



SAPIENZA
UNIVERSITÀ DI ROMA

Mathematical Analysis 1

Volume 3 — Functions

The third volume: relations, functions and graphs. Binary relations (functions are special relations), functions between sets and real functions of a real variable, power, exponential and logarithmic functions, trigonometric functions (with an application: oscillations), integer part and piecewise functions, hyperbolic functions, operations on graphs, composite and inverse functions. Definitions, results, many graphs and worked examples; the exercises of the part, with solutions, are in the volume Exercises 3.

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Online version and downloadable material:
fabiofurini.github.io/mathematical-analysis-1/
github.com/fabiofurini/mathematical-analysis-1

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Binary relations

1.1 Binary relations

Definition 1.1: of binary relation

Given two sets A and B , a **binary relation** R is a subset of the Cartesian product $A \times B$

When we say that R is a binary relation on a single set A , we mean that R is a subset of $A \times A$.

- A binary relation will simply be called a relation (however, non-binary relations also exist).
- If $(a, b) \in R$ we also write $a R b$ and we say that a is **related** to b .

Example 1.1

- The relation “less than” (“ $<$ ”) on the natural numbers is the set:

$$R_{<} = \left\{ (a, b) : a, b \in \mathbb{N} \text{ and } a < b \right\}.$$

- The relation “less than or equal to” (“ $\leq$ ”) on the natural numbers is the set:

$$R_{\leq} = \left\{ (a, b) : a, b \in \mathbb{N} \text{ and } a \leq b \right\}.$$

- The relation “equal to” (“ $=$ ”) on the natural numbers is the set:

$$R_{=} = \left\{ (a, b) : a, b \in \mathbb{N} \text{ and } a = b \right\}.$$

- The relation “is a subset of” $R_{\subseteq}$ on the power set of the natural numbers (denoted by $2^{\mathbb{N}}$) is the set:

$$R_{\subseteq} = \left\{ (A, B) : A, B \in 2^{\mathbb{N}} \text{ and } A \subseteq B \right\}.$$

A relation R on a set of natural numbers can be visualized with a **table**: in row a and column b we write the pair (a, b) if $(a, b) \in R$, and the cell is left empty if $(a, b) \notin R$.

Example 1.2: tables of relations

- The table of the relation $R_{<}$ (rows and columns from 0 to 4):

	0	1	2	3	4	...
0		(0, 1)	(0, 2)	(0, 3)	(0, 4)	
1			(1, 2)	(1, 3)	(1, 4)	
2				(2, 3)	(2, 4)	
3					(3, 4)	
4						
...						

- The table of the relation $R_{=}$:

	0	1	2	3	4	...
0	(0, 0)					
1		(1, 1)				
2			(2, 2)			
3				(3, 3)		
4					(4, 4)	
...						

- The table of the relation $R_{\leq}$, which contains the pairs of $R_{<}$ and those of $R_{=}$:

	0	1	2	3	4	...
0	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	
1		(1, 1)	(1, 2)	(1, 3)	(1, 4)	
2			(2, 2)	(2, 3)	(2, 4)	
3				(3, 3)	(3, 4)	
4					(4, 4)	
...						

Definition 1.2: of congruence modulo n

Given a natural number $n \geq 1$, two natural numbers a and b are **congruent modulo n** ,

written $a \equiv b \pmod{n}$, if their difference is an integer multiple of n :

$$a \equiv b \pmod{n} \iff \exists q \in \mathbb{Z} \text{ such that } b - a = qn.$$

Definition 1.3: of multiple and divisor

Given two natural numbers a and b , we say that a is a **multiple** of b if there exists $k \in \mathbb{N}$ such that $a = kb$. We say that a is a **divisor** of b if there exists $k \in \mathbb{N}$ such that $b = ka$.

- Hence a is a divisor of b if and only if b is a multiple of a .
- With this definition 0 is a multiple of every natural number ($0 = 0 \cdot b$) and 0 is a divisor only of itself (if $b = k \cdot 0$ then $b = 0$).

Example 1.3: other relations on the natural numbers

- The relation “congruent modulo n ” on the natural numbers is the set:

$$R_{\equiv_n} = \left\{ (a, b) : a, b \in \mathbb{N} \text{ and } a \equiv b \pmod{n} \right\}.$$

With $n = 3$ the table is:

	0	1	2	3	4	5	6	...
0	(0, 0)			(0, 3)			(0, 6)	
1		(1, 1)			(1, 4)			
2			(2, 2)			(2, 5)		
3	(3, 0)			(3, 3)			(3, 6)	
4		(4, 1)			(4, 4)			
5			(5, 2)			(5, 5)		
6	(6, 0)			(6, 3)			(6, 6)	
...								

- The relation “is a multiple of” on the natural numbers is the set:

$$R_{\text{mul}} = \left\{ (a, b) : a, b \in \mathbb{N} \text{ and } \exists k \in \mathbb{N} \text{ such that } a = kb \right\}.$$

	0	1	2	3	4	5	6	...
0	(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)	(0, 6)	
1		(1, 1)						
2		(2, 1)	(2, 2)					
3		(3, 1)		(3, 3)				
4		(4, 1)	(4, 2)		(4, 4)			
5		(5, 1)				(5, 5)		
6		(6, 1)	(6, 2)	(6, 3)			(6, 6)	
...								

Example 1.4: other relations on the natural numbers

- The relation “is a divisor of” on the natural numbers is the set:

$$R_{\text{div}} = \left\{ (a, b) : a, b \in \mathbb{N} \text{ and } \exists k \in \mathbb{N} \text{ such that } b = ka \right\}.$$

Since $(a, b) \in R_{\text{div}}$ if and only if $(b, a) \in R_{\text{mul}}$, its table is obtained from that of R_{mul} by swapping the two elements of each pair:

	0	1	2	3	4	5	6	...
0	(0, 0)							
1	(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	
2	(2, 0)		(2, 2)		(2, 4)		(2, 6)	
3	(3, 0)			(3, 3)			(3, 6)	
4	(4, 0)				(4, 4)			
5	(5, 0)					(5, 5)		
6	(6, 0)						(6, 6)	
...								

Example 1.5: other relations on the natural numbers

- The relation “difference of one” on the natural numbers is the set:

$$R_{ab}^1 = \left\{ (a, b) : a, b \in \mathbb{N} \text{ and } a = b - 1 \right\}.$$

	0	1	2	3	4	5	6	...
0		(0, 1)						
1			(1, 2)					
2				(2, 3)				
3					(3, 4)			
4						(4, 5)		
5							(5, 6)	
6								
...								

- The relation “the sum is even” on the natural numbers is the set:

$$R_{\text{even}} = \left\{ (a, b) : a, b \in \mathbb{N} \text{ and } \exists m \in \mathbb{N} \text{ such that } a + b = 2m \right\}.$$

	0	1	2	3	4	5	6	...
0	(0, 0)		(0, 2)		(0, 4)		(0, 6)	
1		(1, 1)		(1, 3)		(1, 5)		
2	(2, 0)		(2, 2)		(2, 4)		(2, 6)	
3		(3, 1)		(3, 3)		(3, 5)		
4	(4, 0)		(4, 2)		(4, 4)		(4, 6)	
5		(5, 1)		(5, 3)		(5, 5)		
6	(6, 0)		(6, 2)		(6, 4)		(6, 6)	
...								

- The table of the relation $R_{\subseteq}$ restricted to the subsets of $\{0, 1, 2\}$ (rows and columns are the sets A and B):

	$\emptyset$	$\{0\}$	$\{1\}$	$\{2\}$	$\{0, 1\}$	$\{0, 2\}$	$\{1, 2\}$	$\{0, 1, 2\}$
$\emptyset$	$(\emptyset, \emptyset)$	$(\emptyset, \{0\})$	$(\emptyset, \{1\})$	$(\emptyset, \{2\})$	$(\emptyset, \{0, 1\})$	$(\emptyset, \{0, 2\})$	$(\emptyset, \{1, 2\})$	$(\emptyset, \{0, 1, 2\})$
$\{0\}$		$(\{0\}, \{0\})$			$(\{0\}, \{0, 1\})$	$(\{0\}, \{0, 2\})$		$(\{0\}, \{0, 1, 2\})$
$\{1\}$			$(\{1\}, \{1\})$		$(\{1\}, \{0, 1\})$		$(\{1\}, \{1, 2\})$	$(\{1\}, \{0, 1, 2\})$
$\{2\}$				$(\{2\}, \{2\})$		$(\{2\}, \{0, 2\})$	$(\{2\}, \{1, 2\})$	$(\{2\}, \{0, 1, 2\})$
$\{0, 1\}$					$(\{0, 1\}, \{0, 1\})$			$(\{0, 1\}, \{0, 1, 2\})$
$\{0, 2\}$						$(\{0, 2\}, \{0, 2\})$		$(\{0, 2\}, \{0, 1, 2\})$
$\{1, 2\}$							$(\{1, 2\}, \{1, 2\})$	$(\{1, 2\}, \{0, 1, 2\})$
$\{0, 1, 2\}$								$(\{0, 1, 2\}, \{0, 1, 2\})$

Definition 1.4: of n -ary relation

Given n sets $A_1, A_2, \dots, A_n$, an n -ary **relation** R is a subset of the Cartesian product $A_1 \times A_2 \times \dots \times A_n$.

- Binary relations are the n -ary relations with $n = 2$.

1.2 Properties of relations

- A relation $R \subseteq A \times A$ is **reflexive** if:

$$\forall a \in A, \quad (a, a) \in R$$

- A relation $R \subseteq A \times A$ is **symmetric** if:

$$\forall a, b \in A, \quad (a, b) \in R \Rightarrow (b, a) \in R$$

- A relation $R \subseteq A \times A$ is **transitive** if:

$$\forall a, b, c \in A, \quad (a, b) \in R \text{ and } (b, c) \in R \Rightarrow (a, c) \in R$$

- A relation $R \subseteq A \times A$ is **antisymmetric** if:

$$\forall a, b \in A, \quad (a, b) \in R \text{ and } (b, a) \in R \Rightarrow a = b$$

Observation 1.5: equivalent form of antisymmetry

A relation $R \subseteq A \times A$ is antisymmetric if and only if:

$$\forall a, b \in A, \quad (a, b) \in R \quad \text{and} \quad a \neq b \quad \Rightarrow \quad (b, a) \notin R$$

Proof. Fix $a, b \in A$. An implication is false only when the antecedent is true and the consequent is false.

- The implication “ $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ ” is false only when $(a, b) \in R$, $(b, a) \in R$ and $a \neq b$.
- The implication “ $(a, b) \in R$ and $a \neq b \Rightarrow (b, a) \notin R$ ” is false only when $(a, b) \in R$, $a \neq b$ and $(b, a) \in R$.

The two implications are therefore false in exactly the same cases: for every pair a, b one is true if and only if the other is, and hence one holds for all $a, b \in A$ if and only if the other does. $\square$

Example 1.6

- The relation $R_{\leq}$ is reflexive, but $R_{<}$ is not.
- The relations $R_{<}$ and $R_{\leq}$ are not symmetric.
- The relations $R_{<}$ and $R_{\leq}$ are transitive, but for example the relation:

$$R_{ab}^1 = \left\{ (a, b) : a, b \in \mathbb{N} \quad \text{and} \quad a = b - 1 \right\}$$

is not, e.g., $(3, 4) \in R_{ab}^1$ and $(4, 5) \in R_{ab}^1$ but $(3, 5) \notin R_{ab}^1$.

Let us now see in detail, for each property, how to prove that a relation has it and how to show, with a counterexample, that it does not.

Example 1.7: reflexive property

- The relation $R_{=}$ is reflexive, since

$$a = a, \quad \forall a \in \mathbb{N}.$$

- The relation $R_{\leq}$ is reflexive, since

$$a \leq a, \quad \forall a \in \mathbb{N}.$$

- The relation $R_{<}$ is not reflexive: a counterexample is $a = 1$, since $1 < 1$ is false and hence $(1, 1) \notin R_{<}$.

Example 1.8: symmetric property

- The relation $R_=_$ is symmetric, since

$$a = b \Rightarrow b = a, \quad \forall a, b \in \mathbb{N}.$$

- The relations $R_<$ and $R_\leq$ are not symmetric: a counterexample is the pair $(1, 2)$, since $1 < 2$ and $1 \leq 2$, but $2 < 1$ and $2 \leq 1$ are false. Hence $(1, 2) \in R_<$ but $(2, 1) \notin R_<$, and $(1, 2) \in R_\leq$ but $(2, 1) \notin R_\leq$.

Example 1.9: transitive property

- The relation $R_=_$ is transitive, since

$$a = b \text{ and } b = c \Rightarrow a = c, \quad \forall a, b, c \in \mathbb{N}.$$

The same holds for $R_<$ and $R_\leq$: if $a < b$ and $b < c$ then $a < c$, and if $a \leq b$ and $b \leq c$ then $a \leq c$.

- The relation R_{ab}^1 is not transitive: a counterexample is given by $(3, 4) \in R_{ab}^1$ and $(4, 5) \in R_{ab}^1$, while $(3, 5) \notin R_{ab}^1$ because $3 \neq 5 - 1$.

Example 1.10: antisymmetric property

- The relation $R_\leq$ is antisymmetric, since $a \leq b$ and $b \leq a$ imply $a = b$.
- The relation $R_=_$ is antisymmetric, since

$$a = b \text{ and } b = a \Rightarrow a = b, \quad \forall a, b \in \mathbb{N}.$$

- The relation $R_<$ is antisymmetric. Indeed, there are no two natural numbers a and b with $a < b$ and $b < a$ (otherwise we would have $a < a$): the antecedent of the implication is never true, and hence the implication is true for all $a, b \in \mathbb{N}$. With the equivalent form of antisymmetry this can be seen directly:

$$a < b \text{ and } a \neq b \Rightarrow b \not< a, \quad \forall a, b \in \mathbb{N}.$$

- The relation $R_{\equiv_3}$ is not antisymmetric: a counterexample is given by $(0, 3) \in R_{\equiv_3}$ (because $3 - 0 = 1 \cdot 3$) and $(3, 0) \in R_{\equiv_3}$ (because $0 - 3 = (-1) \cdot 3$), while $0 \neq 3$.

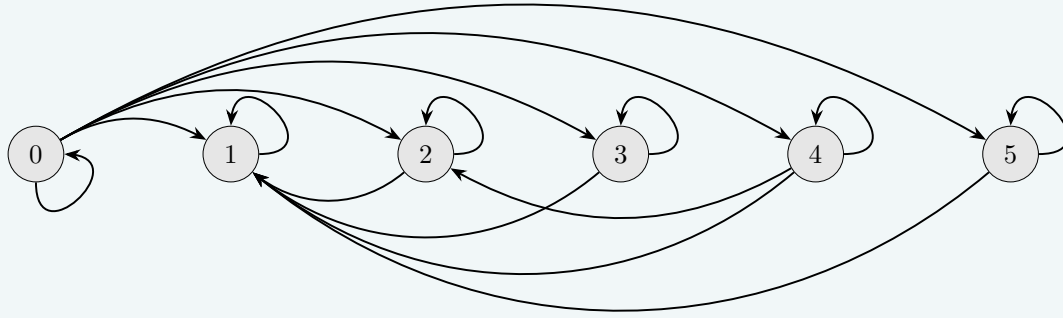
1.3 Representation with directed graphs

Relations can be represented by a **directed graph**, where the **vertices** are the elements of the set on which the relation R is defined and an **arc** (a, b) , drawn as an arrow from a to b , means that $(a, b) \in R$. An arc (a, a) is called a **loop**.

Example 1.11: graphs of relations

- The directed graph of the relation “is a multiple of” on the set $\{0, 1, \dots, 5\}$:

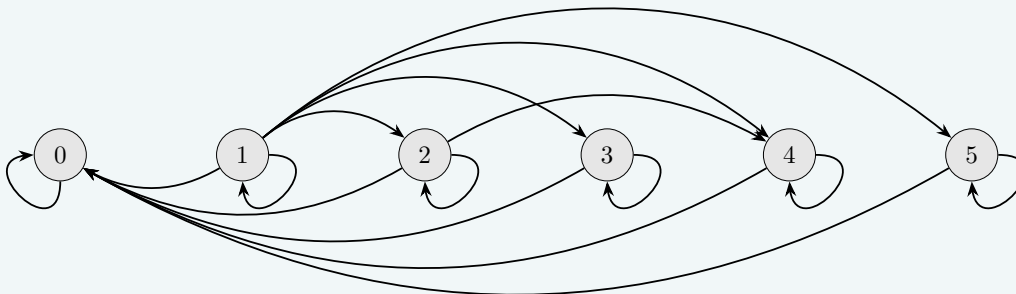
$$R = \left\{ (a, b) : a, b \in \{0, 1, \dots, 5\} \text{ and } \exists k \in \mathbb{N} \text{ such that } a = kb \right\}.$$



- The directed graph of the relation “is a divisor of” on the set $\{0, 1, \dots, 5\}$:

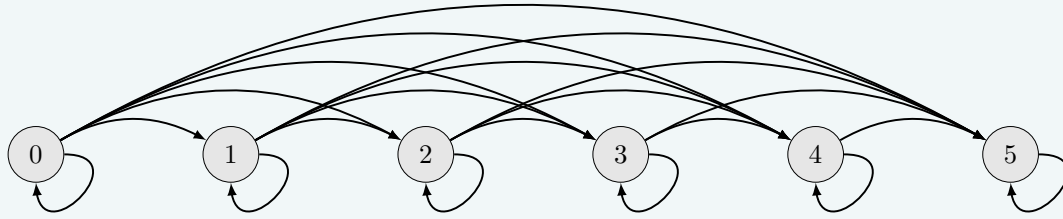
$$R = \left\{ (a, b) : a, b \in \{0, 1, \dots, 5\} \text{ and } \exists k \in \mathbb{N} \text{ such that } b = ka \right\}.$$

It has the same arcs as the previous graph, traversed in the opposite direction.

**Example 1.12: graphs of relations**

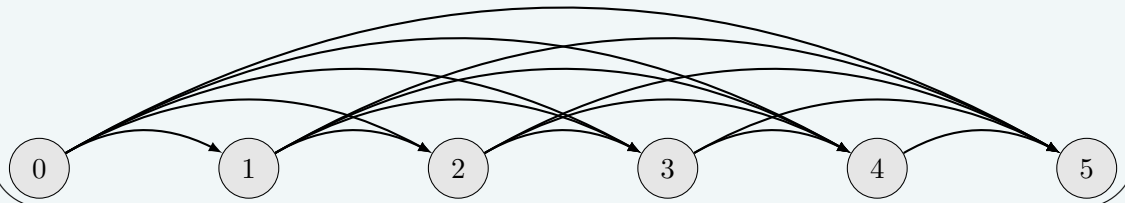
- The directed graph of the relation “less than or equal to” on the set $\{0, 1, \dots, 5\}$:

$$R = \left\{ (a, b) : a, b \in \{0, 1, \dots, 5\} \text{ and } a \leq b \right\}.$$



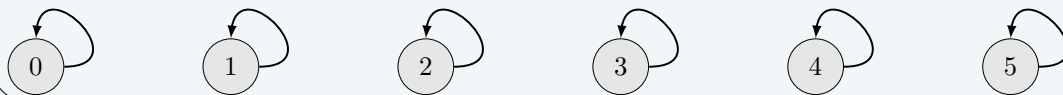
- The directed graph of the relation “less than” on the set $\{0, 1, \dots, 5\}$, obtained from the previous one by removing the loops:

$$R = \left\{ (a, b) : a, b \in \{0, 1, \dots, 5\} \text{ and } a < b \right\}.$$



- The directed graph of the relation “equal to” on the set $\{0, 1, \dots, 5\}$, which has only the loops:

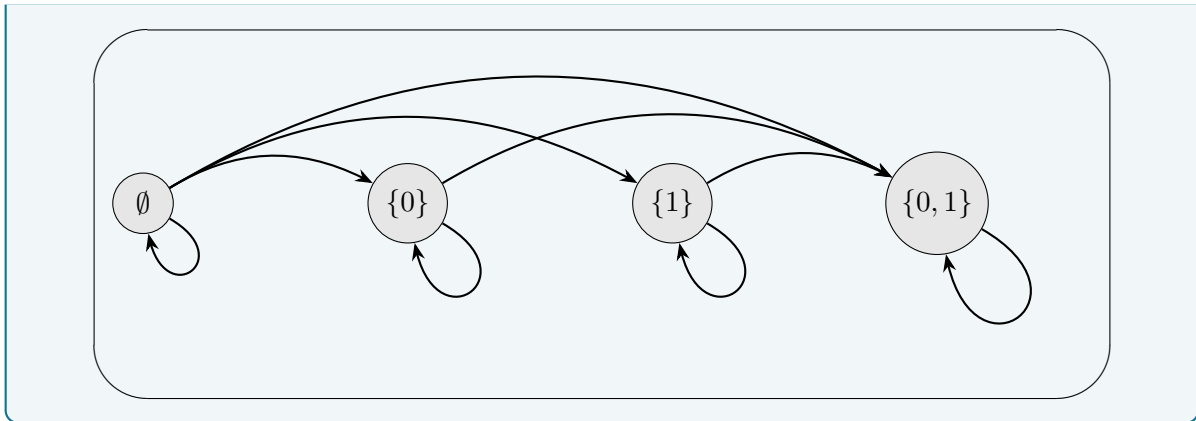
$$R = \left\{ (a, b) : a, b \in \{0, 1, \dots, 5\} \text{ and } a = b \right\}.$$



Example 1.13: graph of the inclusion relation

The directed graph of the relation “is a subset of” on the subsets of $\{0, 1\}$:

$$R = \left\{ (A, B) : A, B \in 2^{\{0,1\}} \text{ and } A \subseteq B \right\}.$$

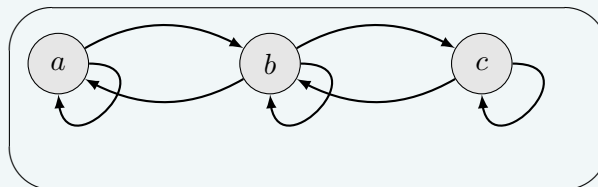


The main properties of a relation can be read on its directed graph:

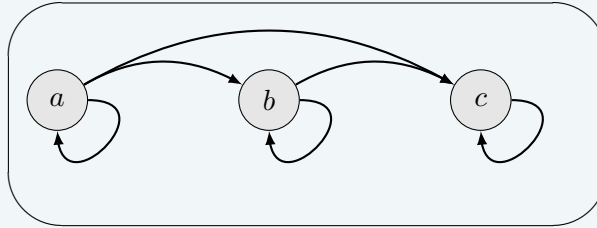
- the relation is **reflexive** if and only if every vertex a has the loop (a, a) ;
 - the relation is **symmetric** if and only if, for every pair of vertices a and b , if there is the arc from a to b then there is also the arc from b to a ;
 - the relation is **transitive** if and only if, for every triple of vertices a , b and c , if there are the arc from a to b and the arc from b to c then there is also the arc from a to c ;
 - the relation is **antisymmetric** if and only if, for every pair of distinct vertices a and b , if there is the arc from a to b then there is no arc from b to a (equivalent form of antisymmetry).
- For example, in the graphs of “is a multiple of”, “is a divisor of”, “less than or equal to” and “equal to” every vertex has the loop (reflexive relations), while in the graph of “less than” no vertex has it. In none of these graphs are there two arcs in opposite directions between two distinct vertices (antisymmetric relations).

Example 1.14: graphs satisfying only two properties

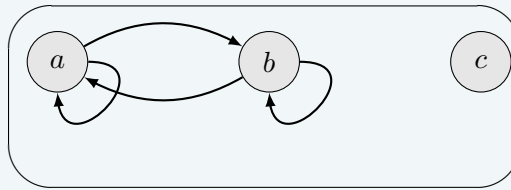
- The relation $\{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\}$ on the set $\{a, b, c\}$ is reflexive and symmetric, but not transitive: there are the arcs (a, b) and (b, c) , but not the arc (a, c) .



- The relation $\{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\}$ on the set $\{a, b, c\}$ is reflexive and transitive, but not symmetric: there is the arc (a, b) , but not the arc (b, a) .



- The relation $\{(a, a), (a, b), (b, a), (b, b)\}$ on the set $\{a, b, c\}$ is symmetric and transitive, but not reflexive: the loop (c, c) is missing. For transitivity it suffices to check the pairs of consecutive arcs: from (a, b) and (b, a) follows (a, a) , from (b, a) and (a, b) follows (b, b) , and the pairs containing a loop give an arc already present.



Observation 1.6: symmetric and transitive relations

Let $R \subseteq A \times A$ be a symmetric and transitive relation. If an element $a \in A$ appears in at least one pair of R , then $(a, a) \in R$.

Proof. If a appears in a pair of R , there exists $b \in A$ such that $(a, b) \in R$ or $(b, a) \in R$. By symmetry, in both cases we have both $(a, b) \in R$ and $(b, a) \in R$. By transitivity, from $(a, b) \in R$ and $(b, a) \in R$ follows $(a, a) \in R$. $\square$

- Hence a symmetric and transitive relation fails to be reflexive only if some element of A appears in no pair of the relation, like the element c of the last graph.

1.4 Equivalence relations

Definition 1.7: of equivalence relation

A reflexive, symmetric and transitive relation is an **equivalence relation**.

Example 1.15: equivalence relations

- The relation $R_{=}$ is an equivalence relation, because it is reflexive, symmetric and transitive.
- The relation $R_{<}$ is not an equivalence relation, because it is not reflexive (nor sym-

metric).

- The relation $R_{\leq}$ is not an equivalence relation, because it is not symmetric.

Example 1.16: congruence modulo n

Given a natural number $n \geq 1$, the relation

$$R_{\equiv_n} = \left\{ (a, b) : a, b \in \mathbb{N} \text{ and } a \equiv b \pmod{n} \right\}$$

is an equivalence relation. Recall that $(a, b) \in R_{\equiv_n}$ if and only if there exists $q \in \mathbb{Z}$ such that $b - a = qn$.

- Reflexive: for every $a \in \mathbb{N}$ we have $a - a = 0 \cdot n$, with $0 \in \mathbb{Z}$, hence $(a, a) \in R_{\equiv_n}$.
- Symmetric: for all $a, b \in \mathbb{N}$, if there exists $q \in \mathbb{Z}$ such that $b - a = qn$, then

$$a - b = (-q)n, \quad \text{with } -q \in \mathbb{Z},$$

hence $(b, a) \in R_{\equiv_n}$.

- Transitive: for all $a, b, c \in \mathbb{N}$, if there exist $q, p \in \mathbb{Z}$ such that $b - a = qn$ and $c - b = pn$, then

$$c - a = (c - b) + (b - a) = pn + qn = (q + p)n, \quad \text{with } q + p \in \mathbb{Z},$$

hence $(a, c) \in R_{\equiv_n}$.

Example 1.17: the sum is even

The relation

$$R_{\text{even}} = \left\{ (a, b) : a, b \in \mathbb{N} \text{ and } \exists m \in \mathbb{N} \text{ such that } a + b = 2m \right\}$$

is an equivalence relation.

- Reflexive: for every $a \in \mathbb{N}$ we have $a + a = 2a$, with $a \in \mathbb{N}$, hence $(a, a) \in R_{\text{even}}$.
- Symmetric: for all $a, b \in \mathbb{N}$, if $a + b = 2m$ with $m \in \mathbb{N}$, then also $b + a = a + b = 2m$, hence $(b, a) \in R_{\text{even}}$.
- Transitive: for all $a, b, c \in \mathbb{N}$, if there exist $p, q \in \mathbb{N}$ such that $a + b = 2p$ and $b + c = 2q$, then

$$(a + b) + (b + c) = 2p + 2q \Rightarrow a + c = 2(p + q - b).$$

The number $p + q - b$ is an integer, and it is not negative because $2(p + q - b) = a + c \geq 0$; hence $p + q - b \in \mathbb{N}$ and $(a, c) \in R_{\text{even}}$.

Definition 1.8: of equivalence class

Given an equivalence relation R on a set A , the **equivalence class** of an element $a \in A$ is the set

$$[a] = \left\{ b \in A : (a, b) \in R \right\}.$$

Lemma 1.9: related elements have the same class

Let R be an equivalence relation on a set A . For all $a, b \in A$:

$$(a, b) \in R \Rightarrow [a] = [b].$$

Proof. Let $(a, b) \in R$. We show the two inclusions.

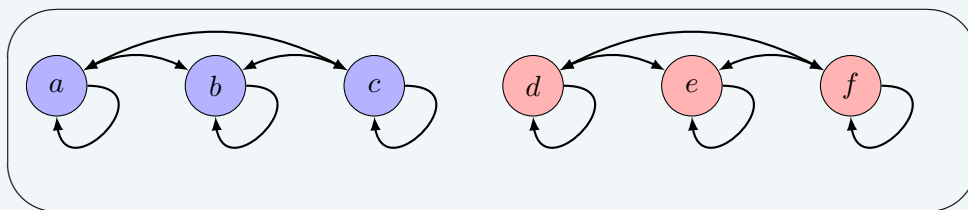
- $[b] \subseteq [a]$: if $c \in [b]$, then $(b, c) \in R$. From $(a, b) \in R$ and $(b, c) \in R$, by transitivity, follows $(a, c) \in R$, that is $c \in [a]$.
- $[a] \subseteq [b]$: by symmetry $(b, a) \in R$. If $c \in [a]$, then $(a, c) \in R$, and from $(b, a) \in R$ and $(a, c) \in R$, by transitivity, follows $(b, c) \in R$, that is $c \in [b]$.

Hence $[a] = [b]$. □

- Any two elements of the same class are related to each other: if $b, c \in [a]$, then $(a, b) \in R$ and $(a, c) \in R$; by symmetry $(b, a) \in R$ and by transitivity $(b, c) \in R$. An equivalence class therefore gathers elements that are all related to each other.

Example 1.18: equivalence classes

The directed graph of an equivalence relation on the set $\{a, b, c, d, e, f\}$ with two equivalence classes, $\{a, b, c\}$ (in blue) and $\{d, e, f\}$ (in red). A double-headed arrow between x and y represents the two arcs (x, y) and (y, x) .


Definition 1.10: of partition

A **partition** of a set A is a family $\mathcal{S}$ of subsets of A such that:

- every set of the family is nonempty: $S \neq \emptyset$ for every $S \in \mathcal{S}$;
- two distinct sets of the family are disjoint: $S \cap T = \emptyset$ for all $S, T \in \mathcal{S}$ with $S \neq T$;

- the union of the sets of the family is A : $\bigcup_{S \in \mathcal{S}} S = A$, that is, every $a \in A$ belongs to at least one $S \in \mathcal{S}$.

- The family can be finite, $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$, or infinite.
- From the second and the third condition it follows that every element of A belongs to **one and only one** set of the partition: to at least one by the third condition, and not to two distinct sets S and T , which are disjoint.

Proposition 1.11: equivalence classes form a partition

Let R be an equivalence relation on a set A . Then the family of the equivalence classes

$$\mathcal{S} = \left\{ [a] : a \in A \right\}$$

is a partition of A .

- The proof checks the three conditions of the definition of partition; the second one follows from the fact that two classes are either equal or disjoint.

Proof. Every class $[a]$ is a subset of A by definition.

- Nonempty classes: by reflexivity $(a, a) \in R$, hence $a \in [a]$ and $[a] \neq \emptyset$.
- Union equal to A : every $a \in A$ belongs to the class $[a]$, hence $A \subseteq \bigcup_{a \in A} [a]$; the other inclusion holds because every class is contained in A .
- Distinct classes are disjoint: we show that if two classes $[a]$ and $[b]$ have an element in common, then they are equal. Let $c \in [a] \cap [b]$. Then $(a, c) \in R$ and $(b, c) \in R$. By symmetry $(c, b) \in R$, and by transitivity, from $(a, c) \in R$ and $(c, b) \in R$, follows $(a, b) \in R$. By Lemma 1.9 we then have $[a] = [b]$. Hence two distinct classes have no element in common.

□

Proposition 1.12: every partition defines an equivalence relation

Let $\mathcal{S}$ be a partition of a set A and let

$$R = \left\{ (a, b) : a, b \in A \text{ and } \exists S \in \mathcal{S} \text{ such that } a \in S \text{ and } b \in S \right\}.$$

Then R is an equivalence relation on A , and its equivalence classes are exactly the sets of the partition $\mathcal{S}$.

Proof. We first show that R is an equivalence relation.

- Reflexive: for every $a \in A$ there exists $S \in \mathcal{S}$ with $a \in S$ (the union of the sets of the partition is A). Then a and a belong to the same set S , that is $(a, a) \in R$.
- Symmetric: if $(a, b) \in R$, there exists $S \in \mathcal{S}$ with $a \in S$ and $b \in S$; the same S contains b and a , hence $(b, a) \in R$.
- Transitive: if $(a, b) \in R$ and $(b, c) \in R$, there exist $S, T \in \mathcal{S}$ with $a, b \in S$ and $b, c \in T$. Then $b \in S \cap T$, hence $S \cap T \neq \emptyset$ and, since two distinct sets of the partition are disjoint, $S = T$. Hence a and c belong to the same set S , that is $(a, c) \in R$.

We now show that the classes are the sets of the partition. Given $a \in A$, let S_a be the only set of $\mathcal{S}$ containing a . If $b \in S_a$, then $(a, b) \in R$, that is $b \in [a]$. Conversely, if $b \in [a]$, there exists $S \in \mathcal{S}$ with $a, b \in S$; since a belongs to only one set of the partition, $S = S_a$ and hence $b \in S_a$. Therefore $[a] = S_a$: every class is a set of the partition. Finally, every $S \in \mathcal{S}$ is nonempty: taking $a \in S$, we have $S = S_a = [a]$, hence every set of the partition is a class. $\square$

Example 1.19: classes modulo 3

The equivalence relation $R_{\equiv_3}$ divides the natural numbers into 3 equivalence classes:

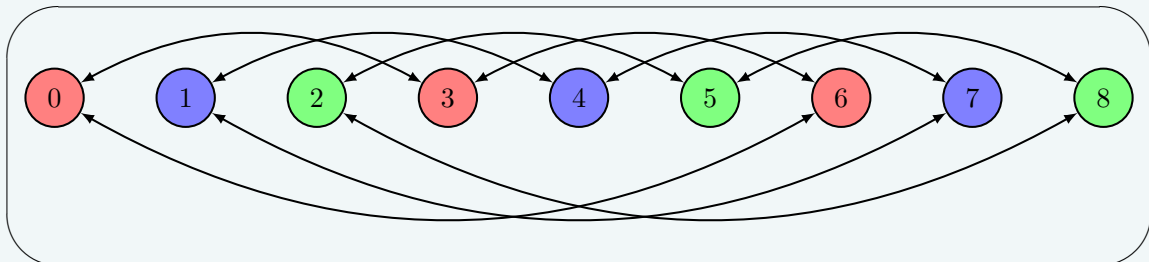
$$[0] = \{0, 3, 6, 9, \dots\}, \quad [1] = \{1, 4, 7, 10, \dots\}, \quad [2] = \{2, 5, 8, 11, \dots\}.$$

Each class contains the natural numbers that have the same remainder in the division by 3. Indeed, write $a = 3q_a + r_a$ and $b = 3q_b + r_b$, with $q_a, q_b \in \mathbb{N}$ and remainders $r_a, r_b \in \{0, 1, 2\}$. Then

$$b - a = 3(q_b - q_a) + (r_b - r_a).$$

- If $r_a = r_b$, then $b - a = 3(q_b - q_a)$ and $(a, b) \in R_{\equiv_3}$.
- If $(a, b) \in R_{\equiv_3}$, that is $b - a = 3q$ with $q \in \mathbb{Z}$, then $r_b - r_a = 3(q - q_b + q_a)$ is a multiple of 3 between -2 and 2 , hence $r_b - r_a = 0$.

Here is the directed graph of the relation on the natural numbers from 0 to 8, with one colour per class. A double-headed arrow represents the two arcs (a, b) and (b, a) ; the loops (a, a) , present for every a , are not drawn.



