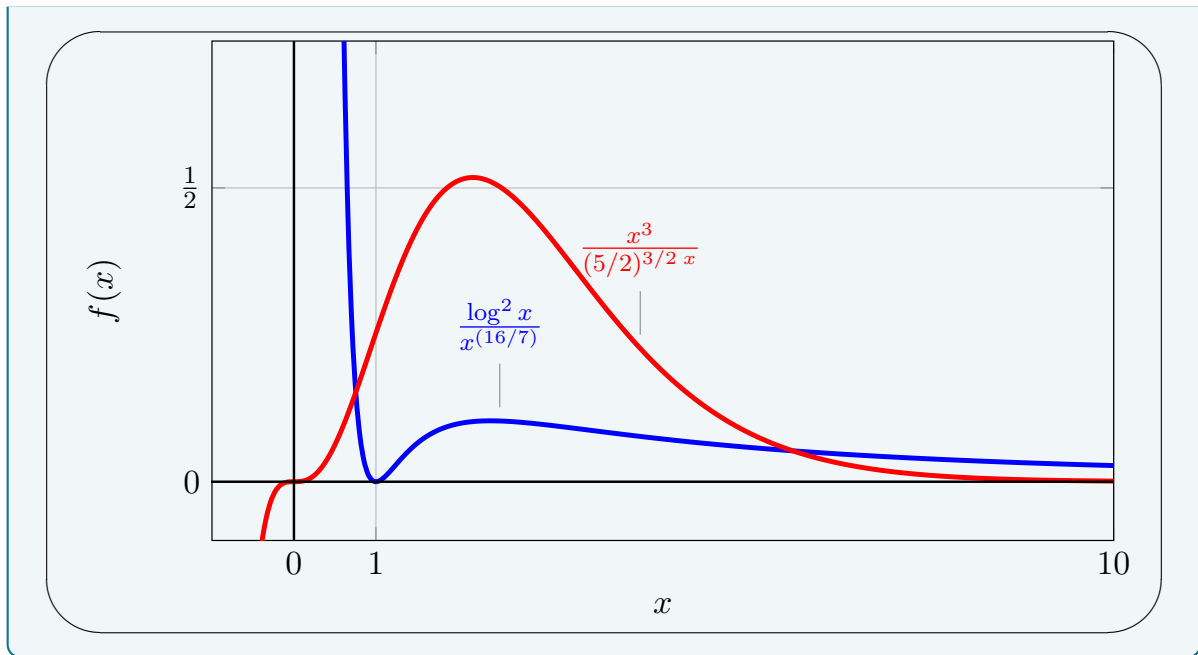
$$\lim_{x \rightarrow +\infty} \frac{\log_a^\beta x}{x^\alpha} = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} \frac{x^\alpha}{b^{\lambda x}} = 0$$

Proof. The theorem will be proved with the tools of differential calculus. □

- Any power with positive exponent is an infinity of higher order than any power of logarithms with base > 1 , and
- Any exponential with base > 1 is an infinity of higher order than any power with positive exponent.

Example 5.1: Comparison of infinities

$$\lim_{x \rightarrow +\infty} \frac{\log^2 x}{x^{(16/7)}} = 0, \quad \lim_{x \rightarrow +\infty} \frac{x^3}{(5/2)^{3/2} x} = 0$$



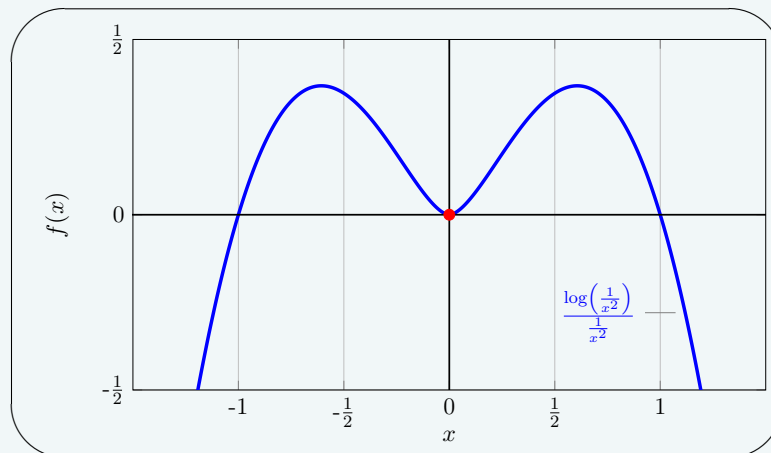
If a function $\eta(x) \rightarrow +\infty$ as $x \rightarrow c \in \mathbb{R}^*$, we have, from the comparison of infinities theorem:

$$\lim_{x \rightarrow c} \frac{(\log_a(\eta(x)))^\beta}{(\eta(x))^\alpha} = 0 \quad \text{and} \quad \lim_{x \rightarrow c} \frac{(\eta(x))^\alpha}{b^\lambda \eta(x)} = 0 \quad \text{with } \alpha, \beta, \lambda > 0, a, b > 1$$

Example 5.2: Comparison of infinities

$$\lim_{x \rightarrow 0} \frac{\log\left(\frac{1}{x^2}\right)}{\frac{1}{x^2}} = 0$$

since, with $\eta(x) = \frac{1}{x^2}$, we have $\eta(x) \rightarrow +\infty$ as $x \rightarrow 0$.



Observation 5.2

$$\lim_{x \rightarrow 0^+} x^\alpha \log_a^\beta x = 0 \quad (\alpha > 0, \beta = \frac{m}{n}, n, m \in \mathbb{N}, n \text{ odd}, a > 1)$$

Proof. Given $\alpha, \beta > 0, a > 1$ we have:

$$\lim_{x \rightarrow +\infty} \frac{\log_a^\beta x}{x^\alpha} = 0 \quad \text{and} \quad \frac{\log_a^\beta x}{x^\alpha} = \left(\frac{1}{x}\right)^\alpha \left(-\log_a \left(\frac{1}{x}\right)\right)^\beta$$

Setting $t = \frac{1}{x}$, $x \rightarrow +\infty$ is equivalent to $t \rightarrow 0^+$, and substituting we have:

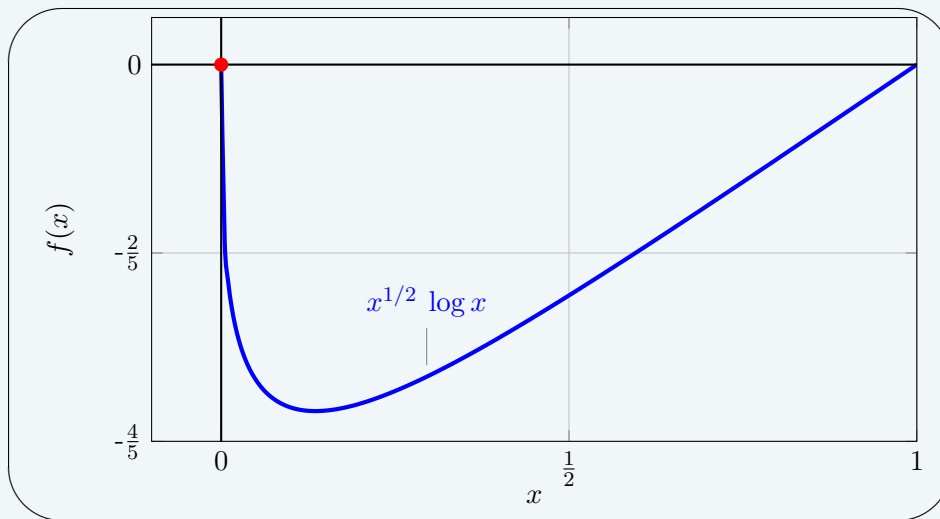
$$\lim_{x \rightarrow +\infty} \frac{\log_a^\beta x}{x^\alpha} = (-1)^\beta \lim_{t \rightarrow 0^+} t^\alpha \log_a^\beta t = 0$$

□

Example 5.3: Comparison of infinities

$$\lim_{x \rightarrow 0^+} x^{1/2} \log x = 0^-$$

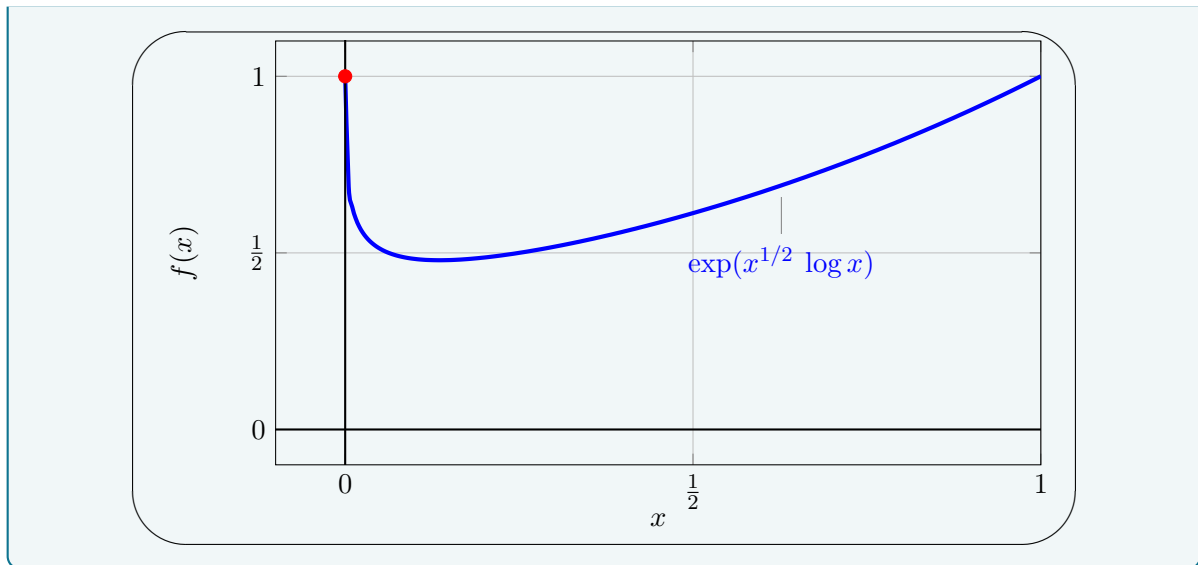
since we have $\alpha = \frac{1}{2} > 0, \beta = 1 > 0$ and $a = e > 1$.



$$\lim_{x \rightarrow 0^+} x^{\sqrt{x}} = 1^-$$

since

$$\lim_{x \rightarrow 0^+} x^{\sqrt{x}} = \lim_{x \rightarrow 0^+} \underbrace{e^{x^{1/2} \log x}}_{=\exp(x^{1/2} \log x)} = e^{0^-} = 1^-$$



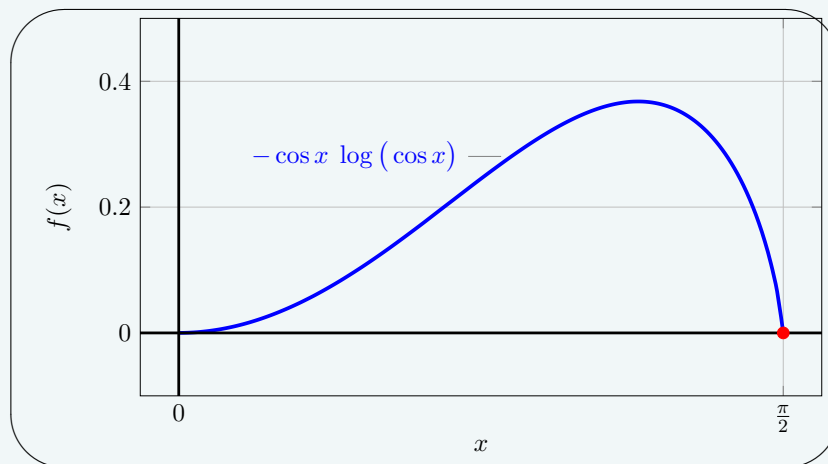
If a function $\varepsilon(x)$ tends to 0^+ as $x \rightarrow c \in \mathbb{R}^*$, it follows from the previous remark that:

$$\lim_{x \rightarrow c} (\varepsilon(x))^\alpha (\log_a(\varepsilon(x)))^\beta = 0 \quad (\alpha > 0, \beta = \frac{m}{n}, n, m \in \mathbb{N}, n \text{ odd}, a > 1)$$

Example 5.4: Comparison of infinities

$$\lim_{x \rightarrow \frac{\pi}{2}^-} -\cos x \log(\cos x) = 0^+$$

since, with $\varepsilon(x) = \cos x$, we have $\varepsilon(x) \rightarrow 0^+$ as $x \rightarrow \frac{\pi}{2}^-$ and $\alpha = \beta = 1$.



From the comparison of infinities theorem it follows that:

$$\lim_{x \rightarrow +\infty} \frac{x^\alpha}{\log_a^\beta x} = +\infty \quad \text{and} \quad \lim_{x \rightarrow +\infty} \frac{b^{\lambda x}}{x^\alpha} = +\infty \quad \text{with} \quad \alpha, \beta, \lambda > 0, \quad a, b > 1$$

Observation 5.3

$$\lim_{x \rightarrow 0^+} x^\alpha b^{\lambda/x} = +\infty \quad (\alpha, \lambda > 0, b > 1)$$

Proof. Given $\alpha, \lambda > 0, b > 1$, we have:

$$\lim_{x \rightarrow +\infty} \frac{b^{\lambda x}}{x^\alpha} = +\infty$$

Setting $t = \frac{1}{x}$, $x \rightarrow +\infty$ is equivalent to $t \rightarrow 0^+$, and substituting we have:

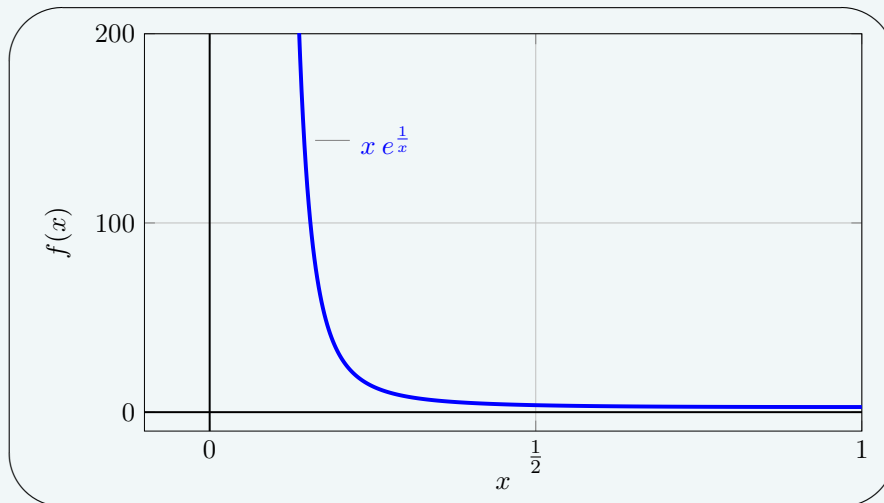
$$\lim_{x \rightarrow +\infty} \frac{b^{\lambda x}}{x^\alpha} = \lim_{x \rightarrow +\infty} \left(\frac{1}{x}\right)^\alpha b^{\lambda x} = \lim_{t \rightarrow 0^+} t^\alpha b^{\lambda/t} = +\infty$$

□

Example 5.5: Comparison of infinities

$$\lim_{x \rightarrow 0^+} x e^{\frac{1}{x}} = +\infty$$

since we have $\alpha = \lambda = 1$ and $b = e > 1$


Observation 5.4

$$\lim_{x \rightarrow 0^+} x^{-\alpha} b^{-\lambda/x} = 0 \quad (\alpha, \lambda > 0, b > 1)$$

Proof. Given $\alpha, \lambda > 0, b > 1$ we have:

$$\lim_{x \rightarrow +\infty} \frac{x^\alpha}{b^{\lambda x}} = 0$$

Setting $t = \frac{1}{x}$, $x \rightarrow +\infty$ is equivalent to $t \rightarrow 0^+$, and substituting we have:

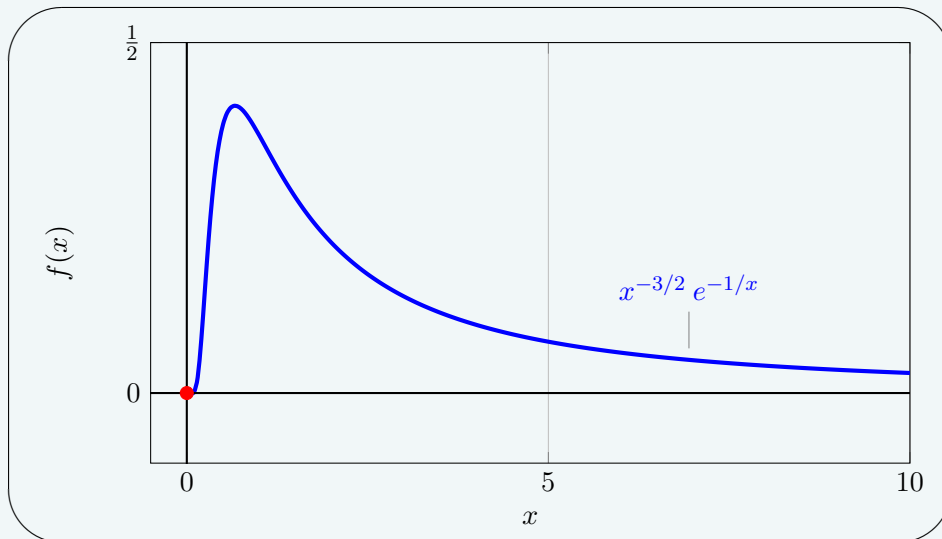
$$\lim_{x \rightarrow +\infty} \frac{x^\alpha}{b^{\lambda x}} = \lim_{x \rightarrow +\infty} \left(\frac{1}{x}\right)^{-\alpha} \frac{1}{b^{\lambda x}} = \lim_{t \rightarrow 0^+} t^{-\alpha} b^{-\lambda/t} = 0$$

□

Example 5.6: Comparison of infinities

$$\lim_{x \rightarrow 0^+} x^{-3/2} e^{-1/x} = 0$$

since we have $\alpha = \frac{3}{2}, \lambda = 1$ and $b = e > 1$.



