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Mathematical Analysis 1

Volume 1 — Logic and proof methods

The first volume: the language and the methods of reasoning in analysis. Basics of logic and proof techniques: logical symbols, universal implications, the law of the contrapositive, necessary and sufficient conditions, proofs by contradiction, logic and sets. The principle of induction and proofs by induction. Definitions, theorems with proofs, worked examples; the exercises of the part, with solutions, are in the volume Exercises 1.

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Basics of logic and proof techniques

1.1 Logical symbols

- a) the symbol “ $\forall$ ” is called the *universal quantifier* and is read “for every”, “for all”, “for each”
- b) the symbol “ $\exists$ ” is called the *existential quantifier* and is read “there exists”, “there exist”
- c) the symbol “ $\Rightarrow$ ” is called *logical implication* and is read “implies” or “if ... then”

- a) the symbol “ $:$ ” is read “such that”
- b) the symbol “ $\in$ ” is read “belongs to”
- c) the symbol “ $\notin$ ” is read “does not belong to”

- a) the symbol “ $\vee$ ” is called *logical disjunction* and is read “or”
- b) the symbol “ $\wedge$ ” is called *logical conjunction* and is read “and”
- c) the symbol “ $\neg$ ” is called *logical negation* and is read “not”.

1.2 Universal implications and proofs

Predicates (or properties) and **propositions** (or statements)

- Consider the following assertion:

“the natural number n is odd” (1.1)

and let us ask whether it is true. Obviously the answer is: “it depends on n ”. Indeed, in (1.1) the symbol n represents a variable that can take different values and make the assertion **true** or **false**.

A sentence of this kind is called a **predicate** (or *property*): its truth or falsity depends on the values of the **variable(s)** appearing in it.

- Now consider the following assertion:

$$\text{“for every natural number } n, \text{ if } n \text{ is odd then } n^2 \text{ is odd”} \quad (1.2)$$

which we can write more formally as follows:

$$\forall n \in \mathbb{N} \ (n \text{ odd} \Rightarrow n^2 \text{ odd}) \quad (1.3)$$

In this case we say that the variable n is not free, but bound by the *quantifier* $\forall$. As a consequence, (1.3) is true or false “once and for all” and is called a **proposition** (or *statement*).

In particular, the components of (1.3) are the set $\mathbb{N}$, two predicates defined on $\mathbb{N}$ given by

$$p(n) : \text{“}n \text{ odd”} \text{ and } q(n) : \text{“}n^2 \text{ odd”}$$

and the implication

$$p(n) \Rightarrow q(n).$$

Definition 1.1: of universal implication

In general, a statement involving a **set** A , **two predicates** $p(x)$ and $q(x)$ whose argument x ranges over A , and the **logical structure**:

$$\forall x \in A \ (p(x) \Rightarrow q(x)) \quad (1.4)$$

is called a **universal implication**.

Most **theorems** consist of **universal implications**, in which the predicate $p(x)$ plays the role of the **hypothesis** and the predicate $q(x)$ plays the role of the **thesis** (conclusion).

- In particular, (1.3) is a proposition (or a statement):

Proposition 1.2

$$\forall n \in \mathbb{N} \ (n \text{ odd} \Rightarrow n^2 \text{ odd}) \quad (1.5)$$

In this case one is easily convinced that the proposition is true, but how can we prove it rigorously?

- For example, is it enough to observe that 3 is odd and $3^2 = 9$ is odd to claim that the proposition is true? Certainly not, since the proposition requires the universal implication to hold for every natural number. However, there are infinitely many odd numbers: how can we prove a universal implication for infinitely many numbers?
- The key procedure is this: we consider a generic n satisfying the hypothesis (being odd) and we prove that n satisfies the thesis (its square is odd).

Let us see how to proceed to prove the previous proposition:

Proof. Let n be odd; we prove that then n^2 is odd.

Any odd number can be written in the form $2k + 1$, for a suitable $k \in \mathbb{N}$. We also observe that $2k$ is an even number for any $k \in \mathbb{N}$.

So let $n = 2k + 1$ be an odd number ($k \in \mathbb{N}$); we therefore need to write n^2 as an even integer plus one. We have:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since $2(2k^2 + 2k)$ is an even integer, n^2 is odd. □

To prove the correctness of a universal implication such as (1.4), we consider a generic x satisfying the hypothesis $p(x)$ and we try to prove that the thesis $q(x)$ is true.

- Let us now prove a similar relation for even numbers:

Proposition 1.3

$$\forall n \in \mathbb{N} \quad (n \text{ even} \Rightarrow n^2 \text{ even}) \quad (1.6)$$

Proof. Let n be even; we prove that then n^2 is even.