Example 1.20: even and odd numbers

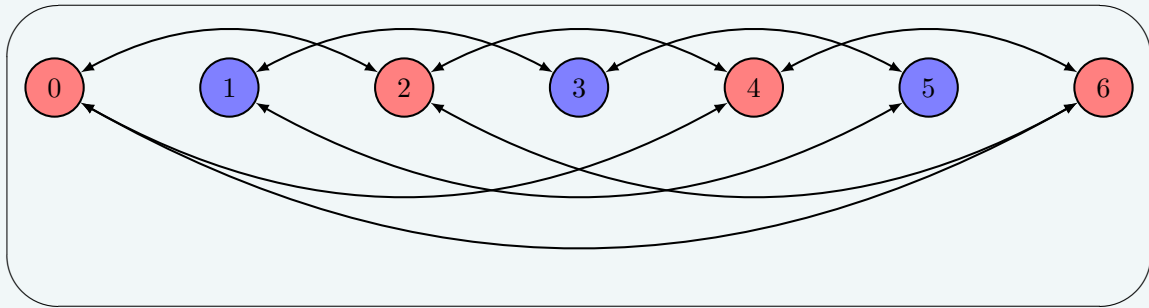
The equivalence relation R_{even} divides the natural numbers into 2 equivalence classes, the even numbers and the odd numbers:

$$[0] = \{0, 2, 4, 6, \dots\}, \quad [1] = \{1, 3, 5, 7, \dots\}.$$

Indeed, write $a = 2q_a + r_a$ and $b = 2q_b + r_b$, with $q_a, q_b \in \mathbb{N}$ and remainders $r_a, r_b \in \{0, 1\}$. Then $a + b = 2(q_a + q_b) + (r_a + r_b)$, with $r_a + r_b \in \{0, 1, 2\}$.

- If $r_a = r_b$, then $r_a + r_b = 2r_a$ and $a + b = 2(q_a + q_b + r_a)$ is even.
- If $r_a \neq r_b$, then $r_a + r_b = 1$ and $a + b = 2(q_a + q_b) + 1$ is not even.

Hence $(a, b) \in R_{\text{even}}$ if and only if a and b are both even or both odd. Here is the directed graph of the relation on the natural numbers from 0 to 6 (even in red, odd in blue; double-headed arrows and loops omitted as in the previous example):


Observation 1.13: equivalence and antisymmetry

If an equivalence relation R on a set A is also antisymmetric, then every equivalence class has a single element: $[a] = \{a\}$ for every $a \in A$.

Proof. Let $a \in A$. By reflexivity $a \in [a]$. If $b \in [a]$, then $(a, b) \in R$; by symmetry also $(b, a) \in R$, and by antisymmetry $a = b$. Hence $[a] = \{a\}$. $\square$

- In this case $(a, b) \in R$ if and only if $b \in [a] = \{a\}$, that is if and only if $a = b$: the only equivalence relation on A that is also antisymmetric is the relation “equal to”.

1.5 Partial order relations

Definition 1.14: of partial order relation

A reflexive, antisymmetric and transitive relation is a **partial order relation**.

Example 1.21

- The relation $R_{\leq}$ is a partial order relation, but the relation $R_{<}$ is not, since it is not reflexive.
- The relation $R_{=}$ is a partial order relation, because it is reflexive, antisymmetric and transitive.

Observation 1.15

The relation $R_{\subseteq}$ is a partial order relation

Proof. We must prove that the relation is reflexive, antisymmetric and transitive.

- For it to be reflexive we must prove that $(S, S) \in R_{\subseteq}$, which is true since $S \subseteq S$.
- For it to be antisymmetric we must prove that if $S_1 \neq S_2$ then $S_1 \not\subseteq S_2$ or $S_2 \not\subseteq S_1$ or both, the negation of the desired property.

Since $S_1 \neq S_2$:

either there exists some element that is in S_1 but not in S_2 , hence $S_1 \not\subseteq S_2$
 or there exists some element that is in S_2 but not in S_1 , hence $S_2 \not\subseteq S_1$
 or both of the previous options hold

- For it to be transitive we must prove that $(S_1, S_2) \in R_{\subseteq}$ and $(S_2, S_3) \in R_{\subseteq}$ imply $(S_1, S_3) \in R_{\subseteq}$. Clearly, since $S_1 \subseteq S_2$ and $S_2 \subseteq S_3$, we have $S_1 \subseteq S_3$.

□

Example 1.22: multiples and divisors

The relation R_{mul} ("is a multiple of") is a partial order relation.

- Reflexive: for every $a \in \mathbb{N}$ we have $a = 1 \cdot a$, hence $(a, a) \in R_{\text{mul}}$.
- Transitive: for all $a, b, c \in \mathbb{N}$, if there exist $q, p \in \mathbb{N}$ such that $a = qb$ and $b = pc$, then

$$a = qb = q(pc) = (qp)c, \quad \text{with } qp \in \mathbb{N},$$

hence $(a, c) \in R_{\text{mul}}$.

- Antisymmetric: let $a, b \in \mathbb{N}$ with $(a, b) \in R_{\text{mul}}$ and $(b, a) \in R_{\text{mul}}$, that is $a = qb$ and $b = pa$ with $q, p \in \mathbb{N}$. Then

$$a = qb = qpa.$$

If $a = 0$, then $b = p \cdot 0 = 0 = a$. If $a \neq 0$, dividing by a we obtain $qp = 1$. Neither factor is 0 (otherwise the product would be 0), hence $q \geq 1$ and $p \geq 1$; if $q \geq 2$ we would have $qp \geq 2p \geq 2$, hence $q = 1$, and in the same way $p = 1$. Therefore $a = 1 \cdot b = b$.

Also R_{div} (“is a divisor of”) is a partial order relation, because $(a, b) \in R_{\text{div}}$ if and only if $(b, a) \in R_{\text{mul}}$:

- reflexive: $(a, a) \in R_{\text{mul}}$, hence $(a, a) \in R_{\text{div}}$;
- transitive: if $(a, b), (b, c) \in R_{\text{div}}$, then $(c, b), (b, a) \in R_{\text{mul}}$, hence $(c, a) \in R_{\text{mul}}$ by transitivity of R_{mul} , that is $(a, c) \in R_{\text{div}}$;
- antisymmetric: if $(a, b), (b, a) \in R_{\text{div}}$, then $(b, a), (a, b) \in R_{\text{mul}}$, hence $a = b$ by antisymmetry of R_{mul} .

Definition 1.16: of partially ordered set

A **partially ordered set** is a pair consisting of a set and a partial order relation defined on it.

Example 1.23

- The sets of natural, rational or real numbers with the relation $R_{\leq}$ are partially ordered sets.
- The relation “is a descendant of” defined on a subset of people is a partial order relation (if we consider individuals as descendants of themselves). Consequently, the subset of people considered with the relation “is a descendant of” is a partially ordered set.
- The set of natural numbers with the relation R_{mul} , or with the relation R_{div} , is a partially ordered set.

Observation 1.17: graph of a partial order

The directed graph of a partial order relation contains no **cycles**, that is, sequences of arcs

$$(a_1, a_2), (a_2, a_3), \dots, (a_{k-1}, a_k), (a_k, a_1)$$

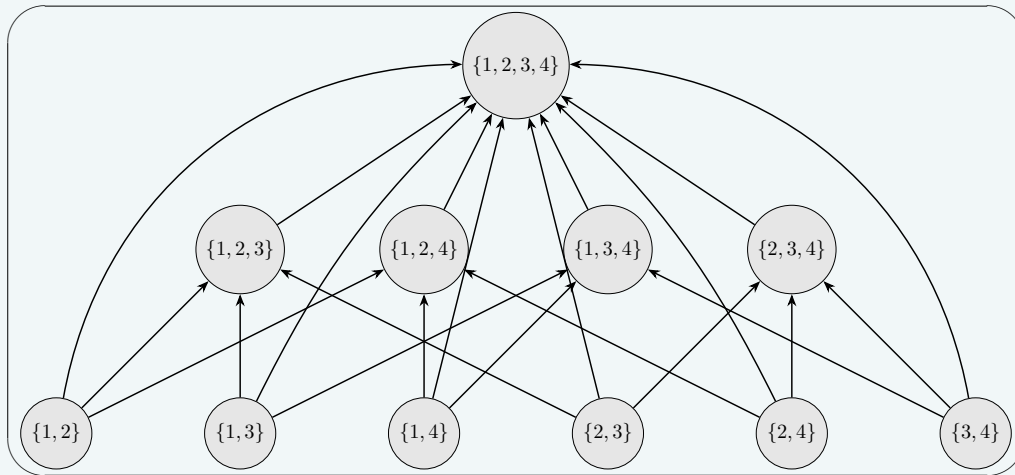
with $k \geq 2$ distinct vertices $a_1, a_2, \dots, a_k$. In other words, once the loops are removed, the graph is **acyclic**.

Proof. Suppose by contradiction that the graph contains a cycle as above. If $k = 2$, the cycle consists of the arcs (a_1, a_2) and (a_2, a_1) . If $k \geq 3$, applying transitivity to the arcs (a_2, a_3) and (a_3, a_4) we obtain $(a_2, a_4) \in R$, then from (a_2, a_4) and (a_4, a_5) we obtain $(a_2, a_5) \in R$, and so on up to the arc (a_k, a_1) , from which we obtain $(a_2, a_1) \in R$. In both cases $(a_1, a_2) \in R$ and $(a_2, a_1) \in R$, and by antisymmetry $a_1 = a_2$, against the assumption that the vertices of the cycle are distinct. $\square$

- The converse does not hold: an acyclic graph does not necessarily represent a partial order relation. For example, the relation $\{(1, 2), (2, 3)\}$ on the set $\{1, 2, 3\}$ has a graph without cycles, but it is neither reflexive ($(1, 1)$ does not belong to the relation) nor transitive ($(1, 3)$ does not belong to the relation).

Example 1.24

The directed graph of the relation $R_{\subseteq}$ on the subsets of the set $\{1, 2, 3, 4\}$ is the following. To keep the figure readable, the empty set, the sets consisting of a single number (and hence the arcs leaving them) and the loops (S, S) are not shown; all the other arcs are drawn.



1.5.1 Maximum, maximal, minimum and minimal elements

Definition 1.18: of maximum, maximal, minimum and minimal element

Let (A, R) be a partially ordered set.

- An element $\tilde{a} \in A$ is the **maximum** of A if:

$$(b, \tilde{a}) \in R, \quad \forall b \in A.$$

- An element $\bar{a} \in A$ is a **maximal element** of A if there is no $b \in A$ such that:

$$(\bar{a}, b) \in R \quad \text{and} \quad b \neq \bar{a}.$$

- An element $\tilde{a} \in A$ is the **minimum** of A if:

$$(\tilde{a}, b) \in R, \quad \forall b \in A.$$

- An element $\bar{a} \in A$ is a **minimal element** of A if there is no $b \in A$ such that:

$$(b, \bar{a}) \in R \quad \text{and} \quad b \neq \bar{a}.$$

- On the directed graph: the maximum is the vertex that receives an arc from every vertex; a maximal element is a vertex from which no arc leaves, apart from the loop. For the minimum and the minimal elements, incoming and outgoing arcs are swapped.

Proposition 1.19: uniqueness of the maximum and of the minimum

In a partially ordered set (A, R) the maximum, if it exists, is unique. The same holds for the minimum.

Proof. Let $\tilde{a}_1$ and $\tilde{a}_2$ be two maxima of A . Since $\tilde{a}_2$ is a maximum, $(\tilde{a}_1, \tilde{a}_2) \in R$; since $\tilde{a}_1$ is a maximum, $(\tilde{a}_2, \tilde{a}_1) \in R$. By antisymmetry $\tilde{a}_1 = \tilde{a}_2$. For the minimum: if $\tilde{a}_1$ and $\tilde{a}_2$ are two minima, then $(\tilde{a}_1, \tilde{a}_2) \in R$ because $\tilde{a}_1$ is a minimum and $(\tilde{a}_2, \tilde{a}_1) \in R$ because $\tilde{a}_2$ is a minimum, and again $\tilde{a}_1 = \tilde{a}_2$. $\square$

Observation 1.20: the maximum is the only maximal element

If a partially ordered set (A, R) has the maximum $\tilde{a}$, then $\tilde{a}$ is a maximal element, and it is the only maximal element of A .

Proof.

- $\tilde{a}$ is maximal: if $b \in A$ and $(\tilde{a}, b) \in R$, since $\tilde{a}$ is the maximum also $(b, \tilde{a}) \in R$, and by antisymmetry $b = \tilde{a}$. Hence there is no $b \neq \tilde{a}$ with $(\tilde{a}, b) \in R$.
- $\tilde{a}$ is the only maximal element: let $\bar{a}$ be a maximal element. Since $\tilde{a}$ is the maximum, $(\bar{a}, \tilde{a}) \in R$; since $\bar{a}$ is maximal, $\bar{a} \neq \tilde{a}$ is impossible. Hence $\bar{a} = \tilde{a}$. $\square$

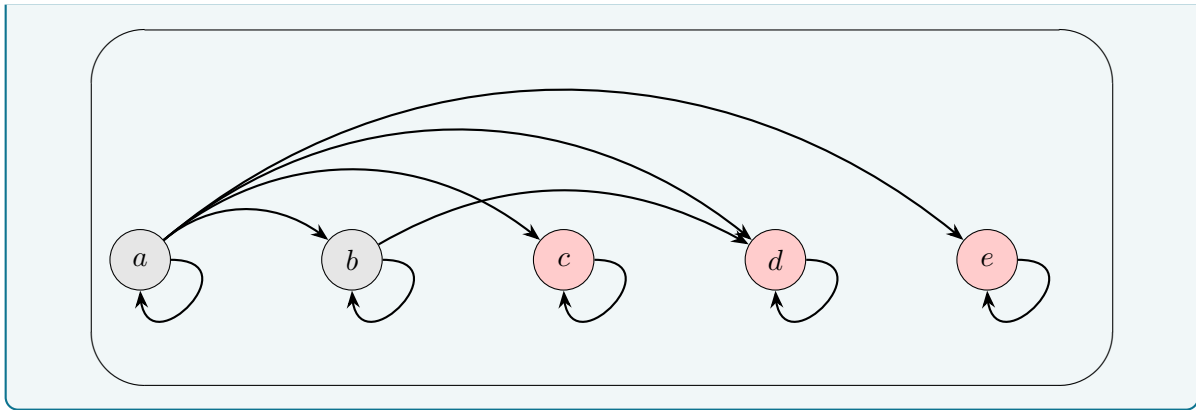
- In a partially ordered set the **maximum element** may not exist, that is, there may be no element a such that:

$$\forall b \in A, \quad (b, a) \in R$$

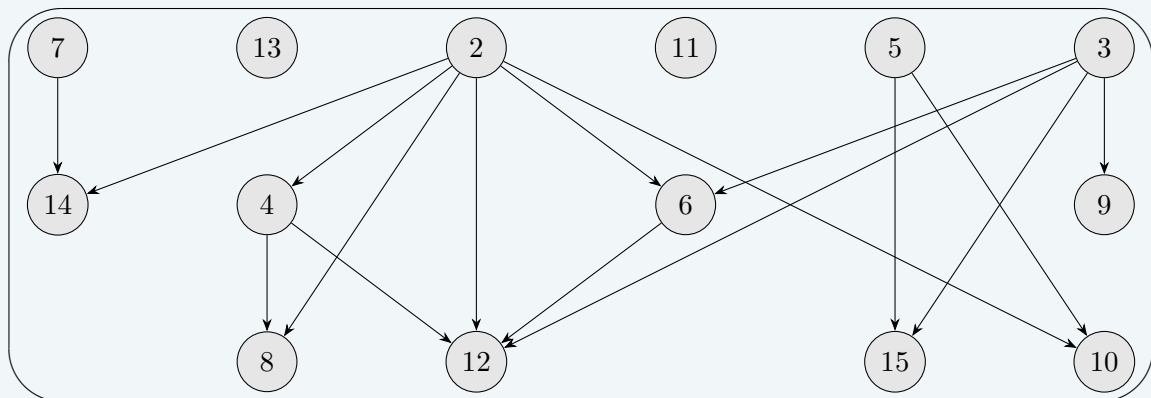
A partially ordered set may therefore contain several maximal elements a such that, for no $b \in A$ with $b \neq a$, we have $(a, b) \in R$.

Example 1.25: maximal elements

The directed graph of a partial order relation on the set $\{a, b, c, d, e\}$ with three maximal elements, c , d and e , in red. The maximum does not exist: for example (d, c) does not belong to the relation. The element a is the minimum.

**Example 1.26**

The directed graph associated with the partial order relation “is a divisor of” on the set $\{2, 3, \dots, 15\}$ is the following (the loops (a, a) , present for every a , are not drawn). The maximal elements are 8, 9, 10, 11, 12, 13, 14 and 15, which divide no other number of the set, and the maximum does not exist. The minimal elements are 2, 3, 5, 7, 11 and 13, and the minimum does not exist.

**Example 1.27**

Given a set of boxes of different sizes, the relation “one box is contained in the other” on the set of boxes may have several maximal boxes, that is, boxes that are not contained in any other box.

1.6 Total order relations**Definition 1.21: of total relation**

A relation R on a set A is a **total relation** if:

$$\forall a, b \in A, \quad (a, b) \in R \quad \text{or} \quad (b, a) \in R \quad (\text{or both})$$

Observation 1.22: a total relation is reflexive

Every total relation R on a set A is reflexive.

Proof. Let $a \in A$. Choosing $b = a$ in the definition of total relation we obtain that $(a, a) \in R$ or $(a, a) \in R$, that is $(a, a) \in R$. $\square$

Example 1.28

- The relation $R_{\leq}$ is a total relation.
- The relation $R_{\subseteq}$ is not a total relation since, taking for example $S_1 = \{1, 2\}$ and $S_2 = \{2, 3\}$, $(S_1, S_2) \notin R_{\subseteq}$ and $(S_2, S_1) \notin R_{\subseteq}$.
- The relation “is a descendant of” is not a total relation since there exist pairs of individuals (a, b) for which neither a descends from b nor b descends from a .
- The relation $R_{<}$ is not a total relation, because it is not reflexive: for $a = b = 1$ the condition of the definition requires $(1, 1) \in R_{<}$, which is false.
- The relation $R_{=}$ is not a total relation: for example $(3, 4) \notin R_{=}$ and $(4, 3) \notin R_{=}$.

Definition 1.23: of total order relation

A partial order relation that is also a total relation is a **total order relation**.

Example 1.29

- The relation $R_{\leq}$ is a total order relation.
- The relation $R_{=}$ is a partial order relation but not a total order relation, because it is not a total relation.
- The relation $R_{<}$ is not a total order relation, because it is not a partial order relation.
- The relation $R_{\subseteq}$ is a partial order relation but not a total order relation: as we have seen, the sets $S_1 = \{1, 2\}$ and $S_2 = \{2, 3\}$ are not comparable.
- The relation R_{mul} is a partial order relation but not a total order relation: for example $(6, 9) \notin R_{\text{mul}}$, because there is no $k \in \mathbb{N}$ with $6 = k \cdot 9$, and $(9, 6) \notin R_{\text{mul}}$, because there is no $k \in \mathbb{N}$ with $9 = k \cdot 6$.

Example 1.30: divisibility among powers of 2

The relation

$$R = \left\{ (a, b) : a, b \in A \text{ and } \exists k \in \mathbb{N} \text{ such that } b = k a \right\}, \quad \text{with } A = \left\{ 2^n : n \in \mathbb{N} \right\},$$

that is, the relation “is a divisor of” on the powers of 2, is a total order relation.

- It is a partial order relation: reflexivity, antisymmetry and transitivity hold for R_{div} on all the natural numbers, hence they hold in particular for the elements of A .
- It is a total relation: let $a = 2^m$ and $b = 2^n$ be two elements of A , with $m, n \in \mathbb{N}$. If $m \leq n$, then

$$b = 2^n = 2^{n-m} \cdot 2^m = 2^{n-m} a, \quad \text{with } 2^{n-m} \in \mathbb{N},$$

hence $(a, b) \in R$. If instead $n < m$, in the same way $a = 2^{m-n} b$ and $(b, a) \in R$.

Definition 1.24: of totally ordered set

A **totally ordered set** is a pair consisting of a set and a total order relation defined on it.

Example 1.31

- The sets of natural, rational or real numbers with the relation $R_{\leq}$ are totally ordered sets.

Proposition 1.25: maximal element and maximum in a totally ordered set

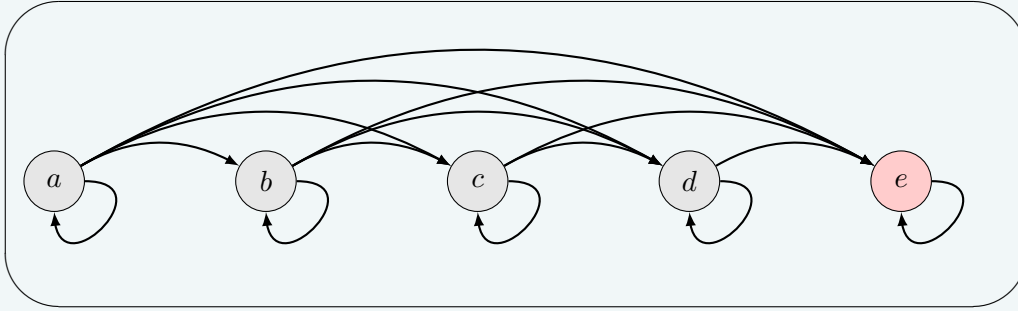
In a totally ordered set (A, R) an element is maximal if and only if it is the maximum. Hence a totally ordered set has at most one maximal element.

Proof. If $\tilde{a}$ is the maximum, then it is maximal, as we have already proved for every partially ordered set. Conversely, let $\bar{a}$ be a maximal element and let $b \in A$. Since the relation is total, $(b, \bar{a}) \in R$ or $(\bar{a}, b) \in R$. In the second case, since $\bar{a}$ is maximal, $b = \bar{a}$, and hence $(b, \bar{a}) = (\bar{a}, \bar{a}) \in R$ by reflexivity. In both cases $(b, \bar{a}) \in R$, hence $\bar{a}$ is the maximum. Since the maximum, if it exists, is unique, there is at most one maximal element. $\square$

- “At most” cannot be replaced by “exactly”: the totally ordered set $(\mathbb{N}, R_{\leq})$ has no maximal elements, because for every $a \in \mathbb{N}$ we have $(a, a+1) \in R_{\leq}$ with $a+1 \neq a$.

Example 1.32: maximum

The directed graph of a total order relation on the set $\{a, b, c, d, e\}$, with the maximum e in red:



1.7 Functions

Definition 1.26: of function

Given two sets A and B , a binary relation f on A and B is a **function** if, for each $a \in A$, there exists one and only one $b \in B$ such that $(a, b) \in f$.

Definition 1.27: of domain and codomain

The set A is called the **domain** of f , and the set B is called the **codomain** of f .

- We write:

$$f : A \rightarrow B$$

and if $(a, b) \in f$, we write:

$$b = f(a)$$

since b is uniquely determined by the choice of a .

- Intuitively, the function f assigns an element of B to each element of A . No element of A is associated with two different elements of B . The same element of B can however be assigned to different elements of A .

Example 1.33

- The binary relation on $\mathbb{N}$ and $\{0, 1\}$:

$$f = \left\{ (a, b) : a \in \mathbb{N}, b \in \{0, 1\} \text{ and } b = a \mod 2 \right\},$$

where $a \mod 2$ is the remainder of the division of a by 2, is a function $f : \mathbb{N} \rightarrow \{0, 1\}$ since for all natural numbers a , there is exactly one value $b \in \{0, 1\}$ such that $b = a \mod 2$. For example,

$$0 = f(0), \quad 1 = f(1), \quad 0 = f(2), \dots$$

Example 1.34

- The binary relation

$$g = \left\{ (a, b) : a, b \in \mathbb{N} \text{ and } a + b \text{ is even} \right\}$$

is not a function, since for example $(1, 3)$ and $(1, 5)$ are both in g . In other words, for $a = 1$, we do not have one and only one b such that $(a, b) \in g$.

Definition 1.28: of argument and value

Given a function $f : A \rightarrow B$, if $b = f(a)$, we say that $a \in A$ is the argument of f and that $b \in B$ is the value of f associated with a .

- We can **define a function** by *directly defining its value* for all the *elements* of its *domain*.

Example 1.35

For example, we can define $f(n) = 2n$ for $n \in \mathbb{N}$, which means:

$$f = \{(n, 2n) : n \in \mathbb{N}\} \text{ and } f : \mathbb{N} \rightarrow \mathbb{N}$$

- The next chapter, “**Functions**”, starts again from this definition: a function $f : A \rightarrow B$ is a binary relation on A and B with the **functional property** (each $a \in A$ corresponds to one and only one $b \in B$). There we study domain, codomain and image, and injective, surjective and bijective functions.

Functions

2.1 The concept of function

The existence of a **variable quantity** implies the existence of a **relation** between **two quantities**, that is, the dependence of one quantity on another.

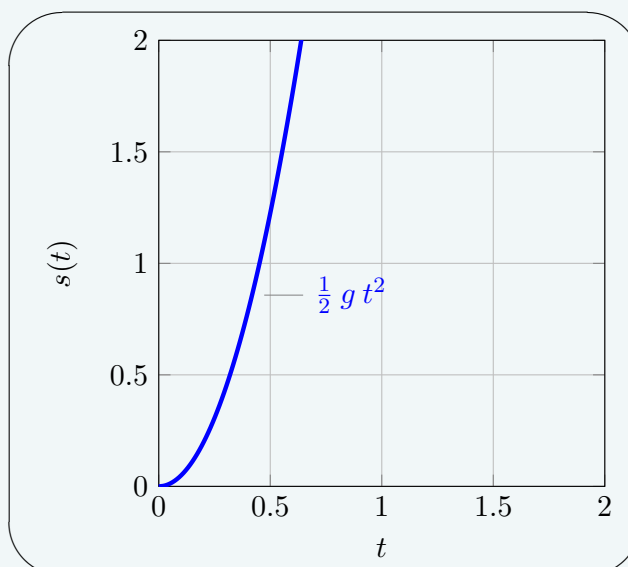
- In each case, this relation follows a certain **law** or **formula**.

Example 2.1: Laws/formulas $\rightarrow$ functions

If a heavy object is dropped from a certain height, the distance traveled by the object varies with time t according to the formula:

$$s(t) = \frac{1}{2} g t^2 \quad \text{with } t \geq 0$$

where $g \approx 9.8$ is the gravitational acceleration constant.



Hence, the time t is associated with the distance traveled $s(t)$: $t \mapsto s(t)$.

Example 2.2: Laws/formulas $\rightarrow$ functions

If a unit capital is invested for one year at the annual rate i , with interest paid monthly at a rate $\frac{i}{12}$, we have:

- after the first month, a capital equal to:

$$1 + \frac{i}{12} \cdot 1 = \underbrace{1 + \frac{i}{12}}_{\alpha}$$

- after the second month, a capital equal to:

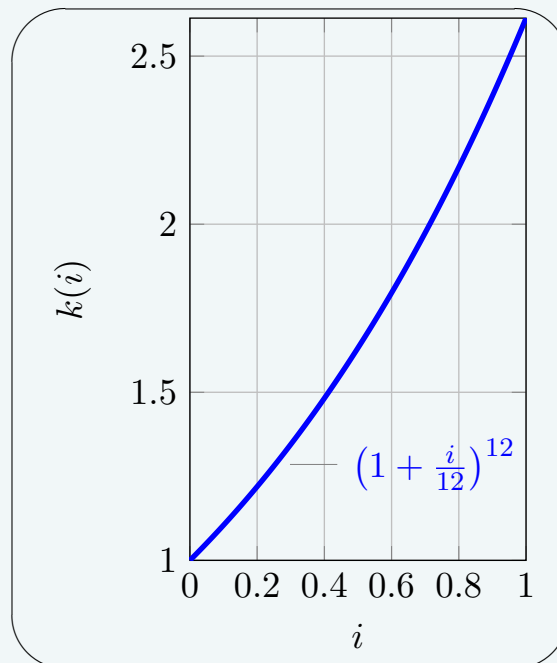
$$\underbrace{1 + \frac{i}{12}}_{\alpha} + \frac{i}{12} \underbrace{\left(1 + \frac{i}{12}\right)}_{\alpha} = \left(1 + \frac{i}{12}\right) \underbrace{\left(1 + \frac{i}{12}\right)}_{\beta} = \underbrace{\left(1 + \frac{i}{12}\right)^2}_{\beta}$$

- after the third month, a capital equal to:

$$\underbrace{\left(1 + \frac{i}{12}\right)^2}_{\beta} + \frac{i}{12} \underbrace{\left(1 + \frac{i}{12}\right)^2}_{\beta} = \underbrace{\left(1 + \frac{i}{12}\right)^2}_{\beta} \underbrace{\left(1 + \frac{i}{12}\right)}_{\beta} = \underbrace{\left(1 + \frac{i}{12}\right)^3}_{\beta}$$

Repeating the argument, the capital at the end of the year therefore depends on the rate i according to the formula:

$$k(i) = \left(1 + \frac{i}{12}\right)^{12} \quad \text{with } i \in [0, 1]$$



Hence, the rate i is associated with the capital $k(i)$ at the end of the year: $i \mapsto k(i)$

- In each of the previous examples, a *real number* (**input**) is associated *uniquely* with another *real number* (**output**). It is precisely the **uniqueness of the relation** that characterizes a function.
- In general, the **admissible inputs** for a given relation (function) are subject to natural restrictions, related to the very nature of the relation.

Example 2.3: Admissible inputs

- In the first example the “starting” real number has the meaning of *time*; imagining that the object is dropped at an initial time $t = 0$, it is clear that we must restrict ourselves to times $t \geq 0$.
- In the second example, clearly we must have $0 \leq i \leq 1$, where 1 corresponds to a rate of 100%.

2.2 Definition of function, domain, codomain and image

Definition 2.1: of domain

The set of admissible inputs for a given function is called the **domain**.

- The expressions **independent variable** (for a generic *input*) and **dependent variable** (for the *output*) are often used.

Example 2.4: Other types of relations

Let us consider the set of students of a given course.

- If A is the set of students and B is the set of their names, the relation that associates each student with his or her name is *unique* (single-valued).
- In this case neither A nor B is a set of numbers, but nevertheless a unique (single-valued) relation between these two sets is well defined.

On the other hand, the converse is not necessarily true: there could be two students with the same name.

Definition 2.2: of function

Given any two sets A , B , a **function** f with **domain** A and values in B (or also “with **codomain** B ”) is any law that associates with each element of A one and only one element of B .

- In the language of the previous chapter, “**Binary relations**”, the law f is identified with the set of pairs $(x, f(x))$ with $x \in A$: a function is thus a **binary relation** on A and B with the **functional property**, that is, each $x \in A$ corresponds to one and only one $y \in B$ with (x, y) in the relation (this is the definition of function given in that chapter).

- The notation:

$$f : A \rightarrow B$$

(which reads “ f defined from A to B ”), indicates the **domain** and the **codomain** of the function f .

- The notation

$$f : x \mapsto f(x)$$

(which reads “ f maps x to $f(x)$ ”) indicates how the function f acts on the elements.

- The **symbol** $f(x)$ **denotes the output or the value** that the function f associates with x , and it must not be confused with the symbol f , which denotes the **function itself**.

Example 2.5: Notation for functions

For the function

$$k(i) = \left(1 + \frac{i}{12}\right)^{12}$$

of the previous example we would have:

$$k : [0, 1] \rightarrow \mathbb{R}$$

$$k : i \mapsto \left(1 + \frac{i}{12}\right)^{12}$$

Definition 2.3: of equality of two functions

Two functions f and g are **equal** if they have the same domain A , the same codomain B and if

$$f(a) = g(a), \quad \text{for every } a \in A.$$

- A function is therefore not just a “formula”: **the domain and the codomain are part of the function**. Changing either of them gives a different function, even if the law is the same.

Example 2.6: equal functions and different functions

Consider the three functions

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto x^2 \quad g : \mathbb{R} \rightarrow [0, +\infty), \quad g : x \mapsto x^2 \quad h : [0, +\infty) \rightarrow \mathbb{R}, \quad h : x \mapsto x^2$$

- $f \neq g$: they have the same domain and the same law, but different codomains;
- $f \neq h$: they have the same codomain and the same law, but different domains.

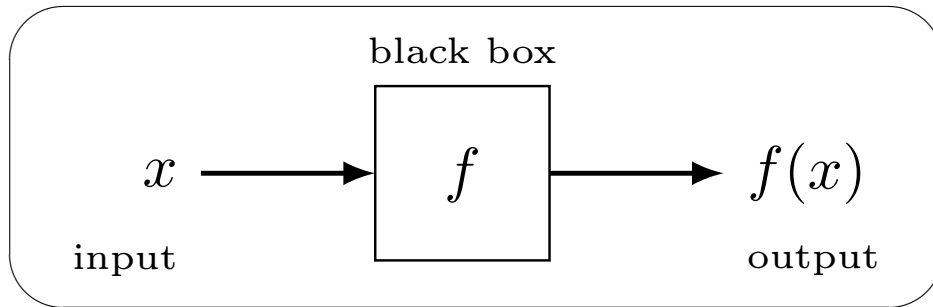
- The notation

$$y = f(x)$$

is also used to indicate that the variable y is a function of x . In this case y is the output or the value that f associates with the input x .

For brevity, sometimes the value of $f(x)$ is defined directly in terms of x , as for example $f(x) = x^2 + 10$.

- In general, we can think of a function as a **black box** that associates with each admissible input x (**input**) a unique output $f(x)$ (**output**):



Definition 2.4: of image and image of the domain

The output corresponding to x is called the **image** of x ; the set of possible outputs is called the **image of the domain A under f** and is denoted by the symbol $f(A)$ or $\text{Im } f$.

- Note that in the notation $f : A \rightarrow B$, the codomain B can be larger than the image $f(A)$, that is, in general we have

$$f(A) \subseteq B.$$

- If f is real-valued, we usually write $f : A \rightarrow \mathbb{R}$ without specifying the actual image of f .

Definition 2.5: of image of a subset of the domain

Given a function $f : A \rightarrow B$ and a subset $A' \subseteq A$, the **image of A' under f** is the set of the outputs produced by the inputs in A' :

$$f(A') = \{b \in B : b = f(a) \text{ for some } a \in A'\}.$$

- Taking $A' = A$ we recover the *image of the domain* (also called the *range* of f), which can therefore be written explicitly as

$$f(A) = \{b \in B : b = f(a) \text{ for some } a \in A\} \subseteq B.$$

Example 2.7: image of the domain and image of a subset

Consider the function

$$f : \mathbb{N} \rightarrow \mathbb{N}, \quad f : n \mapsto 2n$$

- The image of its domain is

$$f(\mathbb{N}) = \{m \in \mathbb{N} : m = 2n \text{ for some } n \in \mathbb{N}\},$$

that is, the set of the **nonnegative even numbers**. In particular $f(\mathbb{N}) \subsetneq \mathbb{N}$: the codomain is strictly larger than the image.

- The image of the subset $A' = \{0, 1, 2, 3\} \subseteq \mathbb{N}$ is instead

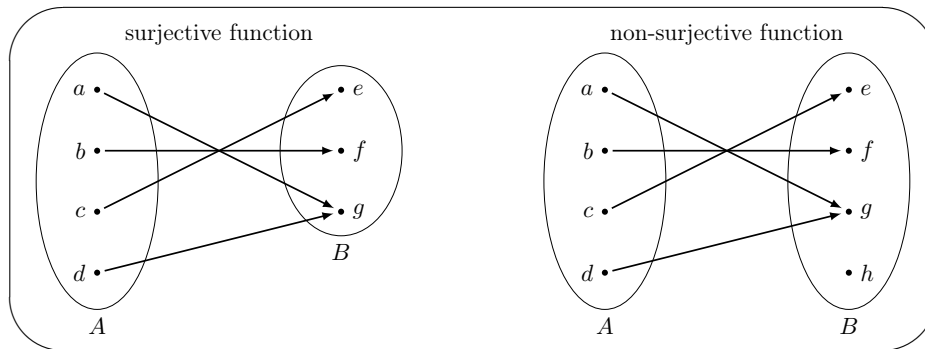
$$f(A') = \{0, 2, 4, 6\}.$$

2.3 Surjections, injections and bijections

Definition 2.6: of surjection (surjective function)

A function f is a **surjection** if the image of its domain coincides with its codomain.