Observation 5.5

$$\lim_{x \rightarrow -\infty} x^\alpha b^{\lambda x} = 0 \quad \left(\alpha = \frac{m}{n} > 0, n, m \in \mathbb{N}, n \text{ odd}, m \neq 0, \lambda > 0, b > 1 \right)$$

Proof. Given $\alpha, \lambda > 0, b > 1$ we have:

$$\lim_{x \rightarrow +\infty} \frac{x^\alpha}{b^{\lambda x}} = 0$$

Setting $t = -x$, $x \rightarrow +\infty$ is equivalent to $t \rightarrow -\infty$, and substituting we have:

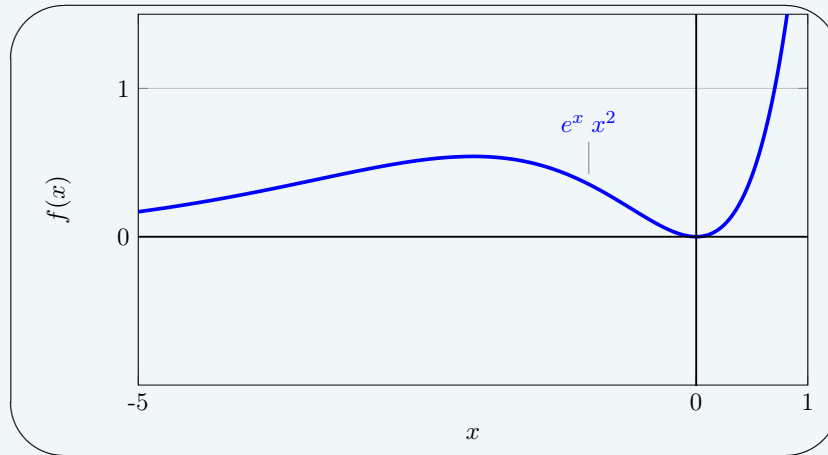
$$\lim_{x \rightarrow +\infty} \frac{x^\alpha}{b^{\lambda x}} = \lim_{t \rightarrow -\infty} \frac{(-t)^\alpha}{b^{-\lambda t}} = (-1)^\alpha \lim_{t \rightarrow -\infty} t^\alpha b^{\lambda t} = 0$$

□

Example 5.7: Comparison of infinities

$$\lim_{x \rightarrow -\infty} x^2 e^x = 0$$

since we have $\alpha = 2, \lambda = 1$ and $b = e > 1$.



Fundamental limits and asymptotic estimates

6.1 Fundamental limits

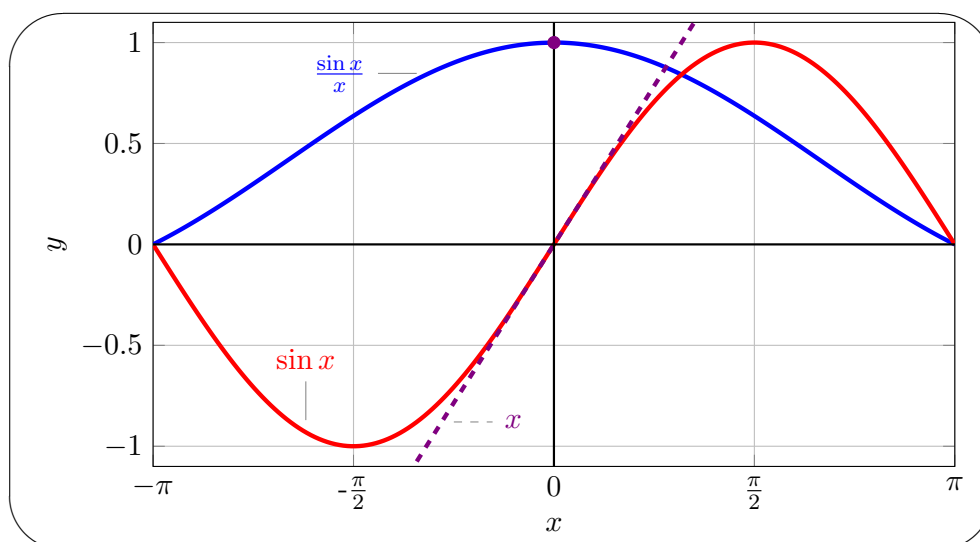
- We now look at some common techniques for computing limits that combine the general theorems on limits with the use of some fundamental limits of certain elementary functions.

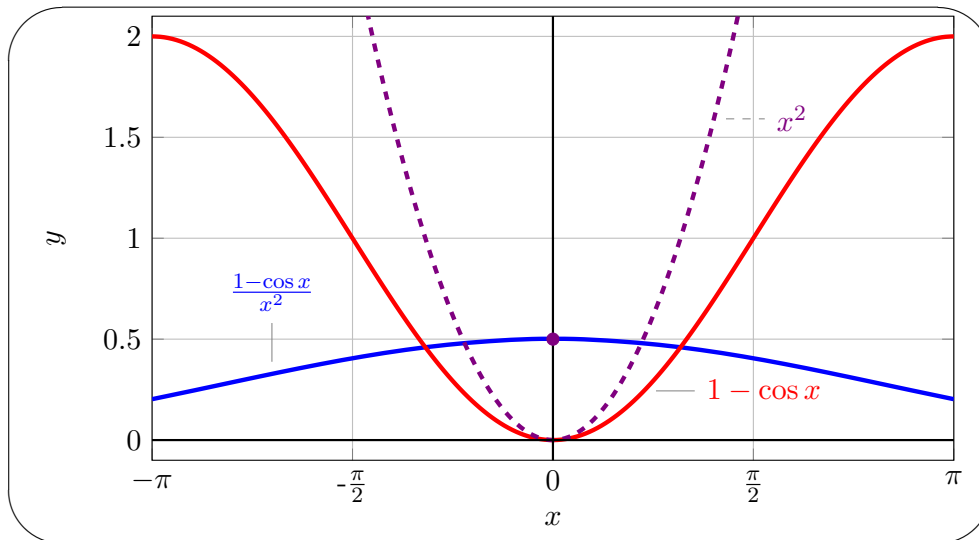
6.1.1 Fundamental limits of sine and cosine

- We want to compute

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \left[\frac{0}{0} \right] \quad \text{and} \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \left[\frac{0}{0} \right]$$

which are indeterminate forms.



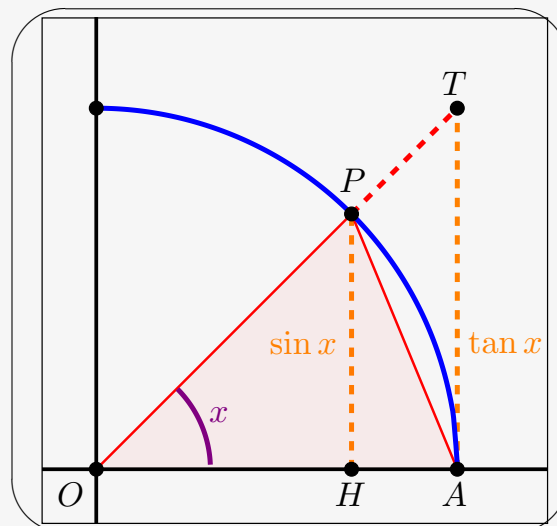

Lemma 6.1

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (6.1)$$

Proof. The functions $\sin x$ and x are odd functions, hence $\frac{\sin x}{x}$ is an even function. Therefore it is sufficient to compute

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x}$$

We have:



$$HP = \sin x \quad AT = \tan x \quad \widehat{AP} = x$$

The area of the triangle OPA is smaller than that of the circular sector OPA , which in

turn is smaller than that of the triangle OTA . It follows that

$$\underbrace{\frac{1}{2} \cdot 1 \cdot \sin x}_{\text{area triangle } OPA} \leq \underbrace{\frac{1}{2} \cdot 1 \cdot x}_{\text{area circ. sector } OPA} \leq \underbrace{\frac{1}{2} \cdot 1 \cdot \tan x}_{\text{area triangle } OTA}$$

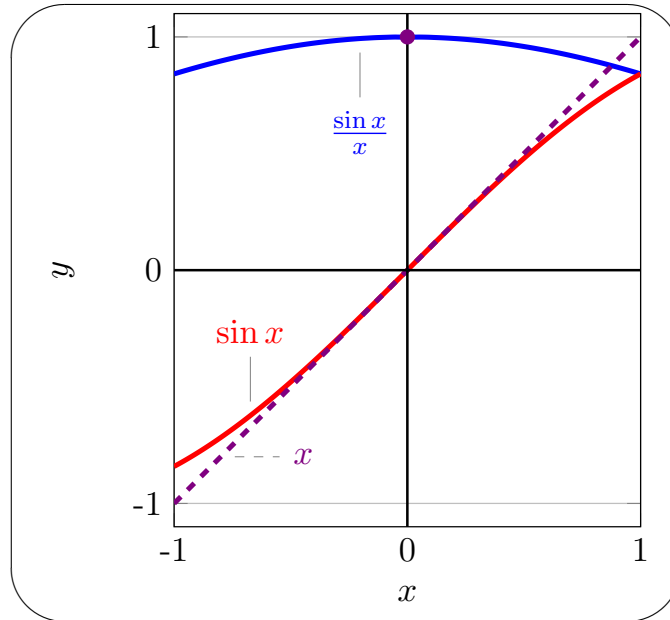
that is, for $x \in (0, \frac{\pi}{2})$:

$$\sin x < x < \tan x$$

Dividing by $\sin x$, which is positive because $x \in (0, \frac{\pi}{2})$, we have

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}, \quad \forall x \in (0, \frac{\pi}{2}) \quad \text{that is} \quad \cos x < \frac{\sin x}{x} < 1, \quad \forall x \in (0, \frac{\pi}{2})$$

By the comparison theorem, since $\lim_{x \rightarrow 0} \cos x = 1$, the lemma follows. $\square$



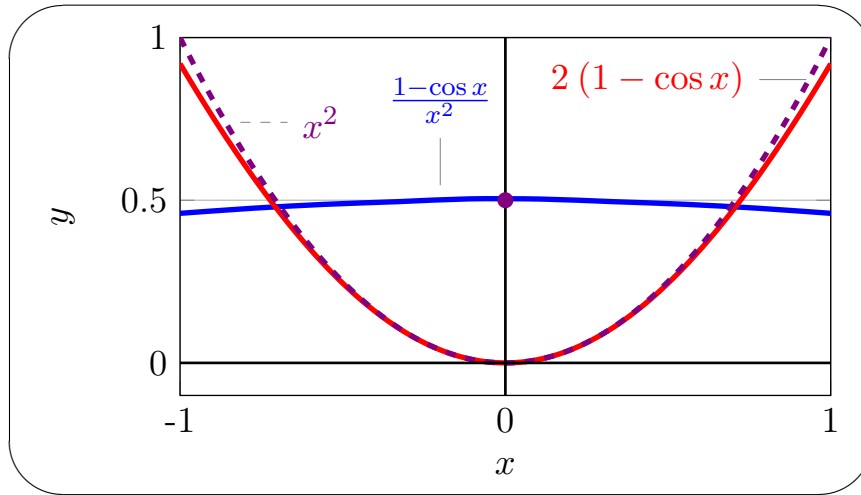
Lemma 6.2

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2} \quad (6.2)$$

Proof. We consider

$$\frac{1 - \cos x}{x^2} = \frac{1 - \cos^2 x}{x^2 (1 + \cos x)} = \left(\frac{\sin x}{x} \right)^2 \frac{1}{1 + \cos x}$$

hence, since $\frac{\sin x}{x} \rightarrow 1$ and $(1 + \cos x) \rightarrow 2$ as $x \rightarrow 0$, the claim follows. $\square$



6.1.2 Continuous extension of a function

- Based on the limit (6.1) just proved, the functions $f(x) = \frac{\sin x}{x}$ and $g(x) = \frac{1-\cos x}{x^2}$, initially not defined at $x = 0$, can be extended by continuity also at $x = 0$, by setting

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \quad g(x) = \begin{cases} \frac{1-\cos x}{x^2} & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$$

The functions f and g defined in this way are continuous also at $x = 0$.

- if a function $f(x)$ is not defined at x_0 but the finite limit exists

$$\lim_{x \rightarrow x_0} f(x) = \ell$$

the function can be extended by continuity also at x_0 , by defining

$$f(x_0) = \ell$$

- If instead the function f has at x_0 a jump discontinuity, a vertical asymptote, or in any case does not have a finite limit, it is not possible to make it continuous at x_0 by changing its definition at a single point.