So let $n = 2k$ be an even number ($k \in \mathbb{N}$); we therefore need to write n^2 as an even integer. We have:

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

Since $2(2k^2)$ is an even integer, n^2 is even. □

1.2.1 Counterexamples

Counterexamples are an important technique for proving the **falsity** of a universal implication.

- Let us ask, for example, whether the following universal implication is true or false:

$$\forall n \in \mathbb{N} \quad (n \text{ prime} \Rightarrow n \text{ odd}) \quad (1.7)$$

A moment's reflection shows that **this proposition is false**. Indeed, the number 2 is prime but it is even.

To claim that a universal implication is true, a proof is needed (an example is not enough), whereas to prove that a universal implication is false, one counterexample is enough.

- The universal implication requires that every x satisfying the hypothesis also satisfies the thesis: therefore, if we find even a single example of x that satisfies the hypothesis but not the thesis, this means that the universal implication is false. Not “false in one case”, but simply “false”, because the universal implication is true or false once and for all.

Definition 1.4: of counterexample

*In general, an example that satisfies the hypothesis but not the thesis of a universal implication, and that therefore proves its falsity, is called a **counterexample**.*

- The formal proof that the previous universal implication is false is the following:

Proof. The number 2 is a counterexample to the universal implication: “For every natural number n , if n is prime then n is odd”.

□

The negation of the proposition

“for every $x \in A$, if $p(x)$ holds then $q(x)$ holds”

$$\forall x \in A \quad (p(x) \Rightarrow q(x))$$

is the proposition

“there exists $x \in A$ for which $p(x)$ holds and $q(x)$ does not hold”

$$\exists x \in A \quad (p(x) \wedge \neg q(x))$$

This particular x is a **counterexample**.

1.3 Law of contraposition

- It is an indirect proof technique

The universal implication

$$\forall x \in A \quad (p(x) \Rightarrow q(x)) \tag{1.8}$$

is logically equivalent to

$$\forall x \in A \quad (\text{not } q(x) \Rightarrow \text{not } p(x)) \tag{1.9}$$

The second implication is called the **contrapositive** of the first.

- For example: since we know that the universal implication (proposition 1.5) holds

$$\forall n \in \mathbb{N} \ (n \text{ odd} \Rightarrow n^2 \text{ odd})$$

the following proposition holds:

Proposition 1.5

$$\forall n \in \mathbb{N} \ (n^2 \text{ even} \Rightarrow n \text{ even}) \quad (1.10)$$

Proof. Let n^2 be even; we prove that then n is even. We prove (1.10) relying on the truth of (1.5).

If n is not even then it is odd and therefore n^2 is odd by (1.5). This case contradicts the hypothesis that n^2 is even (hence it cannot occur).

Consequently n is even and (1.10) is proved. □

- The reasoning used in the previous proof has general validity, and shows precisely that if (1.8) is true then (1.9) is true; moreover, if the second is true then the first is true (because “not not $p(x)$ ” is logically equivalent to $p(x)$), so the two are logically equivalent.

The equivalence between (1.8) and (1.9) is called the **law of contraposition**. It is a method of *indirect proof* that consists in proving (1.9) to show that (1.8) is true (it requires proving that the negation of the thesis implies the negation of the hypothesis).

When using the law of contraposition, one must be able to construct the **correct negation** of a given proposition or property.

- Given any two predicates or properties $p(x)$ and $q(x)$, we list schematically some **rules** for constructing the **negation** of a proposition or property.

The negation of

“for every $x \in A$, $p(x)$ holds”

$$\forall x \in A \ (p(x))$$

is

“there exists $x \in A$ for which $p(x)$ does not hold”

$$\exists x \in A \ (\neg p(x))$$

The negation of

“there exists $x \in A$ for which $p(x)$ holds”

$$\exists x \in A \ (p(x))$$

is

“for every $x \in A$, $p(x)$ does not hold”

$$\forall x \in A \quad (\neg p(x))$$

The negation of

$$\begin{aligned} & \text{"}p(x) \text{ holds and } q(x) \text{ holds"} \\ & (p(x) \wedge q(x)) \end{aligned}$$

is

$$\begin{aligned} & \text{"}p(x) \text{ does not hold or } q(x) \text{ does not hold"} \\ & (\neg p(x) \vee \neg q(x)) \end{aligned}$$

The negation of

$$\begin{aligned} & \text{"}p(x) \text{ holds or } q(x) \text{ holds"} \\ & (p(x) \vee q(x)) \end{aligned}$$

is

$$\begin{aligned} & \text{"}p(x) \text{ does not hold and } q(x) \text{ does not hold"} \\ & (\neg p(x) \wedge \neg q(x)) \end{aligned}$$

1.4 Sufficient conditions and necessary conditions

Definition 1.6: of sufficient condition

A **sufficient condition** is one that, if satisfied, guarantees the truth of the proposition.