- If $f : A \rightarrow B$ is a surjection, we also say that f **maps A onto B** . Note that the word *mapping* denotes an arbitrary function: it is the preposition *onto* that expresses surjectivity.
- Set representation:


Example 2.8: surjective and non-surjective functions

- The function $f(n) = \lfloor \frac{n}{2} \rfloor$ is a surjective function from $\mathbb{N}$ to $\mathbb{N}$, since every element of the codomain $\mathbb{N}$ is the image of some value of the domain.
- The function $f(n) = 2n$ is not a surjective function from $\mathbb{N}$ to $\mathbb{N}$, since no argument of f produces 3 as a value.
- The function $f(n) = 2n$ is, however, a surjective function from the natural numbers to the even numbers.

Observation 2.7

Given two **finite** sets A and B , and a function $f : A \rightarrow B$, if f is surjective then $|A| \geq |B|$.

Proof. Write $B = \{b_1, b_2, \dots, b_m\}$, with $m = |B|$, and for every $i \in \{1, 2, \dots, m\}$ consider the set of the inputs whose output is b_i :

$$A_i = \{a \in A : f(a) = b_i\} \subseteq A.$$

- Every A_i is **nonempty**: since f is surjective, every b_i is the image of at least one element of A , hence $|A_i| \geq 1$.
- The sets $A_1, A_2, \dots, A_m$ are **pairwise disjoint**: if $a \in A_i \cap A_j$ then $b_i = f(a) = b_j$, because f associates with a *one and only one* output, and therefore $i = j$.
- Their **union is the whole of A** : every $a \in A$ belongs to the set A_i with $b_i = f(a)$.

The sets $A_1, A_2, \dots, A_m$ thus form a *partition* of A and, since A is finite, we count the elements of A by summing the cardinalities of the blocks:

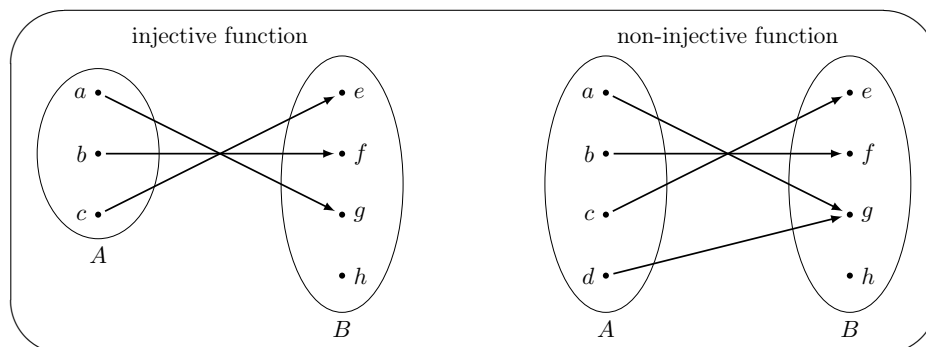
$$|A| = \sum_{i=1}^m |A_i| \geq \sum_{i=1}^m 1 = m = |B|.$$

□

Definition 2.8: of injection (injective function)

A function f is an **injection** if distinct arguments of f produce distinct values, that is, if $a \neq b$ implies $f(a) \neq f(b)$.

- set representation:



Example 2.9

- The function $f(n) = 2n$ is an injective function from $\mathbb{N}$ to $\mathbb{N}$, since each even number b is the image under f of exactly one element of the domain, namely $n = \frac{b}{2}$
- The function $f(n) = \lfloor \frac{n}{2} \rfloor$ is not an injective function, since the value 1 is obtained from two arguments: 2 and 3.

- An injection is also called a **one-to-one** function.

Observation 2.9

Given two **finite** sets A and B , and a function $f : A \rightarrow B$, if f is injective then $|A| \leq |B|$.

Proof. Write $A = \{a_1, a_2, \dots, a_n\}$, with $n = |A|$, and consider the n outputs

$$f(a_1), f(a_2), \dots, f(a_n) \in f(A).$$

- These outputs are **all distinct**: if $i \neq j$ then $a_i \neq a_j$ and, since f is injective, $f(a_i) \neq f(a_j)$.
- Every element of $f(A)$ **appears** in the list: by definition of image of the domain, every $b \in f(A)$ has the form $b = f(a)$ with $a \in A$, that is, $a = a_i$ for some $i \in \{1, 2, \dots, n\}$.

The list $f(a_1), f(a_2), \dots, f(a_n)$ therefore enumerates the elements of $f(A)$ without repetitions, and hence

$$|A| = n = |f(A)|.$$

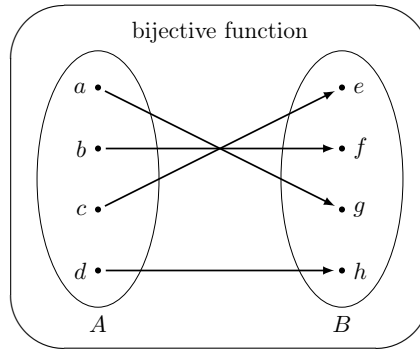
Finally $f(A) \subseteq B$ and B is finite, hence $|f(A)| \leq |B|$. Putting the two relations together we obtain $|A| \leq |B|$. □

The two remarks above are stated for **finite** sets: the proofs *count* elements, and counting only makes sense for finite sets. Comparing the “sizes” of two infinite sets requires a different notion of cardinality, which is introduced in the chapter “[Cardinality of infinite sets](#)” of Part 2.

Definition 2.10: of bijection (bijective function or one-to-one correspondence)

A function f is a **bijection** if: (i) it is injective and (ii) it is surjective.

- set representation:



Example 2.10

- The function $f(n) = (-1)^n \lceil \frac{n}{2} \rceil$ is a bijection from $\mathbb{N}$ to $\mathbb{Z}$. The values of the function are:

$$f(0) = 0, \quad f(1) = -1, \quad f(2) = 1, \quad f(3) = -2, \quad f(4) = 2 \dots$$

The function is injective, since no element of $\mathbb{Z}$ is the image of more than one element of $\mathbb{N}$. The function is surjective, since all elements of $\mathbb{Z}$ are images of some element of $\mathbb{N}$.

- A bijection is also called a **one-to-one** correspondence, since it pairs elements of the domain with elements of the codomain.
- A bijection from a set A to itself is also called a **permutation**.

2.4 Inverse of a bijection

- A bijection $f : A \rightarrow B$ pairs every element of A with an element of B and vice versa: the pairing can therefore be followed in the opposite direction as well, from B to A .

Definition 2.11: of inverse of a bijection

Given a bijection $f : A \rightarrow B$, its **inverse** is the function

$$f^{-1} : B \rightarrow A, \quad f^{-1}(b) = a \iff f(a) = b.$$

- The definition is well posed, that is, f^{-1} really is a function, precisely because f is a bijection: given $b \in B$,
 - the *surjectivity* of f guarantees that there exists *at least one* $a \in A$ with $f(a) = b$;
 - the *injectivity* of f guarantees that there exists *at most one*, because distinct inputs have distinct outputs.

Hence every $b \in B$ corresponds to *one and only one* $a \in A$, as required by the definition of function.

- In other words, if f applied to the input a gives the output b , then f^{-1} applied to b returns a :

$$f^{-1}(f(a)) = a, \quad \forall a \in A \quad \text{and} \quad f(f^{-1}(b)) = b, \quad \forall b \in B.$$

Here the inverse is defined on the **whole** codomain B , and therefore *both* the injectivity *and* the surjectivity of f are needed. In the chapter “[Inverse functions](#)” we study instead functions of a real variable, and there a function is called *invertible* when it is only **injective**: the inverse is built on the *image* $f(D)$ and not on the whole codomain, and with respect to the image surjectivity is automatic.

Example 2.11: inverse of a bijection

Let us go back to the bijection $f : \mathbb{N} \rightarrow \mathbb{Z}$ of the previous example,

$$f(n) = (-1)^n \left\lceil \frac{n}{2} \right\rceil,$$

which pairs $0 \leftrightarrow 0, 1 \leftrightarrow -1, 2 \leftrightarrow 1, 3 \leftrightarrow -2, 4 \leftrightarrow 2, \dots$

Its inverse $f^{-1} : \mathbb{Z} \rightarrow \mathbb{N}$ is:

$$f^{-1}(m) = \begin{cases} 2m & \text{if } m \geq 0 \\ -(2m+1) & \text{if } m < 0 \end{cases}$$

Indeed, if $m \geq 0$, then $n = 2m$ is even and $f(n) = (-1)^{2m} \lceil m \rceil = m$; if instead $m < 0$, then $n = -(2m+1) = -2m-1$ is odd and nonnegative, and

$$f(n) = (-1)^n \left\lceil \frac{-2m-1}{2} \right\rceil = - \left\lceil -m - \frac{1}{2} \right\rceil = -(-m) = m,$$

where we used that $-m - \frac{1}{2}$ has ceiling $-m$, since $-m$ is a positive integer.

Real functions of a real variable

3.1 Real function of a real variable

Definition 3.1: real function of a real variable

A function whose domain D is a subset of $\mathbb{R}$ and whose codomain is $\mathbb{R}$:

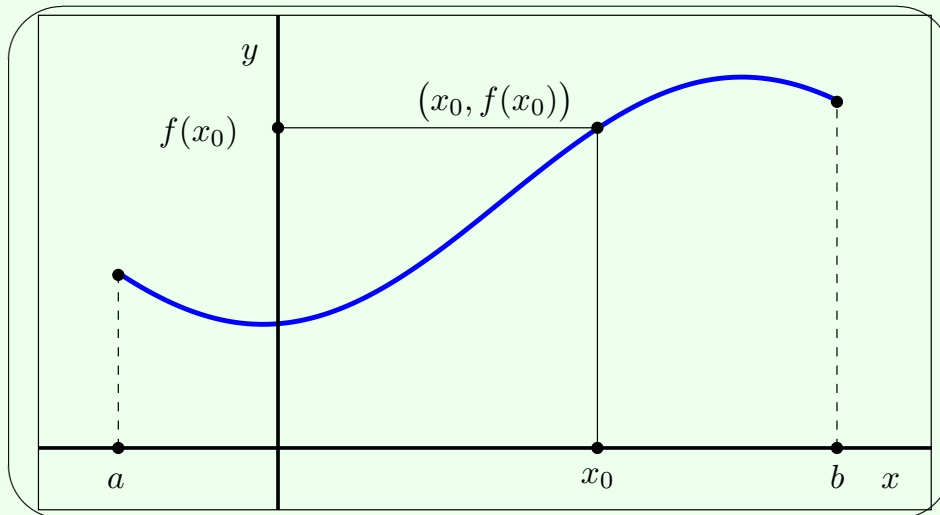
$$f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto f(x)$$

is called a **real function of a real variable**.

- These are functions in which the “input” variable x and the “output” variable $f(x)$ are real numbers.
- The most common real functions of a real variable have as domain D and as image $f(D)$ an **interval** (possibly the whole of $\mathbb{R}$) or the union of a **finite number of intervals**.

The dependence of the output $f(x)$ on the input x is effectively visualized by drawing the **graph** of f , that is, the set of points of the plane with coordinates (x, y) such that $y = f(x)$ and x in the domain D .

Example of the graph of a real function of a real variable with domain $D = [a, b]$ (a closed and bounded interval):



Every line parallel to the y -axis that intersects the x -axis at a point x_0 of the domain D intersects the graph of f at one and only one point.^a **Therefore, not all curves are graphs of functions.**

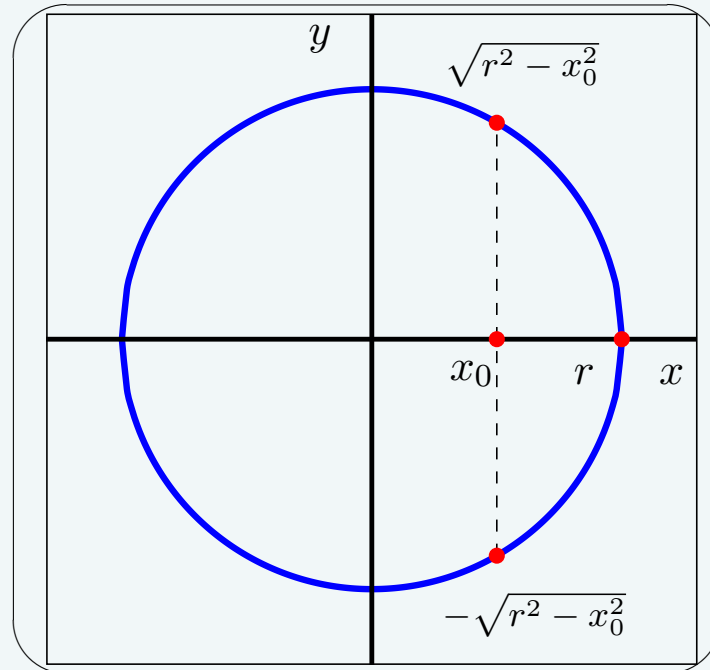
Note, instead, that nothing prevents a line parallel to the x -axis from intersecting the graph of f at several points or at no point at all.

^aIf the line did not intersect the graph, this would mean that no output corresponds to the input x . If the line intersected the graph at more than one point, this would mean that several distinct outputs correspond to the input x , and hence the function would no longer be single-valued.

Example 3.1: curve that does not correspond to the graph of a function

Consider, for example, the curve made of the points of the circle of radius r :

$$x^2 + y^2 = r^2 \iff y = \pm\sqrt{r^2 - x^2}$$



It is not possible to associate a unique output with the input $x_0 \in (-r, r)$, hence this curve does not correspond to the graph of a function!

3.2 Sign of a function

- The properties of a function are almost always studied not only on the whole domain, but also on a **part** of it: one says for instance that $x \mapsto x^2$ is “positive for $x > 0$ ” or that $x \mapsto x^2$ is “decreasing in $(-\infty, 0]$ ”. It is therefore convenient to state the definitions directly **on a subset I of the domain**, and then recover the “global” property as the particular case $I = D$.

Definition 3.2: sign of a function in a subset of the domain

Given a function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and a subset $I \subseteq D$, the function is said to be

$$\begin{cases} \text{nonnegative in } I & \text{if } f(x) \geq 0, \forall x \in I \\ \text{positive in } I & \text{if } f(x) > 0, \forall x \in I \\ \text{nonpositive in } I & \text{if } f(x) \leq 0, \forall x \in I \\ \text{negative in } I & \text{if } f(x) < 0, \forall x \in I \end{cases}$$

When the subset is not specified, the whole domain is understood: a function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called **nonnegative** (**positive**, **nonpositive**, **negative**) if it is nonnegative (positive, nonpositive, negative) in $I = D$.

- Graphically: f is nonnegative in I if no point of the graph with abscissa in I lies *below* the x -axis; it is positive in I if all these points lie *strictly above* the x -axis.
- If f is positive in I then it is also nonnegative in I ; the converse is false, because f may vanish at some point of I .

Example 3.2: sign of a function

- The function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f : x \mapsto x^2$ is *nonnegative* in $\mathbb{R}$, since $x^2 \geq 0$ for every $x \in \mathbb{R}$. It is not, however, *positive* in $\mathbb{R}$, because $f(0) = 0$: it is positive in $I = (0, +\infty)$ and in $I = (-\infty, 0)$.
- The function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f : x \mapsto x^3$ is *positive* in $(0, +\infty)$ and *negative* in $(-\infty, 0)$, since

$$x^3 > 0 \iff x > 0 \quad \text{and} \quad x^3 < 0 \iff x < 0.$$
- The function $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$, $f : x \mapsto \frac{1}{x}$ is *positive* in $(0, +\infty)$ and *negative* in $(-\infty, 0)$: the sign of $\frac{1}{x}$ is the same as the sign of x .

3.3 Bounded functions

Definition 3.3: bounded functions in a subset of the domain

Given a function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and a subset $I \subseteq D$, the function is said to be

$$\begin{cases} \text{bounded above in } I & \text{if } \exists M \in \mathbb{R} : f(x) \leq M, \forall x \in I \\ \text{bounded below in } I & \text{if } \exists m \in \mathbb{R} : f(x) \geq m, \forall x \in I \\ \text{bounded in } I & \text{if } \exists M \in \mathbb{R}_{\geq 0} : |f(x)| \leq M, \forall x \in I \end{cases}$$

- A number $M \in \mathbb{R}$ such that $f(x) \leq M$ for every $x \in I$ is called an **upper bound** for the values of f in I ; a number $m \in \mathbb{R}$ such that $f(x) \geq m$ for every $x \in I$ is called a **lower bound** for the values of f in I . Hence f is bounded above in I if its values in I admit an upper bound, and bounded below in I if they admit a lower bound.

When the subset is not specified, the whole domain is understood: a function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called **bounded above** (**bounded below**, **bounded**) if it is so in $I = D$.

- A function is bounded in I *if and only if* it is bounded both above and below in I . Indeed:
 1. if $|f(x)| \leq M$ for every $x \in I$, then $-M \leq f(x) \leq M$ for every $x \in I$, hence M is an upper bound and $-M$ a lower bound;
 2. conversely, if $m \leq f(x) \leq M$ for every $x \in I$, setting $K = \max\{|m|, |M|\} \geq 0$ we have $-K \leq m \leq f(x) \leq M \leq K$, that is, $|f(x)| \leq K$ for every $x \in I$.
- Graphically, we have:

1. a function is bounded above if its graph is contained in the lower half-plane bounded by a line parallel to the x -axis
2. a function is bounded below if its graph is contained in the upper half-plane bounded by a line parallel to the x -axis
3. a function is bounded if its graph is contained in a horizontal strip

Example 3.3: bounded function

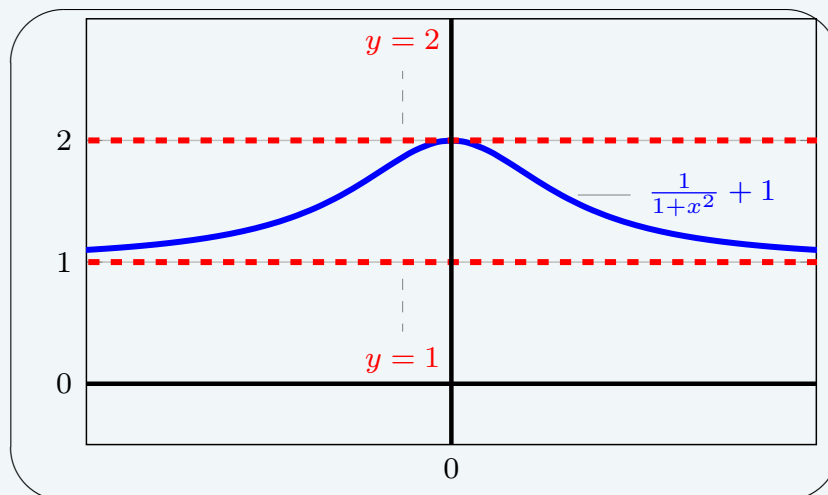
Consider the function:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \frac{1}{1+x^2} + 1$$

we have

$$\frac{1}{1+x^2} + 1 = \frac{1+x^2-x^2}{1+x^2} + 1 = 2 - \frac{x^2}{1+x^2} \quad \text{and} \quad \frac{x^2}{1+x^2} \geq 0, \quad \forall x \in \mathbb{R}$$

$$\text{hence} \quad 1 < \frac{1}{1+x^2} + 1 \leq 2, \quad \forall x \in \mathbb{R}$$

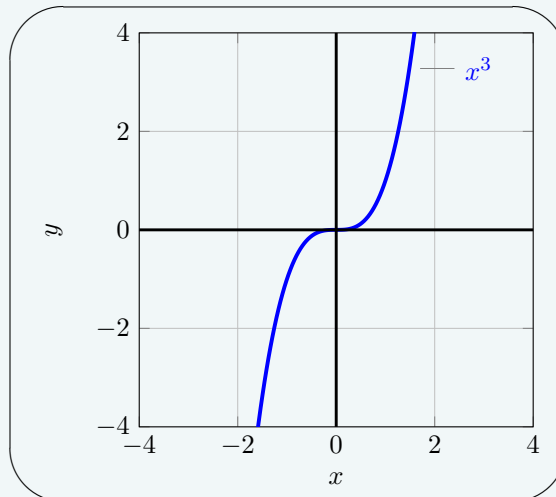


Example 3.4: Unbounded function

The function

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto x^3 \quad (y = x^3)$$

is bounded neither above nor below.

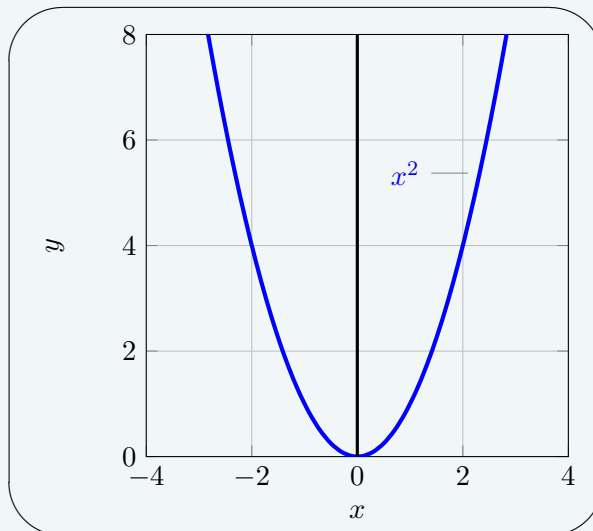


Example 3.5: Function bounded below

The function

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto x^2 \quad (y = x^2)$$

is bounded below; indeed, $x^2 \geq 0, \forall x \in \mathbb{R}$



Example 3.6: boundedness in a subset of the domain

A function may fail to be bounded on its domain and still be bounded on a subset of it.

- The function $f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto x^3$ is not bounded in $\mathbb{R}$, but it is bounded in $I = [-2, 2]$: indeed for $-2 \leq x \leq 2$ we have

$$|x^3| = |x|^3 \leq 2^3 = 8.$$

- The function $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, f : x \mapsto \frac{1}{x}$ is **not** bounded above in $I = (0, +\infty)$: given

any $M \in \mathbb{R}$ with $M > 0$, choosing x with $0 < x < \frac{1}{M}$ we obtain

$$f(x) = \frac{1}{x} > M,$$

hence no real number is an upper bound for the values of f in $(0, +\infty)$. The same function is instead bounded in $I = [1, +\infty)$, where $0 < \frac{1}{x} \leq 1$.

- Equivalently, we can say that a function is *bounded above* (*bounded below*, *bounded*) if, respectively, its **image** is a subset of $\mathbb{R}$ that is *bounded above* (*bounded below*, *bounded*).

3.4 Symmetric functions

Definition 3.4: even function

Functions whose graph is symmetric with respect to the y -axis are called **even**.

- They are characterized by the relation

$$f(-x) = f(x)$$

which expresses the equality of the ordinates corresponding to the points x and $-x$, which are symmetric with respect to $x = 0$.

Definition 3.5: Odd function

Functions whose graph is symmetric with respect to the origin are called **odd**.

- They are characterized by the relation

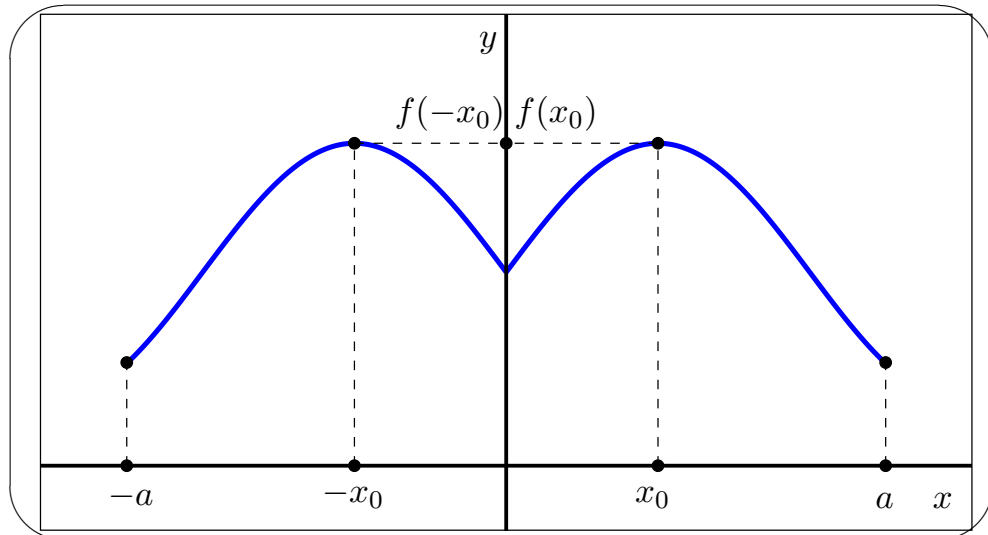
$$f(-x) = -f(x)$$

which expresses the fact that the ordinates corresponding to the points x and $-x$, which are symmetric with respect to $x = 0$, are the opposite of each other.

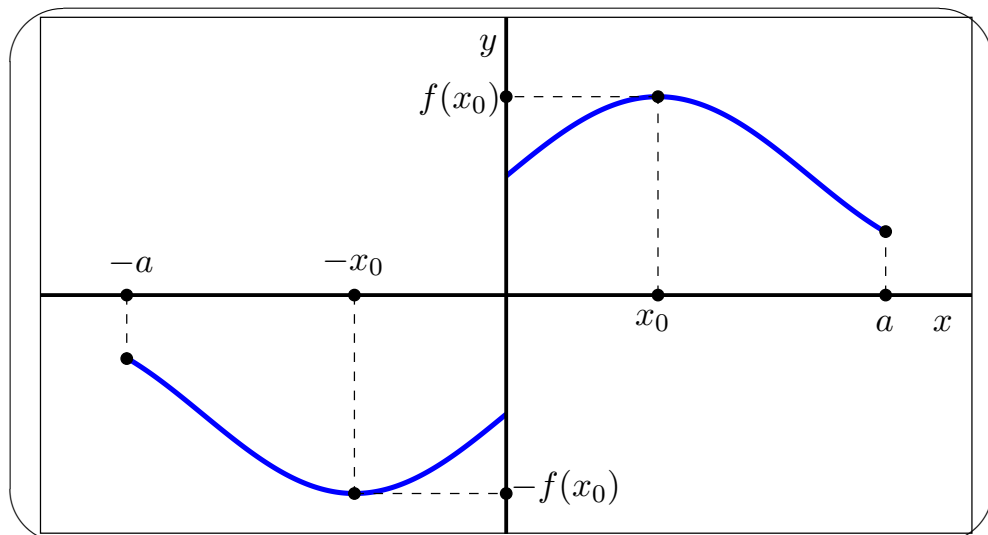
Example 3.7: Even and odd functions

For example, the function $x \mapsto x^2$ is even, while $x \mapsto x^3$ is odd. More generally, powers with integer exponent are even functions if the exponent is even and odd functions if the exponent is odd.

1. **Example of the graph of an even function:**



2. Example of the graph of an odd function:



A function *cannot* have a graph that is symmetric with respect to the x -axis, since the correspondence would no longer be single-valued.

3.5 Monotonic functions

Definition 3.6: Non-decreasing function in a subset of the domain

Given a function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and a subset $I \subseteq D$, the function is called **non-decreasing in I** if for every pair of points $x_1, x_2 \in I$ we have:

$$x_1 < x_2 \implies f(x_1) \leq f(x_2) \quad (3.1)$$

Definition 3.7: Increasing function in a subset of the domain

Given a function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and a subset $I \subseteq D$, the function is called **increasing in I** if for every pair of points $x_1, x_2 \in I$ we have:

$$x_1 < x_2 \implies f(x_1) < f(x_2) \quad (3.2)$$

Definition 3.8: Non-increasing function in a subset of the domain

Given a function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and a subset $I \subseteq D$, the function is called **non-increasing in I** if for every pair of points $x_1, x_2 \in I$ we have:

$$x_1 < x_2 \implies f(x_1) \geq f(x_2) \quad (3.3)$$

Definition 3.9: Decreasing function in a subset of the domain

Given a function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and a subset $I \subseteq D$, the function is called **decreasing in I** if for every pair of points $x_1, x_2 \in I$ we have:

$$x_1 < x_2 \implies f(x_1) > f(x_2) \quad (3.4)$$

- A function f is *non-decreasing* in I if, as x increases in I , the corresponding ordinate on the graph of the function does not decrease (hence it either stays the same or increases); it is *increasing* in I if it always increases;
- A function f is *non-increasing* in I if, as x increases in I , the corresponding ordinate on the graph of the function does not increase (hence it either stays the same or decreases); it is *decreasing* in I if it always decreases.

- A function is called **monotonic in I** if it is *non-decreasing in I* or *non-increasing in I* .
- A function is called **strictly monotonic in I** if it is *increasing in I* or *decreasing in I* .

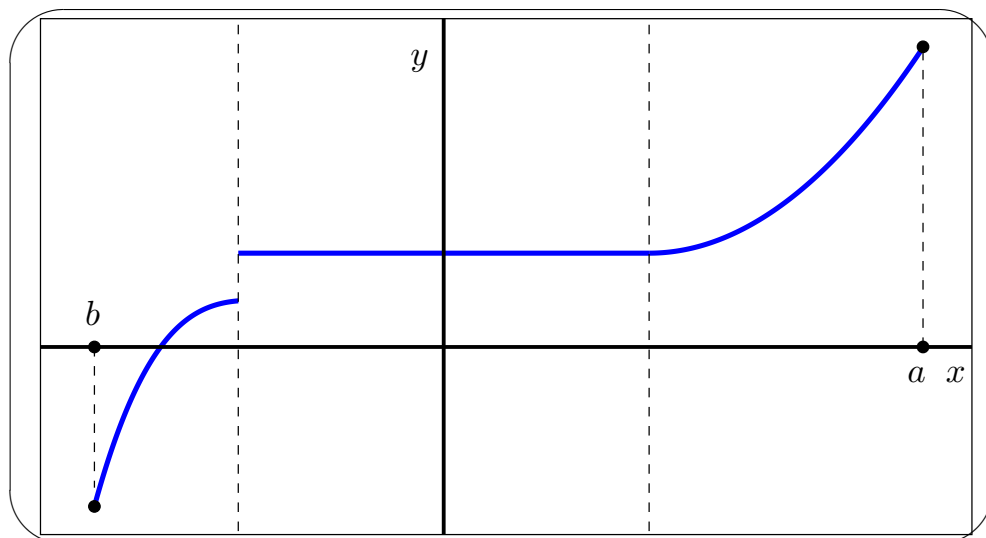
When the subset is not specified, the whole domain is understood: a function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called **non-decreasing (increasing, non-increasing, decreasing, monotonic, strictly monotonic)** if it is so in $I = D$.

- A function increasing in I is in particular non-decreasing in I (because $f(x_1) < f(x_2)$ implies $f(x_1) \leq f(x_2)$), and a function decreasing in I is in particular non-increasing in I . Hence **every function that is strictly monotonic in I is also monotonic in I** ; the converse is false, as the constant function shows.

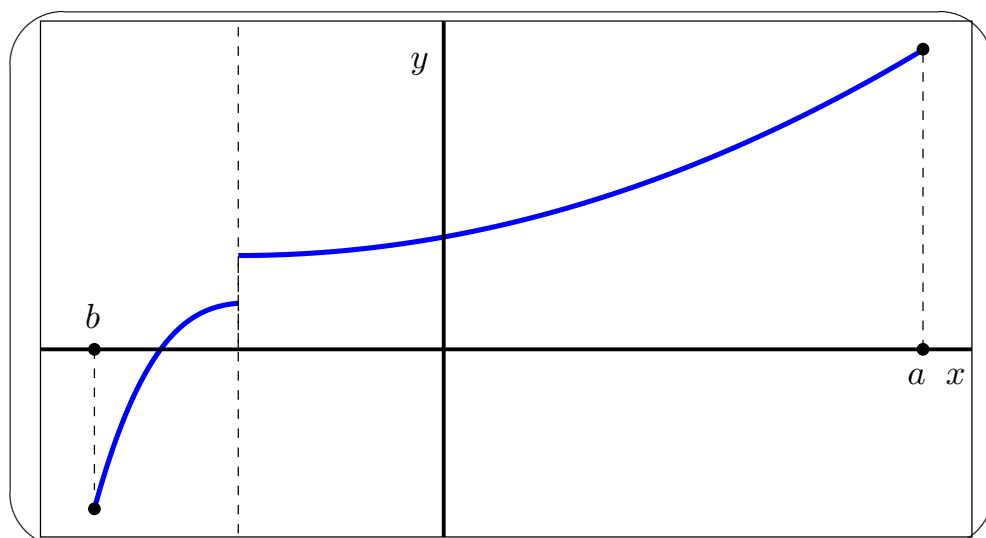
Example 3.8: Monotonic functions

- The function $x \mapsto x^3$ is *increasing* in $\mathbb{R}$ (hence strictly monotonic in $\mathbb{R}$).
- The constant function $x \mapsto k$ (whose graph is the line with equation $y = k$) is both *non-decreasing* and *non-increasing* in $\mathbb{R}$: it is therefore monotonic, but it is neither increasing nor decreasing, and hence it is not strictly monotonic.
- The function $x \mapsto x^2$ is *decreasing* in $(-\infty, 0]$ and *increasing* in $[0, +\infty)$, but it is **not** monotonic in $\mathbb{R}$: indeed $f(-1) = 1 > f(0) = 0$ (hence it is not non-decreasing in $\mathbb{R}$) and $f(0) = 0 < f(1) = 1$ (hence it is not non-increasing in $\mathbb{R}$). It is the typical example of a property that holds *locally* but not on the whole domain.

1. Example of the graph of a non-decreasing function (horizontal segment):



2. Example of the graph of an increasing function:



3.6 Periodic functions

Definition 3.10: Periodic function

A (non-constant) function $f : D \rightarrow \mathbb{R}$ is **periodic** with period T , $T > 0$, if T is the smallest positive real number such that

$$f(x + T) = f(x) \quad \text{for every } x \in D$$

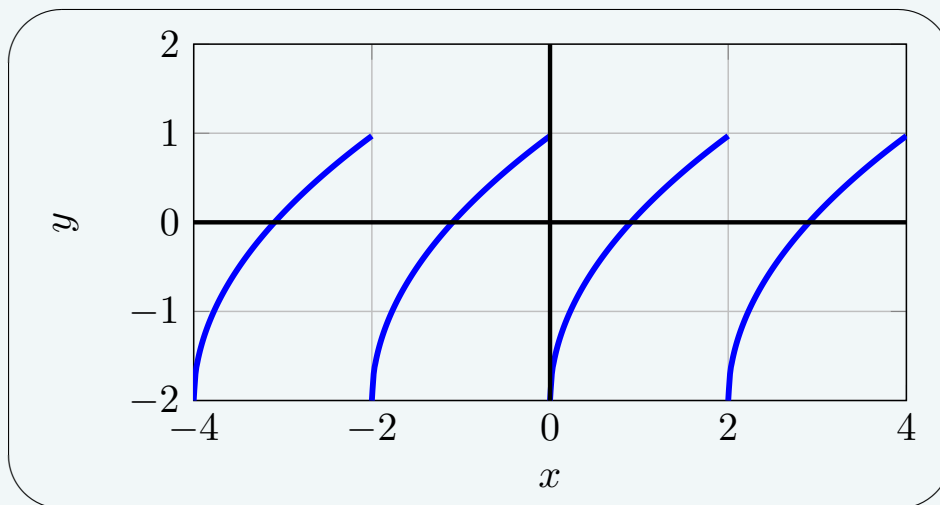
- Every interval of length T contained in D is called a **periodicity interval**.

Example 3.9: Periodic functions

Typical examples of periodic functions are the *trigonometric functions* $x \mapsto \sin(x)$ ($T = 2\pi$), $x \mapsto \cos(x)$ ($T = 2\pi$) and $x \mapsto \tan(x)$ ($T = \pi$).