6.1.3 Other fundamental limits

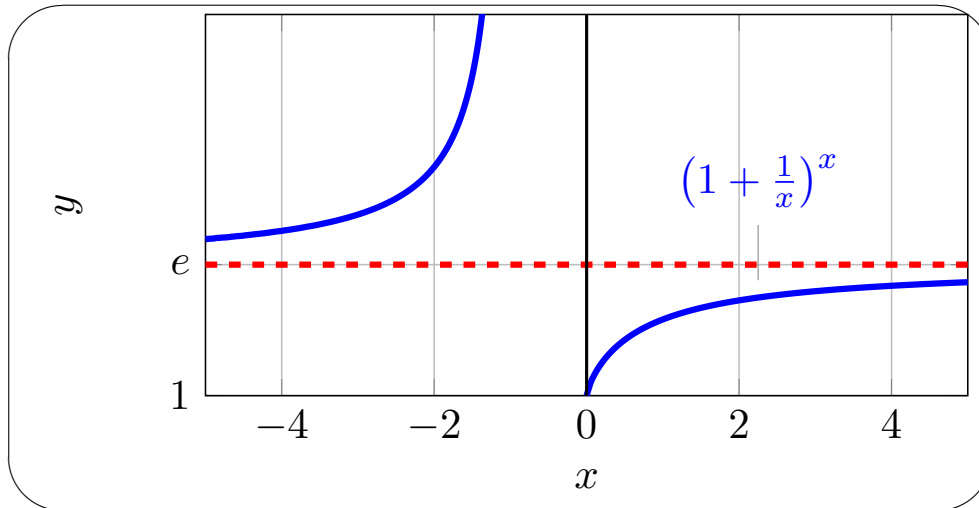
- We know that for every sequence $\{a_n\}$ diverging to $+\infty$ or $-\infty$ we have

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{a_n}\right)^{a_n} = e$$

- By the sequential definition of the limit of a function, this fact immediately implies the next fundamental limit

Lemma 6.3

$$\lim_{x \rightarrow \pm\infty} \left(1 + \frac{1}{x}\right)^x = e \quad (6.3)$$



If $\eta(x)$ is a function that tends to infinity^a (i.e., it is an infinity: $\eta(x) \rightarrow \pm\infty$), we have:

$$\left(1 + \frac{1}{\eta(x)}\right)^{\eta(x)} \rightarrow e \quad (6.4)$$

^ait does not matter what x tends to

Example 6.1: Limits of functions tending to e

1.

$$\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{2x^2 - 10}\right)^{2x^2 - 10} = e$$

since with $\eta(x) = 2x^2 - 10$ we have

$$\eta(x) \rightarrow +\infty \text{ as } x \rightarrow +\infty.$$

2.

$$\lim_{x \rightarrow 0^+} \left(1 + \frac{1}{1/x}\right)^{1/x} = e$$

since with $\eta(x) = \frac{1}{x}$ we have

$$\eta(x) \rightarrow +\infty \text{ as } x \rightarrow 0^+.$$

From the fundamental limit (6.3) three more can be deduced

Corollary 6.4

$$\lim_{y \rightarrow 0} \frac{\log(1+y)}{y} = 1 \quad (6.5)$$

Proof. Taking logarithms in (6.3), we obtain

$$\log \left(1 + \frac{1}{x} \right)^x = x \log \left(1 + \frac{1}{x} \right)$$

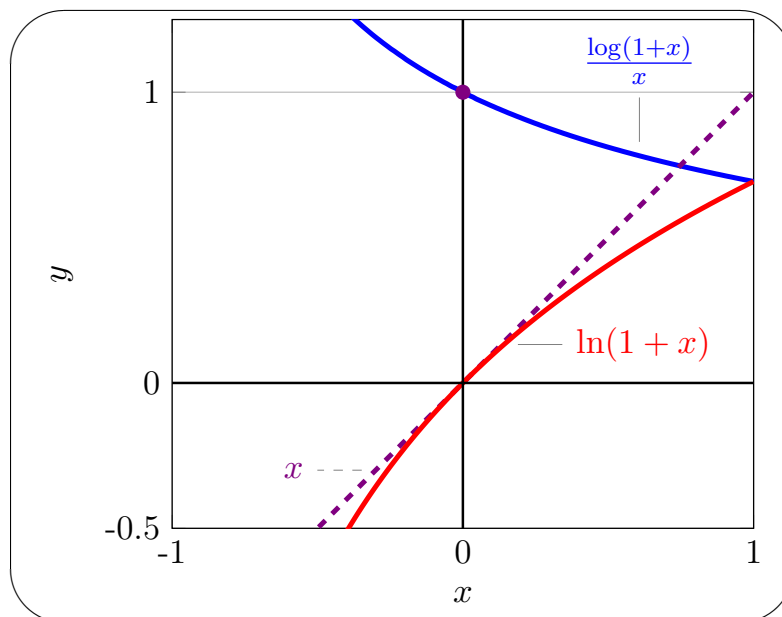
hence

$$\lim_{x \rightarrow \pm\infty} x \log \left(1 + \frac{1}{x} \right) = \log e = 1.$$

Now, if we set $y = \frac{1}{x}$, $x \rightarrow \pm\infty$ is equivalent to $y \rightarrow 0^\pm$, and the last limit can therefore be rewritten in the following form:

$$\frac{\log(1+y)}{y} \rightarrow 1 \text{ as } y \rightarrow 0.$$

□

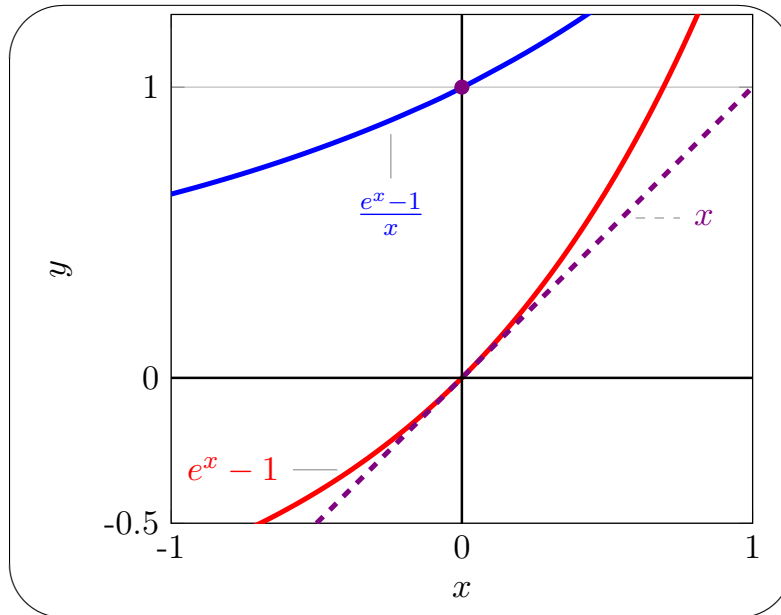
**Corollary 6.5**

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \quad (6.6)$$

Proof. If in (6.5) we instead set $y = e^x - 1$, $y \rightarrow 0$ is equivalent to $x \rightarrow 0$, and substituting we obtain

$$\frac{\log e^x}{e^x - 1} = \frac{x}{e^x - 1} \rightarrow 1 \quad \text{as } x \rightarrow 0.$$

Taking reciprocals we obtain the fundamental limit. □



Example 6.2: Fundamental limit

We compute:

$$\lim_{x \rightarrow 0} \frac{e^{-x} - 1}{x}$$

We set $z = -x$; $x \rightarrow 0$ is equivalent to $z \rightarrow 0$, and substituting we obtain:

$$\lim_{x \rightarrow 0} \frac{e^{-x} - 1}{x} = \lim_{z \rightarrow 0} \frac{e^z - 1}{-z} = - \lim_{z \rightarrow 0} \underbrace{\frac{e^z - 1}{z}}_{\rightarrow 1} = -1$$

Corollary 6.6

$$\lim_{x \rightarrow 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \quad \text{with } \alpha \in \mathbb{R} \quad (6.7)$$

Proof. If in (6.5) we instead set $y = (1+x)^\alpha - 1$, with α any real exponent, then $x \rightarrow 0$ is equivalent to $y \rightarrow 0$ and we have:

$$\begin{aligned} \frac{\log(1+y)}{y} &= \frac{\log[1+(1+x)^\alpha - 1]}{(1+x)^\alpha - 1} = \frac{\alpha \log(1+x)}{(1+x)^\alpha - 1} \\ &= \frac{\alpha x}{(1+x)^\alpha - 1} \cdot \frac{\log(1+x)}{x} \rightarrow 1 \quad \text{as } x \rightarrow 0. \end{aligned}$$

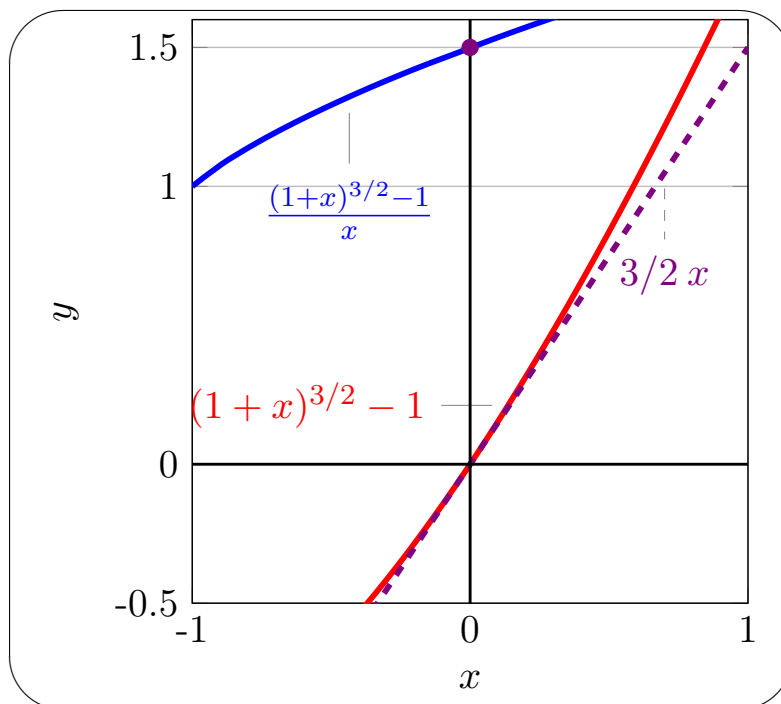
But since also

$$\frac{\log(1+x)}{x} \rightarrow 1 \quad \text{as } x \rightarrow 0,$$

then

$$\frac{\alpha x}{(1+x)^\alpha - 1} \rightarrow 1 \quad \text{as } x \rightarrow 0.$$

Taking reciprocals and then multiplying by α we obtain the fundamental limit. □



6.2 Asymptotic estimates

Definition 6.7: Asymptotic functions

Two functions f , g are said to be **asymptotic** as $x \rightarrow c$ if

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = 1$$

and we write $f \sim g$ as $x \rightarrow c$.

- The asymptotic symbol for functions enjoys all the properties stated for sequences

As $x \rightarrow 0$ we have (from the fundamental limits):

$$\sin x \sim x \quad (6.8a)$$

$$\log(1+x) \sim x \quad (6.8b)$$

$$\cos x \sim 1 - \frac{1}{2}x^2 \quad (6.8c)$$

$$e^x \sim 1 + x \quad (6.8d)$$

$$(1+x)^\alpha \sim 1 + \alpha x \quad \text{with } \alpha \in \mathbb{R} \quad (6.8e)$$

- The functions $\sim x$ behave, to a first approximation or to first order, like x as $x \rightarrow 0$.

If $\varepsilon(x)$ is a function that tends to zero^a (i.e., it is an infinitesimal: $\varepsilon(x) \rightarrow 0$), we have:

$$\sin(\varepsilon(x)) \sim \varepsilon(x) \quad (6.9a)$$

$$\log(1 + \varepsilon(x)) \sim \varepsilon(x) \quad (6.9b)$$

$$\cos(\varepsilon(x)) \sim 1 - \frac{1}{2}\varepsilon^2(x) \quad (6.9c)$$

$$e^{\varepsilon(x)} \sim 1 + \varepsilon(x) \quad (6.9d)$$

$$(1 + \varepsilon(x))^\alpha \sim 1 + \alpha \varepsilon(x) \quad \text{with } \alpha \in \mathbb{R} \quad (6.9e)$$

^ait does not matter what x tends to

- Formulas (6.1.3) follow from formulas (6.8) simply by a change of variable: replace x with $\varepsilon(x)$.

Example 6.3: Limits with asymptotic functions

$$\lim_{x \rightarrow 1} \frac{(x-1)^2}{e^{3(x-1)^2} - 1} = \left[\frac{0}{0} \right]$$

We use the estimate

$$e^{\varepsilon(x)} - 1 \sim \varepsilon(x)$$

with

$$\varepsilon(x) = 3(x-1)^2 \quad \text{and} \quad \varepsilon(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow 1$$

Then, as $x \rightarrow 1$, we have

$$e^{3(x-1)^2} - 1 \sim 3(x-1)^2.$$

and

$$\lim_{x \rightarrow 1} \frac{(x-1)^2}{e^{3(x-1)^2} - 1} = \lim_{x \rightarrow 1} \frac{(x-1)^2}{3(x-1)^2} = \frac{1}{3}$$

Example 6.4: Limits with asymptotic functions

$$\lim_{x \rightarrow 0} \frac{\log(1+2x)}{\sin 3x} = \left[\frac{0}{0} \right]$$

We use the estimate

$$\log(1+\varepsilon(x)) \sim \varepsilon(x)$$

with

$$\varepsilon(x) = 2x \quad \text{and} \quad \varepsilon(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow 0.$$

Hence, as $x \rightarrow 0$ we have

$$\log(1+2x) \sim 2x$$

Now we also use the estimate

$$\sin \varepsilon(x) \sim \varepsilon(x)$$

with

$$\varepsilon(x) = 3x \quad \text{and} \quad \varepsilon(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow 0$$

Hence, as $x \rightarrow 0$ we have

$$\sin 3x \sim 3x$$

Then, as $x \rightarrow 0$, we have

$$\frac{\log(1+2x)}{\sin 3x} \sim \frac{2x}{3x}$$

and

$$\lim_{x \rightarrow 0} \frac{\log(1+2x)}{\sin 3x} = \lim_{x \rightarrow 0} \frac{2x}{3x} = \frac{2}{3}$$

Example 6.5: Limits with asymptotic functions

$$\lim_{x \rightarrow +\infty} \sqrt[3]{x^3 + 2x^2 + 1} - x = [+ \infty - \infty]$$

We rewrite,

$$\sqrt[3]{x^3 + 2x^2 + 1} - x = x \left(\sqrt[3]{1 + \left(\frac{2}{x} + \frac{1}{x^3} \right)} - 1 \right)$$

We use the estimate

$$\sqrt[3]{1 + \varepsilon(x)} - 1 \sim \frac{1}{3}\varepsilon(x)$$

with

$$\varepsilon(x) = \left(\frac{2}{x} + \frac{1}{x^3} \right) \quad \text{and} \quad \varepsilon(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow +\infty.$$

Then, as $x \rightarrow +\infty$, we have

$$\sqrt[3]{x^3 + 2x^2 + 1} - x \sim x \left[\frac{1}{3} \left(\frac{2}{x} + \frac{1}{x^3} \right) \right]$$

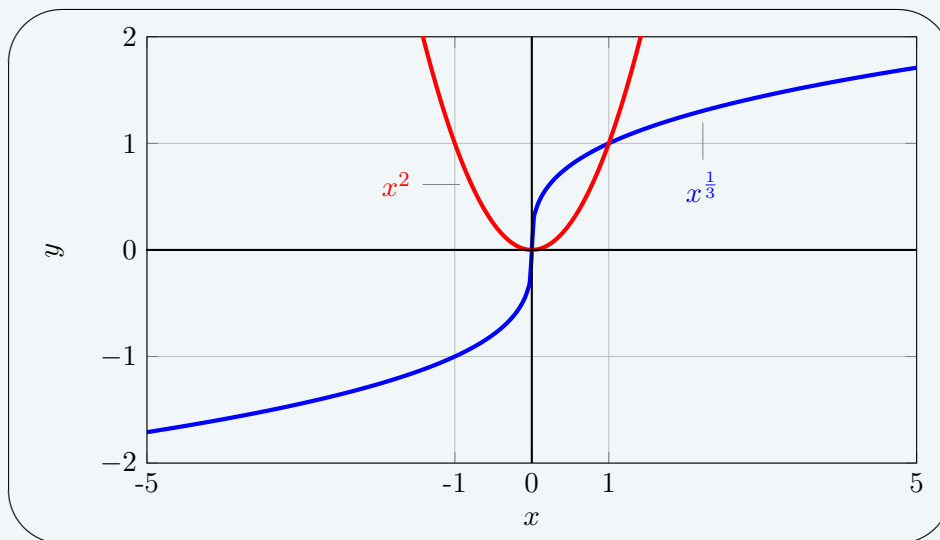
and

$$\lim_{x \rightarrow +\infty} \sqrt[3]{x^3 + 2x^2 + 1} - x = \lim_{x \rightarrow +\infty} x \left[\frac{1}{3} \left(\frac{2}{x} + \frac{1}{x^3} \right) \right] = \frac{2}{3}$$

6.3 Asymptotic estimates and graphs

- Asymptotic estimates are useful not only for computing limits, but also for sketching the qualitative graph of a function in a neighborhood of a given point, or as $x \rightarrow \pm\infty$.

Example 6.6: Known graphs



Example 6.7: Qualitative graph

We now want to study the qualitative graph of the function (the sum of the previous two):

$$f(x) = \sqrt[3]{x} + x^2 = x^{\frac{1}{3}} + x^2$$

- the function is defined and continuous on all of $\mathbb{R}$;
- as $x \rightarrow \pm\infty$, $f(x) \sim x^2$, since

$$\lim_{x \rightarrow \pm\infty} \frac{x^{\frac{1}{3}} + x^2}{x^2} = 1$$

Therefore $f(x) \rightarrow +\infty$ as $x \rightarrow \pm\infty$; moreover, for x large in absolute value, its graph will be similar to that of x^2 .