Definition 1.7: of necessary condition

A **necessary condition** is one that must be satisfied for the proposition to be true.

Let $p(x)$ and $q(x)$ be any two predicates or properties. If $p(x)$ implies $q(x)$, formally:

$$p(x) \Rightarrow q(x)$$

then:

- $p(x)$ is a **sufficient condition** for $q(x)$
- $q(x)$ is a **necessary condition** for $p(x)$

- We have seen that the following universal implication is true (proposition (1.5)):

$$\forall n \in \mathbb{N} \quad (n \text{ odd} \Rightarrow n^2 \text{ odd})$$

Hence “ n odd” is a sufficient condition for “ n^2 odd” and “ n^2 odd” is a necessary condition for “ n odd”.

- Let us now prove that the following proposition is also true:

Proposition 1.8

$$\forall n \in \mathbb{N} \quad (n^2 \text{ odd} \Rightarrow n \text{ odd}) \quad (1.11)$$

Proof. Let n^2 be odd; we prove that then n is odd.

So let $n^2 = 2(2k^2 + 2k) + 1$ be an odd number ($k \in \mathbb{N}$), since $2(2k^2 + 2k)$ is an even number). We therefore need to write n as an even integer plus one. We have:

$$n = \sqrt{2(2k^2 + 2k) + 1} = \sqrt{4k^2 + 4k + 1} = \sqrt{(2k + 1)^2} = 2k + 1.$$

Since $2k + 1$ is an odd integer, n is odd.

□

In this way we have proved that “ n^2 odd” is a necessary and sufficient condition for “ n odd” and also that “ n odd” is a necessary and sufficient condition for “ n^2 odd”:

Proposition 1.9

$$\forall n \in \mathbb{N} \quad (n \text{ odd} \iff n^2 \text{ odd}) \quad (1.12)$$

- We have seen that the following universal implication is true (proposition (1.6)):

$$\forall n \in \mathbb{N} \quad (n \text{ even} \Rightarrow n^2 \text{ even})$$

Hence “ n even” is a sufficient condition for “ n^2 even” and “ n^2 even” is a necessary condition for “ n even”.

- Let us now prove that the following proposition is also true (direct proof, without using the law of contraposition as seen previously in (1.10)):

$$\forall n \in \mathbb{N} \quad (n^2 \text{ even} \Rightarrow n \text{ even})$$

Proof. Let n^2 be even; we prove that then n is even.

So let $n^2 = 2(2k^2)$ be an even number ($k \in \mathbb{N}$); we therefore need to write n as an even integer. We have:

$$n = \sqrt{2 (2 k^2)} = \sqrt{4 k^2} = 2 k.$$

Since $2 k$ is an even integer, n is even. □

In this way we have proved that “ n^2 even” is a necessary and sufficient condition for “ n even” and also that “ n even” is a necessary and sufficient condition for “ n^2 even”:

Proposition 1.10

$$\forall n \in \mathbb{N} \quad (n \text{ even } \iff n^2 \text{ even }) \quad (1.13)$$

Example 1.1: necessary and sufficient conditions

For example, for a square matrix of real numbers, the fact that its determinant is different from zero is a necessary and sufficient condition for it to be invertible.

Proposition 1.11

$$\forall n \in \mathbb{N} \quad (n \text{ prime } > 2 \Rightarrow n \text{ odd }) \quad (1.14)$$

Proof. Let $n > 2$ be a prime number; we prove that then n is odd.

If n is not odd, it is even. But no even number greater than two is prime, a fact that contradicts the hypothesis that n is prime and greater than two (hence this case cannot occur).

Consequently n is odd. □

- Hence “ n odd” is a necessary condition for “ n prime > 2 ” and “ n prime > 2 ” is a sufficient condition for “ n odd”.
- However, “ n odd” does not imply “ n prime > 2 ”, since for example the number 9 is not prime (counterexample). That is:

$$\forall n \in \mathbb{N} \quad (n \text{ odd } \nRightarrow n \text{ prime number greater than } 2)$$

Hence “ n odd” is a necessary but not sufficient condition for “ n prime > 2 ” and “ n prime > 2 ” is a sufficient but not necessary condition for “ n odd”.

Proposition 1.12

$$\forall n \in \mathbb{N} \quad (n \text{ divisible by } 6 \Rightarrow n \text{ even}) \quad (1.15)$$

Proof. Let n be a number divisible by six; we prove that then n is even.

If n is not even, it is odd. But no odd number is divisible by six, a fact that contradicts the hypothesis that n is divisible by six (hence this case cannot occur).