Example 3.10: Graph of periodic functions

Graph of a periodic function with period $T = 2$:



- Lines parallel to the x -axis have equation

$$y = k \quad k \in \mathbb{R}$$

Power functions

4.1 Power functions

Definition 4.1: power functions

Given $\alpha \in \mathbb{R}, \alpha \neq 0$, the function:

$$f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto x^\alpha \quad (4.1)$$

is called the **power function** with exponent α . The domain D depends on the value of α :

- with rational exponent $\alpha = \frac{m}{n} \in \mathbb{Q}$, $m \in \mathbb{Z}$ and $n \in \mathbb{N}_+$ coprime, we have four cases:

$$D = \begin{cases} \mathbb{R} & \text{if } n \text{ odd and } \frac{m}{n} > 0 \\ \mathbb{R} \setminus \{0\} & \text{if } n \text{ odd and } \frac{m}{n} < 0 \\ [0, +\infty) & \text{if } n \text{ even and } \frac{m}{n} > 0 \\ (0, +\infty) & \text{if } n \text{ even and } \frac{m}{n} < 0 \end{cases}$$

- with real, non-rational exponent $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we have two cases:

$$D = \begin{cases} [0, +\infty) & \text{if } \alpha > 0 \\ (0, +\infty) & \text{if } \alpha < 0 \end{cases}$$

4.1.1 Rational exponent

With $\alpha = \frac{m}{n} \in \mathbb{Q}$, $m \in \mathbb{Z}$ and $n \in \mathbb{N}_+$ coprime, we have a power function with rational exponent:

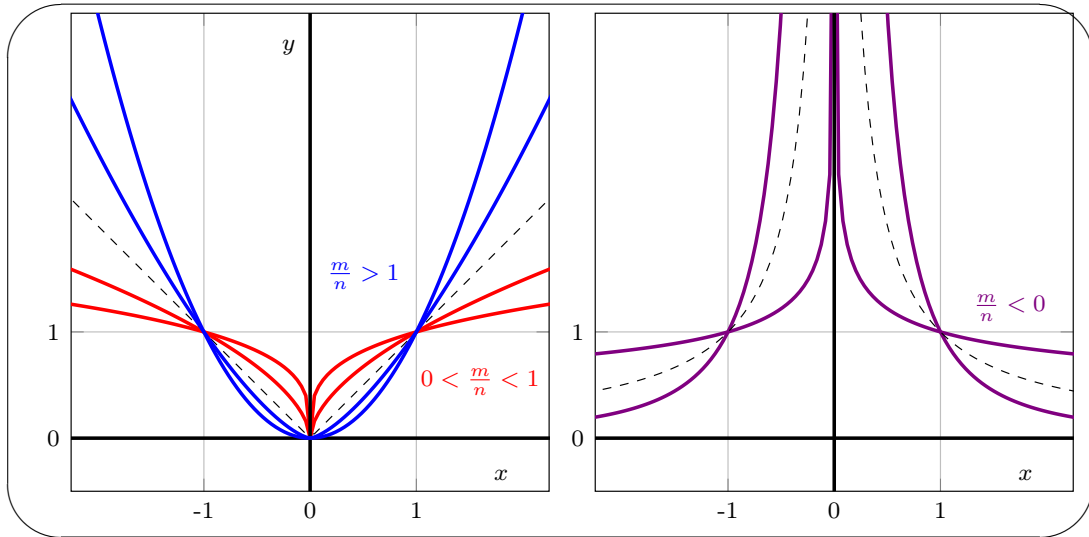
$$f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

We have the following three cases, depending on whether m and n are even or odd.

1. With n odd and m even, it is an **even function**, i.e., $(-x)^{\frac{m}{n}} = x^{\frac{m}{n}}$. For the sign and the monotonicity we have:

$$x^{\frac{m}{n}} > 0, \quad \forall x \in \mathbb{R} \setminus \{0\}; \quad x^{\frac{m}{n}} = 0 \iff x = 0 \quad \text{and} \quad \frac{m}{n} > 0$$

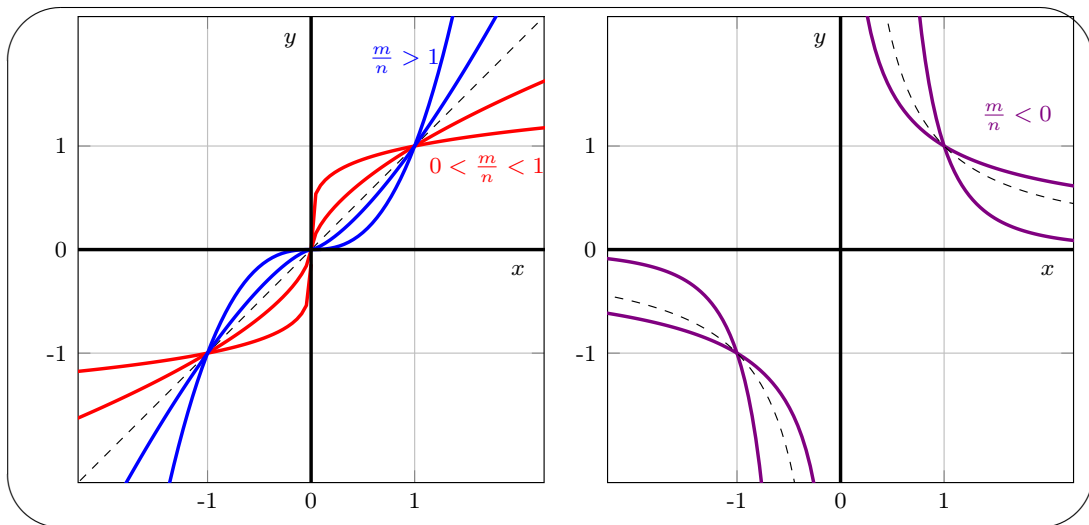
$$\begin{cases} \text{if } \frac{m}{n} > 0 & \text{then } f \text{ is decreasing in } (-\infty, 0] \text{ and increasing in } [0, +\infty) \\ \text{if } \frac{m}{n} < 0 & \text{then } f \text{ is increasing in } (-\infty, 0) \text{ and decreasing in } (0, +\infty) \end{cases}$$



2. With n odd and m odd, it is an **odd function**, i.e., $(-x)^{\frac{m}{n}} = -x^{\frac{m}{n}}$. For the sign and the monotonicity we have:

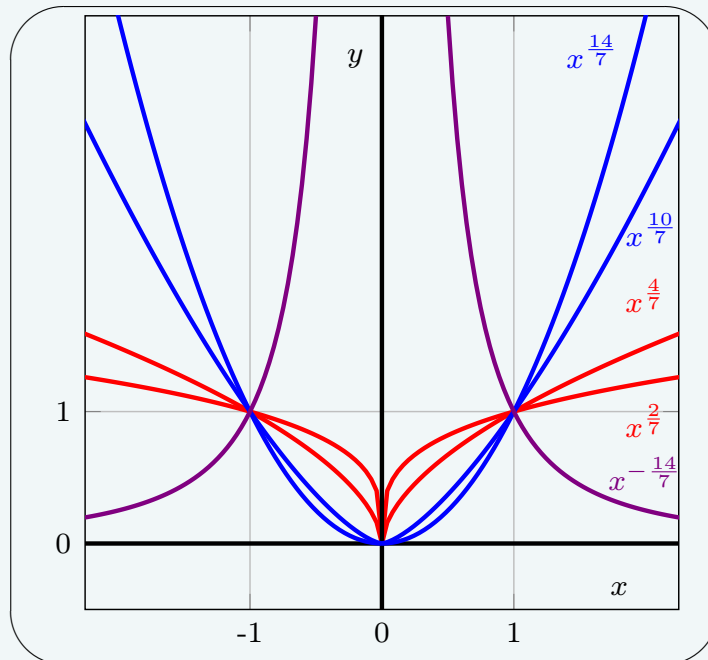
$$x^{\frac{m}{n}} < 0, \quad \forall x < 0, \quad x^{\frac{m}{n}} > 0, \quad \forall x > 0; \quad x^{\frac{m}{n}} = 0 \iff x = 0 \quad \text{and} \quad \frac{m}{n} > 0$$

$$\begin{cases} \text{if } \frac{m}{n} > 0 & \text{then } f \text{ is increasing in } \mathbb{R} \\ \text{if } \frac{m}{n} < 0 & \text{then } f \text{ is decreasing in } (-\infty, 0) \text{ and in } (0, +\infty) \end{cases}$$

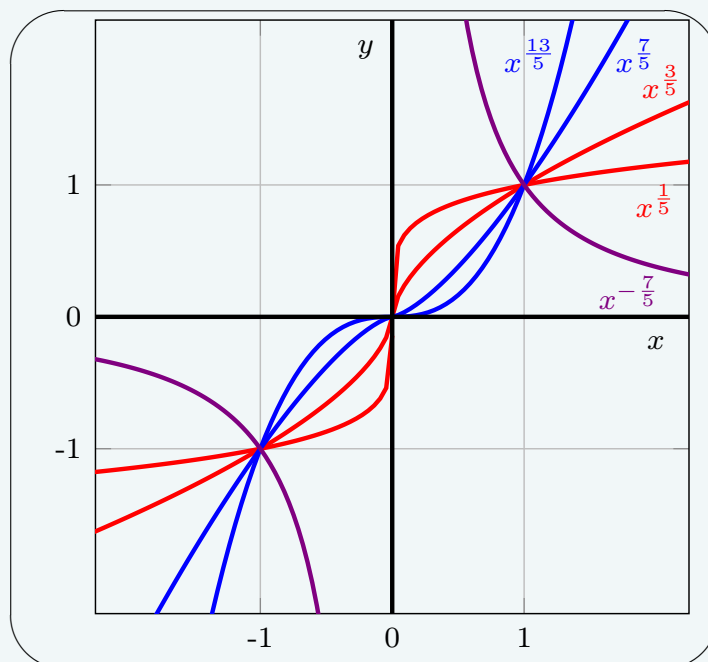


Example 4.1: graphs of power functions with rational exponent $\frac{m}{n}$ and n odd

With m even:



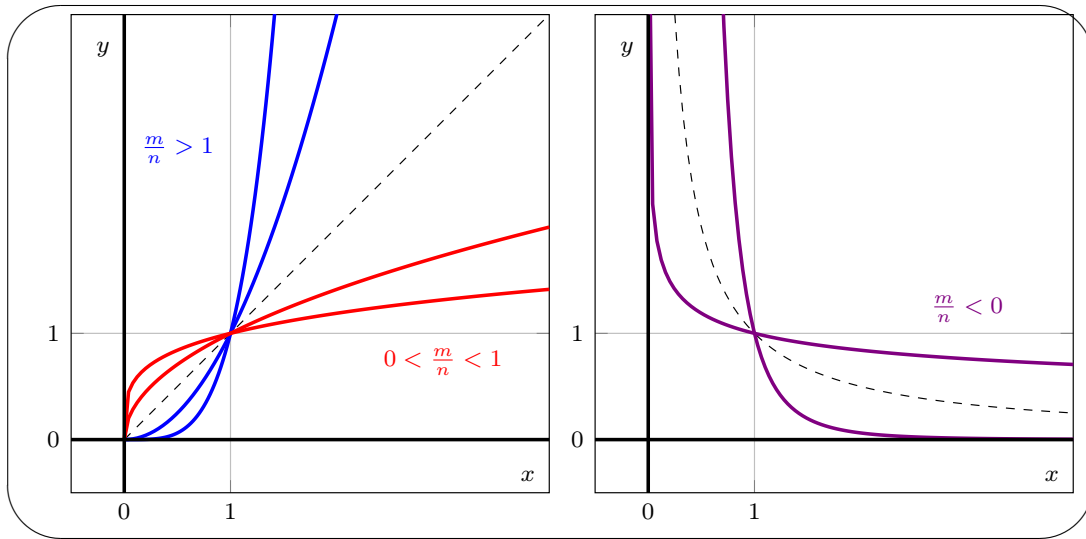
With m odd:



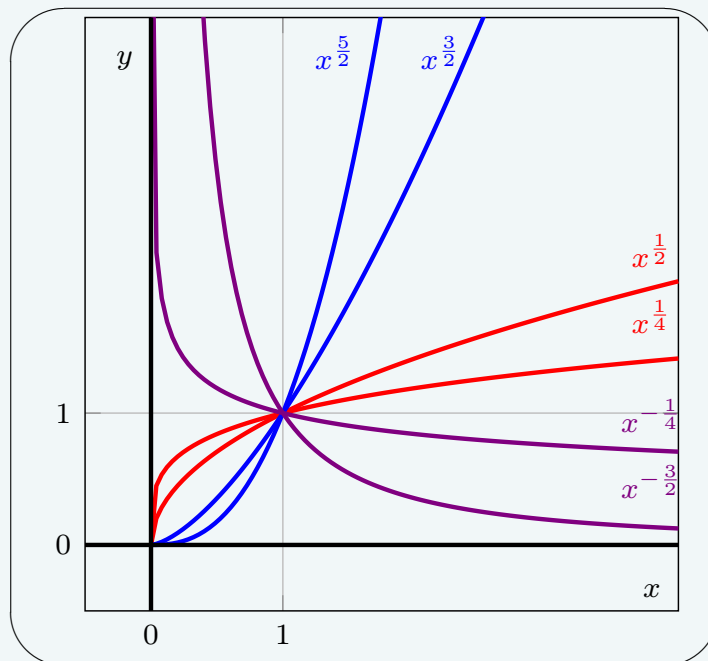
3. With n even and m odd (m cannot be even since they are coprime), for the sign and the monotonicity we have:

$$x^{\frac{m}{n}} > 0, \quad \forall x > 0; \quad x^{\frac{m}{n}} = 0 \iff x = 0 \quad \text{and} \quad \frac{m}{n} > 0$$

$$\begin{cases} \text{if } \frac{m}{n} > 0 & \text{then } f \text{ is increasing in } [0, +\infty) \\ \text{if } \frac{m}{n} < 0 & \text{then } f \text{ is decreasing in } (0, +\infty) \end{cases}$$



Example 4.2: graphs of power functions with rational exponent $\frac{m}{n}$ and n even

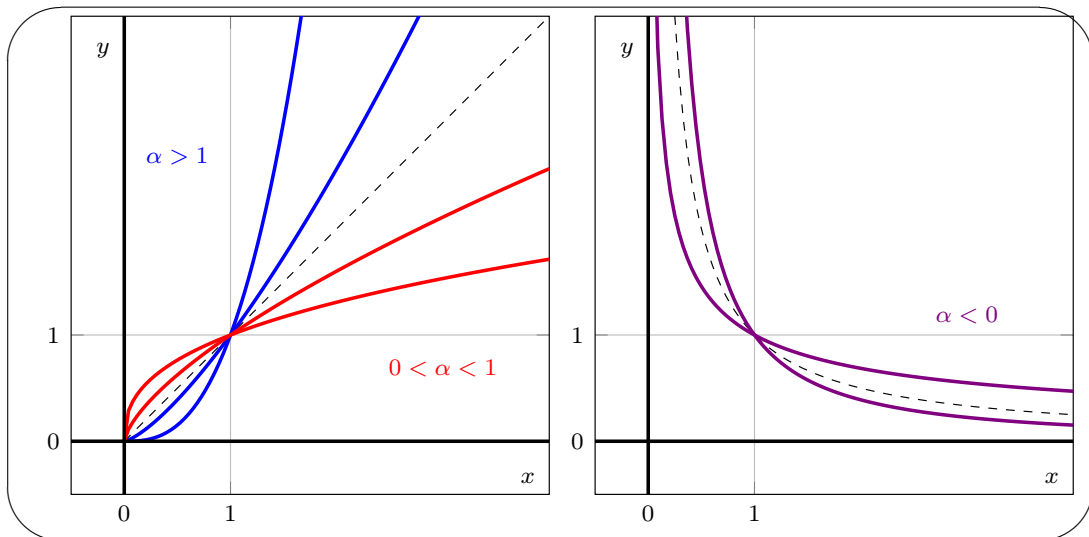


4.1.2 Real exponent

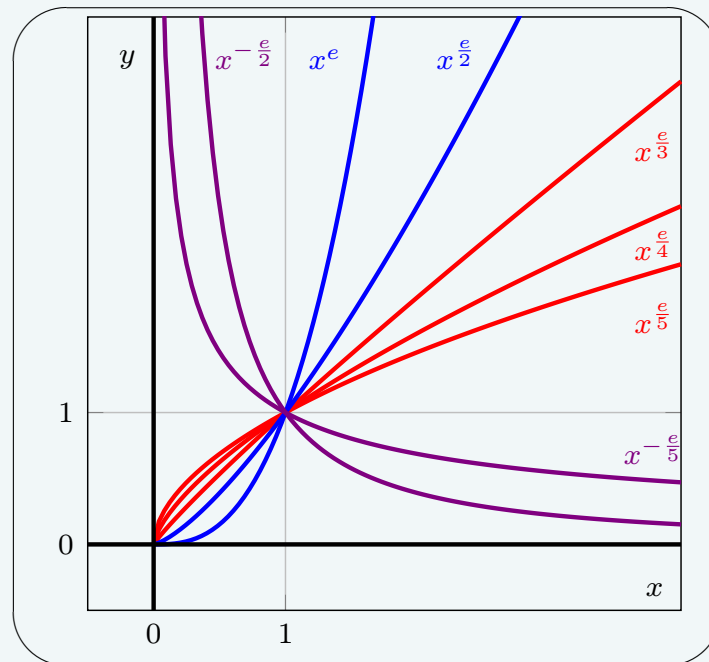
- With $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, for the sign and the monotonicity we have:

$$x^\alpha > 0, \quad \forall x > 0; \quad x^\alpha = 0 \iff x = 0 \text{ and } \alpha > 0$$

$$\begin{cases} \text{if } \alpha > 0 & \text{then } f \text{ is increasing in } [0, +\infty) \\ \text{if } \alpha < 0 & \text{then } f \text{ is decreasing in } (0, +\infty) \end{cases}$$



Example 4.3: graphs of power functions with real exponent



4.1.3 Polynomials

Definition 4.2: polynomial

Given $n + 1$ values $a_i \in \mathbb{R}$ with $i \in \{0, 1, \dots, n\}$ and $a_n \neq 0$, the function:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \underbrace{\sum_{i=0}^n a_i x^i}_{=P_n(x)} \quad (4.2)$$

is called a **polynomial** of degree n .

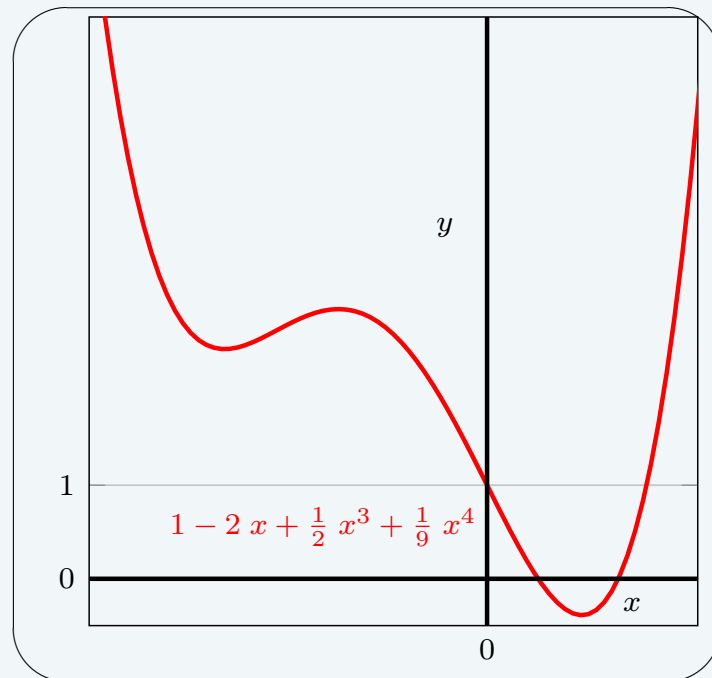
- For each **monomial** $i \in \{0, 1, \dots, n\}$:
 1. the value a_i is the **coefficient of the monomial**
 2. the power function x^i with integer exponent i is the **literal part** of the monomial

The value a_0 is the **constant term** of the polynomial since $x^0 = 1$.

Example 4.4: polynomial

For example, with $n = 4$, $a_0 = 1$, $a_1 = -2$, $a_2 = 0$, $a_3 = \frac{1}{2}$ and $a_4 = \frac{1}{9}$ we have the following polynomial of degree four:

$$P_4(x) = \sum_{i=0}^4 a_i x^i = 1 - 2x + \frac{1}{2}x^3 + \frac{1}{9}x^4$$

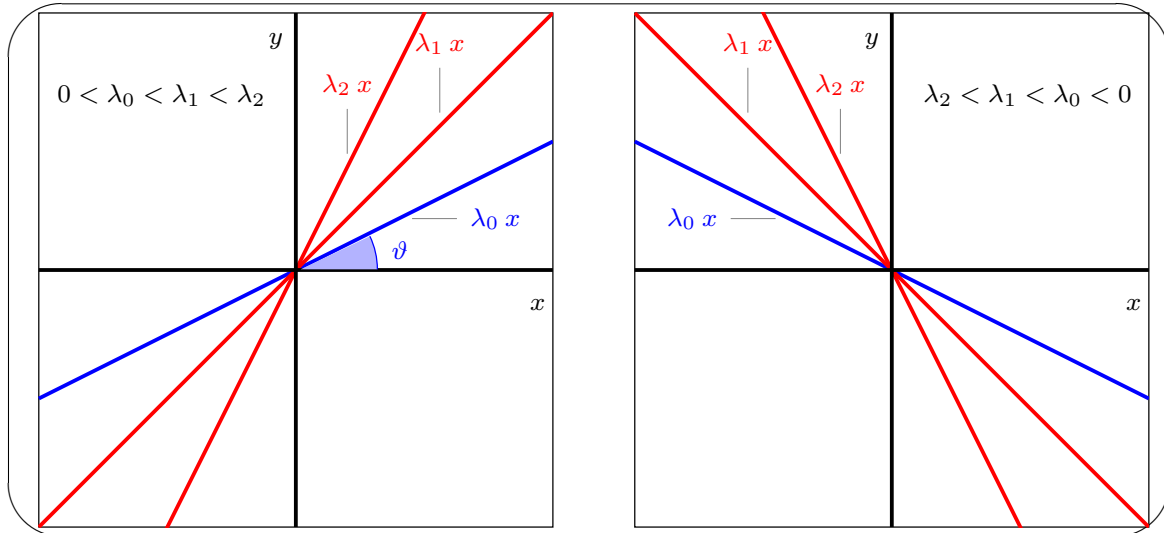


4.2 Direct and inverse proportionality functions

With $\alpha = 1$ and given $\lambda \in \mathbb{R}$, we have the family of **direct proportionality (or linear) functions**:

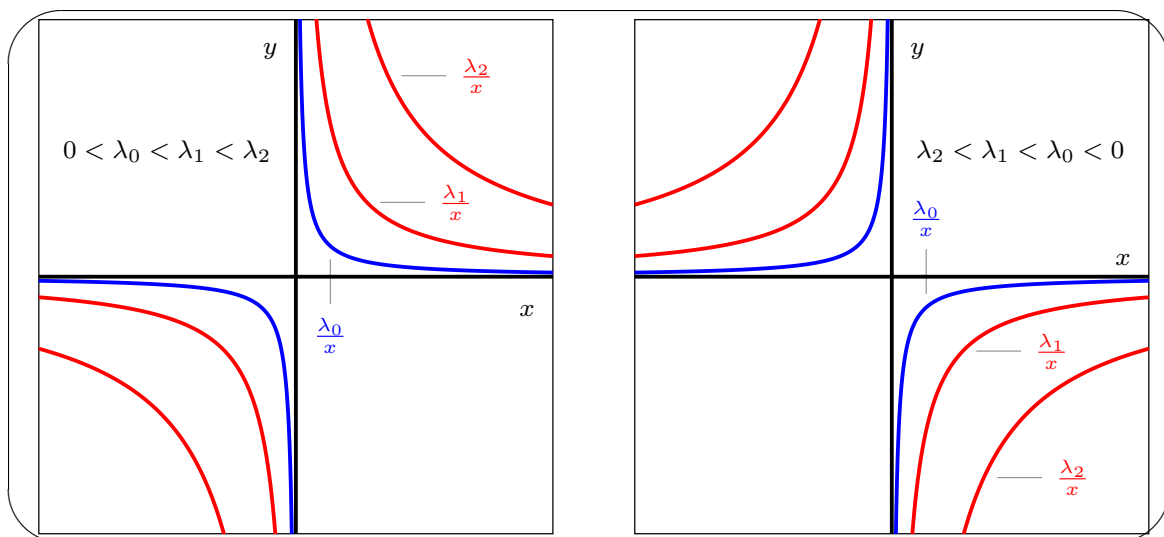
$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \lambda x \quad \text{where } \lambda \text{ is the constant of direct proportionality}$$

Moreover, $\lambda = \tan \vartheta$ and ϑ is the angle between the line $y = \lambda x$ and the x -axis.



With $\alpha = -1$ and given $\lambda \in \mathbb{R}$, we have the family of **inverse proportionality functions (or rectangular hyperbolas)**:

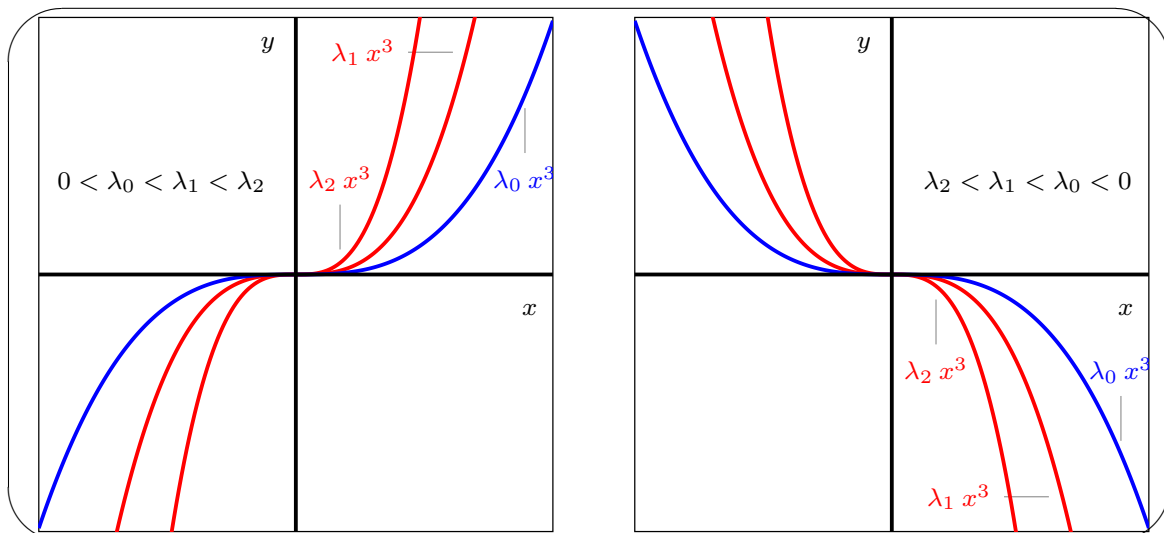
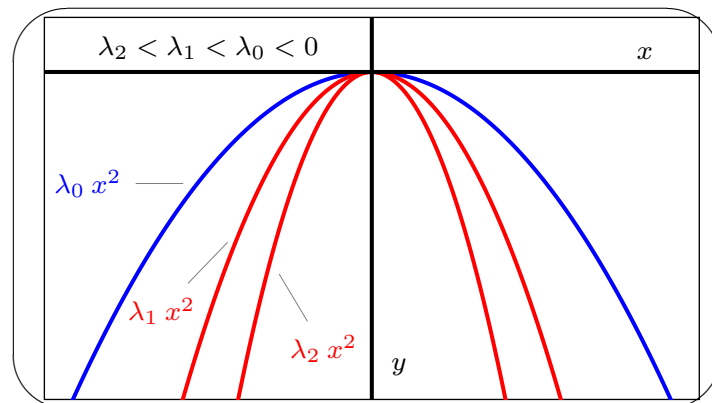
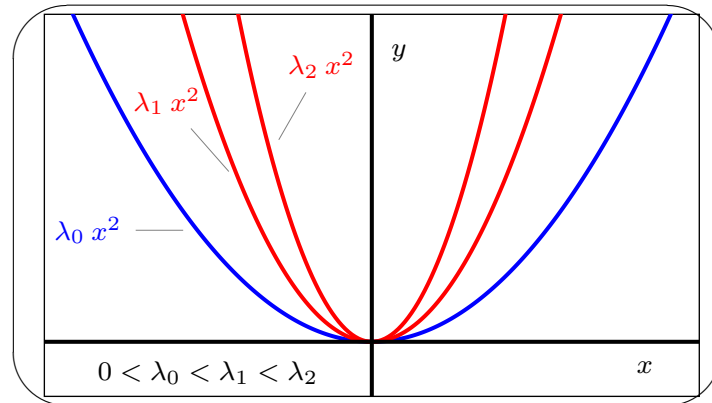
$$f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, \quad f : x \mapsto \frac{\lambda}{x} \quad \text{where } \lambda \text{ is the constant of inverse proportionality}$$



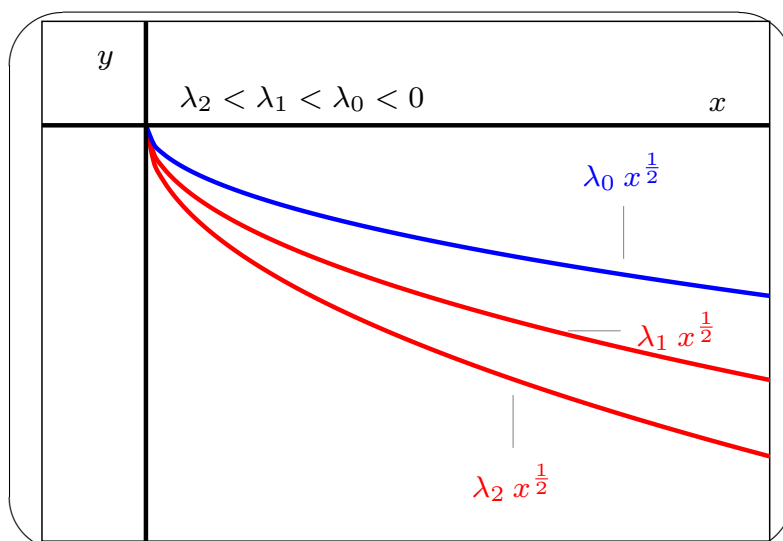
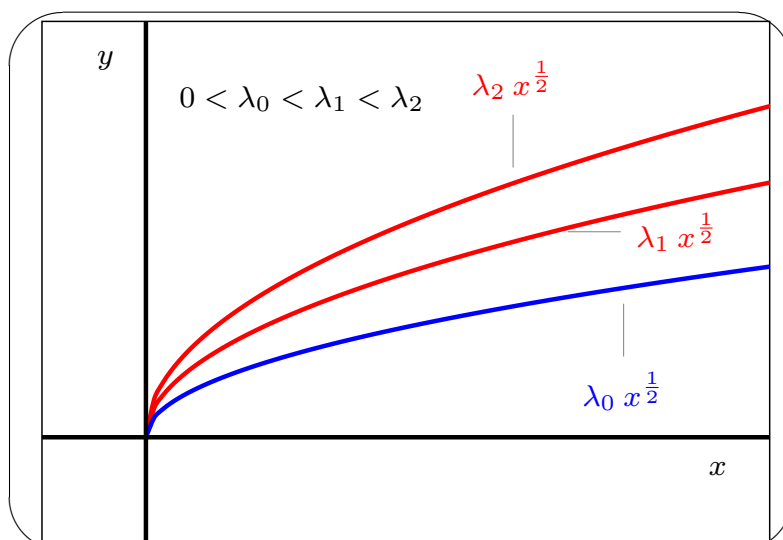
Given $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, we have the family of **power functions multiplied by a constant**:

$$f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \lambda x^\alpha \quad \text{where } \lambda \text{ is the multiplicative constant}$$

- For example, with α equal to 2 or equal to 3 and $\lambda \in \mathbb{R}$ we have:



- For example, with $\alpha = \frac{1}{2}$ and $\lambda \in \mathbb{R}$ we have:



Exponential and logarithmic functions

5.1 Exponential functions

Definition 5.1: exponential functions

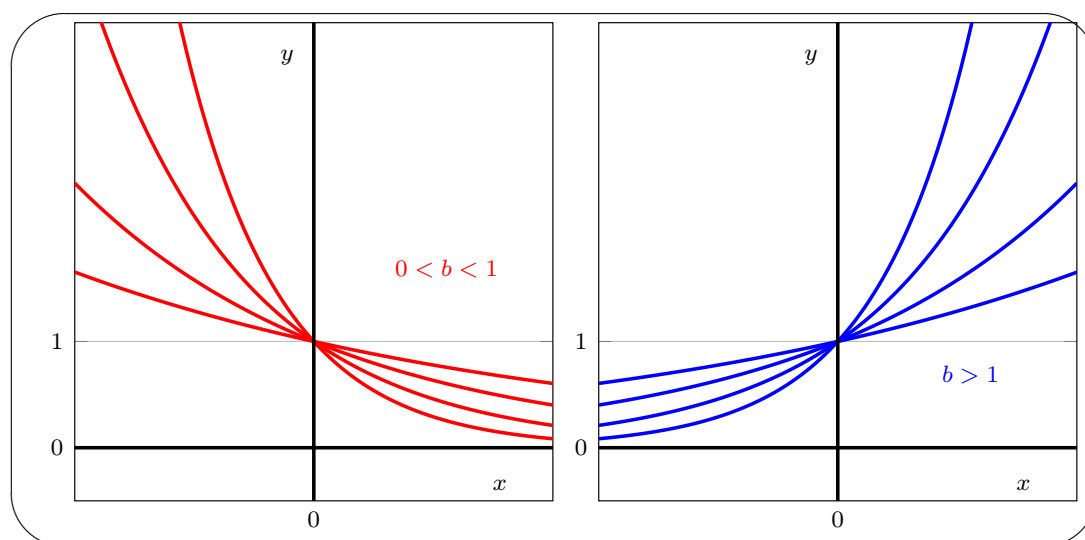
Given $b \in \mathbb{R}_+ \setminus \{1\}$, the function:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto b^x \quad (5.1)$$

is called the **exponential function** with base b .

- Regarding positivity and monotonicity, we have:

$$b^x > 0, \quad \forall x \in \mathbb{R} \quad \text{and} \quad \begin{cases} \text{if } 0 < b < 1 & \text{then } f \text{ is decreasing on } \mathbb{R} \\ \text{if } b > 1 & \text{then } f \text{ is increasing on } \mathbb{R} \end{cases}$$



5.2 Logarithmic functions

Definition 5.2: logarithmic functions

Given $a \in \mathbb{R}_{>0} \setminus \{1\}$, the function:

$$f : (0, +\infty) \rightarrow \mathbb{R}, \quad f : x \mapsto \log_a x \quad (5.2)$$

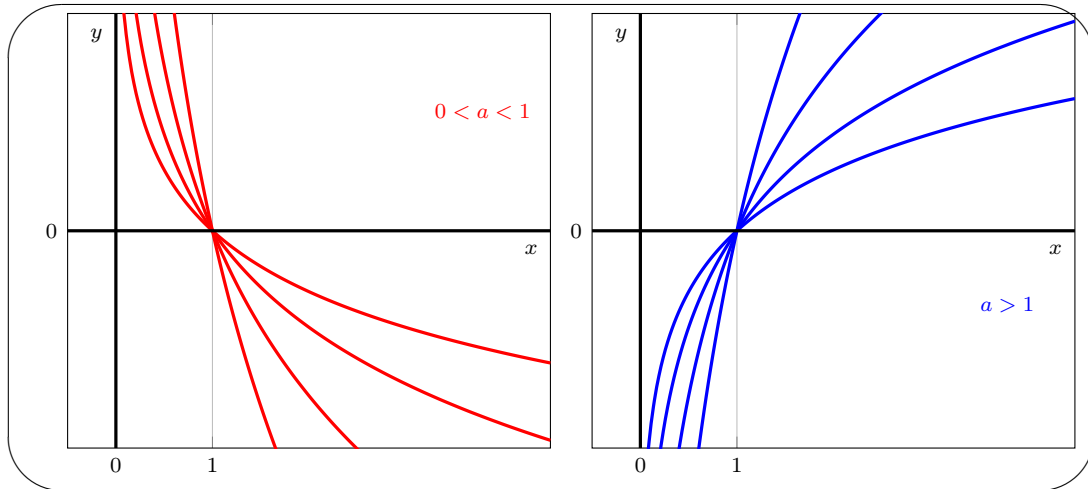
is called the **logarithmic function** with base a .