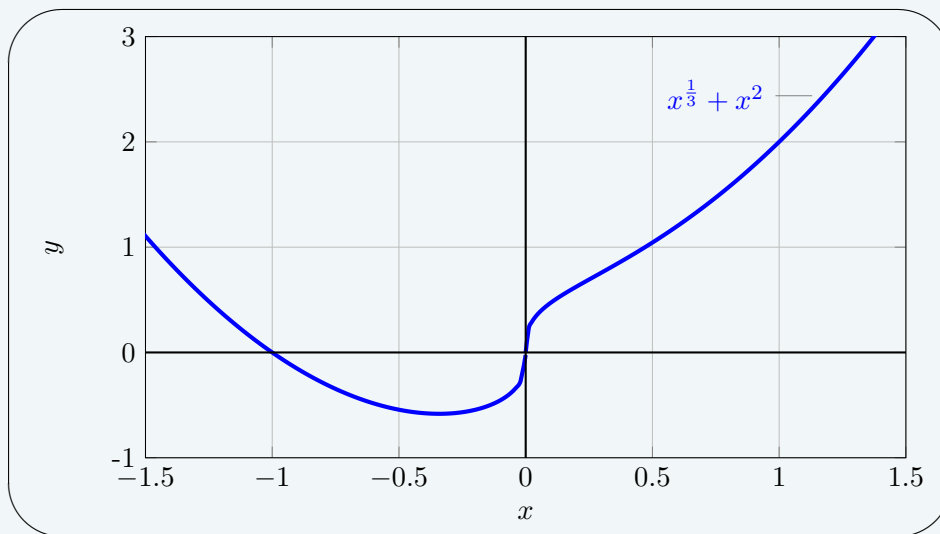
- Moreover, the function vanishes at $x = 0$ and, as $x \rightarrow 0$, $f(x) \sim x^{\frac{1}{3}}$, since

$$\lim_{x \rightarrow 0} \frac{x^{\frac{1}{3}} + x^2}{x^{\frac{1}{3}}} = 1$$

therefore, in a neighborhood of $x = 0$, its graph will be similar to that of $x^{\frac{1}{3}}$; in particular, it will have a vertical tangent at the origin.

Example 6.8: Actual graph

$$f(x) = \sqrt[3]{x} + x^2 = x^{\frac{1}{3}} + x^2$$



- Often the behavior of a function in a neighborhood of a point (for example, the fact that it has a vertical or horizontal tangent) can be predicted from a suitable asymptotic estimate.
- The asymptotic estimate allows us to sketch the qualitative graph of f (in a neighborhood of the point) by comparison with that of a known function (for example, a power with rational exponent).
- Analogous estimates are useful as $x \rightarrow \pm\infty$.

Growth of a function at infinity

- Suppose we want to sketch the graph of a function f that, as $x \rightarrow +\infty$ (or $-\infty$), tends to $+\infty$ (or $-\infty$).
- To describe the speed at which the function tends to infinity, the following notions are useful: we say that, as $x \rightarrow +\infty$,

$$f \text{ has } \begin{cases} \text{superlinear growth} \\ \text{linear growth} \\ \text{sublinear growth} \end{cases} \quad \text{if} \quad \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \begin{cases} \pm\infty \\ m \text{ (finite and different from 0)} \\ 0 \end{cases}$$

- Analogous definitions are given for $x \rightarrow -\infty$.
- Only when a function has linear growth is it possible for it to have an oblique asymptote

Example 6.9: Growth of a function at infinity

For example, as $x \rightarrow +\infty$

- Superlinear growth: exponentials a^x and powers x^a with $a > 1$.
- Sublinear growth: logarithms $\log_a x$ and powers x^a with $0 < a < 1$.

Example 6.10: Growth of a function at infinity

The function

$$f(x) = 2x + e^x + e^{\frac{1}{x}}$$

- as $x \rightarrow +\infty$ it is asymptotic to e^x ; therefore it tends to $+\infty$ with superlinear growth;
- as $x \rightarrow -\infty$ it is asymptotic to $2x$; therefore it tends to $-\infty$ linearly
- since

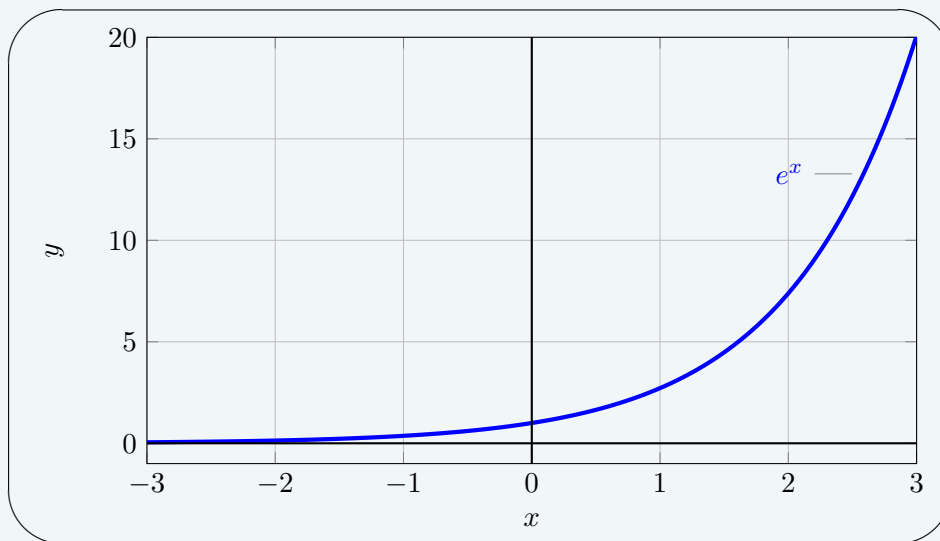
$$\lim_{x \rightarrow -\infty} [f(x) - 2x] = 1$$

the function has the oblique asymptote

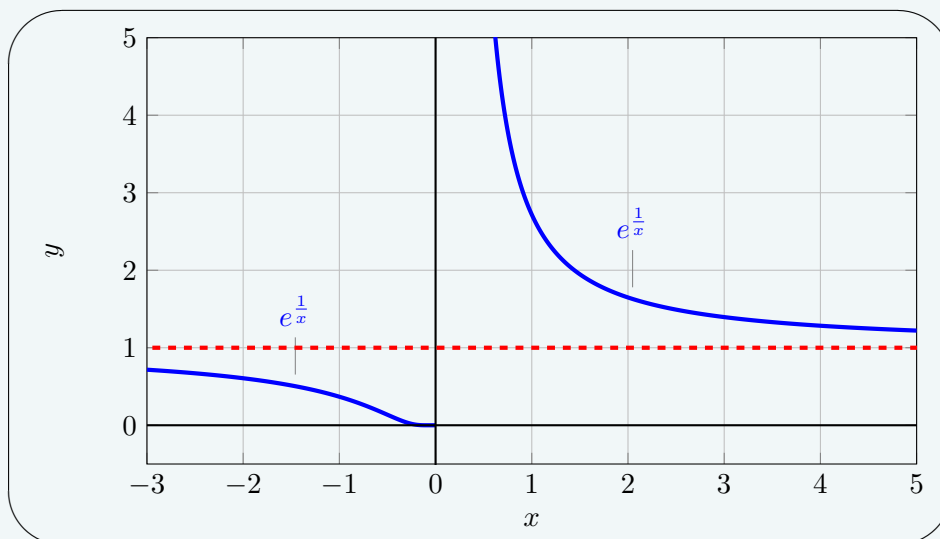
$$y = 2x + 1 \quad \text{as } x \rightarrow -\infty$$

Example 6.11: Graphs

$$f(x) = e^x, \quad \lim_{x \rightarrow +\infty} e^x = +\infty, \quad \lim_{x \rightarrow -\infty} e^x = 0, \quad \lim_{x \rightarrow 0} e^x = 1$$

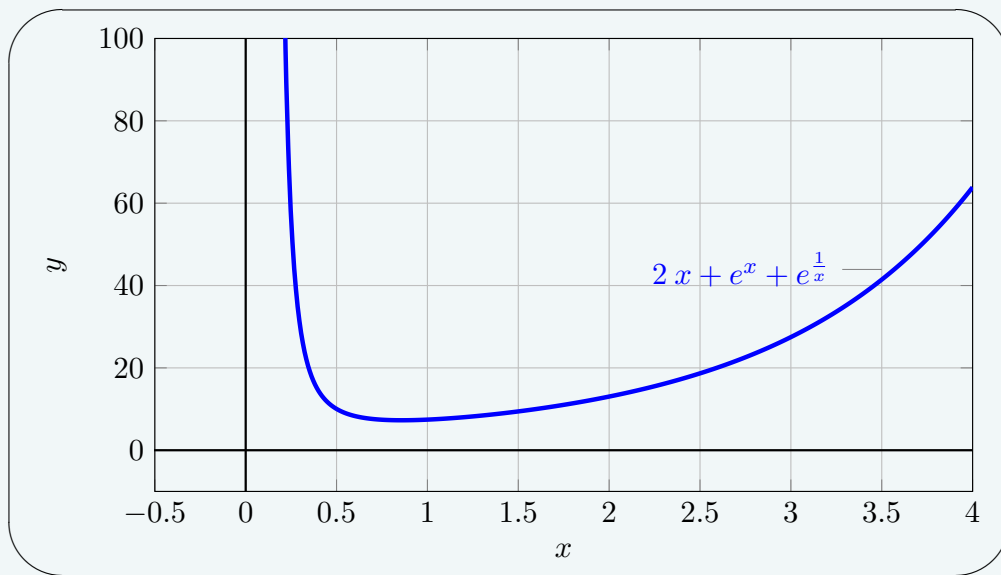


$$f(x) = e^{\frac{1}{x}}, \quad \lim_{x \rightarrow +\infty} e^{\frac{1}{x}} = 1, \quad \lim_{x \rightarrow -\infty} e^{\frac{1}{x}} = 1, \quad \lim_{x \rightarrow 0^-} e^{\frac{1}{x}} = 0, \quad \lim_{x \rightarrow 0^+} e^{\frac{1}{x}} = +\infty$$

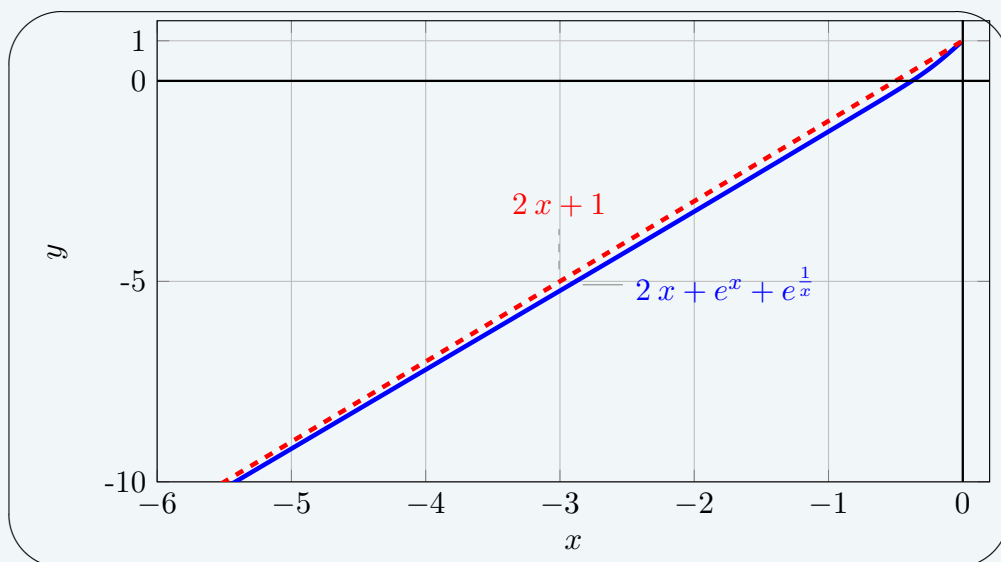


Example 6.12: Actual graph

$$f(x) = 2x + e^x + e^{\frac{1}{x}}, \quad f(x) \sim e^x \text{ as } x \rightarrow +\infty, \quad \lim_{x \rightarrow 0^+} f(x) = +\infty$$



$$f(x) = 2x + e^x + e^{\frac{1}{x}}, \quad f(x) \sim 2x \text{ as } x \rightarrow -\infty, \quad \lim_{x \rightarrow 0^-} f(x) = 1$$



Asymptotic expansions

7.1 The “little o ” symbol and asymptotic expansions

Definition 7.1: of little o

Given two functions $f(x)$ and $g(x)$, defined in a neighborhood of $c \in \mathbb{R}^*$, we say that

$$f(x) = o(g(x)) \quad \text{as } x \rightarrow c$$

(read “ $f(x)$ is little o of $g(x)$ ” as $x \rightarrow c$) if and only if

$$\frac{f(x)}{g(x)} \rightarrow 0 \quad \text{as } x \rightarrow c$$

The symbol $o(g(x))$ as x tends to c does not denote a particular function $f(x)$, but any function $f(x)$ such that the ratio between $f(x)$ and $g(x)$ tends to 0 as x tends to c .

Example 7.1: Little o

For example:

$$x^2 = o(x) \quad \text{as } x \rightarrow 0 \quad \text{since} \quad \frac{x^2}{x} \rightarrow 0 \quad \text{as } x \rightarrow 0$$

$$x^3 = o(x) \quad \text{as } x \rightarrow 0 \quad \text{since} \quad \frac{x^3}{x} \rightarrow 0 \quad \text{as } x \rightarrow 0$$

$$x^3 = o(x^2) \quad \text{as } x \rightarrow 0 \quad \text{since} \quad \frac{x^3}{x^2} \rightarrow 0 \quad \text{as } x \rightarrow 0$$

$$e^{-1/x^2} = o(x^4) \quad \text{as } x \rightarrow 0 \quad \text{since} \quad \frac{e^{-1/x^2}}{x^4} \rightarrow 0 \quad \text{as } x \rightarrow 0$$

- From the definition of asymptotically equivalent functions, we have:

$$f(x) \sim g(x) \quad \text{as } x \rightarrow c \iff \frac{f(x)}{g(x)} \rightarrow 1 \quad \text{as } x \rightarrow c$$

in these cases we have:

$$\frac{f(x)}{g(x)} - 1 \rightarrow 0 \text{ as } x \rightarrow c \quad \text{and} \quad \frac{f(x) - g(x)}{g(x)} \rightarrow 0 \text{ as } x \rightarrow c$$

from the definition of little o we can then write:

$$f(x) - g(x) = o(g(x)) \text{ as } x \rightarrow c$$

We have the following *asymptotic expansion*:

$$f(x) \sim g(x) \text{ as } x \rightarrow c \iff f(x) = g(x) + o(g(x)) \text{ as } x \rightarrow c$$

- Moreover, we have:

$$g(x) + f(x) \sim g(x) \text{ as } x \rightarrow c \iff \frac{f(x)}{g(x)} \rightarrow 0 \text{ as } x \rightarrow c,$$

since:

$$\frac{g(x) + f(x)}{g(x)} = 1 + \frac{f(x)}{g(x)}$$

which tends to 1 as $x \rightarrow c$ if and only if $f(x)/g(x)$ tends to 0 as $x \rightarrow c$.

In such situations we say that $g(x)$ is the **principal part** of the sum $g(x) + f(x)$ and that $f(x)$ is **negligible** with respect to $g(x)$ as $x \rightarrow c$; that is:

$$f(x) = o(g(x)) \text{ as } x \rightarrow c$$

We have the following *asymptotic expansion*:

$$g(x) + f(x) \sim g(x) \text{ as } x \rightarrow c \iff f(x) = o(g(x)) \text{ as } x \rightarrow c$$

- It follows from the definition of little o that with three functions we have:

$$f(x) = h(x) + o(g(x)) \text{ as } x \rightarrow c \iff \frac{f(x) - h(x)}{g(x)} \rightarrow 0 \text{ as } x \rightarrow c$$

- The “little o ” symbol behaves as follows with products:

$$f \cdot o(g) = o(f \cdot g)$$

$$o(f) \cdot o(g) = o(f \cdot g)$$

Example 7.2: Little o and products

For example, as $x \rightarrow c$:

$$x \cdot o(x^2) = o(x^3), \quad \frac{o(x^3)}{x} = o(x^2), \quad o(x) \cdot o(x^2) = o(x^3)$$

Some properties of the little o symbol, shown through examples:

$$\begin{aligned} o(x) \pm o(x) &= o(x), & \text{as } x \rightarrow c \\ o(ax) &= o(x), & \forall a \in \mathbb{R}, a \neq 0, \text{ as } x \rightarrow c \\ a o(x) &= o(x), & \forall a \in \mathbb{R}, \text{ as } x \rightarrow c \\ o(x) + o(x^2) &= o(x), & \text{as } x \rightarrow 0 \\ o(x) + o(x^2) &= o(x^2), & \text{as } x \rightarrow \infty \end{aligned}$$

Definition 7.2: of $o(1)$

A function f is said to be infinitesimal as $x \rightarrow c$, and we write

$$f(x) = o(1) \text{ as } x \rightarrow c \text{ if } f(x) \rightarrow 0 \text{ as } x \rightarrow c$$

Example 7.3: $o(1)$

For example:

$$\begin{aligned} \sqrt{x} &= o(1) \text{ as } x \rightarrow 0^+ \text{ since } \frac{\sqrt{x}}{1} \rightarrow 0 \text{ as } x \rightarrow 0^+ \\ x^{-2} &= o(1) \text{ as } x \rightarrow +\infty \text{ since } \frac{x^{-2}}{1} \rightarrow 0 \text{ as } x \rightarrow +\infty \end{aligned}$$

- We have:

$$\lim_{x \rightarrow c} f(x) = \ell \iff \lim_{x \rightarrow c} (f(x) - \ell) = 0 \iff \lim_{x \rightarrow c} |f(x) - \ell| = 0$$

hence

$$\lim_{x \rightarrow c} f(x) = \ell \iff f(x) = \ell + o(1) \text{ as } x \rightarrow c$$

the last expression reads “ $f(x)$ is the sum of ℓ and of a function that is infinitesimal as $x \rightarrow c$ ”

Example 7.4: $o(1)$

For example:

$$\begin{aligned} x^2 &= x(1 + o(1)) \text{ as } x \rightarrow 1 \text{ since } \frac{x^2}{x} = x \rightarrow 1 \text{ as } x \rightarrow 1 \\ x^2 + 3x &= x(3 + o(1)) \text{ as } x \rightarrow 0 \text{ since } \frac{x^2 + 3x}{x} = x + 3 \rightarrow 3 \text{ as } x \rightarrow 0 \\ x^2 + 3x &= x^2(1 + o(1)) \text{ as } x \rightarrow +\infty \text{ since } \frac{x^2 + 3x}{x^2} = 1 + \frac{3}{x} \rightarrow 1 \text{ as } x \rightarrow +\infty \end{aligned}$$

$$\sqrt{x+5} = \sqrt{x}(1+o(1)) \quad \text{as } x \rightarrow +\infty \quad \text{since} \quad \frac{\sqrt{x+5}}{\sqrt{x}} = \sqrt{1+\frac{5}{x}} \rightarrow 1 \quad \text{as } x \rightarrow +\infty$$

Some properties of the $o(1)$ symbol:

$$o(1) \pm o(1) = o(1), \quad \text{as } x \rightarrow c$$

$$o(1) \cdot o(1) = o(1), \quad \text{as } x \rightarrow c$$

$$c \cdot o(1) = o(1), \quad \text{as } x \rightarrow c, \quad c \neq 0$$

$$(1+o(1)) \cdot (1+o(1)) = 1+o(1), \quad \text{as } x \rightarrow c$$

$$\frac{1}{1+o(1)} = 1+o(1), \quad \text{as } x \rightarrow c$$

$$\frac{c}{1+o(1)} - c = o(1), \quad \text{as } x \rightarrow c, \quad c \neq 0$$

Since

$$\frac{c}{1+o(1)} - c = c \left(\frac{1}{1+o(1)} - 1 \right) = c(1+o(1) - 1) = o(1)$$

Example 7.5: $o(1)$

We have

$$(x+o(1))^2 = x^2(1+o(1)) \quad \text{as } x \rightarrow +\infty \quad \text{since} \quad \frac{(x+o(1))^2}{x^2} \rightarrow 1 \quad \text{as } x \rightarrow +\infty$$

7.2 Asymptotic expansions derived from the fundamental limits

We have seen that the following fundamental limits hold:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}, \quad \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1, \quad \lim_{x \rightarrow 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$$

and consequently the following **asymptotic equivalences** hold as $x \rightarrow 0$:

$$\sin x \sim x, \quad 1 - \cos x \sim \frac{1}{2}x^2, \quad e^x - 1 \sim x, \quad \log(1+x) \sim x, \quad (1+x)^\alpha - 1 \sim \alpha x$$

Let us now see how to obtain the **asymptotic expansions** starting from the fundamental limits:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Hence, as $x \rightarrow 0$:

$$\frac{\sin x}{x} - 1 \rightarrow 0, \quad \frac{\sin x - x}{x} \rightarrow 0 \quad \text{hence} \quad \sin x - x = o(x)$$

which gives the asymptotic expansion:

$$\sin x = x + o(x)$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

Hence, as $x \rightarrow 0$:

$$\frac{1 - \cos x}{x^2} - \frac{1}{2} \rightarrow 0, \quad \frac{2(1 - \cos x) - x^2}{2x^2} \rightarrow 0 \quad \text{hence} \quad 2(1 - \cos x) - x^2 = o(x^2)$$

which gives the asymptotic expansion:

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$$

$$\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

Hence, as $x \rightarrow 0$:

$$\frac{\log(1+x)}{x} - 1 \rightarrow 0, \quad \frac{\log(1+x) - x}{x} \rightarrow 0 \quad \text{hence} \quad \log(1+x) - x = o(x)$$

which gives the asymptotic expansion:

$$\log(1+x) = x + o(x)$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

Hence, as $x \rightarrow 0$:

$$\frac{e^x - 1}{x} - 1 \rightarrow 0, \quad \frac{e^x - 1 - x}{x} \rightarrow 0 \quad \text{hence} \quad e^x - 1 - x = o(x)$$

which gives the asymptotic expansion:

$$e^x = 1 + x + o(x)$$

$$\lim_{x \rightarrow 0} \frac{(1+x)^\alpha - 1}{x} = \alpha \quad \text{with} \quad \alpha \in \mathbb{R}$$

Hence, as $x \rightarrow 0$:

$$\frac{(1+x)^\alpha - 1}{x} - \alpha \rightarrow 0 \quad \frac{(1+x)^\alpha - 1 - \alpha x}{x} \rightarrow 0 \quad \text{hence} \quad (1+x)^\alpha - 1 - \alpha x = o(x)$$

which gives the asymptotic expansion:

$$(1+x)^\alpha = 1 + \alpha x + o(x)$$

- These asymptotic expansions can be generalized

If $\varepsilon(x)$ is a function that tends to zero, i.e., it is an infinitesimal (it does not matter what x tends to), we have the following **asymptotic expansions** derived from the fundamental limits.

As $\varepsilon(x) \rightarrow 0$, we have:

$$\sin(\varepsilon(x)) = \varepsilon(x) + o(\varepsilon(x)) \quad (7.1a)$$

$$\cos(\varepsilon(x)) = 1 - \frac{1}{2} \varepsilon^2(x) + o(\varepsilon^2(x)) \quad (7.1b)$$

$$\log(1 + \varepsilon(x)) = \varepsilon(x) + o(\varepsilon(x)) \quad (7.1c)$$

$$e^{\varepsilon(x)} = 1 + \varepsilon(x) + o(\varepsilon(x)) \quad (7.1d)$$

$$(1 + \varepsilon(x))^\alpha = 1 + \alpha \varepsilon(x) + o(\varepsilon(x)) \quad \text{with } \alpha \in \mathbb{R} \quad (7.1e)$$

Intermediate zero theorem and bisection method

8.1 Zeros of a function

Given a function f , we are interested in solving the equation:

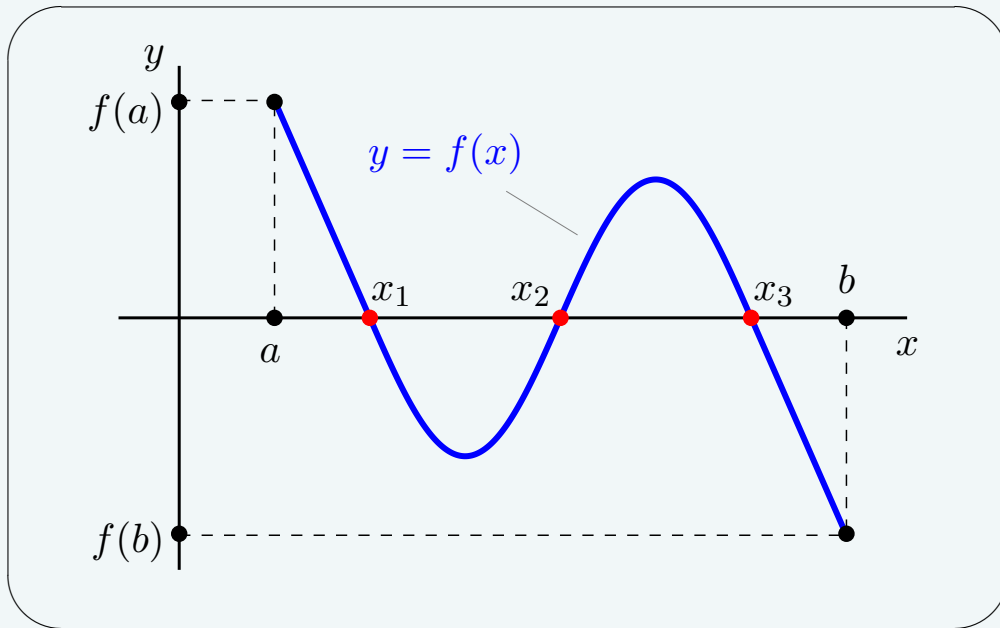
$$f(x) = 0 \tag{8.1}$$

that is, in finding the **zeros** of f . The zeros are the solutions of equation (8.1), that is, the points c of the domain of the function where $f(c) = 0$.

- When f is a polynomial of degree ≤ 4 there are formulas that give the solutions of (8.1). However, if f is a polynomial of degree > 4 or a more complicated function, except in particularly lucky cases, there are no formulas for the solutions of equation (8.1).
- Geometrically, solving equation (8.1) means determining the x -coordinates of the intersection points between the graph of $y = f(x)$ and the x -axis. There can be: **infinitely many solutions, a finite number of solutions, no solutions**.

Example 8.1: Zeros of a function

Consider the function f with the following graph:

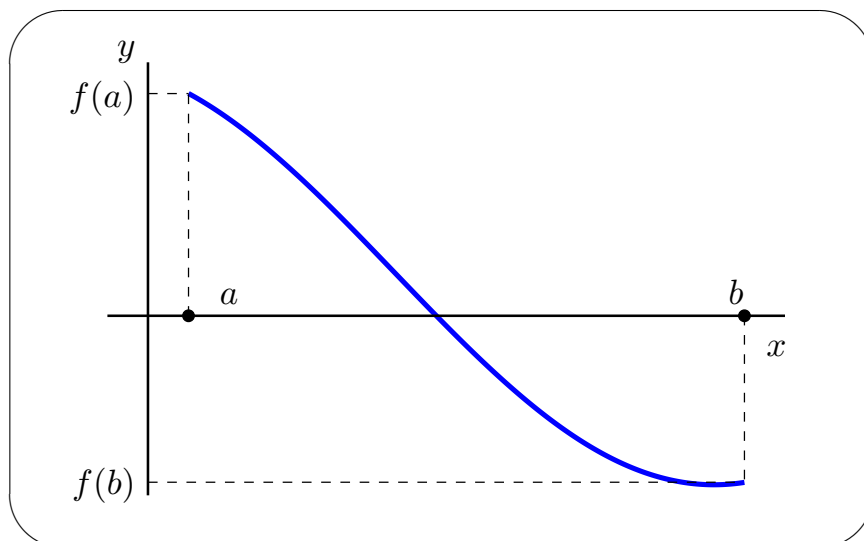


the equation $f(x) = 0$ has 3 solutions in the interval $[a, b]$, that is, the function f has 3 zeros, x_1 , x_2 and x_3 .

Theorem 8.1: Intermediate zero theorem (Bolzano's theorem)

If a function $f : [a, b] \rightarrow \mathbb{R}$ is continuous on the interval $[a, b]$ and $f(a) \cdot f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$. If f is also strictly monotonic then the zero c is unique.

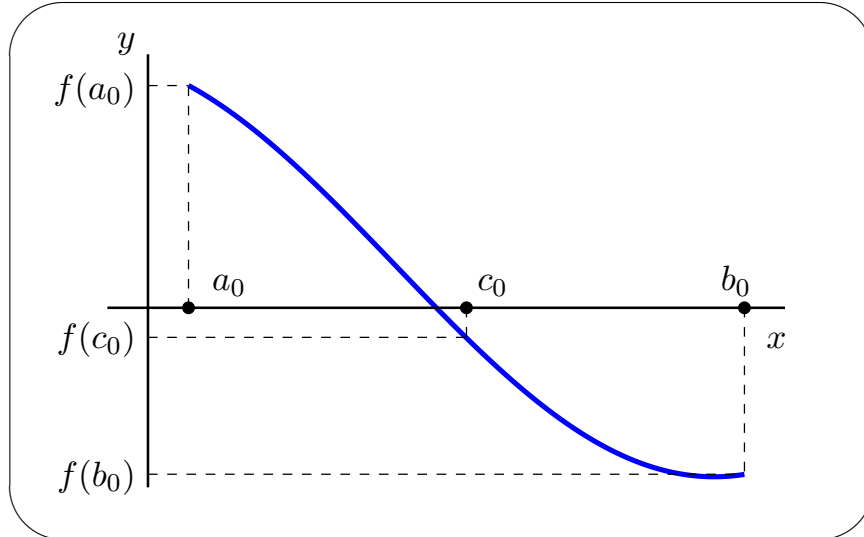
- The idea of the proof is to construct two sequences that converge to the zero of the function. To understand how to construct them, consider for example a function with $f(a) > 0$ and $f(b) < 0$:



- We set $a_0 = a$ and $b_0 = b$ and compute

$$c_0 = \frac{a_0 + b_0}{2}, \text{ the midpoint of the interval } [a_0, b_0].$$

In the previous example we have:



If $f(c_0) = 0$ we have found a zero. If $f(c_0) \neq 0$ we look at the sign of $f(a_0) \cdot f(c_0)$ and proceed as follows:

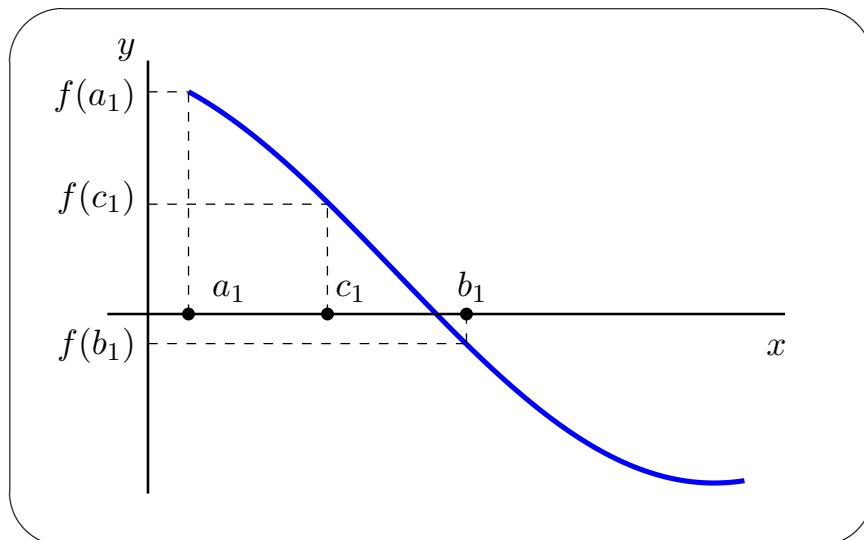
$$\begin{cases} \text{if } f(a_0) \cdot f(c_0) < 0 \\ \text{if } f(a_0) \cdot f(c_0) > 0 \end{cases} \quad \text{we create } [a_1, b_1] \quad \begin{cases} \text{with } a_1 = a_0, \quad b_1 = c_0 \\ \text{with } a_1 = c_0, \quad b_1 = b_0 \end{cases}$$

In the previous example we have $f(a_0) \cdot f(c_0) < 0$, hence we have $a_1 = a_0$, $b_1 = c_0$.

- With a_1 and b_1 , we compute:

$$c_1 = \frac{a_1 + b_1}{2}, \text{ the midpoint of the interval } [a_1, b_1].$$

In the previous example we have:



If $f(c_1) = 0$ we have found a zero. If $f(c_1) \neq 0$ we look at the sign of $f(a_1) \cdot f(c_1)$ and proceed as follows:

$$\begin{cases} \text{if } f(a_1) \cdot f(c_1) < 0 \\ \text{if } f(a_1) \cdot f(c_1) > 0 \end{cases} \quad \text{we create } [a_2, b_2] \quad \begin{cases} \text{with } a_2 = a_1, \quad b_2 = c_1 \\ \text{with } a_2 = c_1, \quad b_2 = b_1 \end{cases}$$

In the previous example we have $f(a_1) \cdot f(c_1) > 0$, hence we have $a_2 = c_1, \quad b_2 = b_1$.