- Regarding positivity and monotonicity, we have:

$$\left\{ \begin{array}{l} \text{if } 0 < a < 1 \quad \log_a x > 0, \quad \forall x \in (0, 1), \quad \log_a x < 0, \quad \forall x \in (1, +\infty) \\ \quad \text{and } f \text{ is decreasing on } \mathbb{R}_{>0} \\ \\ \text{if } a > 1 \quad \log_a x < 0, \quad \forall x \in (0, 1), \quad \log_a x > 0, \quad \forall x \in (1, +\infty) \\ \quad \text{and } f \text{ is increasing on } \mathbb{R}_{>0} \end{array} \right.$$

Moreover, we have:

$$\log_a x = 0 \iff x = 1, \quad \forall a \in \mathbb{R}_{>0} \setminus \{1\}$$



5.3 Change of base

Given any base $c \in \mathbb{R}_{>0} \setminus \{1\}$, all exponential and logarithmic functions can be rewritten in terms of another base $d \in \mathbb{R}_{>0} \setminus \{1\}$ as follows:

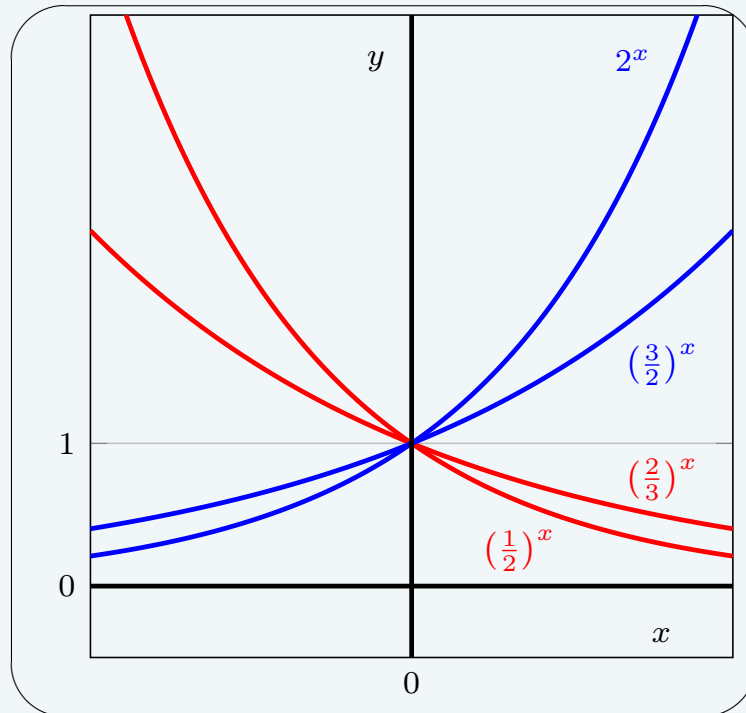
$$c^x = d^{\log_d c^x} = d^{x \log_d c} \quad \text{hence with } \lambda = \log_d c \quad \text{we get} \quad c^x = d^{\lambda x}$$

$$\log_c x = \frac{\log_d x}{\log_d c} \quad \text{hence with } \lambda = \frac{1}{\log_d c} \quad \text{we get} \quad \log_c x = \lambda \log_d x$$

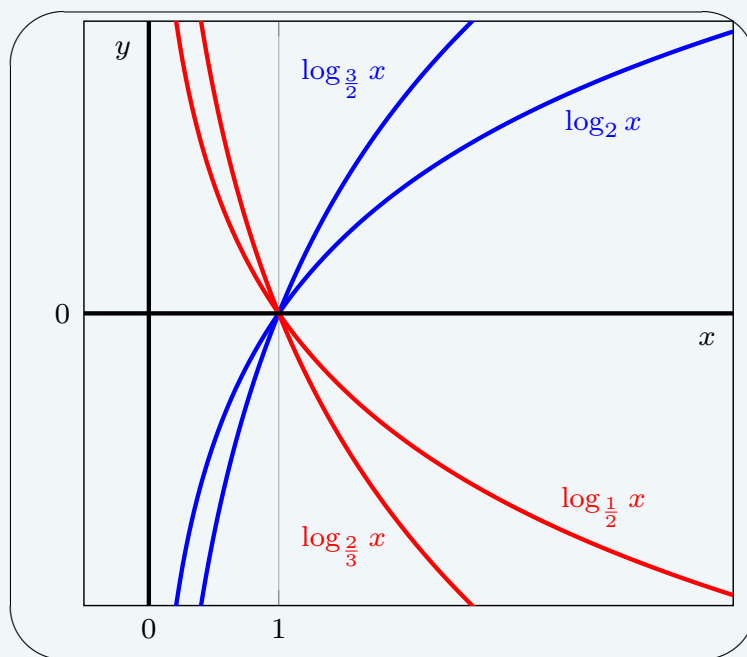
The logarithmic function with base e is also written $\log x$ or $\ln x$, while the one with base 2 is also written $\lg x$.

Example 5.1: graphs of exponential and logarithmic functions

Exponential functions:



Logarithmic functions:



Trigonometric functions

6.1 Trigonometric functions

Definition 6.1: sine function and cosine function

The trigonometric functions:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \sin x \quad (6.1)$$

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \cos x \quad (6.2)$$

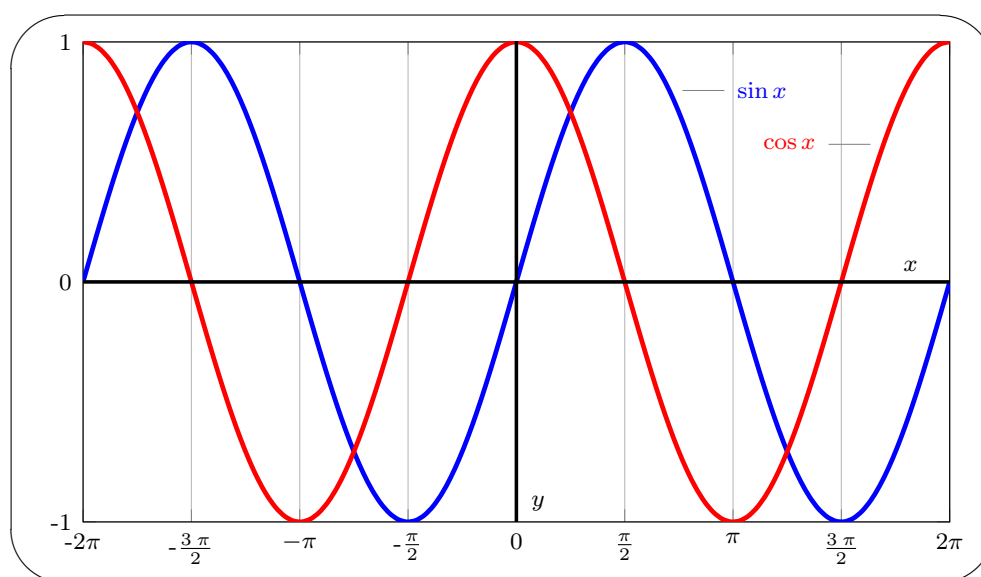
are called **sine** and **cosine**.

- The sine is an **odd** function, while the cosine is an **even** function. Hence we have:

$$\sin(-x) = -\sin x, \quad \forall x \in \mathbb{R} \quad \text{and} \quad \cos(-x) = \cos x, \quad \forall x \in \mathbb{R}$$

- The sine and the cosine are periodic functions with period:

$$T = 2\pi \quad \text{and we have the fundamental identity} \quad \sin^2 x + \cos^2 x = 1, \quad \forall x \in \mathbb{R}$$



We have

$$\cos\left(x - \frac{\pi}{2}\right) = \sin x, \quad \forall x \in \mathbb{R}$$

- From the sign and the monotonicity of the sine function we have:

$$\sin x = 0 \iff x = k\pi, \quad \forall k \in \mathbb{Z} \quad \begin{cases} \sin x > 0 & \text{if } x \in (2k\pi, 2k\pi + \pi), \quad \forall k \in \mathbb{Z} \\ \sin x < 0 & \text{if } x \in (2k\pi - \pi, 2k\pi), \quad \forall k \in \mathbb{Z} \end{cases}$$

$$\begin{cases} \text{if } x \in (2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}), \quad \forall k \in \mathbb{Z} & f \text{ is increasing} \\ \text{if } x \in (2k\pi + \frac{\pi}{2}, 2k\pi + \frac{3\pi}{2}), \quad \forall k \in \mathbb{Z} & f \text{ is decreasing} \end{cases}$$

- From the sign and the monotonicity of the cosine function we have:

$$\cos x = 0 \iff x = k\pi - \frac{\pi}{2}, \quad \forall k \in \mathbb{Z}$$

$$\begin{cases} \cos x > 0 & \text{if } x \in (2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}), \quad \forall k \in \mathbb{Z} \\ \cos x < 0 & \text{if } x \in (2k\pi + \frac{\pi}{2}, 2k\pi + \frac{3\pi}{2}), \quad \forall k \in \mathbb{Z} \end{cases}$$

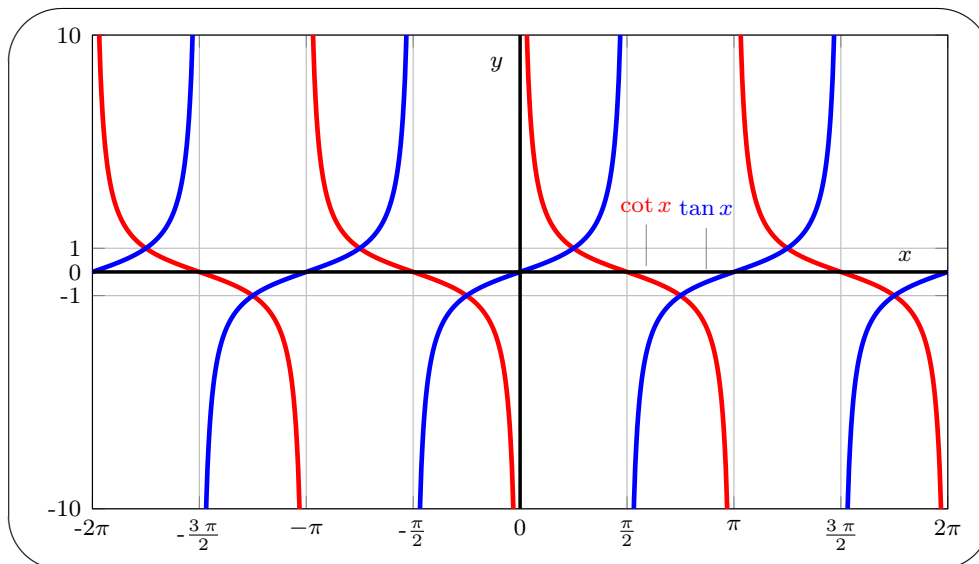
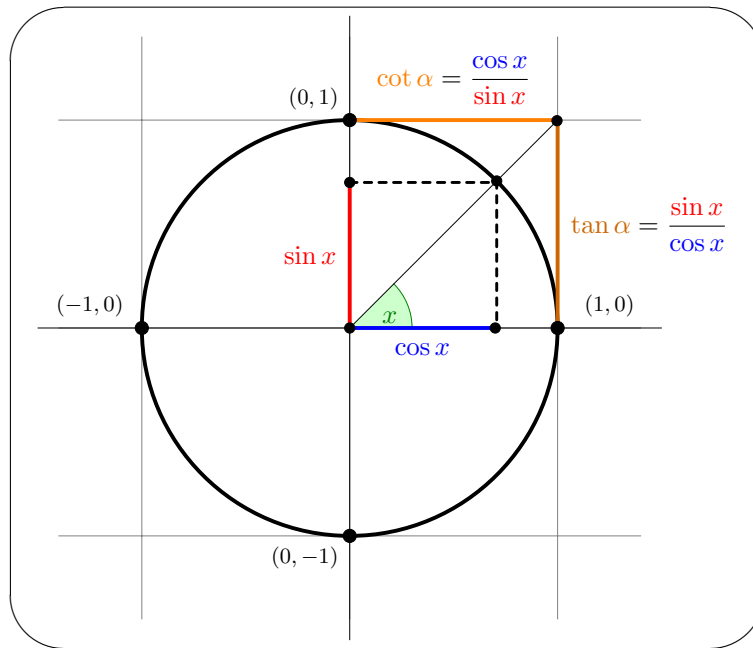
$$\begin{cases} \text{if } x \in (2k\pi + \pi, 2k\pi + 2\pi), \quad \forall k \in \mathbb{Z} & f \text{ is increasing} \\ \text{if } x \in (2k\pi, 2k\pi + \pi), \quad \forall k \in \mathbb{Z} & f \text{ is decreasing} \end{cases}$$

The trigonometric functions:

$$f : \mathbb{R} \setminus \{k\pi - \frac{\pi}{2}, \quad \forall k \in \mathbb{Z}\} \rightarrow \mathbb{R}, \quad f : x \mapsto \tan x = \frac{\sin x}{\cos x} \quad (6.3)$$

$$f : \mathbb{R} \setminus \{k\pi, \quad \forall k \in \mathbb{Z}\} \rightarrow \mathbb{R}, \quad f : x \mapsto \cot x = \frac{\cos x}{\sin x} \quad (6.4)$$

are called **tangent** and **cotangent**; they are both odd and periodic with period $T = \pi$.

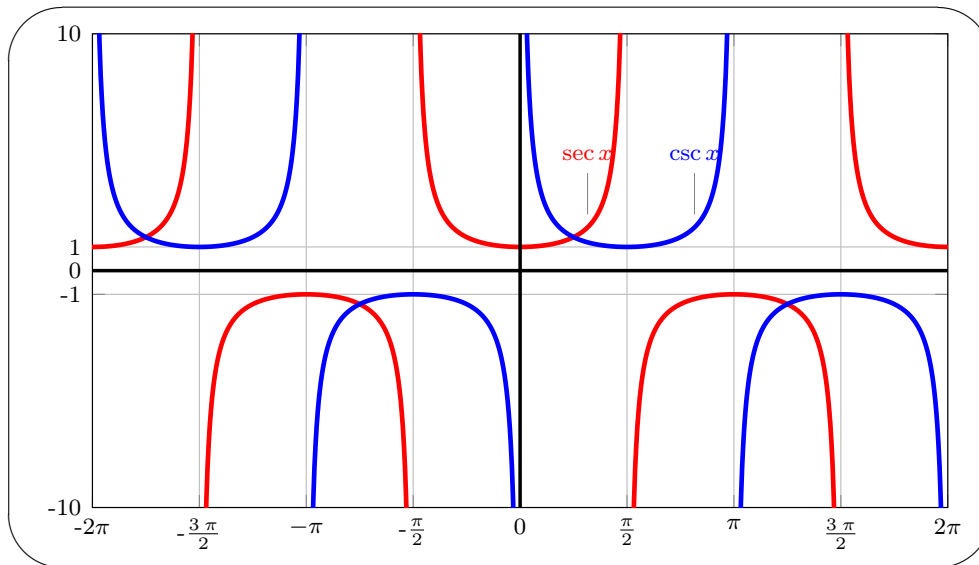


The trigonometric functions:

$$f : \mathbb{R} \setminus \{k\pi - \frac{\pi}{2}, \forall k \in \mathbb{Z}\} \rightarrow \mathbb{R}, \quad f : x \mapsto \sec x = \frac{1}{\cos x} \quad (6.5)$$

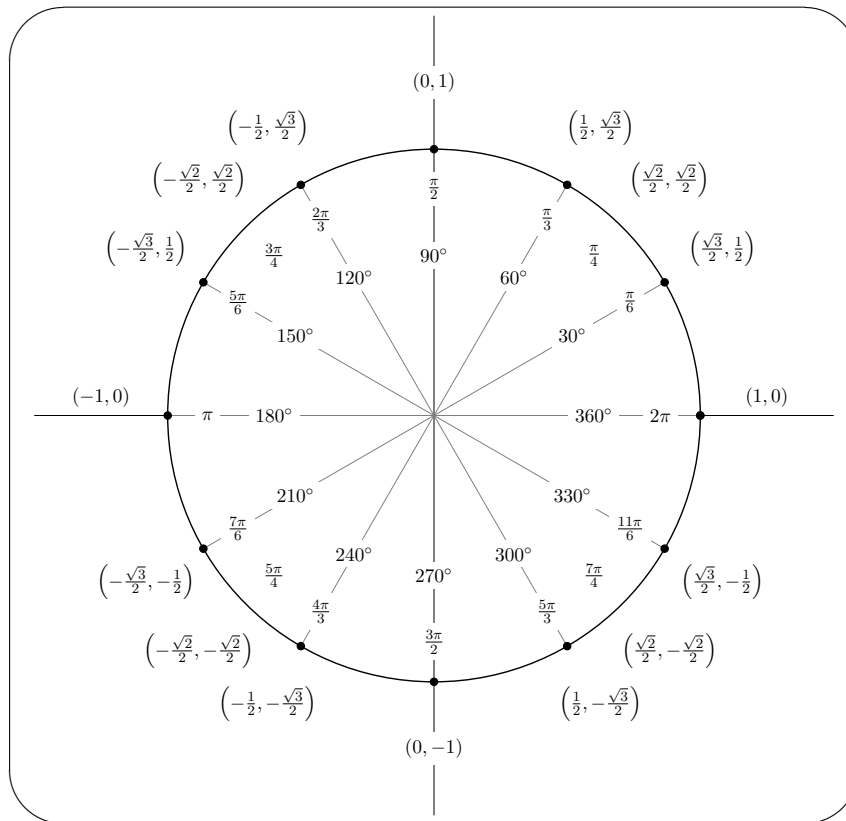
$$f : \mathbb{R} \setminus \{k\pi, \forall k \in \mathbb{Z}\} \rightarrow \mathbb{R}, \quad f : x \mapsto \csc x = \frac{1}{\sin x} \quad (6.6)$$

are called **secant** and **cosecant** and they are periodic with period $T = 2\pi$. The secant function is even, while the cosecant function is odd.



6.2 Values of the trigonometric functions

	cos	sin	tan	cot	sec	csc
0	1	0	0	$\pm\infty$	1	$\pm\infty$
$x = \frac{\pi}{6}$ (30°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
$x = \frac{\pi}{4}$ (45°)	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$x = \frac{\pi}{3}$ (60°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2}{3}\sqrt{3}$
$x = \frac{\pi}{2}$ (90°)	0	1	$\pm\infty$	0	$\pm\infty$	1
$x = \pi$ (180°)	-1	0	0	$\pm\infty$	-1	$\pm\infty$



	cos	sin	tan	cot
x	$\cos x$	$\sin x$	$\tan x$	$\cot x$
$-x$	$\cos x$	$-\sin x$	$-\tan x$	$-\cot x$
$\frac{\pi}{2} + x$	$-\sin x$	$\cos x$	$-\cot x$	$-\tan x$
$\frac{\pi}{2} - x$	$\sin x$	$\cos x$	$\cot x$	$\tan x$
$\pi + x$	$-\cos x$	$-\sin x$	$\tan x$	$\cot x$
$\pi - x$	$-\cos x$	$\sin x$	$-\tan x$	$-\cot x$

6.3 Main trigonometric formulas

Relations among the trigonometric functions

	$\sin x$	$\cos x$	$\tan x$	$\cot x$
$\sin x$	-	$\pm\sqrt{1 - \cos^2 x}$	$\pm\sqrt{\frac{\tan^2 x}{1 + \tan^2 x}}$	$\pm\sqrt{\frac{1}{1 + \cot^2 x}}$
$\cos x$	$\pm\sqrt{1 - \sin^2 x}$	-	$\pm\sqrt{\frac{1}{1 + \tan^2 x}}$	$\pm\sqrt{\frac{\cot^2 x}{1 + \cot^2 x}}$
$\tan x$	$\pm\sqrt{\frac{\sin^2 x}{1 - \sin^2 x}}$	$\pm\sqrt{\frac{1 - \cos^2 x}{\cos^2 x}}$	-	$\frac{1}{\cot x}$
$\cot x$	$\pm\sqrt{\frac{1 - \sin^2 x}{\sin^2 x}}$	$\pm\sqrt{\frac{\cos^2 x}{1 - \cos^2 x}}$	$\frac{1}{\tan x}$	-

where the symbol $\pm$ means that the sign depends on the quadrant in which x lies

Addition and subtraction

$$\sin(x_1 \pm x_2) = \sin x_1 \cos x_2 \pm \sin x_2 \cos x_1 \quad (6.7a)$$

$$\cos(x_1 \pm x_2) = \cos x_1 \cos x_2 \mp \sin x_1 \sin x_2 \quad (6.7b)$$

$$\tan(x_1 \pm x_2) = \frac{\tan x_1 \pm \tan x_2}{1 \mp \tan x_1 \tan x_2} \quad (6.8a)$$

$$\cot(x_1 \pm x_2) = \frac{\cot x_1 \cot x_2 \mp 1}{\cot x_2 \pm \cot x_1} \quad (6.8b)$$

Double-angle formulas

$$\sin(2x) = 2 \sin x \cos x \quad (6.9a)$$

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ &= 2 \cos^2 x - 1 \\ &= 1 - 2 \sin^2 x \end{aligned} \quad (6.9b)$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x} \quad (6.10a)$$

$$\cot(2x) = \frac{\cot^2 x - 1}{2 \cot x} \quad (6.10b)$$

Triple-angle formulas

$$\sin(3x) = 3 \sin x - 4 \sin^3 x \quad (6.11a)$$

$$\cos(3x) = 4 \cos^3 x - 3 \cos x \quad (6.11b)$$

Half-angle formulas

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}} \quad (6.12a)$$

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}} \quad (6.12b)$$

$$\begin{aligned}
 \tan \frac{x}{2} &= \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} \\
 &= \frac{\sin x}{1 + \cos x} \\
 &= \frac{1 - \cos x}{\sin x}
 \end{aligned} \tag{6.13a}$$

$$\begin{aligned}
 \cot \frac{x}{2} &= \pm \sqrt{\frac{1 + \cos x}{1 - \cos x}} \\
 &= \frac{\sin x}{1 - \cos x} \\
 &= \frac{1 + \cos x}{\sin x}
 \end{aligned} \tag{6.14a}$$

Parametric formulas (tangent half-angle substitution)

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \tag{6.15a}$$

$$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \tag{6.15b}$$

Sum-to-product formulas (prosthaphaeresis)

$$\sin x_1 + \sin x_2 = 2 \sin \frac{x_1 + x_2}{2} \cos \frac{x_1 - x_2}{2} \tag{6.16a}$$

$$\sin x_1 - \sin x_2 = 2 \sin \frac{x_1 - x_2}{2} \cos \frac{x_1 + x_2}{2} \tag{6.16b}$$

$$\cos x_1 + \cos x_2 = 2 \cos \frac{x_1 + x_2}{2} \cos \frac{x_1 - x_2}{2} \tag{6.16c}$$

$$\cos x_1 - \cos x_2 = -2 \sin \frac{x_1 + x_2}{2} \sin \frac{x_1 - x_2}{2} \tag{6.16d}$$

Product-to-sum formulas (Werner's formulas)

$$\sin x_1 \sin x_2 = \frac{1}{2} \left(\cos(x_1 - x_2) - \cos(x_1 + x_2) \right) \tag{6.17a}$$

$$\sin x_1 \cos x_2 = \frac{1}{2} \left(\sin(x_1 - x_2) + \sin(x_1 + x_2) \right) \tag{6.17b}$$

$$\cos x_1 \cos x_2 = \frac{1}{2} \left(\cos(x_1 - x_2) + \cos(x_1 + x_2) \right) \tag{6.17c}$$

Additional formulas

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad (6.18a)$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \quad (6.18b)$$

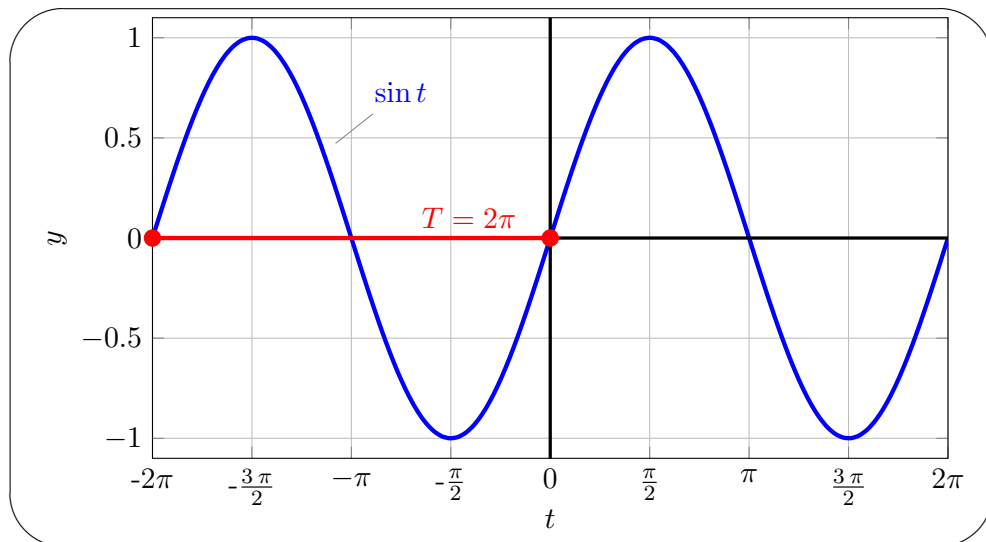
$$\sin x \cos x = \frac{\sin(2x)}{2} \quad (6.18c)$$

6.4 An application: oscillations

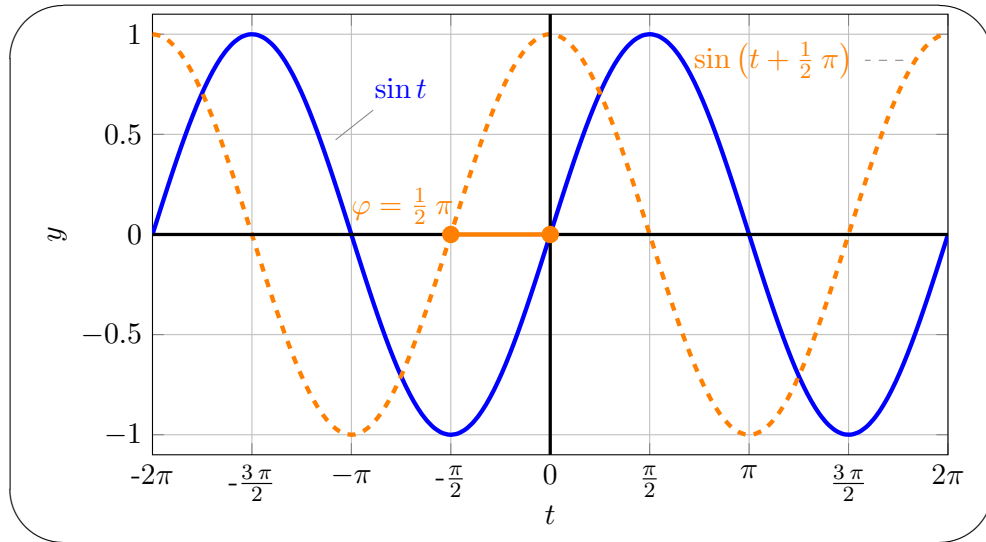
The sine and cosine functions are periodic: this is why they are the natural tool to describe phenomena that repeat in time. In this section we apply them to **oscillations** (vibration phenomena).

6.4.1 Vibration phenomena

- We have seen that the sine and cosine functions are periodic with **period** 2π . We will use the variable t to denote time.



- Recall that $\sin(t + \varphi)$ corresponds to a **phase shift** φ (horizontal translation).



The functions:

$$t \mapsto a \sin(\omega t), \quad t \mapsto b \cos(\omega t) \quad (6.19)$$

where a , b and ω are positive real numbers, are called **elementary vibrations**. They are periodic functions with

$$\text{period } T = \frac{2\pi}{\omega}$$

Example 6.1: period

For example, considering $t' = t + \frac{2\pi}{\omega}$:

$$\sin \left[\omega \left(\underbrace{t + \frac{2\pi}{\omega}}_{t'} \right) \right] = \sin(\omega t + 2\pi) = \sin(\omega t)$$

- Clearly we have:

$$|a \sin(\omega t)| \leq a \quad \text{and} \quad |b \cos(\omega t)| \leq b$$

The functions (6.19) describe elementary vibrations characterized by

amplitude a and b (respectively)

$$\text{angular frequency } \omega = \frac{2\pi}{T}$$

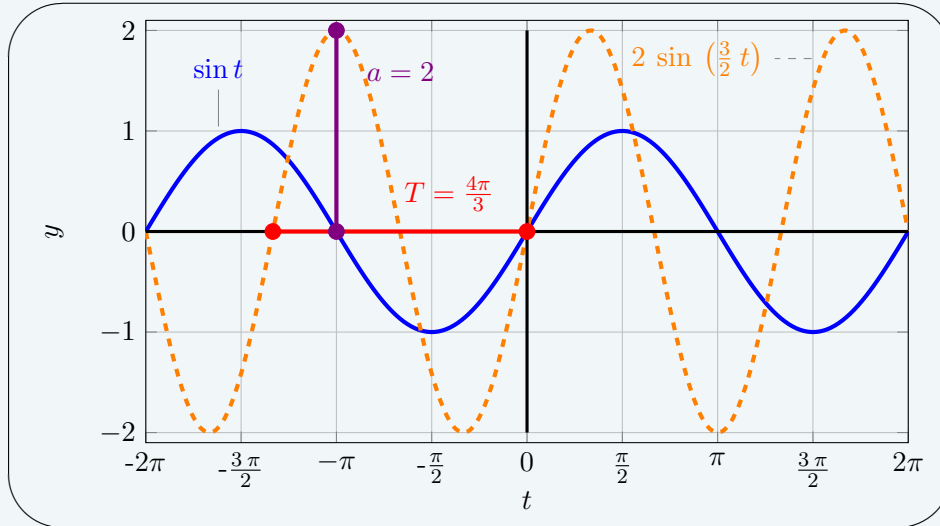
also called *angular velocity*, which indicates how many periods there are in an interval of length 2π . Moreover, they are characterized by the

$$\text{frequency } \nu = \frac{\omega}{2\pi}$$

which indicates how many times the function repeats itself in an interval of length 1

Example 6.2: of amplitude, angular frequency and frequency

$$t \mapsto 2 \sin \left(\frac{3}{2} t \right) \quad a = 2, \quad \omega = \frac{3}{2}, \quad T = \frac{4\pi}{3}$$



The angular frequency $\omega = \frac{3}{2}$ means that there are 1.5 periods in an interval of length 2π . The frequency ν is $\frac{3}{4\pi}$.

- The function

$$h(t) = a \sin(\omega t) + b \cos(\omega t) \quad (6.20)$$

describes the **superposition** of the two elementary vibrations with the **same angular frequency** ω . The latter is again an elementary vibration, **phase-shifted** with respect to the previous ones.

- Indeed, setting

$$A = \sqrt{a^2 + b^2}$$

we can write

$$h(t) = A \underbrace{\frac{a}{\sqrt{a^2 + b^2}}}_{\alpha} \sin(\omega t) + A \underbrace{\frac{b}{\sqrt{a^2 + b^2}}}_{\beta} \cos(\omega t) \quad (6.21)$$

- Now observe that the numbers

$$\alpha = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \beta = \frac{b}{\sqrt{a^2 + b^2}}$$

satisfy the conditions

$$-1 \leq \alpha \leq 1 \quad -1 \leq \beta \leq 1 \quad \alpha^2 + \beta^2 = 1$$

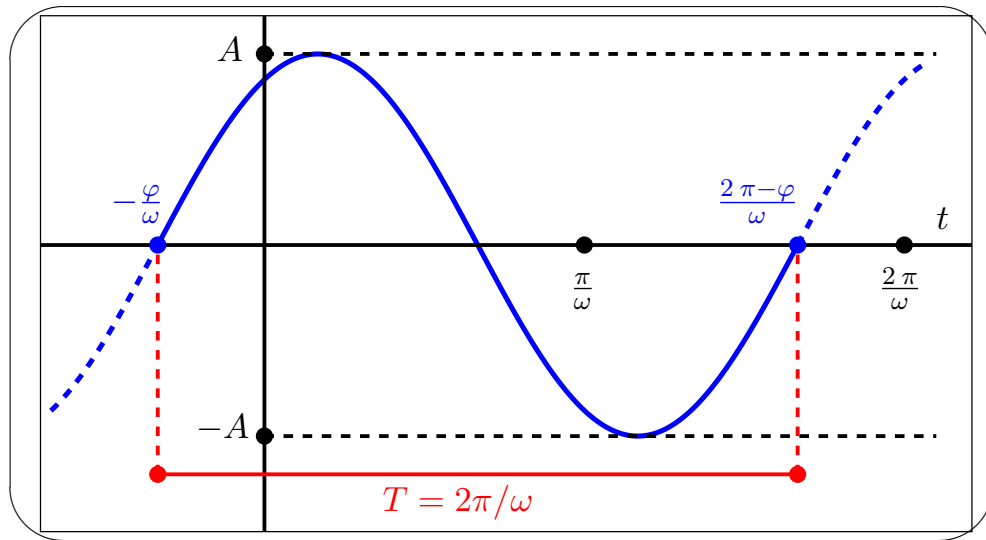
Hence there exists a **unique angle** φ such that

$$\cos \varphi = \alpha \quad \sin \varphi = \beta$$

so that (6.21) can be rewritten in the following form

$$\begin{aligned} h(t) &= A \cos \varphi \sin(\omega t) + A \sin \varphi \cos(\omega t) \\ &= A \sin(\omega t + \varphi) \end{aligned} \quad (6.22)$$

Therefore $h(t)$ represents an elementary vibration with amplitude A , angular frequency ω , phase-shifted by an angle φ .



In summary:

$$a \sin(\omega t) + b \cos(\omega t) = A \sin(\omega t + \varphi)$$

$$A = \sqrt{a^2 + b^2} \quad \begin{cases} a = A \cos \varphi \\ b = A \sin \varphi \end{cases}$$

- Under fairly general conditions, a periodic natural phenomenon can be written as the superposition of a finite or infinite number of elementary vibrations with different frequencies (**Fourier series**).

6.4.2 Damping and amplification effects

- By multiplying an elementary vibration by **powers** or **exponentials**, we can model **damping** or **amplification** effects.

First example

- For example, consider the function

$$h(t) = t \sin(\omega t)$$

which models an **amplified vibration**.

- Since $-1 \leq \sin(\omega t) \leq 1$, we have

$$-t \leq t \sin(\omega t) \leq t$$

and hence the graph of $h(t)$ lies between the graphs of the lines with equations $y = -t$, $y = t$.

- At the points where

$$\sin(\omega t) = 1$$

i.e.,

$$t = \frac{\pi}{2\omega} + k \frac{2\pi}{\omega} \quad (k = 0, 1, 2, \dots)$$

the graph of $h(t)$ touches that of $y = t$.

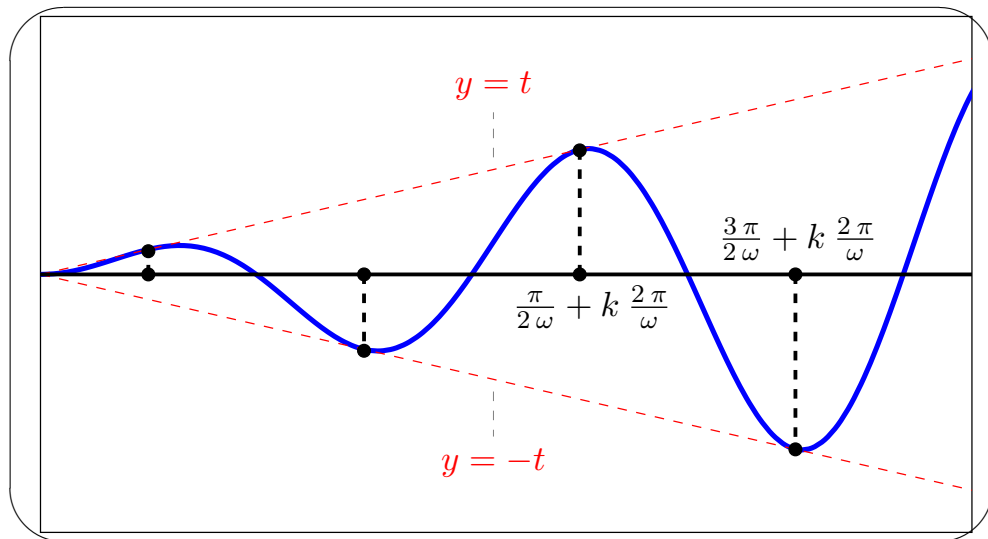
- At the points where

$$\sin(\omega t) = -1$$

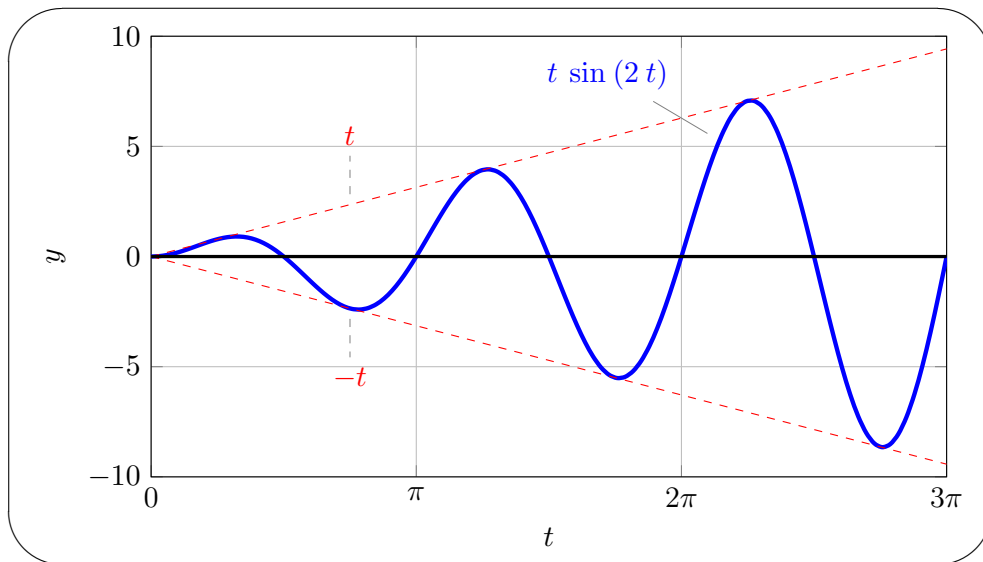
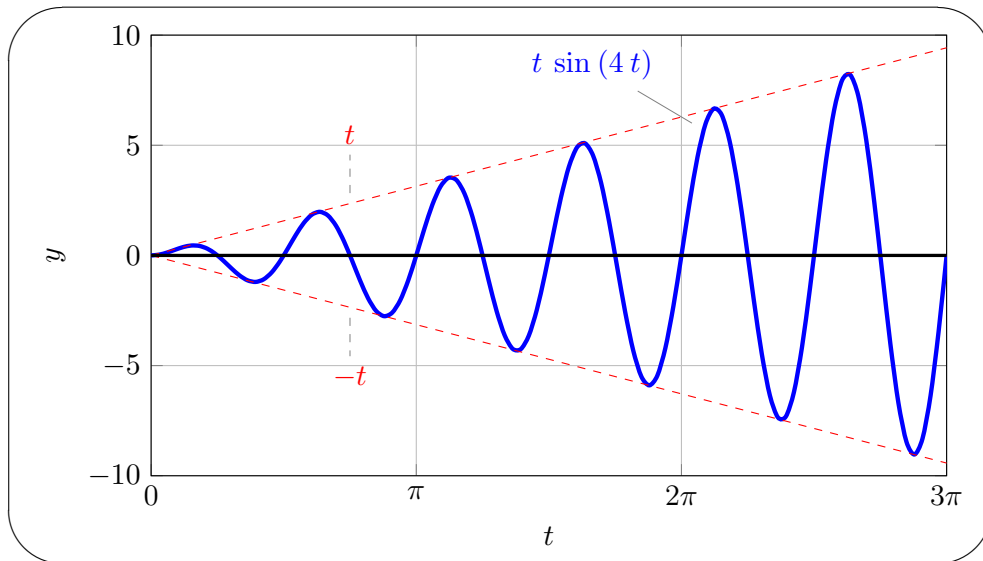
i.e.,

$$t = \frac{3\pi}{2\omega} + k \frac{2\pi}{\omega} \quad (k = 0, 1, 2, \dots)$$

the graph of $h(t)$ touches that of $y = -t$.



- From the graph we can see that multiplication by t has the effect of amplifying the vibration as t increases.



Second example

- For example, consider the function

$$k(t) = e^{-\alpha t} \sin(\omega t) \quad (\alpha > 0)$$

which models a **damped vibration**.

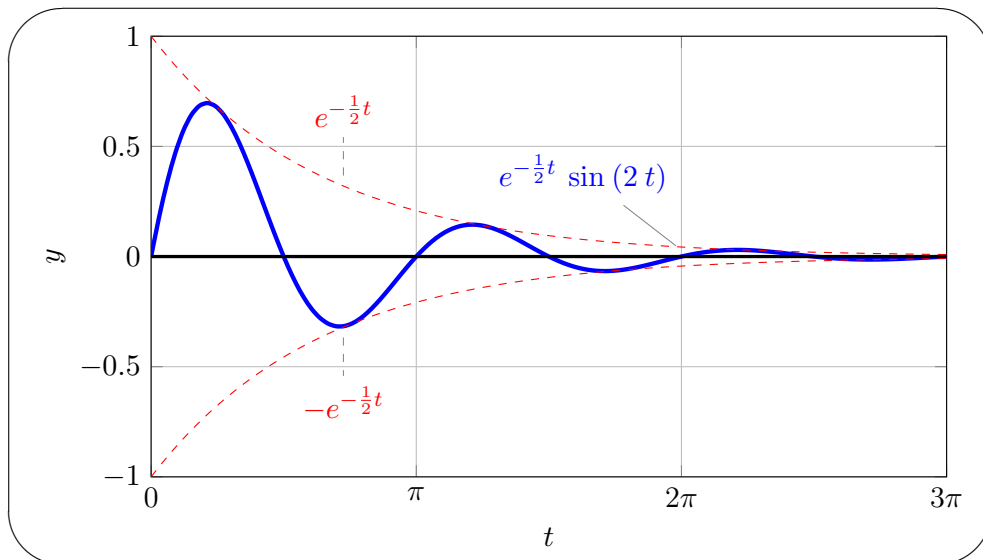
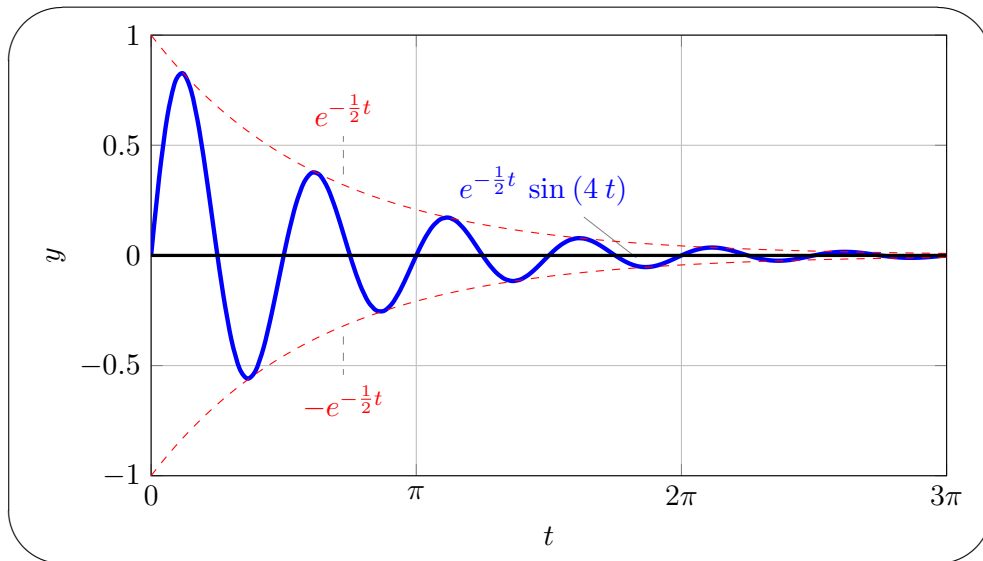
- Recalling that

$$e^{-\alpha t} = \left(\frac{1}{e^\alpha}\right)^t$$

is an exponential with base less than 1, considerations analogous to those made for the function $h(t)$ show that the graph of k lies between the graphs of the functions

$$y_1 = e^{-\alpha t} \quad \text{and} \quad y_2 = -e^{-\alpha t}$$

as shown in the figure:



Integer part and fractional part functions

7.1 Integer part and fractional part functions

- Two functions that are typically encountered when writing *algorithms* are the integer part (floor) function and the fractional part (or decimal part) function.

Definition 7.1: of the integer part function

The *integer part (floor) function* is:

$$f : \mathbb{R} \rightarrow \mathbb{Z}, x \mapsto [x] \quad (\text{or } x \mapsto \lfloor x \rfloor) \quad (7.1)$$

Definition 7.2: of the upper integer part function

The *upper integer part (ceiling) function* is:

$$f : \mathbb{R} \rightarrow \mathbb{Z}, x \mapsto \lceil x \rceil \quad (7.2)$$

Definition 7.3: of the fractional part function

The *fractional part function* is:

$$f : \mathbb{R} \rightarrow [0, 1), x \mapsto (x) \quad (7.3)$$

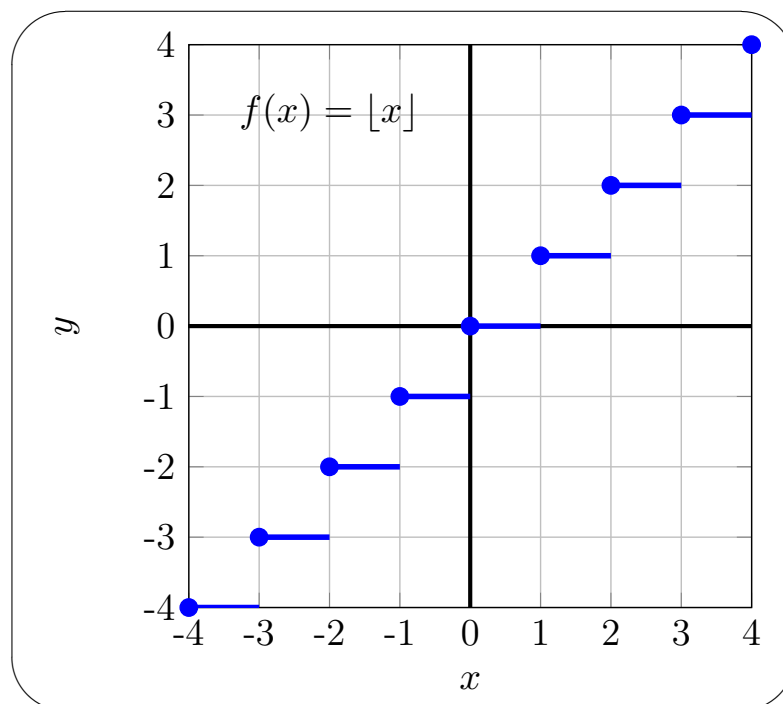
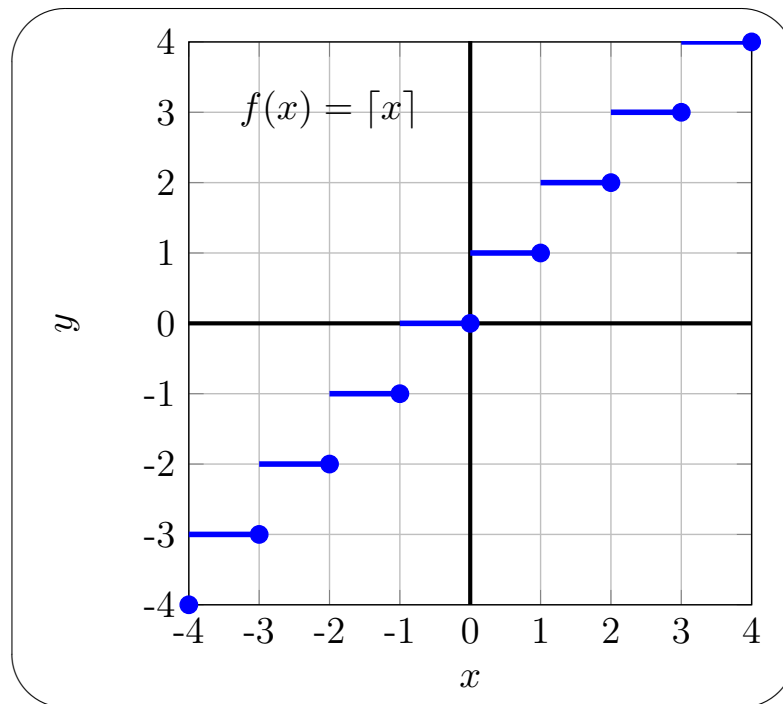
- The integer part function $f(x) = [x]$ and the upper integer part function $f(x) = \lceil x \rceil$ are **non-decreasing** in $\mathbb{R}$, but they are **not** increasing.
 - They are non-decreasing: let $x_1 < x_2$. Then $[x_1] \leq x_1 < x_2$ and, since $[x_1]$ is an integer less than or equal to x_2 , while $[x_2]$ is the *largest* integer less than or equal to x_2 (see the chapter “[Roots, powers, logarithms and modular arithmetic](#)” of Part 2), we have $[x_1] \leq [x_2]$. In the same way $\lceil x_2 \rceil \geq x_2 > x_1$ and, since $\lceil x_1 \rceil$ is the *smallest* integer greater than or equal to x_1 , we have $\lceil x_1 \rceil \leq \lceil x_2 \rceil$.

- They are not increasing: both are *constant* on every interval between two consecutive integers, for instance

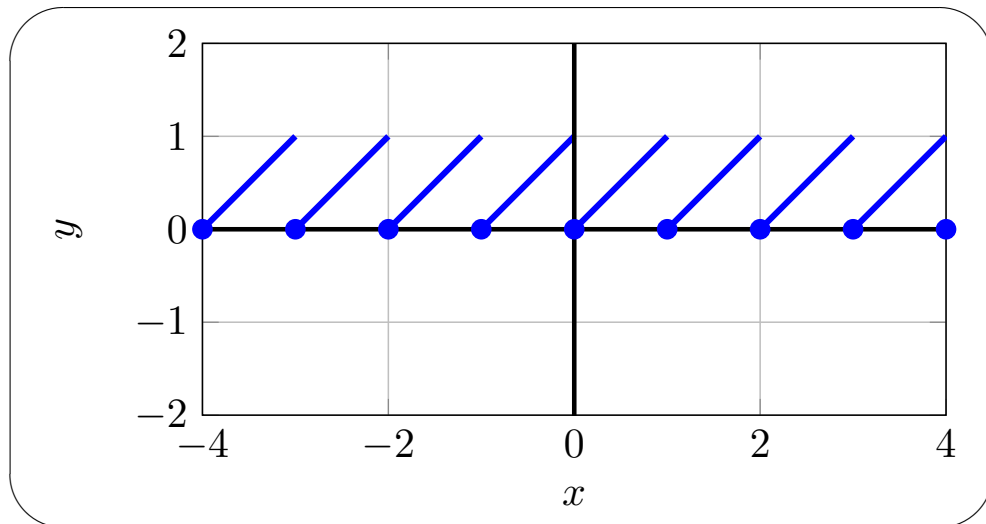
$$\left\lfloor \frac{1}{3} \right\rfloor = \left\lfloor \frac{2}{3} \right\rfloor = 0 \quad \text{and} \quad \left\lceil \frac{1}{3} \right\rceil = \left\lceil \frac{2}{3} \right\rceil = 1,$$

hence there exist $x_1 < x_2$ with $f(x_1) = f(x_2)$.

- Graphs of the functions $f(x) = \lceil x \rceil$ and $f(x) = \lfloor x \rfloor$ on the interval $[-4, 4]$.



- Note that the fractional part is a periodic function with period 1:



7.2 Piecewise-defined functions

- Starting from the elementary functions, new ones can be built by using **different analytic definitions on different intervals**.

Definition 7.4: of piecewise-defined function

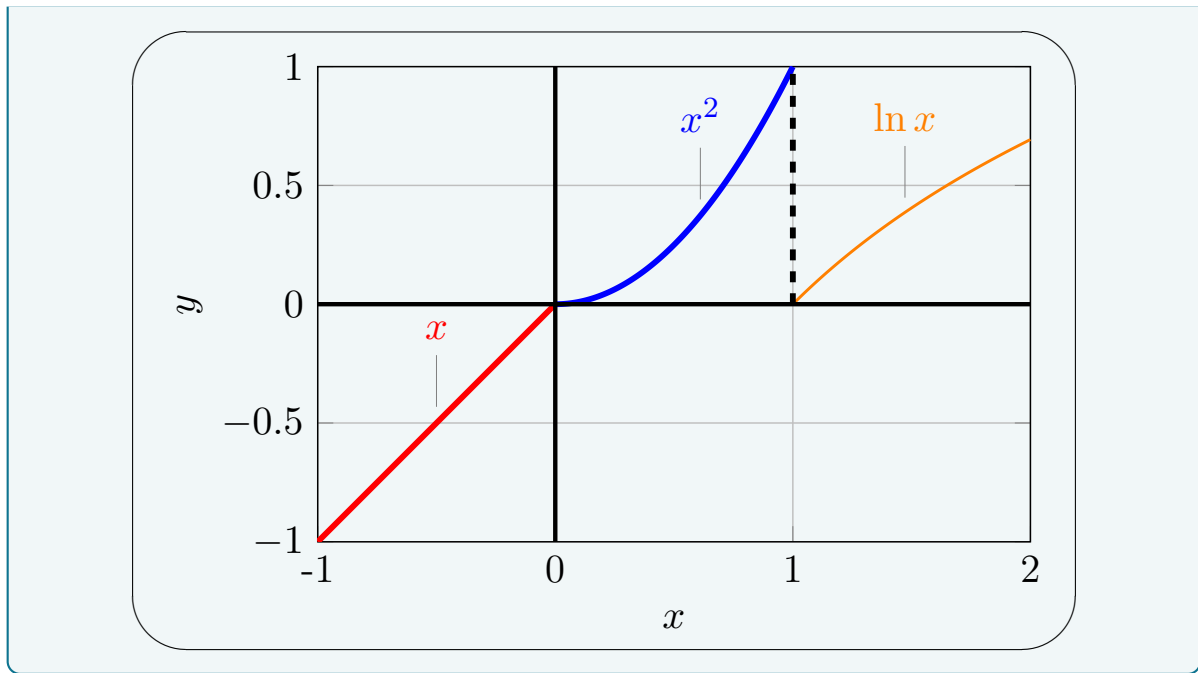
A function f whose value $f(x)$ is computed through different “instructions” depending on the interval in which x lies is called a **piecewise-defined function**.

Example 7.1: Piecewise-defined functions

Consider for example

$$f(x) = \begin{cases} \ln x & \text{if } x > 1, \\ x^2 & \text{if } 0 < x \leq 1, \\ x & \text{if } x \leq 0 \end{cases}$$

its graph is:



- From the *logical / algorithmic* point of view, one can say that a function of this kind has the peculiarity of being built using not only mathematical functions but also the **logical function** “if ... then”.

7.3 The absolute-value function

- In the chapter “[Factorials, binomial coefficients and triangle inequality](#)” of Part 2 the **absolute value** of a real number a was defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Associating with every $x \in \mathbb{R}$ the number $|x|$ we obtain a function of a real variable, which is the most important example of a *piecewise defined function*.

Definition 7.5: of absolute-value function

The **absolute-value function** is:

$$f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto |x| \quad (7.4)$$

- **Sign.** The absolute-value function is *nonnegative* in $\mathbb{R}$ and *positive* in $\mathbb{R} \setminus \{0\}$:

$$|x| \geq 0, \quad \forall x \in \mathbb{R} \quad \text{and} \quad |x| = 0 \iff x = 0.$$

Indeed, if $x \geq 0$ then $|x| = x \geq 0$, and if $x < 0$ then $|x| = -x > 0$; in both cases $|x| = 0$ forces $x = 0$.

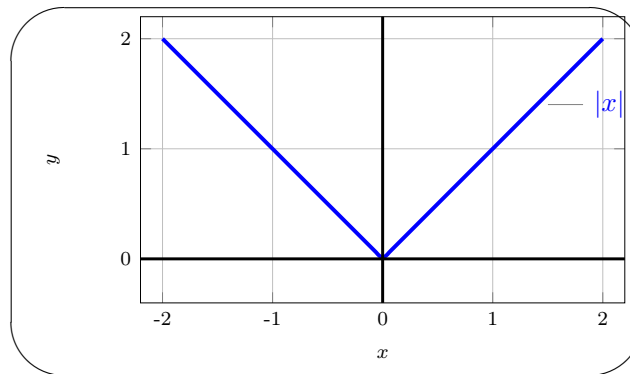
- **Monotonicity.** The absolute-value function is *decreasing* in $(-\infty, 0]$ and *increasing* in $[0, +\infty)$, but it is **not** monotonic in $\mathbb{R}$.

- If $x_1 < x_2 \leq 0$, then $|x_1| = -x_1 > -x_2 = |x_2|$.
- If $0 \leq x_1 < x_2$, then $|x_1| = x_1 < x_2 = |x_2|$.
- It is not monotonic in $\mathbb{R}$ because $f(-1) = 1 > f(0) = 0$ (hence it is not non-decreasing in $\mathbb{R}$) and $f(0) = 0 < f(1) = 1$ (hence it is not non-increasing in $\mathbb{R}$).
- **Boundedness and image.** The function is *bounded below* in $\mathbb{R}$ (the number 0 is a lower bound for its values), but it is **not** bounded above: for every $M \in \mathbb{R}$ with $M > 0$ we have $f(M + 1) = M + 1 > M$. Its image is

$$f(\mathbb{R}) = [0, +\infty),$$

because $|x| \geq 0$ for every x and, conversely, every $y \geq 0$ is the image of y itself: $|y| = y$.

- **Symmetry.** The absolute-value function is *even*, since $|-x| = |x|$ for every $x \in \mathbb{R}$: its graph is symmetric with respect to the y -axis.
- The graph consists of the two half-lines $y = -x$ for $x \leq 0$ and $y = x$ for $x \geq 0$:



Hyperbolic functions

8.1 Hyperbolic functions

Definition 8.1: hyperbolic sine function and hyperbolic cosine function

The hyperbolic functions:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \text{Sh } x = \frac{e^x - e^{-x}}{2} \quad (8.1)$$

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \text{Ch } x = \frac{e^x + e^{-x}}{2} \quad (8.2)$$

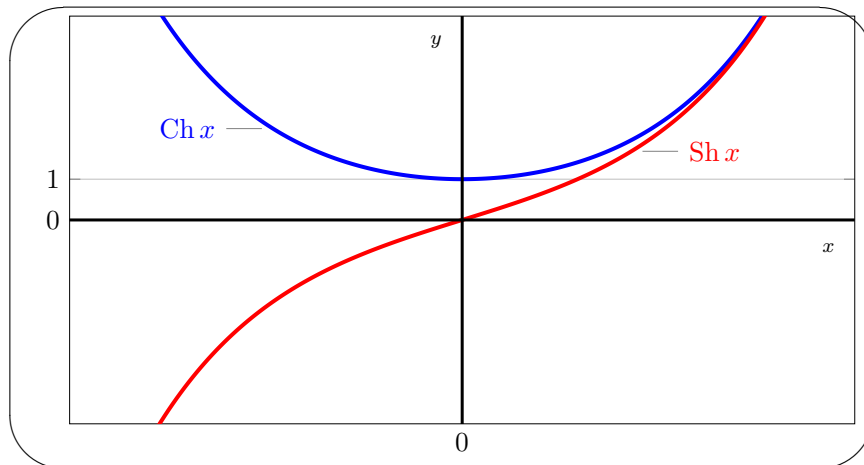
are called **hyperbolic sine** and **hyperbolic cosine**.

- The hyperbolic sine is an **odd** function, while the hyperbolic cosine is an **even** function. Hence we have:

$$\text{Sh}(-x) = -\text{Sh } x, \quad \forall x \in \mathbb{R} \quad \text{and} \quad \text{Ch}(-x) = \text{Ch } x, \quad \forall x \in \mathbb{R}$$

- We have the fundamental relations:

$$\text{Ch}^2 x - \text{Sh}^2 x = 1, \quad \forall x \in \mathbb{R} \quad \text{and} \quad \text{Sh } x \leq \frac{e^x}{2} \leq \text{Ch } x, \quad \forall x \in \mathbb{R} \quad (8.3)$$



- For the sign and monotonicity of the hyperbolic sine function we have:

$$\operatorname{Sh} x = 0 \iff x = 0 \quad \begin{cases} \operatorname{Sh} x > 0 & \text{if } x > 0 \\ \operatorname{Sh} x < 0 & \text{if } x < 0 \end{cases}$$

$$f \text{ is increasing} \quad \forall x \in \mathbb{R}$$

- For the sign and monotonicity of the hyperbolic cosine function we have:

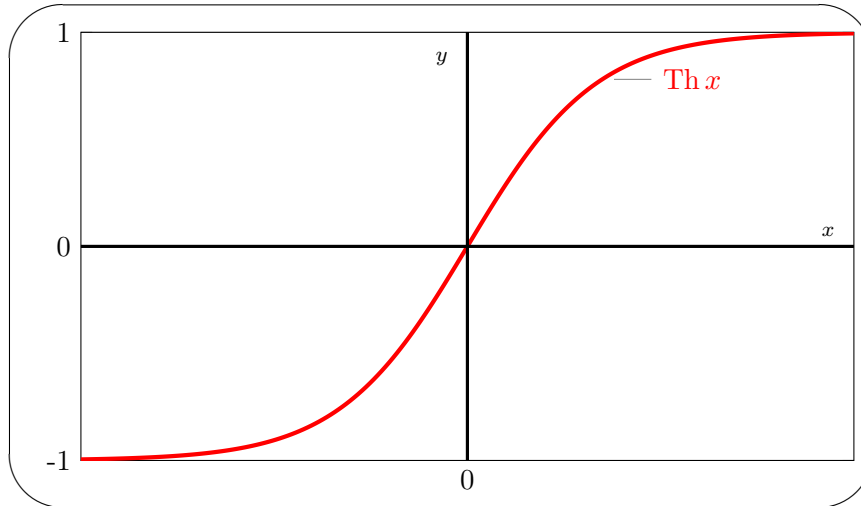
$$\operatorname{Ch} x > 0, \quad \forall x \in \mathbb{R}$$

$$\begin{cases} \text{if } x < 0 & f \text{ is decreasing} \\ \text{if } x > 0 & f \text{ is increasing} \end{cases}$$

The hyperbolic function:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto \operatorname{Th} x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad (8.4)$$

is called the **hyperbolic tangent** and it is an odd function.



8.2 Main hyperbolic formulas

Addition

$$\operatorname{Sh}(x_1 + x_2) = \operatorname{Sh} x_1 \operatorname{Ch} x_2 + \operatorname{Sh} x_2 \operatorname{Ch} x_1 \quad (8.5a)$$

$$\operatorname{Ch}(x_1 + x_2) = \operatorname{Ch} x_1 \operatorname{Ch} x_2 + \operatorname{Sh} x_1 \operatorname{Sh} x_2 \quad (8.5b)$$

$$\operatorname{Th}(x_1 + x_2) = \frac{\operatorname{Th} x_1 + \operatorname{Th} x_2}{1 + \operatorname{Th} x_1 \operatorname{Th} x_2} \quad (8.5c)$$

Double-argument formulas

$$\operatorname{Sh}(2x) = 2 \operatorname{Sh} x \operatorname{Ch} x \quad (8.6a)$$

$$\operatorname{Ch}(2x) = \operatorname{Ch}^2 x + \operatorname{Sh}^2 x \quad (8.6b)$$

Operations on graphs

9.1 Operations on graphs

- Knowing the graph of a function $y = f(x)$, by means of simple geometric transformations it is possible to draw the graph of the following functions:

$$y_1 = f(x) + a, \quad a \in \mathbb{R} \quad (9.1a)$$

$$y_2 = f(x + a), \quad a \in \mathbb{R} \quad (9.1b)$$

$$y_3 = k f(x), \quad k \in \mathbb{R} \quad (9.1c)$$

$$y_4 = f(kx), \quad k \in \mathbb{R} \quad (9.1d)$$

$$y_5 = |f(x)| \quad (9.1e)$$

$$y_6 = f(|x|) \quad (9.1f)$$

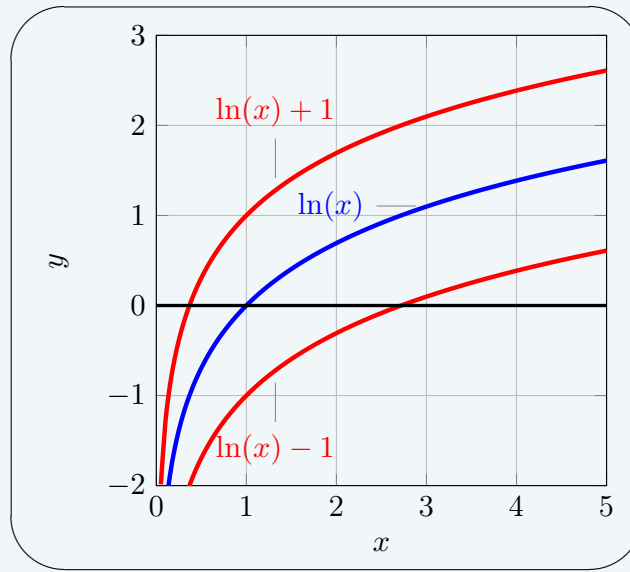
- Therefore, starting from the elementary functions, it is possible to build, by means of these operations, a large variety of new functions. These operations create **translations**, **dilations** and **reflections** of the original function.

9.1.1 Operations related to $y_1 = f(x) + a$

Example 9.1: Operations on graphs related to $y_1 = f(x) + a$

Consider $y = \ln x$. Then

$$y_1 = \ln(x) + a \quad (x > 0).$$



- The graph of y_1 is obtained from that of y by a **translation** of a units **upward** if $a > 0$, **downward** if $a < 0$.

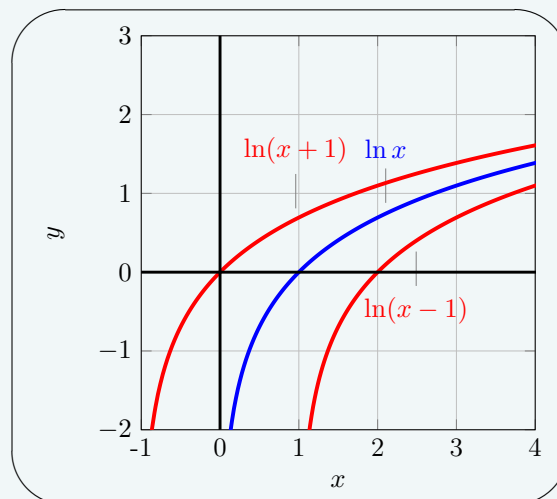
9.1.2 Operations related to $y_2 = f(x + a)$

Example 9.2: Operations on graphs related to $y_2 = f(x + a)$

Consider again $y = \ln x$. Then

$$y_2 = \ln(x + a).$$

Since $\ln x$ is defined for $x > 0$, $\ln(x + a)$ will be defined for $x + a > 0$, i.e., for $x > -a$. Since $\ln x = 0$ if $x = 1$, we have $\ln(x + a) = 0$ for $x + a = 1$, i.e., $x = 1 - a$.



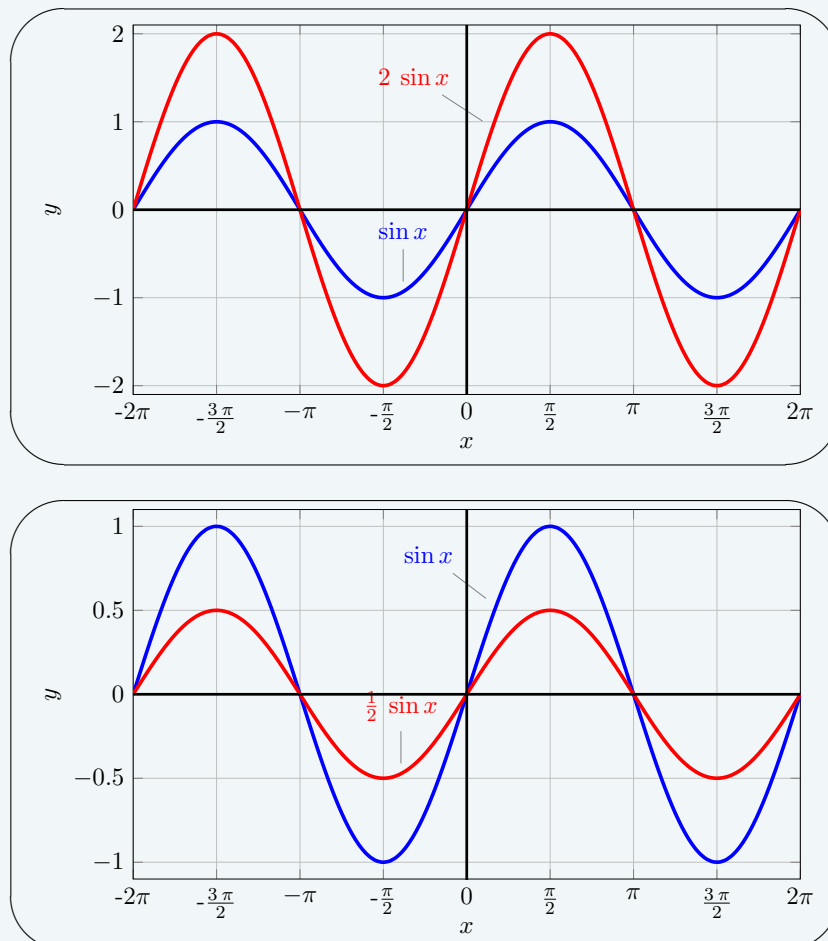
- The graph of y_2 is obtained from that of y by a **translation** of a units to the **left** if $a > 0$, to the **right** if $a < 0$.

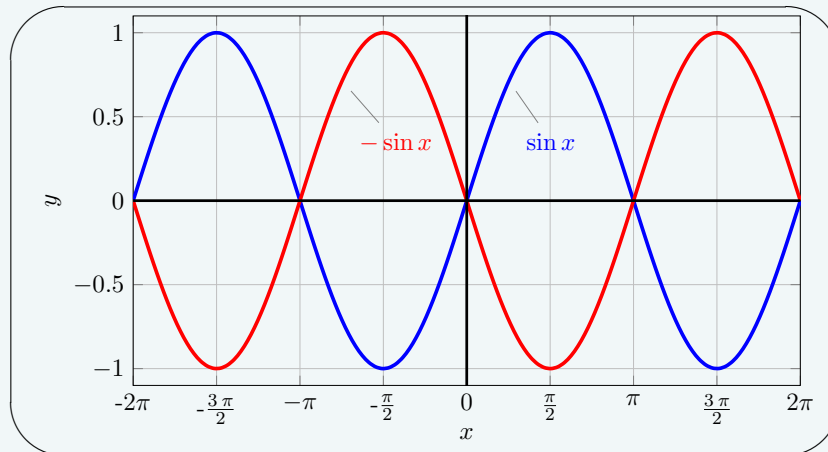
9.1.3 Operations related to $y_3 = k f(x)$

- The graph of $y_3 = k f(x)$ is obtained from that of f by multiplying all the ordinates $f(x)$ by k .
- In particular, if $k = -1$ the ordinates simply change sign, so that the graph of y_3 is symmetric to that of f with respect to the x -axis.
- Note that if $k > 1$, the graph is “**stretched**” in the **vertical direction**, dilating the **positive** ordinates **upward** and the **negative** ones **downward**.
- Conversely, if $0 < k < 1$ the graph “**shrinks**”, again in the vertical direction.
- Therefore, the operation of multiplying $f(x)$ by k has the geometric meaning of a **dilation** (if $|k| > 1$) or a **contraction** (if $|k| < 1$) along the y -axis, possibly accompanied by a **reflection with respect to the x -axis**, if $k < 0$.