- We now generalize this idea in the proof of the theorem.

Proof. Consider the following two recursively defined sequences:

$$\begin{aligned} a_0 &= a, & a_{n+1} &= \begin{cases} a_n & \text{if } f(a_n) \cdot f(c_n) < 0 \\ c_n & \text{if } f(a_n) \cdot f(c_n) > 0 \end{cases} & \forall n \in \mathbb{N} \\ b_0 &= b, & b_{n+1} &= \begin{cases} c_n & \text{if } f(a_n) \cdot f(c_n) < 0 \\ b_n & \text{if } f(a_n) \cdot f(c_n) > 0 \end{cases} & \forall n \in \mathbb{N} \end{aligned}$$

where

$$c_n = \frac{a_n + b_n}{2}, \quad \forall n \in \mathbb{N}$$

The two sequences create a sequence of intervals $[a_n, b_n]$ with the following properties:

- 1) We have $a_n \leq a_{n+1}$, hence the sequence $\{a_n\}$ is increasing and, since $a_n \leq b, \forall n$, it is also bounded. Moreover, we have $b_n \geq b_{n+1}$, hence the sequence $\{b_n\}$ is decreasing and, since $b_n \geq a, \forall n$, it is also bounded.
- 2) $b_n - a_n = \frac{b-a}{2^n}$ (each interval is half as long as the previous one)
- 3) $f(a_n) \cdot f(b_n) < 0$ (because of how a_n and b_n were chosen at each step)

By point 1), we can then deduce that the sequences $\{a_n\}$ and $\{b_n\}$ have a finite limit, thanks to the monotone sequence theorem. Hence:

$$a_n \rightarrow \ell_1 \in \mathbb{R} \quad \text{and} \quad b_n \rightarrow \ell_2 \in \mathbb{R} \quad \text{as } n \rightarrow +\infty.$$

From point 2) we deduce that:

$$b_n - a_n = \frac{b-a}{2^n} \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \quad \text{and therefore} \quad \ell_2 = \ell_1 = \ell$$

By the continuity of f , we then have that:

$$f(a_n) \cdot f(b_n) \rightarrow (f(\ell))^2 \quad \text{as } n \rightarrow +\infty$$

On the other hand, from point 3) and the sign-preservation theorem we deduce $(f(\ell))^2 \leq 0$. Therefore it must be $f(\ell) = 0$ and thus ℓ is the zero we are looking for, that is, $c = \ell$.

□

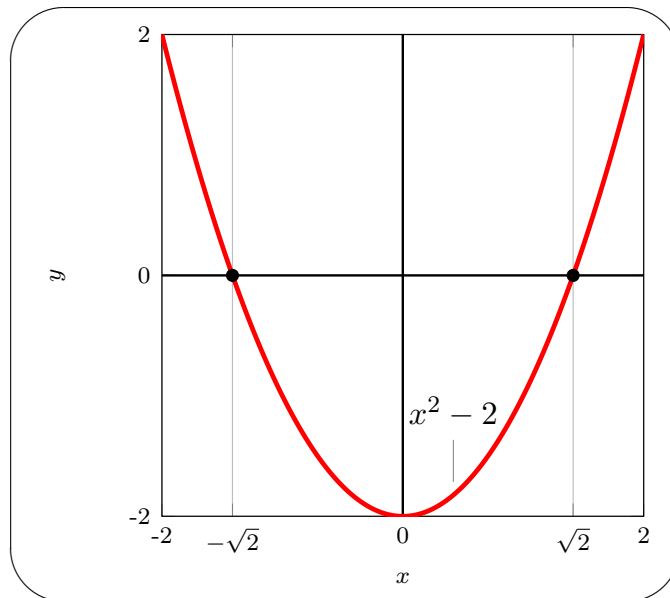
- We have proved:

$$\begin{aligned} f : [a, b] \rightarrow \mathbb{R} \text{ continuous on } [a, b] \text{ and } f(a) \cdot f(b) < 0 \\ \Rightarrow \text{there exists } c \in [a, b] \text{ such that } f(c) = 0 \end{aligned}$$

hence “ $f : [a, b] \rightarrow \mathbb{R}$ continuous on $[a, b]$ and $f(a) \cdot f(b) < 0$ ” is a sufficient condition for “there exists $c \in [a, b]$ such that $f(c) = 0$ ” and “there exists $c \in [a, b]$ such that $f(c) = 0$ ” is a necessary condition for “ $f : [a, b] \rightarrow \mathbb{R}$ continuous on $[a, b]$ and $f(a) \cdot f(b) < 0$ ”. The implication does not work in the other direction:

$$\begin{aligned} \text{there exists } c \in [a, b] \text{ such that } f(c) = 0 \\ \nRightarrow f : [a, b] \rightarrow \mathbb{R} \text{ continuous on } [a, b] \text{ and } f(a) \cdot f(b) < 0 \end{aligned}$$

It suffices, for example, to consider $f(x) = x^2 - 2$ on the interval $[-2, 2]$. We have $f(-2) = 2$ and $f(2) = 2$, hence $f(a) \cdot f(b) \not< 0$, but the function does have zeros:



Hence the theorem provides **sufficient** (but not necessary) conditions for the existence of a zero of a function.

- The proof of the theorem is constructive, that is, we have constructed two sequences that tend to a zero of the function (**constructive proof/algorithm**). This algorithm is called the **bisection method**.

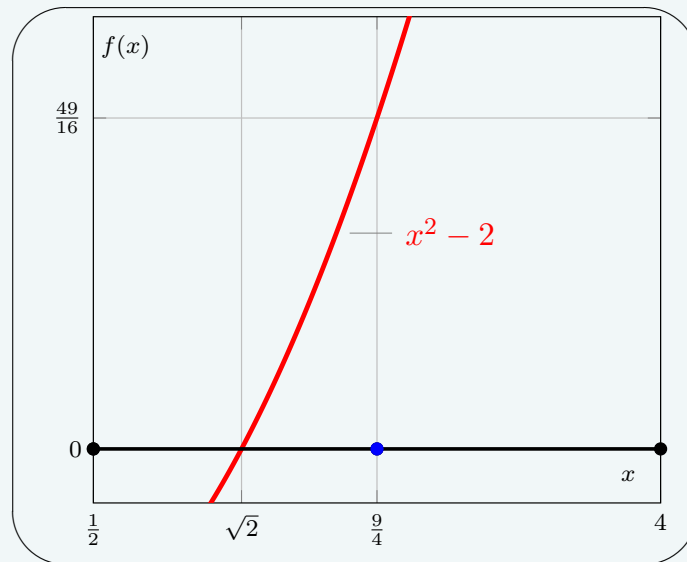
If the function f has several zeros in $[a, b]$, the procedure does not indicate which of them will be determined. The zero found depends on the interval $[a, b]$ given as input, and finding it may require infinitely many iterations. However, by stopping the procedure, as we will see, we obtain an estimate of the zero and of the error made.

8.2 Bisection method

Example 8.2: Bisection method

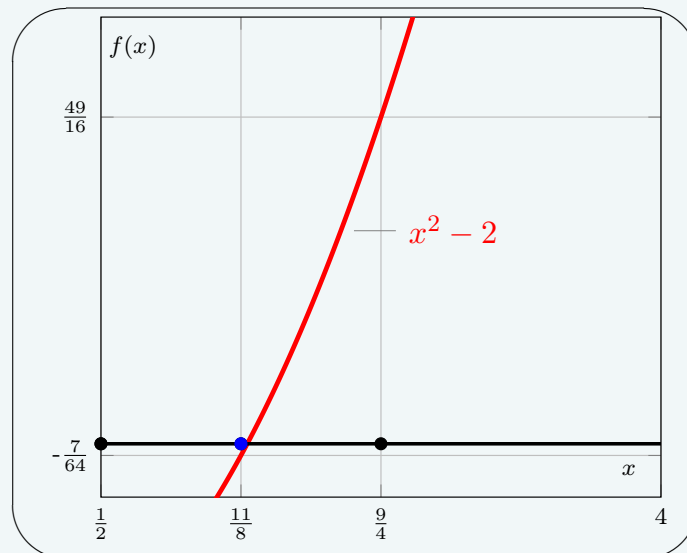
- We look for a zero of the continuous function $f(x) = x^2 - 2$ on the interval $[\frac{1}{2}, 4]$, that is, we look for the value of $\sqrt{2}$. Initialization, with $n = 0$ we have:

$$[a_0, b_0] = \left[\frac{1}{2}, 4\right], \quad f(a) = -\frac{7}{4}, \quad f(b) = 14 \quad \Rightarrow \quad c_0 = \frac{9}{4} \quad \text{and} \quad f(c_0) = \frac{49}{16}$$



- Iteration $n = 1$, we have:

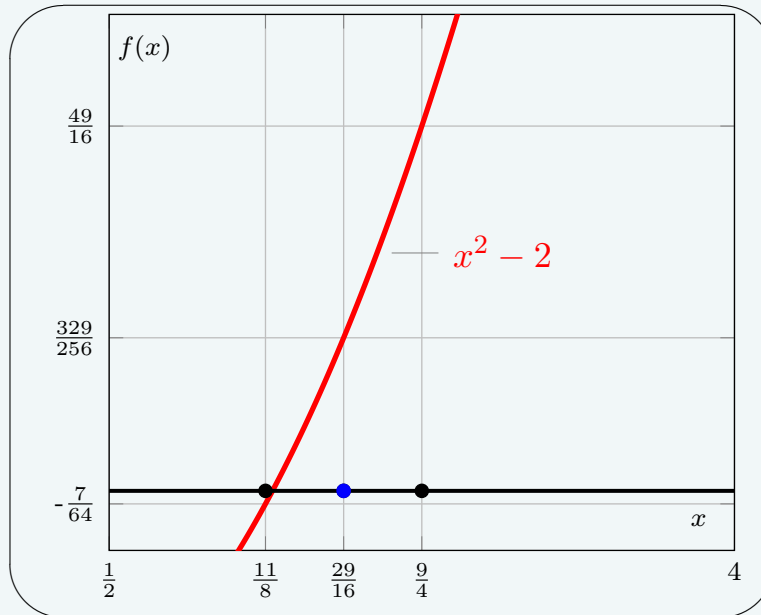
$$[a_1, b_1] = \left[\frac{1}{2}, \frac{9}{4}\right], \quad f(a_1) = -\frac{7}{4}, \quad f(b_1) = \frac{49}{16} \quad \Rightarrow \quad c_1 = \frac{11}{8} \quad \text{and} \quad f(c_1) = -\frac{7}{64}$$



Example 8.3: Bisection method

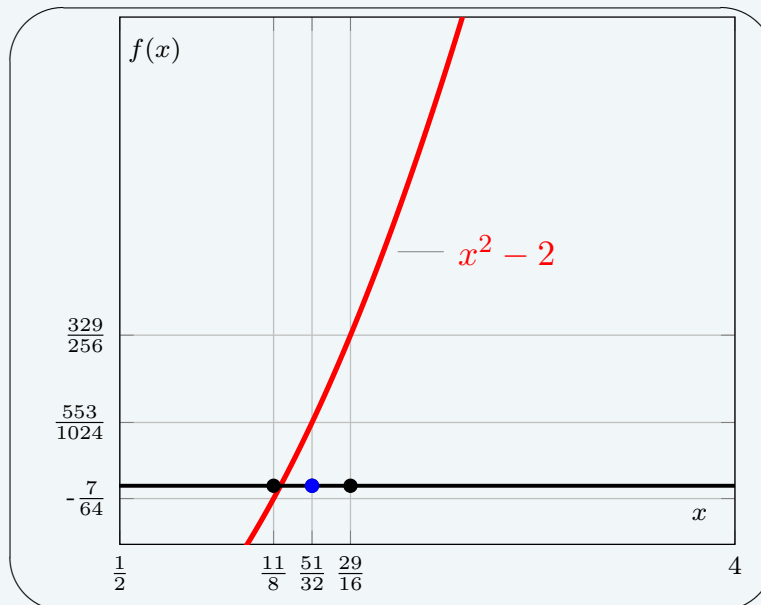
- Iteration $n = 2$, we have:

$$[a_2, b_2] = \left[\frac{11}{8}, \frac{9}{4} \right], \quad f(a_2) = -\frac{7}{64}, \quad f(b_2) = \frac{49}{16} \Rightarrow c_2 = \frac{29}{16} \text{ and } f(c_2) = \frac{329}{256}$$



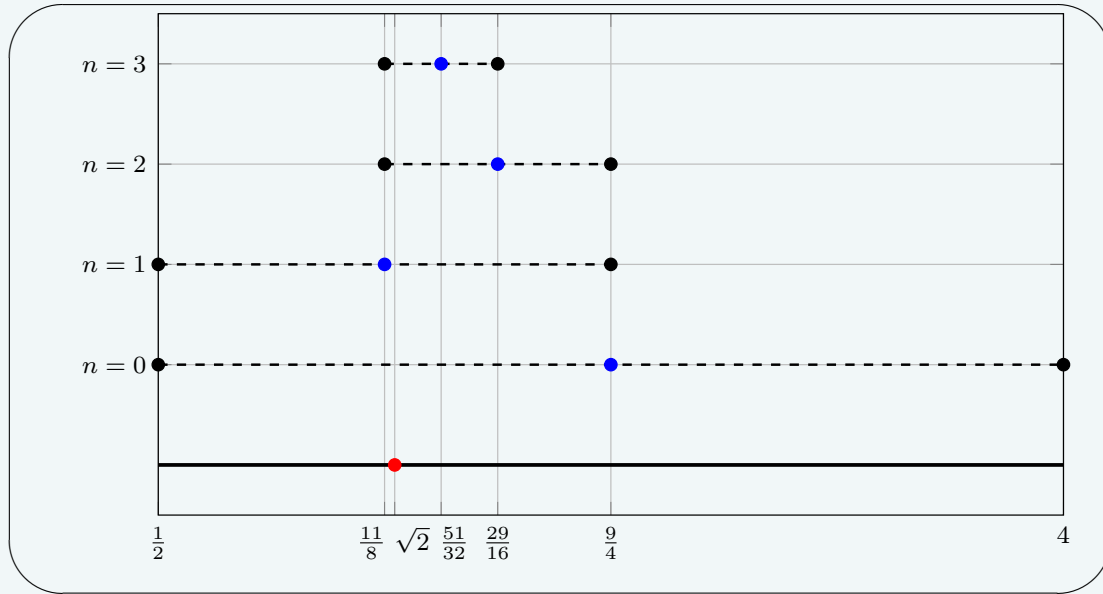
- Iteration $n = 3$, we have:

$$[a_3, b_3] = \left[\frac{11}{8}, \frac{29}{16} \right], \quad f(a_3) = -\frac{7}{64}, \quad f(b_3) = \frac{329}{256} \Rightarrow c_3 = \frac{51}{32} \text{ and } f(c_3) = \frac{553}{1024}$$



Example 8.4: Intervals of the bisection method

Plotting the values obtained in the previous exercise, we have the following intervals and estimates of $\sqrt{2}$:


8.2.1 Error estimates for the bisection method

- At iteration n , we have:

$$c \in [a_n, b_n]$$

that is, the zero lies inside the n -th interval. Hence, stopping the procedure after n steps, a_n approximates the zero from below and b_n approximates the zero from above. Therefore:

$$a_n \leq c \leq b_n$$

- The estimate of the value c at iteration n is:

$$c_n = \frac{a_n + b_n}{2}$$

that is, the midpoint of the interval $[a_n, b_n]$.

- The absolute error ε_n made at iteration n is:

$$\varepsilon_n = |c_n - c|$$

The value of c is unknown, hence we need a way to analyze the values of the errors that is independent of c .

- We can estimate the error at iteration n as follows:

$$\varepsilon_n = |c_n - c| \leq \frac{b - a}{2^n}$$

this follows from:

- the length of the interval at iteration n is $\frac{b-a}{2^n}$

- the midpoint of this interval is c_n and c lies inside the interval itself
- the distance of any point c from the center of the interval is less than or equal to the length of the interval itself

If $n \rightarrow +\infty$ then $\varepsilon_n \rightarrow 0$, that is, the error tends to zero as n tends to infinity.

- At iteration n , we can improve the error estimate as follows:

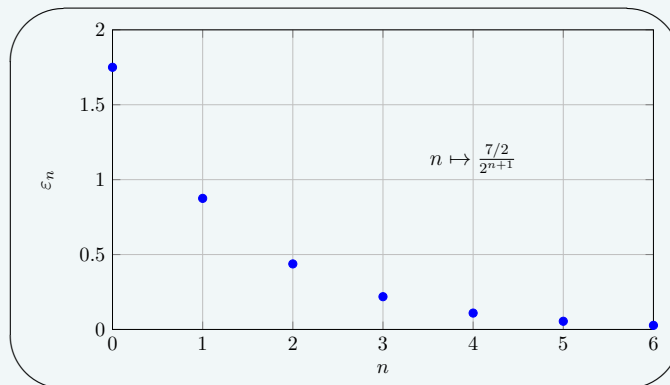
$$\varepsilon_n = |c_n - c| \leq \frac{1}{2} \frac{b - a}{2^n} = \frac{b - a}{2^{n+1}}$$

that is, half the length of the interval $[a_n, b_n]$, since $c \in (a_n, c_n)$ or $c \in (c_n, b_n)$.

Example 8.5: Error estimates for the bisection method ($\sqrt{2} = 1.414213562\dots$)

Let us go back to the previous exercise with $b - a = \frac{7}{2}$; in the first 6 iterations we have:

iteration	a_n	c_n	b_n	estimate of $\sqrt{2}$	ε_n
$n = 0$	$\frac{1}{2}$	$\frac{9}{4}$	4	2.25	$\leq \frac{7/2}{2^1} = 1.75$
$n = 1$	$\frac{1}{2}$	$\frac{11}{8}$	$\frac{9}{4}$	1.375	$\leq \frac{7/2}{2^2} = 0.875$
$n = 2$	$\frac{11}{8}$	$\frac{29}{16}$	$\frac{9}{4}$	1.8125	$\leq \frac{7/2}{2^3} = 0.4375$
$n = 3$	$\frac{11}{8}$	$\frac{51}{32}$	$\frac{29}{16}$	1.59375	$\leq \frac{7/2}{2^4} = 0.21875$
$n = 4$	$\frac{11}{8}$	$\frac{95}{64}$	$\frac{51}{32}$	1.484375	$\leq \frac{7/2}{2^5} = 0.109375$
$n = 5$	$\frac{11}{8}$	$\frac{183}{128}$	$\frac{95}{64}$	1.4296875	$\leq \frac{7/2}{2^6} = 0.0546875$



- In order to guarantee that the error made does not exceed a given tolerance δ , that is, to impose $\varepsilon_n \leq \delta$, we need to perform $n(\delta)$ iterations, where $n(\delta)$ is the smallest integer that satisfies the inequality:

$$n(\delta) > \log_2 \left(\frac{b - a}{\delta} \right) - 1$$

obtained by solving for $n(\delta)$ in the maximum error estimate:

$$\delta = \frac{b-a}{2^{(n(\delta)+1)}}$$

hence

$$2^{(n(\delta)+1)} = \frac{b-a}{\delta} \quad \text{and} \quad n(\delta) + 1 = \log_2 \left(\frac{b-a}{\delta} \right)$$

Clearly, if $\delta \rightarrow 0$ then $n(\delta) \rightarrow +\infty$, and the smaller the error tolerance, the larger the number of iterations needed to guarantee it.

- If we wanted to reduce the tolerance by one decimal digit, that is, to go from δ to $\frac{\delta}{10}$, we would have:

$$n(\delta) \approx \log_2 \left(\frac{b-a}{\delta} \right) - 1 \quad \text{and} \quad n \left(\frac{\delta}{10} \right) \approx \log_2 \left(\frac{b-a}{\frac{\delta}{10}} \right) - 1$$

since

$$\log_2 \left(\frac{b-a}{\frac{\delta}{10}} \right) - 1 = \log_2 \left(\frac{b-a}{\delta} \cdot 10 \right) - 1 = \underbrace{\log_2 \left(\frac{b-a}{\delta} \right) - 1}_{\approx n(\delta)} + \log_2 10$$

the number of additional iterations is independent of the interval $[a, b]$, and it is equal to

$$\log_2 10 \approx 3.32$$

On average, more than three bisections are needed to improve the accuracy of the estimate by one significant digit; **convergence is therefore slow**.

Example 8.6: Bisection method (error estimates)

Again for the previous example:

- if we want a tolerance (maximum error) $\delta = 0.00001 = 10^{-5}$:

$$n(10^{-5}) > \log_2 \left(\frac{7/2}{10^{-5}} \right) - 1 = 17.416 \dots \quad \text{hence} \quad n(10^{-5}) = 18$$

- if we want a tolerance (maximum error) $\delta = 0.000001 = 10^{-6}$:

$$n(10^{-6}) > \log_2 \left(\frac{7/2}{10^{-6}} \right) - 1 = 20.738 \dots \quad \text{hence} \quad n(10^{-6}) = 21$$

We now want to understand how “fast” the error decreases by comparing the errors of two consecutive iterations. This estimate gives us information about the quality of the algorithm, that is, we want to determine the “**rate of convergence**” of the algorithm.

- Consider the maximum error, which we denote by ε_n^* ; we have:

$$\varepsilon_n^* = \frac{b-a}{2^{n+1}} \quad \text{and} \quad \varepsilon_n \leq \varepsilon_n^*$$

hence

$$\frac{\varepsilon_{n+1}^*}{\varepsilon_n^*} = \frac{1}{2} \quad \text{since} \quad \frac{\frac{b-a}{2^{n+2}}}{\frac{b-a}{2^{n+1}}} = \frac{1}{2}$$

that is, the maximum error is halved at each iteration. Consequently, we have:

$$\underbrace{\varepsilon_{n+1}^*}_{\geq \varepsilon_{n+1}} = \frac{1}{2} \underbrace{\varepsilon_n^*}_{\geq \varepsilon_n}$$

The maximum absolute error of the bisection method at each step is proportional to the maximum absolute error at the previous step. **The rate of convergence is linear.**

- For every triple c_{n+1}, c_n and c_{n-1} , with $n \geq 1$, we have the following relation:

$$c_{n+1} = \frac{c_{n-1} + c_n}{2} \tag{8.2}$$

that is, the estimate c_{n+1} is the midpoint of the interval with endpoints c_{n-1} and c_n . We now write:

$$c_{n+1} = c + \varepsilon_{n+1}, \quad c_n = c + \varepsilon_n, \quad c_{n-1} = c + \varepsilon_{n-1}$$

now, substituting into equation (8.2), we obtain:

$$\begin{aligned} c + \varepsilon_{n+1} &= \frac{c + \varepsilon_n + c + \varepsilon_{n-1}}{2} = \frac{2c + \varepsilon_n + \varepsilon_{n-1}}{2} \\ &= c + \frac{\varepsilon_n + \varepsilon_{n-1}}{2} \end{aligned}$$

hence

$$\varepsilon_{n+1} = \frac{\varepsilon_n + \varepsilon_{n-1}}{2} = \frac{1}{2} \varepsilon_n \left(\frac{\varepsilon_{n-1}}{\varepsilon_n} + 1 \right)$$

The term $\varepsilon_{n-1}/\varepsilon_n$, that is, the ratio between the errors of two consecutive iterations, can be an arbitrarily small or large number (indeed, the error at each iteration can increase or decrease).

Weierstrass theorem and intermediate value theorem

9.1 Weierstrass theorem

- The following theorem establishes **sufficient but not necessary** conditions for a function to have a maximum and a minimum.

Theorem 9.1: Weierstrass theorem

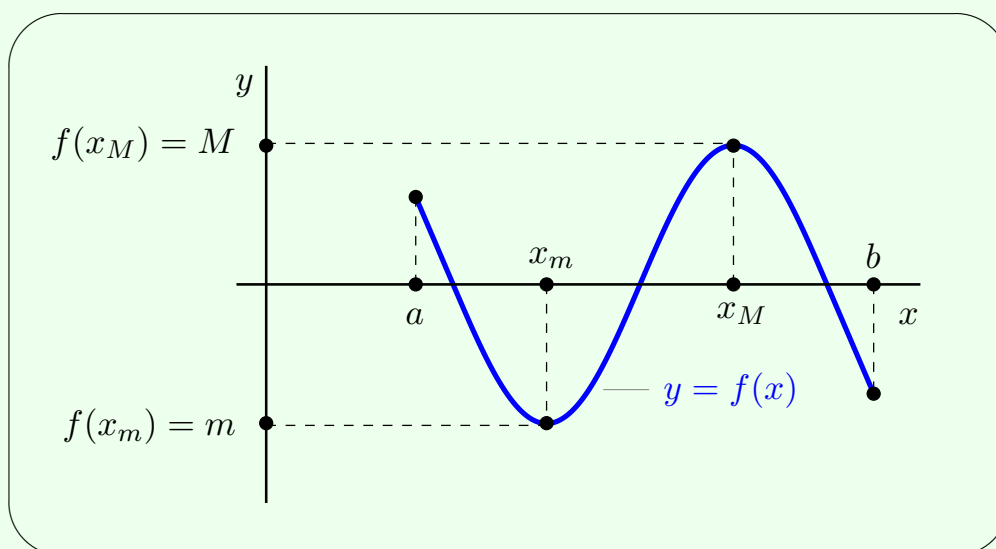
If a function $f : [a, b] \rightarrow \mathbb{R}$ is continuous on the interval $[a, b]$, then it has a maximum M and a minimum m in $[a, b]$.

Under the hypotheses of the theorem there exist:

$$x_m \text{ and } x_M \in [a, b] \text{ such that } f(x_m) \leq f(x) \leq f(x_M) \text{ for every } x \in [a, b]$$

We say that x_m is a **minimum point** of f and $m = f(x_m)$ is the **minimum** of f .

We say that x_M is a **maximum point** of f and $M = f(x_M)$ is the **maximum** of f .



Properties of the supremum/infimum

- Given two non-empty subsets E_1, E_2 of $\mathbb{R}$ we have:

$$\sup (E_1 \cup E_2) = \max (\sup E_1, \sup E_2)$$

This property holds for both bounded and unbounded sets. If one or both sets are unbounded above we have: $\sup(E_1 \cup E_2) = +\infty$. For the infimum we have:

$$\inf (E_1 \cup E_2) = \min (\inf E_1, \inf E_2)$$

Proof.

We prove that f has a maximum in $[a, b]$. Consider the function:

$$f : [a, b] \rightarrow \mathbb{R}, \quad \text{and set} \quad \ell = \sup_{[a, b]} f$$

We now need to prove that:

- i) ℓ is finite, that is, $\ell \in \mathbb{R}$;
- ii) ℓ is equal to $f(x_0)$ for some $x_0 \in [a, b]$ (and hence ℓ is the maximum and x_0 is the maximum point).

We split the interval $[a, b]$ into two equal intervals: I_1 and I_2 ; by the properties of the supremum we have:

$$\sup_{[a, b]} f = \max \left(\sup_{I_1} f, \sup_{I_2} f \right) \quad \text{that is} \quad \sup_{[a, b]} f = \sup_{I_1} f \quad \text{or} \quad \sup_{[a, b]} f = \sup_{I_2} f$$

Hence for one of the two intervals, which we call $[a_1, b_1]$, it will be true that:

$$\ell = \sup_{[a_1, b_1]} f$$

We now split $[a_1, b_1]$ into two equal intervals; for one of them, which we call $[a_2, b_2]$, it will be true that:

$$\ell = \sup_{[a_2, b_2]} f$$

Proceeding by bisection in this way, we construct a sequence of intervals $[a_n, b_n]$, each contained in the previous ones, with the properties:

- 1) the sequence $\{a_n\}$ is monotone non-decreasing and bounded, and the sequence $\{b_n\}$ is monotone non-increasing and bounded;
- 2) $b_n - a_n = \frac{b-a}{2^n} \rightarrow 0$ as $n \rightarrow +\infty$;
- 3) $\ell = \sup_{[a_n, b_n]} f$ (the supremum of f is attained in the interval $[a_n, b_n]$)

By the same reasoning used in the proof of the intermediate zero theorem (which uses the monotonicity theorem for sequences), from points 1) and 2) it follows that the sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit $x_0 \in [a, b]$:

$$a_n \rightarrow x_0 \text{ and } b_n \rightarrow x_0 \text{ as } n \rightarrow +\infty.$$

We now proceed by cases. □

Proof. Case 1: $\ell \in \mathbb{R}$

If $\ell \in \mathbb{R}$, for every n there exists a point $t_n \in [a_n, b_n]$ such that:

$$\ell - \frac{1}{n} < f(t_n) \leq \ell \quad (9.1)$$

Indeed, since $\ell - \frac{1}{n}$ is less than ℓ , which is the least upper bound of the values of $f(x)$ in $[a_n, b_n]$, $\ell - \frac{1}{n}$ is not an upper bound; hence there exists $t_n \in [a_n, b_n]$ with property (9.1).

Since $t_n \in [a_n, b_n]$, and we have:

$$a_n \rightarrow x_0 \text{ and } b_n \rightarrow x_0 \text{ as } n \rightarrow +\infty \text{ then } t_n \rightarrow x_0 \text{ as } n \rightarrow +\infty$$

by the comparison (squeeze) theorem. Again by the comparison theorem, (9.1) then gives

$$\lim_{n \rightarrow +\infty} f(t_n) = \ell$$

On the other hand, since f is continuous and $t_n \rightarrow x_0$, we have:

$$\lim_{n \rightarrow +\infty} f(t_n) = f(x_0) \quad \text{therefore} \quad f(x_0) = \ell.$$

Then, since ℓ is the sup of the values of f in $[a, b]$, ℓ is the **maximum** of f in $[a, b]$, and it is attained at the point x_0 , the **maximum point**. Therefore, in this case the theorem is proved.

Case 2: $\ell = +\infty$

If $\ell = +\infty$, for every n there exists a point $t_n \in [a_n, b_n]$ such that:

$$f(t_n) \geq n \quad (9.2)$$

Reasoning as above, one proves that $t_n \rightarrow x_0$ for some $x_0 \in [a, b]$. Since f is continuous at x_0 :

$$\lim_{n \rightarrow +\infty} f(t_n) = f(x_0)$$

but by (9.2) we have

$$\lim_{n \rightarrow +\infty} f(t_n) = +\infty$$

which is a **contradiction**, since at x_0 the function must have a finite value. Hence this case cannot occur.

Similarly, one proves that the function has a minimum in $[a, b]$. □

The proof is a constructive existence proof (similar to that of the intermediate zero theorem), which, however, cannot be used algorithmically since, after each bisection, it is not known a priori which interval the supremum or infimum is associated with.

The hypotheses of the Weierstrass theorem are all essential

1. The interval must be closed

Consider as a counterexample:

$$f(x) = x \quad \text{with } x \in (0, 1)$$

The function is continuous on a bounded, but not closed, interval. In this case, the function has neither a maximum nor a minimum (its supremum, 1, and its infimum, 0, are not attained by the function).

2. The interval must be bounded

Consider as a counterexample:

$$f(x) = x \quad \text{with } x \in \mathbb{R}$$

The function is continuous on an unbounded interval but has neither a maximum nor a minimum (it is not even bounded).

3. The function must be continuous

Consider as a counterexample:

$$f(x) = \begin{cases} x & \text{for } x \in (0, 1) \\ \frac{1}{2} & \text{for } x = 0 \text{ and } x = 1 \end{cases}$$

The function is defined on a closed and bounded interval $[0, 1]$ but it is not continuous. The function has neither a maximum nor a minimum (its supremum, 1, and its infimum, 0, are not attained by the function).

9.2 Intermediate value theorem

Theorem 9.2: Intermediate value theorem

If a function $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$, then it has a maximum M and a minimum m in $[a, b]$ and it takes all the values between m and M .

We are under the hypotheses of the Weierstrass theorem, hence we have:

$$x_m \text{ and } x_M \in [a, b] \quad \text{such that} \quad f(x_m) \leq f(x) \leq f(x_M) \quad \text{for every } x \in [a, b]$$

The intermediate value theorem further tells us that:

$$\forall \lambda \in (m, M), \quad \text{there exists } x(\lambda) \in [x_m, x_M] \quad \text{such that } f(x(\lambda)) = \lambda$$

This property is called the **intermediate value property**.

Proof. From the Weierstrass theorem we have a maximum point x_M and a minimum point x_m such that $f(x_M) = M$ (maximum) and $f(x_m) = m$ (minimum) in $[a, b]$. Let

$$m < \lambda < M$$

and consider the function

$$g(x) = f(x) - \lambda \quad \text{with } x \in [x_m, x_M]$$

which is continuous since $f(x)$ is continuous on $[a, b]$ and $x_m, x_M \in [a, b]$.