Example 9.3: Operations on graphs related to $y_3 = k f(x)$

Consider $y = \sin x$, then $y_3 = k \sin(x)$.



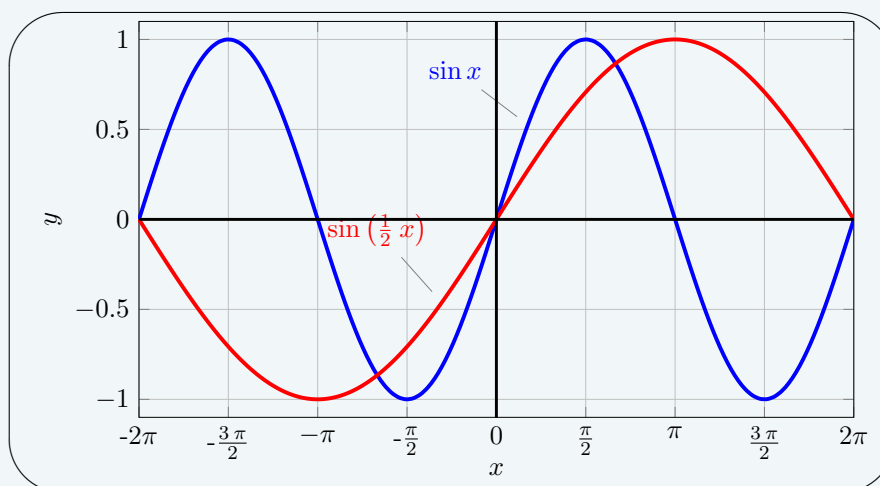


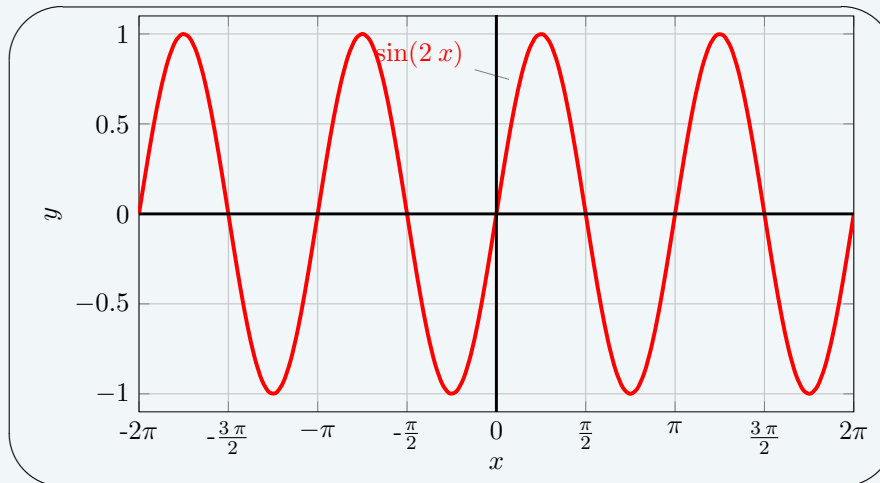
9.1.4 Operations related to $y_4 = f(kx)$

- The graph of $y_4 = f(kx)$ is obtained from that of $f(x)$ by a **change of scale** along the x -axis.
- If $k > 1$, kx grows faster than x , and therefore the graph of y_4 will be similar to that of f but with faster oscillations, that is, it will be “**compressed**” in the **horizontal direction** by a factor $\frac{1}{k}$.
- Similarly, if $0 < k < 1$ the graph will appear “**dilated**” in the **horizontal direction**, with gentler oscillations.
- If $k < 0$, in addition to a compression (if $|k| > 1$) or a dilation (if $|k| < 1$) along the x -axis, there will be a **reflection with respect to the y -axis**.

Example 9.4: Operations on graphs related to $y_4 = f(kx)$

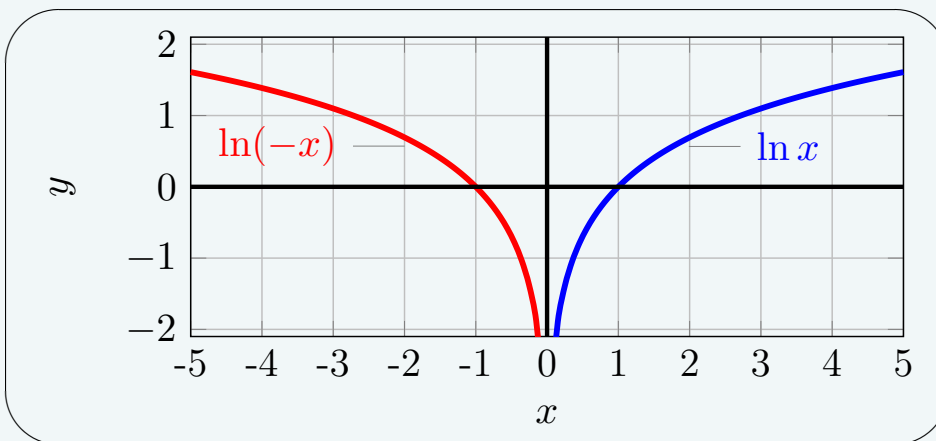
Consider $y = \sin x$, then $y_4 = \sin(kx)$. With $k = \frac{1}{2}$ and $k = 2$ we have:





Example 9.5: Operations on graphs related to $y_4 = f(kx)$

Consider $y = \ln x$, then $y_4 = \ln(kx)$. With $k = -1$ we have:



9.1.5 Operations related to $y_5 = |f(x)|$

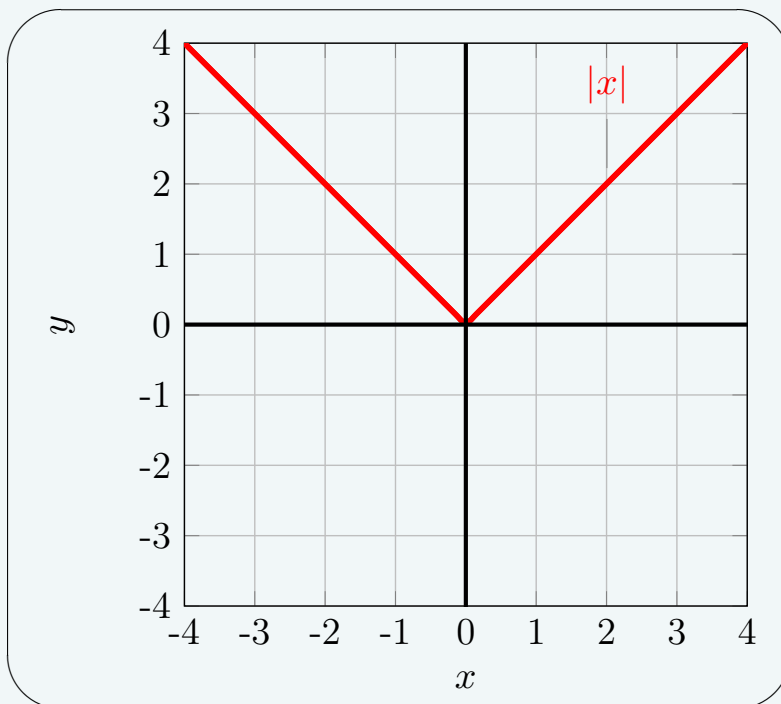
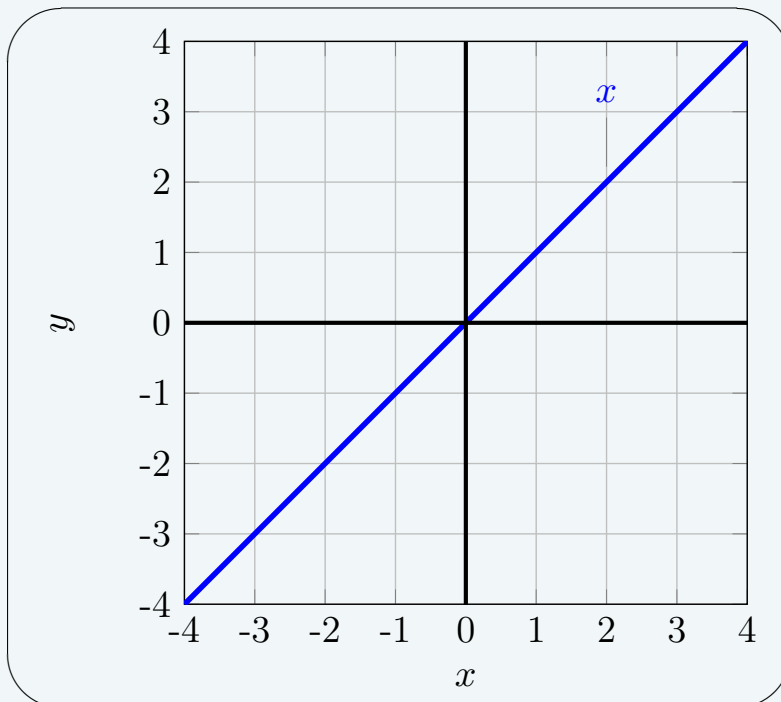
Absolute value of $f(x)$:

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

- When passing from the graph of $y = f(x)$ to that of $y_5 = |f(x)|$, the points with **non-negative ordinate remain unchanged**, while those with **negative ordinate are transformed into their symmetric points with respect to the x -axis**.
- The graph of $y_5 = |f(x)|$ is obtained from that of f by “flipping” symmetrically with respect to the x -axis the part of the graph of f that lies in the lower half-plane, and leaving the rest unchanged.

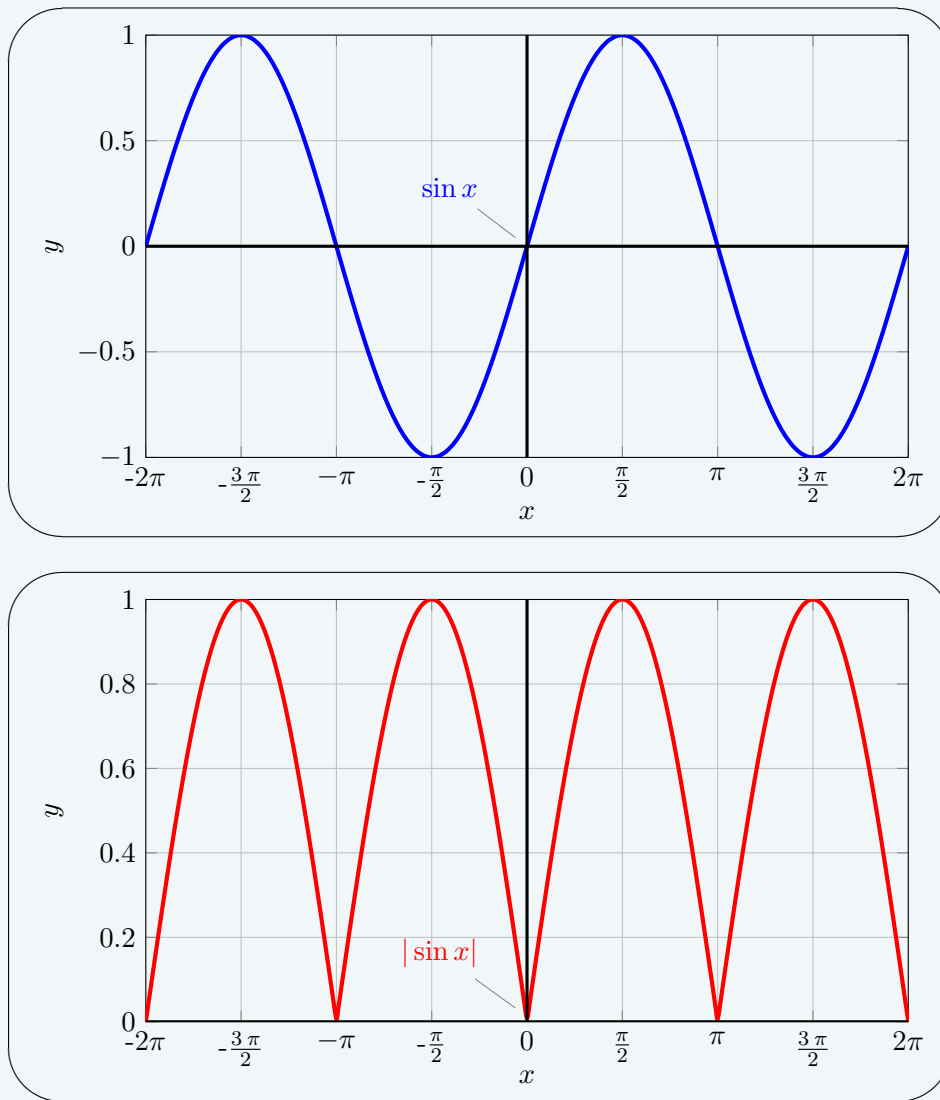
Example 9.6: Operations on graphs related to $y_5 = |f(x)|$

Consider $y = x$, then $y_5 = |x|$



Example 9.7: Operations on graphs related to $y_5 = |f(x)|$

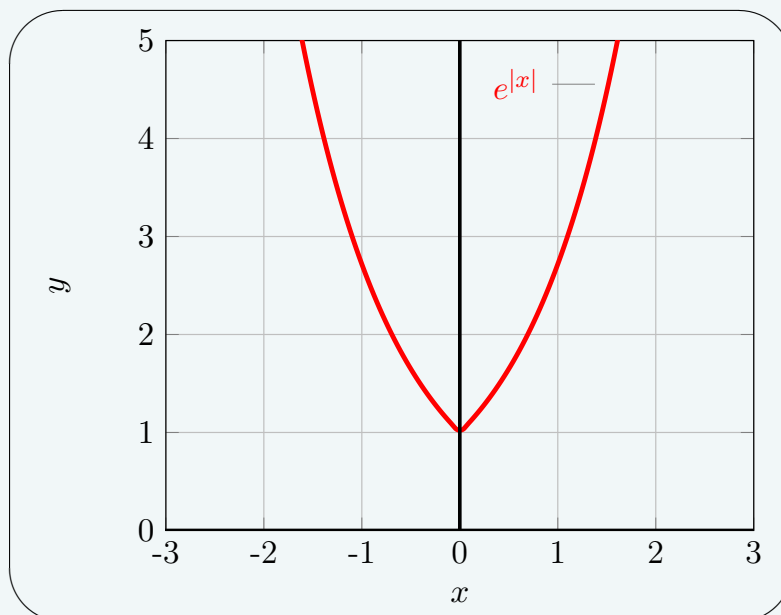
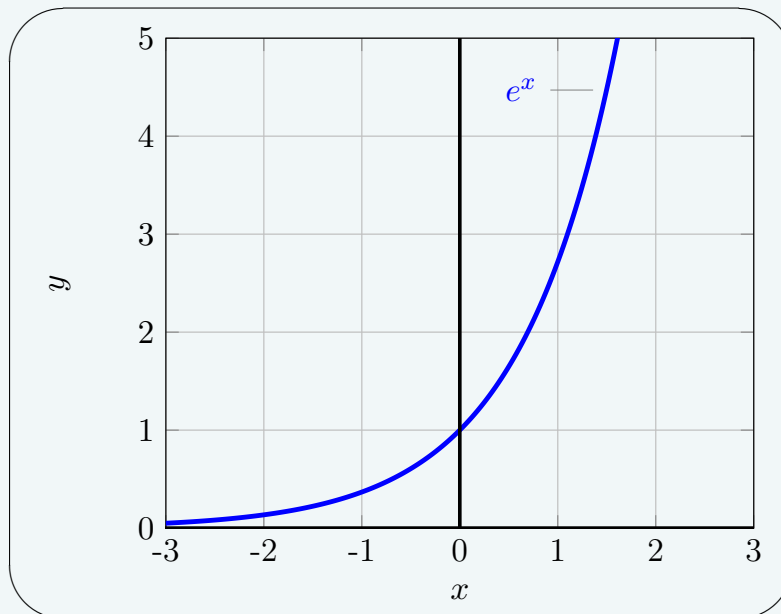
Consider $y = \sin x$, then $y_5 = |\sin x|$

**9.1.6 Operations related to $y_6 = f(|x|)$**

- Finally, to draw the graph of $y_6 = f(|x|)$, we observe that $|x| = x$ for $x \geq 0$, and hence in the right half-plane the two graphs coincide.
- We have $|-x| = |x|$, and hence y_6 is an even function, therefore symmetric with respect to the y -axis.
- Consequently, the graph of $y_6 = f(|x|)$ is drawn by **leaving the graph of f unchanged in the right half-plane and flipping it symmetrically with respect to the y -axis.**

Example 9.8: Operations on graphs related to $y_6 = f(|x|)$

Consider $y = e^x$, then $y_6 = e^{|x|}$



Composite functions

10.1 Composite functions

Definition 10.1: of composite function

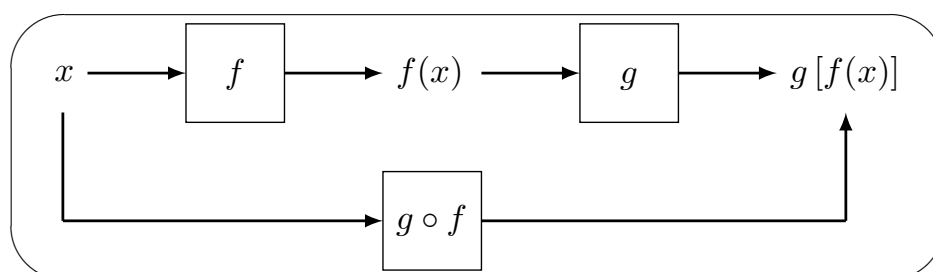
Given two functions:

$$f : E \rightarrow \mathbb{R} \quad \text{and} \quad g : F \rightarrow \mathbb{R},$$

if $f(E) \subseteq F$ (i.e., if for every $x \in E$ we have $f(x) \in F$), one can define the function $h : E \rightarrow \mathbb{R}$, the **composite** of f and g (in this order), denoted by the symbol $g \circ f$, through the formula

$$h(x) = (g \circ f)(x) = g[f(x)]$$

- The composition scheme is the following:

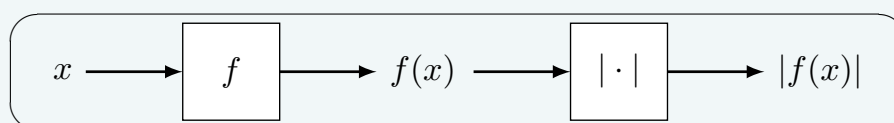


Example 10.1: of composite function

The function $x \mapsto |f(x)|$ is actually “composed” of two functions:

1. given x , we compute $f(x)$
2. once $f(x)$ is computed, we compute $|f(x)|$

This amounts to operating in series with two black boxes, the first corresponding to f , the second to its absolute value, according to the following scheme:



- It may happen that both compositions $(g \circ f)$ and $(f \circ g)$ are well defined, but in general

$$(f \circ g) \neq (g \circ f)$$

In other words, **the commutative property does not hold**.

Example 10.2: of composition of functions

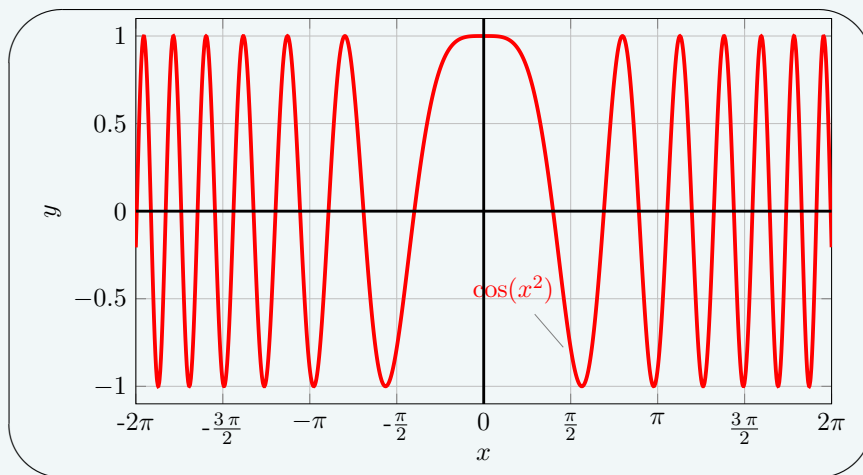
Consider the functions:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto x^2 \quad \text{and} \quad g : \mathbb{R} \rightarrow \mathbb{R}, \quad g : x \mapsto \cos x$$

that is, $f(x) = x^2$ and $g(x) = \cos x$.

1. Since g is defined on all of $\mathbb{R}$, $h = g \circ f$ is well defined on $\mathbb{R}$ and the following formula holds

$$h(x) = (g \circ f)(x) = g[f(x)] = \cos(x^2)$$



2. Since f is defined on all of $\mathbb{R}$, $k = f \circ g$ is well defined on $\mathbb{R}$ and the following formula holds

$$k(x) = (f \circ g)(x) = f[g(x)] = \cos^2 x$$



Example 10.3: of composition of functions

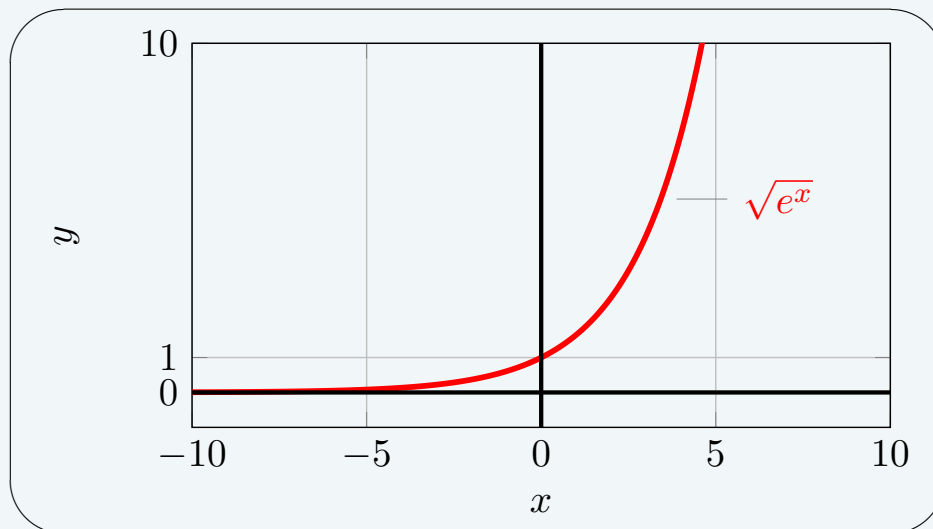
Consider the functions:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto e^x \quad g : [0, +\infty) \rightarrow \mathbb{R}, \quad g : x \mapsto \sqrt{x}$$

that is, $f(x) = e^x$ and $g(x) = \sqrt{x}$.

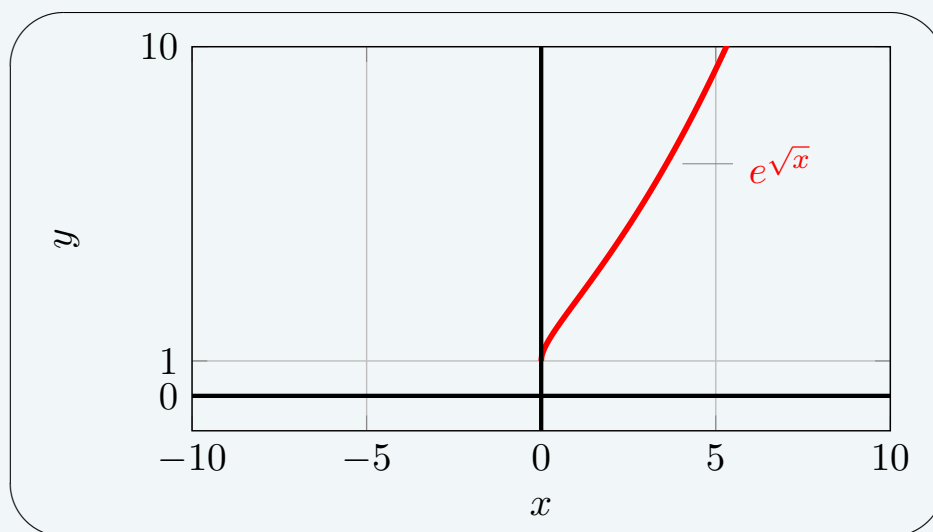
1. Since g is defined on $[0, +\infty)$ and $f(x) > 0$ for every $x \in \mathbb{R}$, $h = g \circ f$ is well defined on $\mathbb{R}$ and the following formula holds

$$h(x) = (g \circ f)(x) = g[f(x)] = \sqrt{e^x}$$



2. Since g is defined only on $[0, +\infty)$, $k = f \circ g$ is well defined only on $\mathbb{R}_+$ and the following formula holds

$$k(x) = (f \circ g)(x) = f[g(x)] = e^{\sqrt{x}}$$



- The composition operation can be extended to three or more factors. One can verify that

if the composition $(f \circ g) \circ r$ exists, then $f \circ (g \circ r)$ also exists and they are equal

$$(f \circ g) \circ r = f \circ (g \circ r)$$

In other words, **the associative property holds**.

- If a function $f : D \rightarrow \mathbb{R}$ is such that $f(D) \subseteq D$, then it can be composed with itself

$$f^2 = f \circ f \quad \text{that is} \quad f^2(x) = f[f(x)]$$

f^2 is called the **second iterate** of f . Similarly, f^n is called the **n -th iterate function** of f .

Inverse functions

11.1 Invertible functions and inverse functions

- Given a real function of a real variable $f : D \rightarrow \mathbb{R}$, for every input x in the domain D there exists a unique output $y = f(x)$ in the image of the domain $f(D)$.
- If, in addition, for every output $y = f(x) \in f(D)$ there exists a unique input $x \in D$, then f is called **invertible**, and it establishes a **one-to-one correspondence** between the domain D of f and the image of the domain $f(D)$.

Definition 11.1: of invertible function

A function $f : D \rightarrow \mathbb{R}$ is **invertible** on the domain D if one of the following equivalent conditions holds:

$$\forall x_1, x_2 \in D, \quad x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

$$\forall x_1, x_2 \in D, \quad f(x_1) = f(x_2) \implies x_1 = x_2$$

$$\forall y \in f(D), \quad \exists! x \in D : y = f(x)$$

that is, if f is **injective**.

WARNING: the definition of invertible function given here for real functions of a real variable is different from the usual one given for functions between arbitrary sets, which also requires surjectivity. In this case *injectivity is enough* because we define the inverse on the image of f and not on its whole codomain.

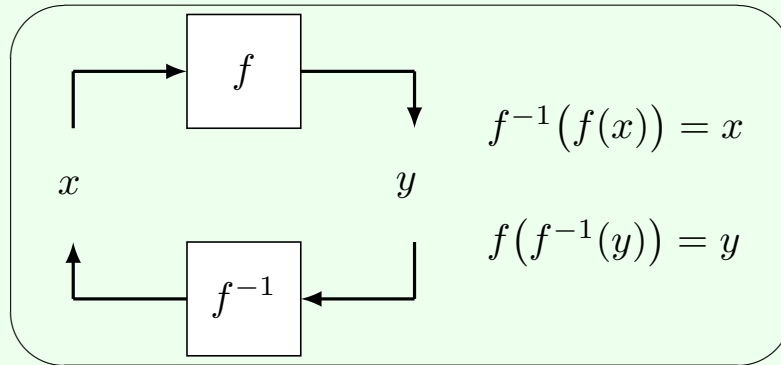
Definition 11.2: of inverse function

Given an invertible function $f : D \rightarrow \mathbb{R}$, the function that associates to each output $y \in f(D)$ the unique input $x \in D$ such that $f(x) = y$ is called the **inverse function** of f and is denoted by the symbol f^{-1} .

The pair f and f^{-1} is written as:

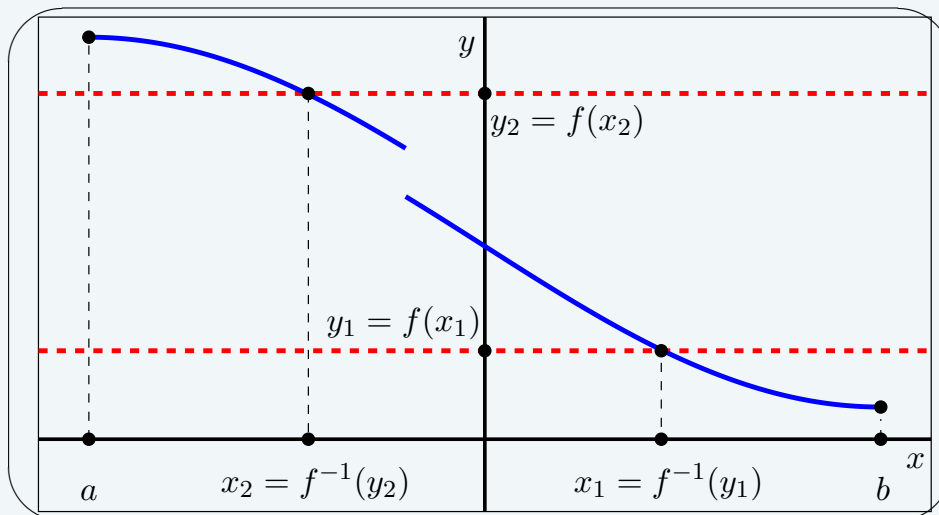
$$f : \begin{cases} y = f(x) \\ x \in D \end{cases} \quad f^{-1} : \begin{cases} x = f^{-1}(y) \\ y \in f(D) \end{cases} \quad (11.1)$$

The black box of f^{-1} works backwards with respect to that of f , according to the following scheme:



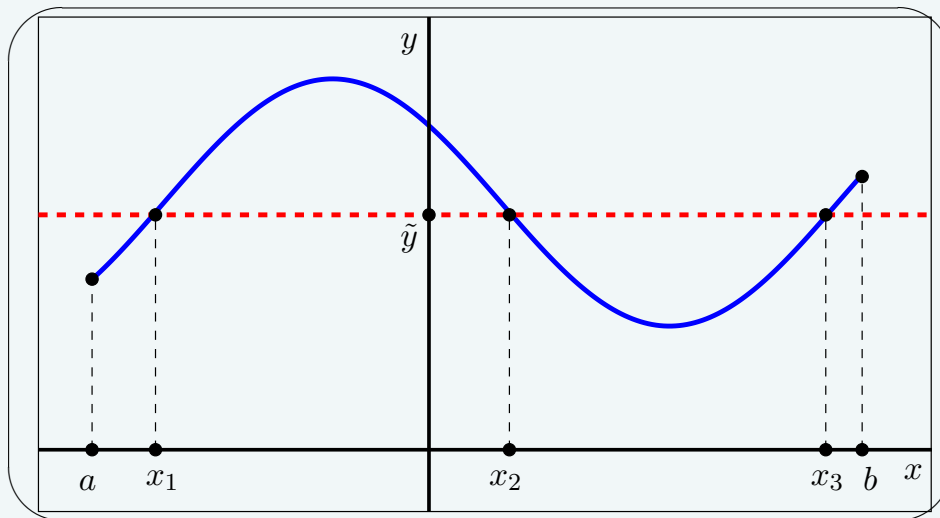
The invertibility condition is equivalent to requiring that the graph of f be intersected in at most one point by every line parallel to the x -axis.

Example 11.1: graph of an invertible function



Graph of a function that is **invertible** on $[a, b]$, since every line parallel to the x -axis either does not intersect the graph of f or intersects it in exactly one point.

Example 11.2: graph of a non-invertible function

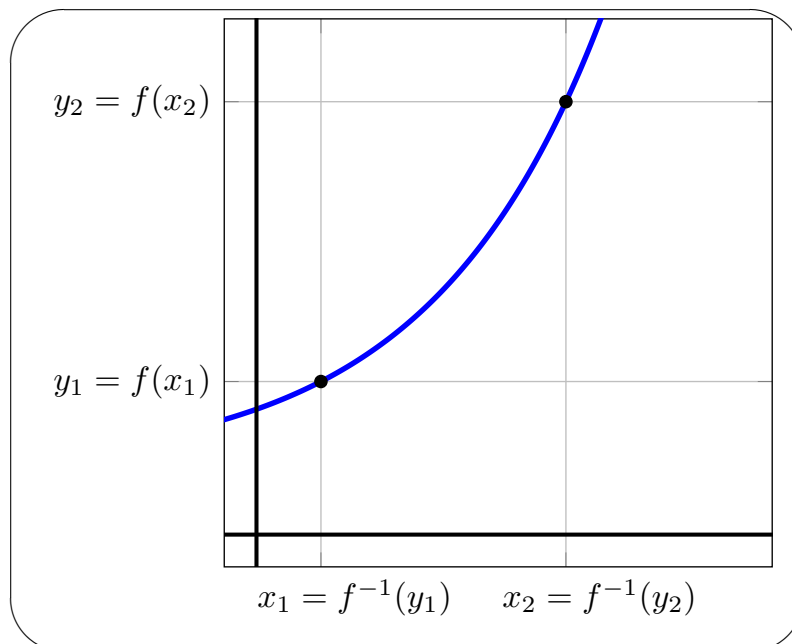


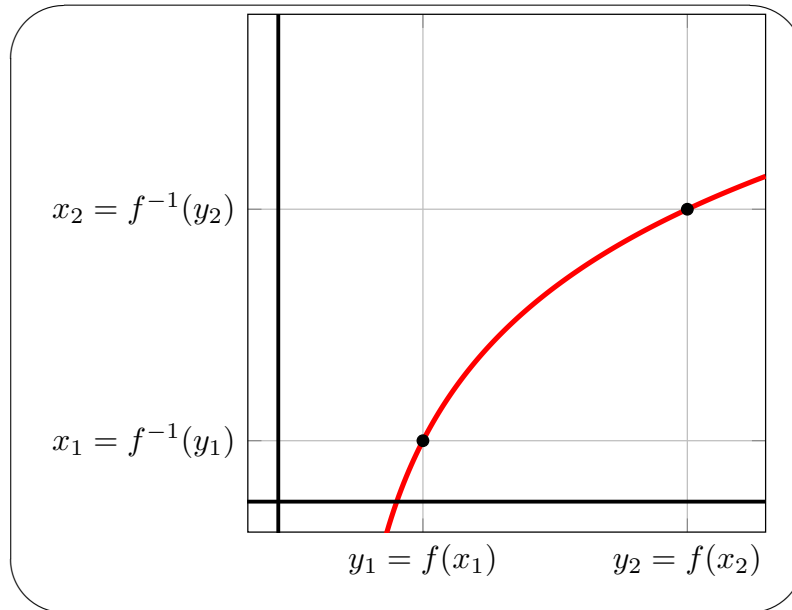
Graph of a function that is **not invertible** on $[a, b]$, since for the indicated value $\tilde{y}$ there are several values x (precisely x_1 , x_2 and x_3) whose image is $\tilde{y}$.

Theorem 11.3

If a function $f : D \rightarrow \mathbb{R}$ is strictly increasing (decreasing) on D , then it is invertible on D . Moreover, its inverse function is strictly increasing (decreasing).

For the strictly increasing case, graphically we have:





Proof. Consider the case of a function that is strictly increasing on D and two arbitrary values $x_1, x_2 \in D$ (the strictly decreasing case is analogous).