Hence:

$$g(x_M) = f(x_M) - \lambda = M - \lambda > 0$$

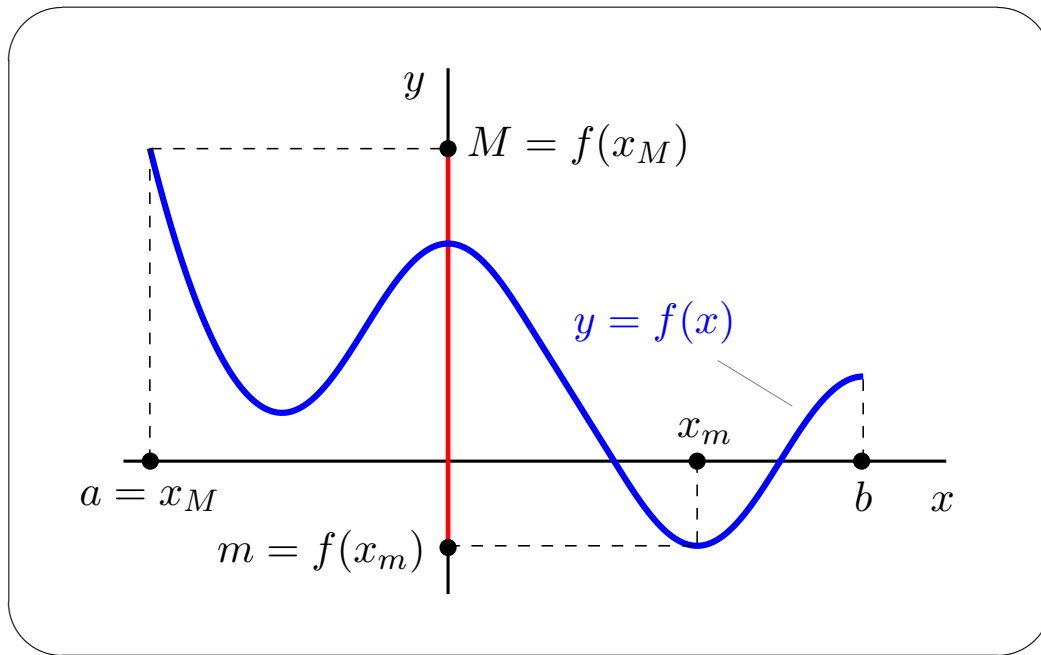
$$g(x_m) = f(x_m) - \lambda = m - \lambda < 0$$

Then, by the intermediate zero theorem, there exists $x(\lambda) \in (x_m, x_M)$ such that:

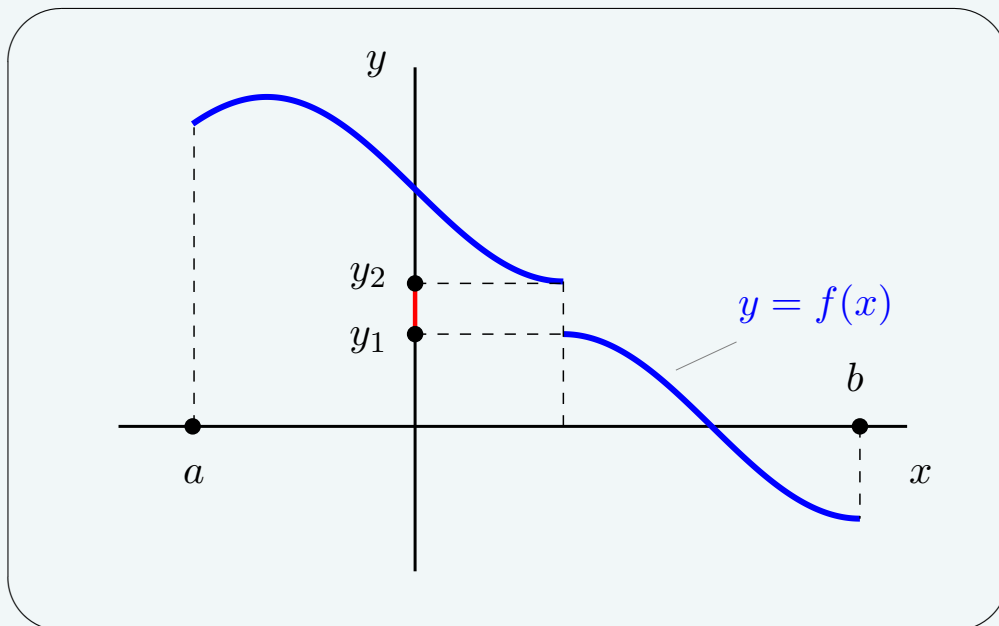
$$g(x(\lambda)) = 0 \quad \text{that is} \quad f(x(\lambda)) = \lambda$$

□

- Given a function continuous on an interval $[a, b]$, graphically we have:


Example 9.1: Discontinuous function without the intermediate value property

Consider for example the graph of the following function, discontinuous on $[a, b]$:



This function does not have the intermediate value property, that is, the values $\lambda \in (y_1, y_2)$ are not outputs of f .

- The properties of the two previous theorems can be summarized in the following single statement:

Corollary 9.3

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then

$$f([a, b]) = [m, M];$$

i.e., the image of an interval $[a, b]$ is the interval with endpoints:

$$m = \min_{[a, b]} f \quad \text{and} \quad M = \max_{[a, b]} f.$$

Proof. It follows from the Weierstrass theorem and the intermediate value theorem. $\square$

Example 9.2: Failure of the intermediate value theorem in $\mathbb{Q}$

- Let

$$f(x) = x^2$$

and consider f as a function from the set $\mathbb{Q}$ of rational numbers to $\mathbb{Q}$ itself (this is legitimate because the square of a rational number is rational).

- Then f does not have the intermediate value property.
- Indeed, for example,

$$f(1) = 1, f(2) = 4,$$

but f does not take all the rational values between 1 and 4: for example, it never takes the value 2, or 3.

- In other words, the intermediate value property holds for continuous functions **thanks to the properties of the set of real numbers**.
- This is a further reason why it is useful to work in the set of real numbers rather than in the set of rational numbers.

9.3 Existence theorem for the n -th root**Theorem 9.4**

For every $y \in \mathbb{R}$, $y > 0$ and $n \in \mathbb{N}$, $n \geq 1$, there exists one and only one $x \in \mathbb{R}$, $x > 0$, such that $x^n = y$.

This number x is called the n -th root of y

Proof. Consider the function $f(x) = x^n$.

This function is continuous on all of $\mathbb{R}$, because it is the product of n continuous functions $g(x) = x$.

We show that there exist

$$x_2 > x_1 > 0 \text{ such that } x_1^n < y < x_2^n.$$

If $y = 1$ the n -th root exists and is unique; moreover:

1. If $y > 1$: it suffices to choose

$$x_1 = 1, x_2 = y \text{ and we have } x_1^n < y < x_2^n \quad (1 < y < y^n)$$

2. If $y < 1$: it suffices to choose

$$x_1 = y, x_2 = 1 \text{ and we have } x_1^n < y < x_2^n \quad (y^n < y < 1)$$

Then we can apply the intermediate value theorem to the continuous function $f(x) = x^n$ on the interval $[x_1, x_2]$. Here its minimum is x_1^n and its maximum is x_2^n .

Since:

$$x_1^n < y < x_2^n, \text{ there exists } x_0 \in [x_1, x_2] \text{ such that } x_0^n = y.$$

Uniqueness follows from the fact that the function is strictly increasing.

□

Monotone functions on an interval and invertibility

10.1 Monotone functions on an interval

- We now deal with monotone functions on an interval (and **not necessarily continuous**), and the next theorem (based on the completeness axiom of $\mathbb{R}$) can be seen as an extension to functions of the monotonicity theorem for sequences.

Theorem 10.1: Monotonicity theorem for functions

Let $f : (a, b) \rightarrow \mathbb{R}$ be a monotone function. Then for every $c \in (a, b)$ the right and left limits as $x \rightarrow c$ exist and are finite; at the two endpoints a, b the right limit (at a) and the left limit (at b) exist, possibly infinite.

Proof. Suppose f is increasing in (a, b) and let $c \in (a, b)$, that is, a point inside the interval.

We prove that the following limit exists and is finite:

$$\lim_{x \rightarrow c^-} f(x) = \sup \{f(x) : x \in (a, c)\}.$$

We set

$$\ell = \sup \{f(x) : x \in (a, c)\}$$

Note that ℓ exists and is finite by the supremum property, since $f(c)$ is an upper bound of the set $\{f(x) : x \in (a, c)\}$.

Hence we need to prove that

$$\lim_{x \rightarrow c^-} f(x) = \ell$$

So let $\{x_n\}$ be any sequence in (a, c) such that $x_n \rightarrow c$, and let us prove that $f(x_n) \rightarrow \ell$, i.e., that for every $\varepsilon > 0$ we eventually have

$$\ell - \varepsilon < f(x_n) < \ell + \varepsilon.$$

The second inequality is obvious, since $f(x_n) \leq \ell$ by definition of ℓ , because $x_n \in (a, c)$.

□

Proof. To prove the first one, we observe that, since $\ell - \varepsilon$ is less than ℓ , i.e., less than the least upper bound of $\{f(x) : x \in (a, c)\}$, it is not an upper bound of this set; therefore there exists a point

$$\tilde{x} \in (a, c) \text{ such that } f(\tilde{x}) > \ell - \varepsilon$$

Since f is increasing, it follows that

$$f(x) > \ell - \varepsilon, \forall x \in (\tilde{x}, c)$$

On the other hand, by hypothesis

$$x_n < c, \forall n \text{ and } x_n \rightarrow c \text{ as } n \rightarrow +\infty$$

therefore $x_n \in (\tilde{x}, c)$ eventually. Hence

$$f(x_n) \geq \ell - \varepsilon \text{ (eventually)}$$

which is what we needed to prove.

Similarly, one can prove that the following limit exists and is finite:

$$\lim_{x \rightarrow c^+} f(x) = \inf \{f(x) : x \in (c, b)\}.$$

As for the limits at the two endpoints of the interval, we prove that the following limit exists:

$$\lim_{x \rightarrow b^-} f(x)$$

We set (as before)

$$\ell = \sup \{f(x) : x \in (a, b)\}$$

however, in this case ℓ could also be $+\infty$ (we cannot claim that $f(b)$ is an upper bound of the set, because the function is not defined at b). We therefore proceed by cases.

In the case $\ell < \infty$ the previous proof can be repeated.

□

Proof. If instead $\ell = \infty$, the argument changes. From the hypothesis

$$\sup \{f(x) : x \in (a, b)\} = +\infty$$

it follows that for every $M > 0$ there exists $\tilde{x} \in (a, b)$ such that $f(\tilde{x}) > M$.

By the monotonicity of f , then

$$f(x) > M, \quad \forall x \in (\tilde{x}, b)$$

hence, taking any sequence $x_n \rightarrow b$, we have $x_n \in (\tilde{x}, b)$ eventually, and therefore

$$f(x_n) > M, \quad \text{eventually,}$$

therefore

$$\lim_{n \rightarrow +\infty} f(x_n) = +\infty$$

In the same way one proves that the following limit exists:

$$\lim_{x \rightarrow a^+} f(x)$$

□

A consequence of the monotonicity theorem is that if a function is monotone on an interval (a, b) , its possible points of discontinuity in (a, b) are necessarily jump discontinuities, except for the endpoints a, b , where there may also be a vertical asymptote.

10.2 Continuity and invertibility

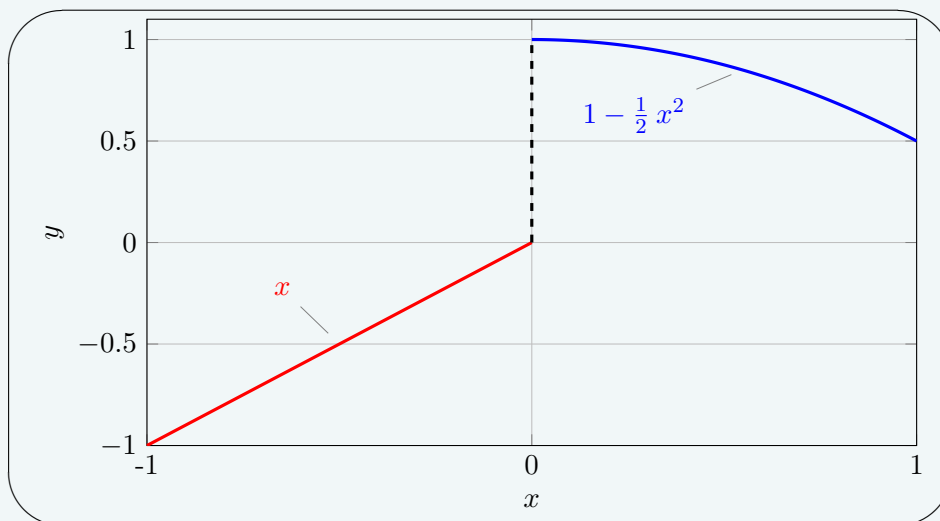
- We have seen the theorem stating that if a generic function with domain D is strictly monotone, then it is invertible.
- We also know that the converse is not true in general: there exist functions that are invertible on an interval but not monotone.

Example 10.1: Invertible but not monotone function

Consider for example

$$f(x) = \begin{cases} 1 - \frac{1}{2}x^2 & \text{if } 0 < x \leq 1, \\ x & \text{if } x \leq 0 \end{cases}$$

its graph is:



This function satisfies the invertibility condition, which requires the graph of f to be intersected at most at one point by every line parallel to the x -axis, but it is not monotone.

- If we add the hypothesis of continuity and require the domain to be an interval, being strictly monotone becomes a necessary and sufficient condition for invertibility, as stated by the following theorem.

Theorem 10.2: Invertibility of monotone and continuous functions

Let $f : I \rightarrow \mathbb{R}$ be a function defined on an interval I .

If the function f is continuous, then it is invertible on the interval I if and only if it is strictly monotone.

In this case its inverse function is also strictly monotone and continuous.

Proof. We already know that if f is strictly monotone, it is invertible (regardless of the hypotheses that f is continuous and that it is defined on an interval).

We show that the converse holds, i.e., that if f is continuous and invertible, then it is strictly monotone.

Suppose by contradiction that the function is not strictly monotone; then there exist three points

$$x_1 < x_2 < x_3 \quad (\text{in the interval } I)$$

such that

$$f(x_1) < f(x_2) \quad \text{and} \quad f(x_2) > f(x_3)$$

or such that

$$f(x_1) > f(x_2) \quad \text{and} \quad f(x_2) < f(x_3).$$

Suppose the first of the two alternatives holds (the other case is handled similarly). Let us compare the values $f(x_1)$ and $f(x_3)$; they cannot be equal because f is invertible by hypothesis, hence

$$f(x_1) < f(x_3) \quad \text{or} \quad f(x_1) > f(x_3).$$

Again, suppose the first of the two alternatives holds (the other case is handled similarly). So we know that:

$$x_1 < x_2 < x_3 \quad \text{and} \quad f(x_1) < f(x_3) < f(x_2).$$

Since f is continuous, by the intermediate value theorem there exists

$$x_0 \in (x_1, x_2) \quad \text{such that} \quad f(x_0) = f(x_3).$$

Since $x_0 \neq x_3$ (because $x_1 < x_2 < x_3$), it follows that f cannot be invertible, a **contradiction**. This proves the first part of the theorem.

Now let f be a continuous, strictly monotone and hence invertible function on I , and let g be its inverse function, which is also strictly monotone and invertible. We prove that g is continuous.

- By what we observed after the monotonicity theorem, the strictly monotone function g is either continuous or has jump discontinuity points.
- In the latter case the image of g is not an interval (but the union of at least two disjoint intervals), which is a **contradiction** because this image is I . Hence g is continuous.

□

- Note that in the proof of the previous theorem we used both the intermediate value theorem and the monotonicity theorem for functions. Moreover, we implicitly used the completeness axiom of $\mathbb{R}$.

- The theorem just proved means, in particular, that:

a continuous and invertible function on an interval has a continuous inverse function.

- This fact completes the proof of the continuity theorem for elementary functions:
 1. the continuity of the function a^x implies the continuity of the function $\log_a x$.
 2. the continuity of the functions $\sin x$, $\cos x$ and $\tan x$ implies the continuity of the functions $\arcsin x$, $\arccos x$ and $\arctan x$.