If $x_1 \neq x_2$, then either $x_1 < x_2$ or $x_1 > x_2$. Since f is strictly increasing, we have:

$$\text{if } x_1 < x_2, \text{ then } f(x_1) < f(x_2)$$

$$\text{if } x_1 > x_2, \text{ then } f(x_1) > f(x_2)$$

In both cases $f(x_1) \neq f(x_2)$, therefore f is invertible.

Consider the inverse function f^{-1} , that is, $x = f^{-1}(y)$, and let us prove that it is strictly increasing. Consider two arbitrary values $y_1, y_2 \in f(D)$, with $y_1 < y_2$.

If we had $f^{-1}(y_1) = x_1 \geq x_2 = f^{-1}(y_2)$, since f is strictly increasing we would have $y_1 = f(x_1) \geq f(x_2) = y_2$, that is, $y_1 \geq y_2$. This cannot happen, since it contradicts $y_1 < y_2$ (**contradiction**).

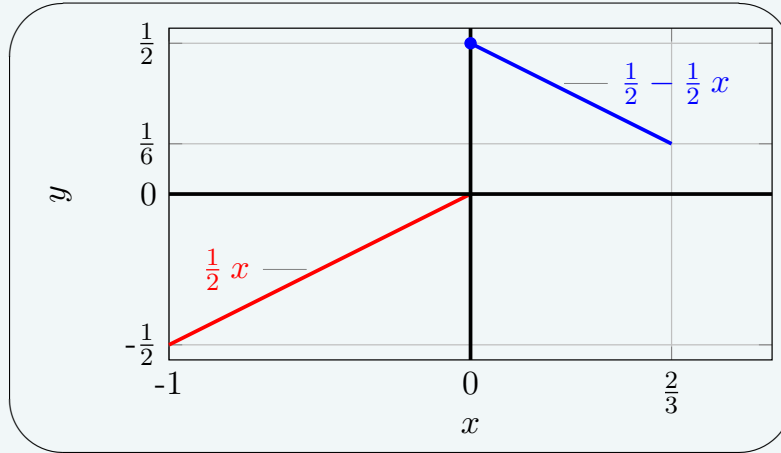
Consequently, we have $f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2)$, that is, $f^{-1}(y_1) < f^{-1}(y_2)$, and hence f^{-1} is strictly increasing. □

However, a function can be invertible even without being strictly increasing or decreasing.

Example 11.3: invertible function that is neither strictly increasing nor strictly decreasing

For example, the following piecewise-defined function is invertible:

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } -1 \leq x < 0, \\ \frac{1}{2} - \frac{1}{2}x & \text{if } 0 \leq x \leq \frac{2}{3} \end{cases}$$



- Two classes of functions that are certainly not invertible are:

1. even (symmetric) functions, since:

$$f(-x) = f(x)$$

2. periodic functions, since:

$$f(x + T) = f(x)$$

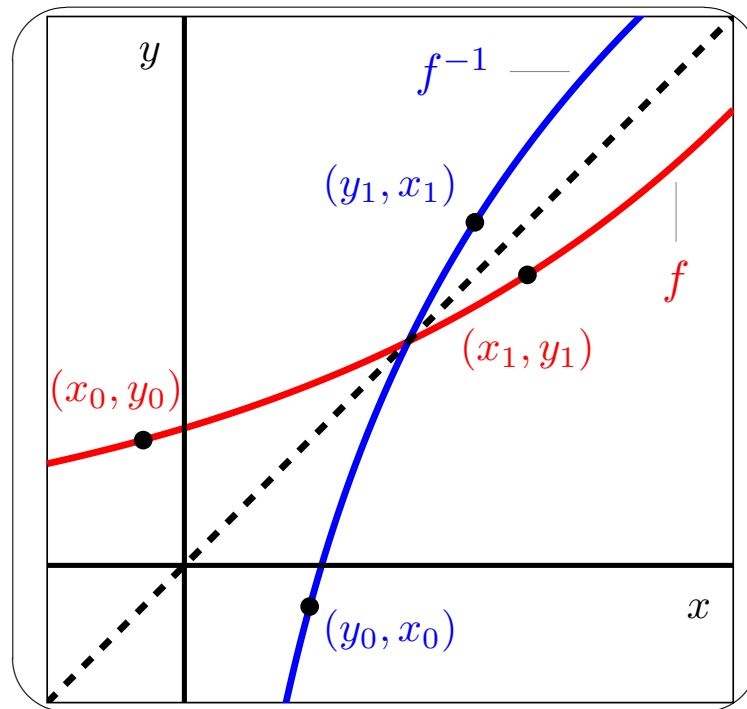
11.1.1 Graph of the inverse function

- The relations between an invertible function f and its inverse function f^{-1} :

$$f : \begin{cases} y = f(x) \\ x \in D \end{cases} \quad f^{-1} : \begin{cases} x = f^{-1}(y) \\ y \in f(D) \end{cases}$$

indicate that if the point (x_0, y_0) lies on the graph of f , then the point (y_0, x_0) lies on the graph of f^{-1} .

- Since the points (x_0, y_0) and (y_0, x_0) are **symmetric with respect to the bisector** with equation $y = x$, the graph of f^{-1} is obtained from that of f by symmetry with respect to the bisector.



If the analytic expression of f is known and f is invertible, the **analytic expression** of f^{-1} is found by trying to solve for x the equation:

$$f(x) = y \quad \text{that is, find the input } x \text{ that produces the output } y$$

Note that it is not always possible to find it, even in cases where the inverse function exists!

Given $a, b \in \mathbb{R}, a \neq 0$, we have:

$$f : \begin{cases} y = a x + b \\ x \in \mathbb{R} \end{cases} \quad f^{-1} : \begin{cases} x = \frac{y-b}{a} \\ y \in \mathbb{R} \end{cases}$$

Example 11.4: inverse function of an affine function

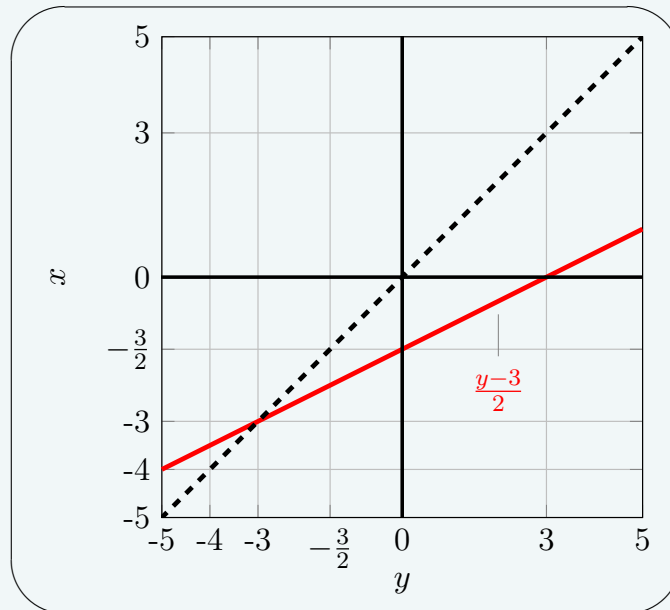
The function $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto 2x + 3$ is strictly increasing, hence invertible on $\mathbb{R}$, and the equation

$$2x + 3 = y \quad \text{solved for } x, \quad \text{that is } x = \frac{y-3}{2}$$

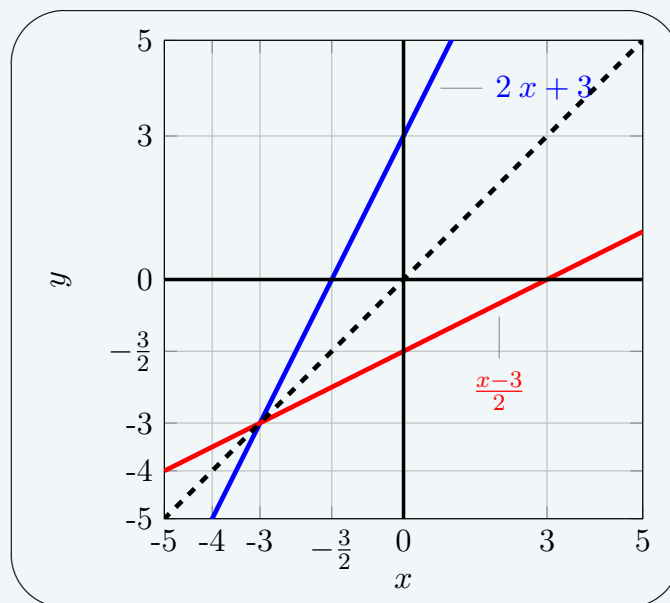
gives for f^{-1} the analytic expression:

$$f^{-1}(y) = \frac{y-3}{2}$$

that is, the function: $f^{-1} : \mathbb{R} \rightarrow \mathbb{R}, y \mapsto \frac{y-3}{2}$.



The graphs $y = 2x + 3$ and $y = \frac{x-3}{2}$ are symmetric with respect to the bisector $y = x$



We have:

$$f : \begin{cases} y = x^2 \\ x \geq 0 \end{cases} \quad f^{-1} : \begin{cases} x = \sqrt{y} \\ y \geq 0 \end{cases}$$

Example 11.5: inverse function of $f(x) = x^2$

The function $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2$ is strictly increasing on the interval $[0, +\infty)$, hence invertible there; the equation

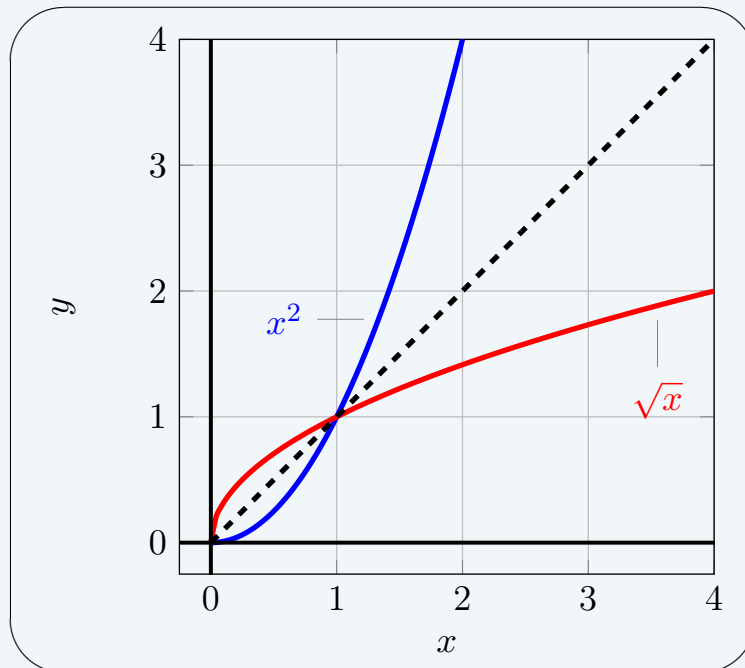
$$x^2 = y \text{ solved for } x, \text{ that is } x = \sqrt{y}$$

gives for f^{-1} the analytic expression

$$f^{-1}(y) = \sqrt{y}$$

that is, the function: $f^{-1} : [0, +\infty) \rightarrow \mathbb{R}, y \mapsto \sqrt{y}$.

The graphs $y = x^2$ and $y = \sqrt{x}$ are symmetric with respect to the bisector $y = x$



Given $a \in \mathbb{R}_+, a \neq 1$, we have:

$$f : \begin{cases} y = a^x \\ x \in \mathbb{R} \end{cases} \quad f^{-1} : \begin{cases} x = \log_a y \\ y > 0 \end{cases}$$

Example 11.6: inverse function of $f(x) = e^x$

The function $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto e^x$ is strictly increasing on all of $\mathbb{R}$ and hence invertible; the equation

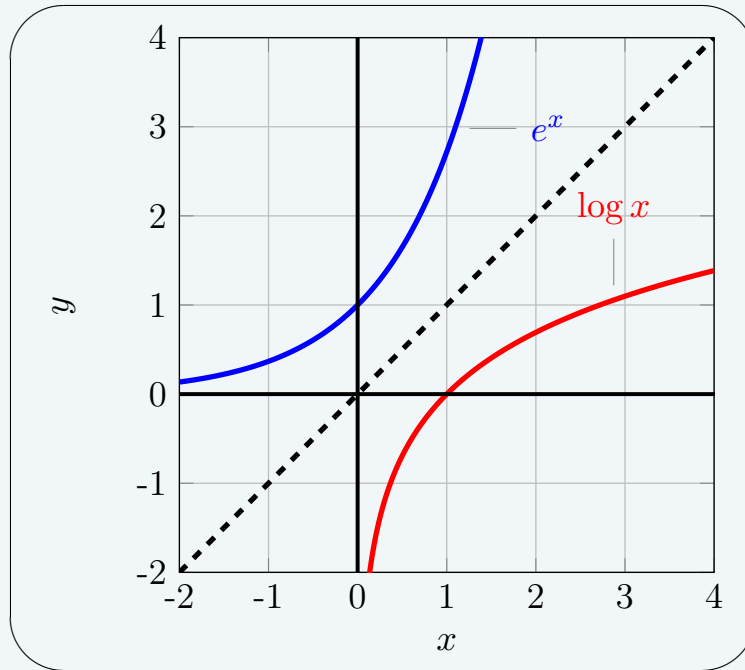
$$e^x = y \text{ solved for } x, \text{ that is } x = \log y$$

gives for f^{-1} the analytic expression

$$f^{-1}(y) = \log y$$

that is, the function: $f^{-1} : (0, +\infty) \rightarrow \mathbb{R}, y \mapsto \log y$.

The graphs $y = e^x$ and $y = \log x$ are symmetric with respect to the bisector $y = x$



- Now consider the function:

$$f(x) = x + e^x$$

Being the sum of two functions that are strictly increasing on all of $\mathbb{R}$, $f(x)$ is strictly increasing and hence invertible on all of $\mathbb{R}$ (see Theorem (11.3)). However, we would try in vain to solve the equation $x + e^x = y$ for x . In other words, f^{-1} exists, but we cannot write it explicitly.

11.1.2 Inverse power functions

Given $\alpha \in \mathbb{R}, \alpha \neq 0$, we have:

$$f : \begin{cases} y = x^\alpha \\ x > 0 \end{cases} \quad f^{-1} : \begin{cases} x = y^{\frac{1}{\alpha}} \\ y > 0 \end{cases}$$

If $\alpha > 0$ we have:

$$f : \begin{cases} y = x^\alpha \\ x \geq 0 \end{cases} \quad f^{-1} : \begin{cases} x = y^{\frac{1}{\alpha}} \\ y \geq 0 \end{cases}$$

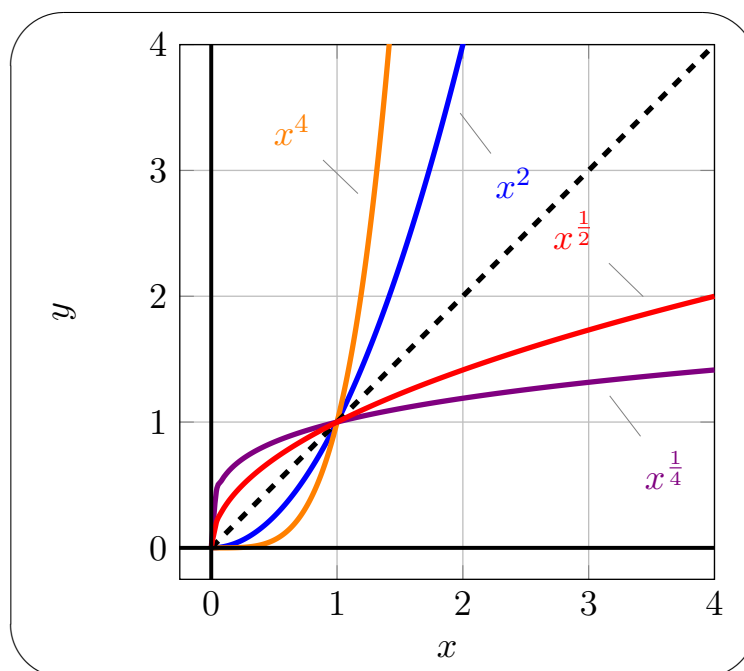
If $\alpha = \frac{m}{n} \in \mathbb{Q}$, with $m \in \mathbb{Z}, n \in \mathbb{N}_+$ odd and coprime, we have:

$$f : \begin{cases} y = x^{\frac{m}{n}} \\ x \in \mathbb{R} \end{cases} \quad f^{-1} : \begin{cases} x = y^{\frac{n}{m}} \\ y \in \mathbb{R} \end{cases}$$

- The **even powers**:

$$x^{2n} \quad \text{with } n = 1, 2, \dots$$

are not invertible on the whole real line, but only on the half-line $x \geq 0$.



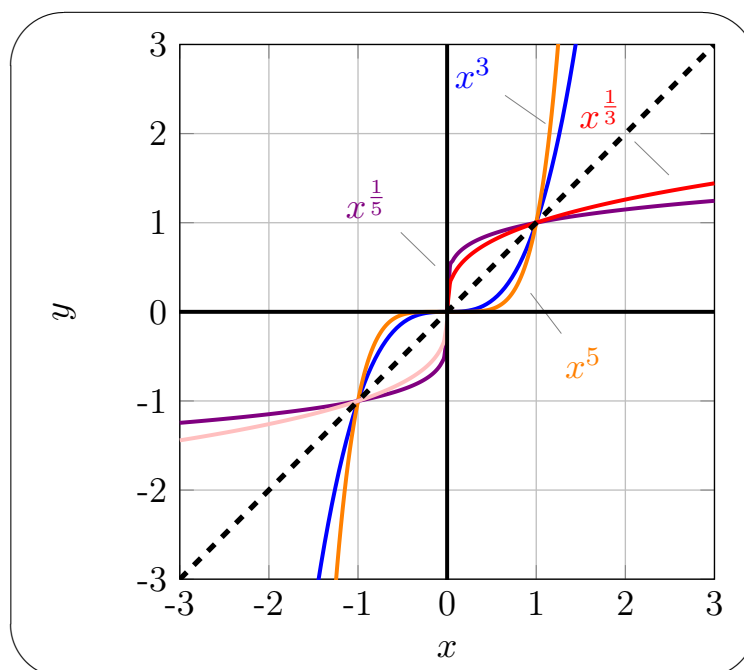
- The **odd powers**:

$$x^{2n+1} \quad \text{with } n = 1, 2, \dots$$

and the **powers with rational exponent**

$$x^{\frac{m}{n}} \quad \text{with } n, m \text{ positive odd integers}$$

being strictly increasing monotone functions, are invertible from $-\infty$ to $+\infty$.



11.1.3 Inverse trigonometric functions

- Being periodic, the trigonometric functions cannot be invertible. Indeed, for example, the equation

$$y = \sin x$$

has infinitely many solutions if $-1 \leq y \leq 1$ (the output y corresponds to infinitely many inputs), or has no real solutions if $|y| > 1$.

- To speak of inverse functions of sine, cosine and tangent, we will need to restrict ourselves to intervals on which these functions are strictly monotone and therefore invertible.
- An interval on which the sine function is invertible is $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Definition 11.4: of arcsine

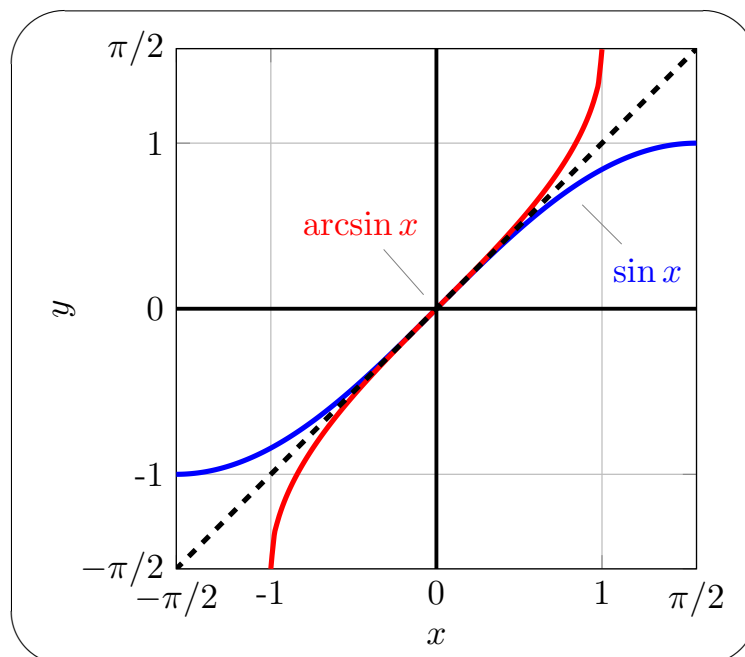
The inverse function of the sine on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is the **arcsine**:

$$f : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad f : y \mapsto \arcsin y$$

We have:

$$f : \begin{cases} y = \sin x \\ x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{cases} \quad f^{-1} : \begin{cases} x = \arcsin y \\ y \in [-1, 1] \end{cases} \quad (11.2)$$

- The graph of the arcsine is obtained from the arc of the sine curve restricted to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (strictly monotone), by symmetry with respect to the bisector $y = x$,



- An interval on which the cosine function is invertible is $[0, \pi]$.

Definition 11.5: of arccosine

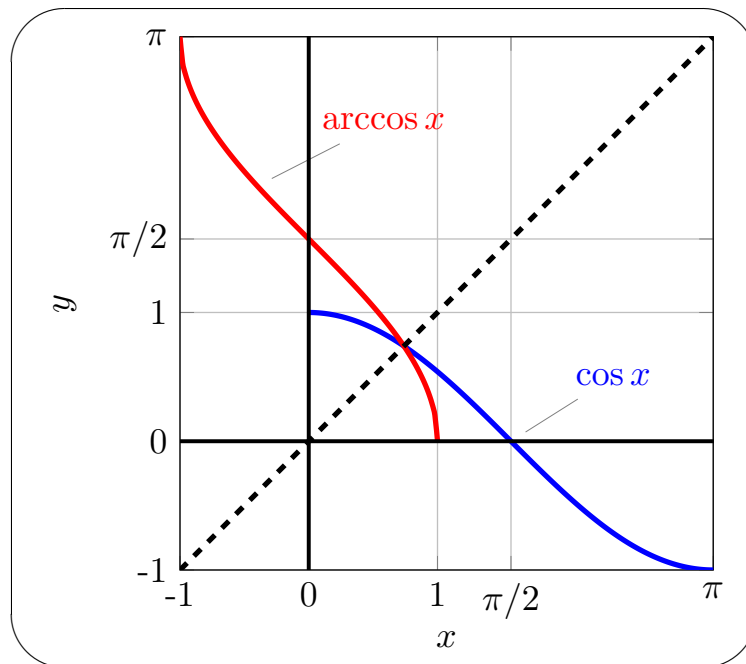
The inverse function of the cosine on the interval $[0, \pi]$ is the **arccosine**:

$$f : [-1, 1] \rightarrow [0, \pi], \quad f : y \mapsto \arccos y$$

We have:

$$f : \begin{cases} y = \cos x \\ x \in [0, \pi] \end{cases} \quad f^{-1} : \begin{cases} x = \arccos y \\ y \in [-1, 1] \end{cases} \quad (11.3)$$

- The graph of the arccosine is obtained from the arc of the cosine curve restricted to the interval $[0, \pi]$ (strictly monotone), by symmetry with respect to the bisector $y = x$,



- Note that on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ the tangent is strictly monotone and hence invertible. Its inverse function is called the arctangent (arctan), and it is defined on $\mathbb{R}$.

Definition 11.6: of arctangent

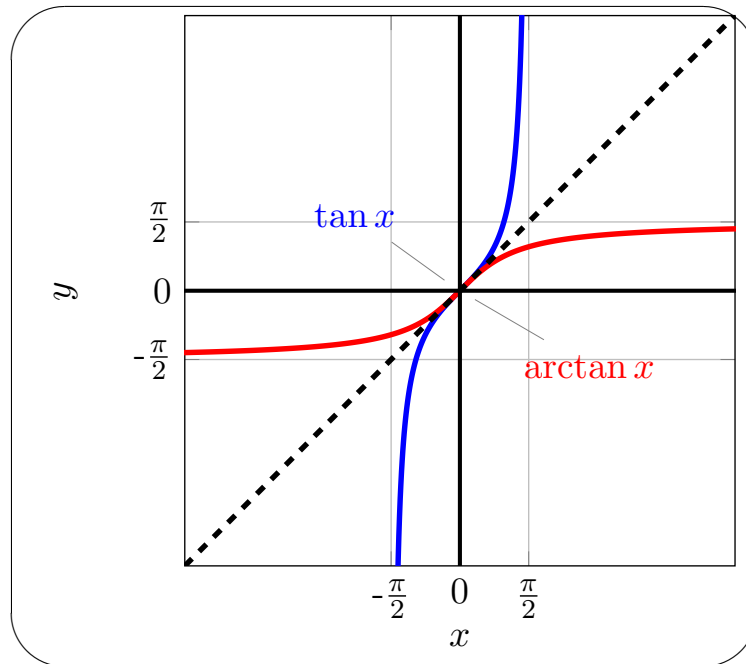
The inverse function of the tangent on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ is the **arctangent**:

$$f : \mathbb{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad f : y \mapsto \arctan y$$

We have:

$$f : \begin{cases} y = \tan x \\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{cases} \quad f^{-1} : \begin{cases} x = \arctan y \\ y \in \mathbb{R} \end{cases} \quad (11.4)$$

- The graph of the arctangent is obtained from that of the tangent restricted to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ (strictly monotone), by symmetry with respect to the bisector $y = x$,



- By means of the inverse trigonometric functions, we can express the solutions of a **trigonometric equation or inequality** when it involves angles that are not standard angles.

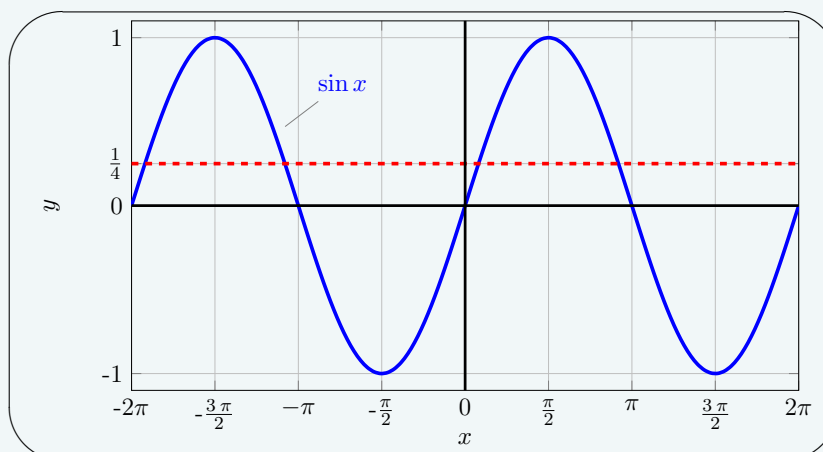
Example 11.7: Trigonometric equations/inequalities

- The solutions of:

$$\sin x = \frac{1}{4}$$

are:

$$(i) \quad x = \arcsin \frac{1}{4} + 2k\pi; \quad (ii) \quad x = \pi - \arcsin \frac{1}{4} + 2k\pi \quad (k \in \mathbb{Z})$$

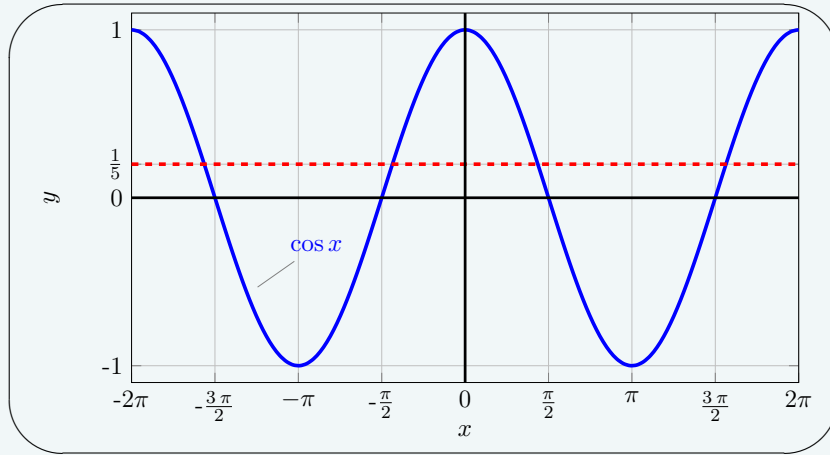


- The solutions of:

$$\cos x < \frac{1}{5}$$

are:

$$\arccos \frac{1}{5} + 2k\pi < x < 2\pi - \arccos \frac{1}{5} + 2k\pi \quad (k \in \mathbb{Z})$$



- The solutions of:

$$\tan x \geq 3$$

are:

$$\arctan 3 + k\pi \leq x < \frac{\pi}{2} + k\pi \quad (k \in \mathbb{Z})$$

11.1.4 Inverse hyperbolic functions

- Consider the **hyperbolic sine** function:

$$y = \operatorname{Sh} x = \frac{e^x - e^{-x}}{2}$$

It is defined and strictly increasing on all of $\mathbb{R}$, therefore it is invertible.

- To solve the equation for x , multiplying both sides by e^x we obtain:

$$\begin{aligned} 0 &= e^x y - e^x \frac{e^x - e^{-x}}{2} \\ &= 2e^x y - (e^x e^x - \underbrace{e^x e^{-x}}_{=e^{x-x}=e^0}) \\ &= 2ye^x - e^{2x} + 1 \\ &= e^{2x} - 2ye^x - 1 \end{aligned}$$

which is a quadratic equation in the unknown e^x . Setting $t = e^x$ we obtain

$$0 = t^2 - 2yt - 1$$

We obtain:

$$t = y \pm \sqrt{y^2 + 1} \quad \text{and substituting back} \quad e^x = y \pm \sqrt{y^2 + 1}$$

since $e^x > 0$, the solution with the minus sign must be discarded. Hence we are left with:

$$e^x = \underbrace{y + \sqrt{y^2 + 1}}_{>0, \forall y \in \mathbb{R}} \quad \text{and hence} \quad x = \log \left(y + \sqrt{y^2 + 1} \right)$$

- This is the analytic expression of the inverse function of $\text{Sh } x$, which is called the **inverse hyperbolic sine** (area hyperbolic sine), and is also denoted by SettSh . It is defined for every real y .

Definition 11.7: of inverse hyperbolic sine

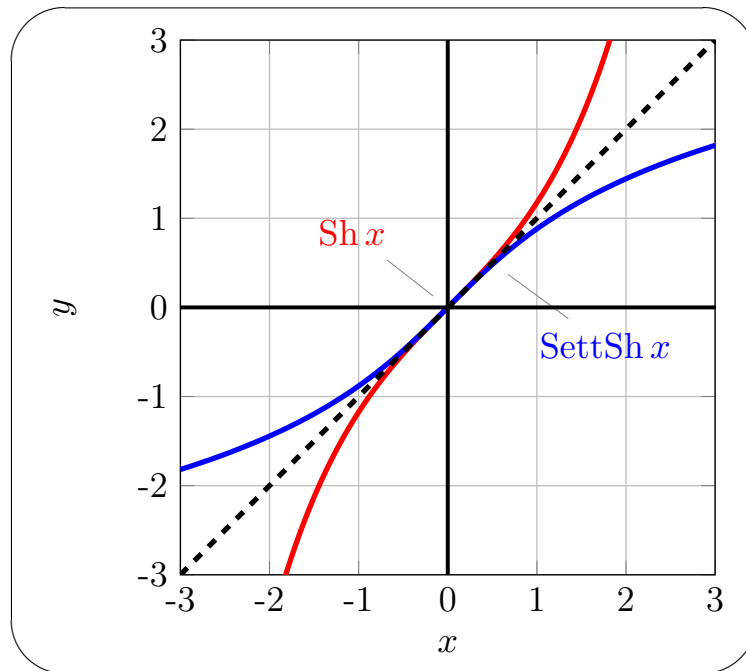
The inverse function of the hyperbolic sine is the **inverse hyperbolic sine**:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : y \mapsto \text{SettSh } y$$

We have:

$$f : \begin{cases} y = \text{Sh } x \\ x \in \mathbb{R} \end{cases} \quad f^{-1} : \begin{cases} x = \text{SettSh } y \\ y \in \mathbb{R} \end{cases} \quad (11.5)$$

- The graph of the inverse hyperbolic sine is obtained from that of the hyperbolic sine on $\mathbb{R}$ (strictly monotone), by symmetry with respect to the bisector $y = x$



- Consider the **hyperbolic cosine** function:

$$y = \text{Ch } x = \frac{e^x + e^{-x}}{2}$$

It is defined on all of $\mathbb{R}$, strictly increasing for $x \geq 0$, decreasing for $x \leq 0$. Therefore it is not invertible on all of $\mathbb{R}$.

- Its restriction to $x \geq 0$, however, is invertible. We want to determine the analytic expression of the inverse function of this restriction.
- Proceeding as before, we obtain:

$$e^x = y \pm \sqrt{y^2 - 1}$$

- This time both numbers $y \pm \sqrt{y^2 - 1}$ are positive; recall, however, that we are reasoning only for $x \geq 0$, which is equivalent to choosing the plus sign. Therefore:

$$x = \log \left(y + \sqrt{y^2 - 1} \right)$$

This is the analytic expression of the inverse function of $\text{Ch } x$, which is called the inverse hyperbolic cosine (area hyperbolic cosine), and is also denoted by SettCh . Note that it is defined for $y \geq 1$.

Definition 11.8: of inverse hyperbolic cosine

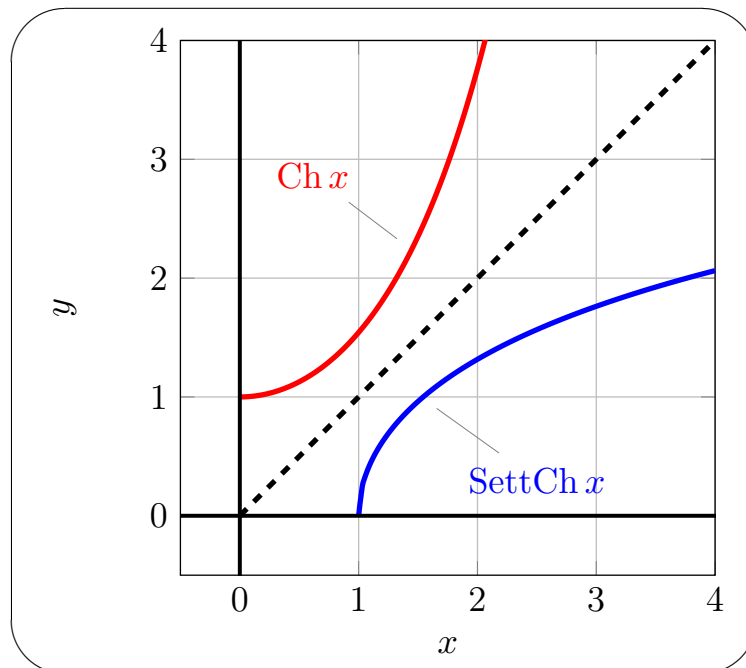
The inverse function of the hyperbolic cosine on the interval $[0, +\infty)$ is the **inverse hyperbolic cosine**:

$$f : [1, +\infty) \rightarrow [0, +\infty), \quad f : y \mapsto \text{SettCh } y$$

We have:

$$f : \begin{cases} y = \text{Ch } x \\ x \in [0, +\infty) \end{cases} \quad f^{-1} : \begin{cases} x = \text{SettCh } y \\ y \in [1, +\infty) \end{cases}$$

- The graph of the inverse hyperbolic cosine is obtained from that of the hyperbolic cosine on $[0, +\infty)$ (strictly monotone), by symmetry with respect to the bisector $y = x$.



Example 11.8: Hyperbolic equations

- The equation:

$$\operatorname{Sh} x = 2$$

has the unique solution:

$$x = \operatorname{SettSh} 2 = \log(2 + \sqrt{5})$$

- The equation:

$$\operatorname{Ch} x = 3$$

has two solutions:

$$x = \pm \operatorname{SettCh} 3 = \pm \log(3 + 2\sqrt{2})$$