Consequently n is even. □

- Hence “ n even” is a necessary condition for “ n divisible by 6” and “ n divisible by 6” is a sufficient condition for “ n even”.
- However, “ n even” does not imply “ n divisible by 6”, since for example the number 2 is even but not divisible by six (counterexample). That is:

$$\forall n \in \mathbb{N} \quad (n \text{ even} \not\Rightarrow n \text{ divisible by } 6)$$

Hence “ n even” is a necessary but not sufficient condition for “ n divisible by 6” and “ n divisible by 6” is a sufficient but not necessary condition for “ n even”.

Example 1.2: necessary/sufficient but not sufficient/necessary conditions

Being a square implies being a rectangle:

$$\text{being a square} \Rightarrow \text{being a rectangle}$$

since all squares are rectangles.

Hence “being a rectangle” is a necessary condition for “being a square” and “being a square” is a sufficient condition for “being a rectangle”.

But being a rectangle does not imply being a square

$$\text{being a rectangle} \not\Rightarrow \text{being a square}$$

because there are rectangles that are not squares.

Hence “being a rectangle” is not a sufficient condition (but it is a necessary one) for “being a square” and “being a square” is not a necessary condition (but it is a sufficient one) for “being a rectangle”.

1.5 Proofs by contradiction

- It is an indirect proof technique

Definition 1.13: of proof by contradiction

In general, a **proof by contradiction** consists in assuming that the hypothesis of the theorem and the negation of the thesis are true, and deducing from these facts a contradiction of any kind.

- We illustrate **proof by contradiction** with the following theorem.

Theorem 1.14

There is no rational number whose square is 2.

Proof. Suppose, by contradiction, that there exists a number $r \in \mathbb{Q}$ such that $r^2 = 2$.

We can write $r = \frac{n}{m}$ with $n, m \in \mathbb{Z}$, $m \neq 0$.

Moreover, we can assume that the fraction $\frac{n}{m}$ is already reduced to lowest terms, that is, “simplified” (in other words: n, m have no common factors).

We then have the chain of implications:

$$\left(\frac{n}{m}\right)^2 = 2$$

$$n^2 = 2 m^2$$

so n^2 is even; but then by (1.13) n is also even and we can write $n = 2k$ for some $k \in \mathbb{Z}$.

Hence the relation $n^2 = 2 m^2$ can be rewritten as:

$$(2k)^2 = 2 m^2$$

$$4 k^2 = 2 m^2$$

$$m^2 = 2 k^2.$$

So m^2 is even. But then by (1.13) m is also even.

Therefore both n and m are even, and this is a **contradiction**, because we had assumed that the fraction $\frac{n}{m}$ had already been simplified.

The proof can be found in Euclid’s Elements (around 300 BC). □

1.6 Logic and sets

Sets are treated in detail in the chapter “[Sets](#)” of Part 2 (volume 2); here we only use their basic notions.

The language of logic and the language of sets are two sides of the same coin.

1.6.1 Logical implication and set inclusion

- There is a parallel between the relation of set inclusion and logical implication. To explain it, consider the universal implication:

$$\forall n \in \mathbb{N} \quad (n \text{ divisible by } 4 \Rightarrow n \text{ divisible by } 2) \quad (1.16)$$

If we denote by:

$$D_4 = \{ n \in \mathbb{N} : n \text{ is divisible by } 4 \} \quad \text{and} \quad D_2 = \{ n \in \mathbb{N} : n \text{ is divisible by } 2 \}$$

we can observe that the universal implication written above is equivalent to the assertion:

$$D_4 \subseteq D_2$$

Indeed, this inclusion means that every element belonging to D_4 also belongs to D_2 , that is, that every natural number divisible by 4 is also divisible by 2.

The universal implication:

$$\text{“for every } x \in A, \text{ if } p(x) \text{ holds then } q(x) \text{ holds”}$$

is equivalent to the set inclusion:

$$\{ x \in A : p(x) \text{ is true} \} \subseteq \{ x \in A : q(x) \text{ is true} \}.$$

1.6.2 Equality of sets and universal implications

- Proving the equality of two sets, i.e., $A = B$, requires proving two universal implications. Formally:

$$\forall x \quad (x \in A \Rightarrow x \in B) \quad \text{and} \quad \forall x \quad (x \in B \Rightarrow x \in A). \quad (1.17)$$

- Stating that $A \subsetneq B$ means stating that “Every element that belongs to A also belongs to B , and there exists an element of B that does not belong to A ”. Formally:

$$\forall x \quad (x \in A \Rightarrow x \in B) \quad \text{and} \quad \exists x \in B : x \notin A. \quad (1.18)$$

1.6.3 Set operations and logical operations

There is a relation between operations on sets and logical operations. Precisely:

1. **Set intersection** is defined by means of “and” (**logical conjunction**).
2. **Set union** is defined by means of “or” (**logical disjunction**).
3. **Set difference** and the **complement operation** are defined by means of “not” (**logical negation**).

- The distributive properties of union and intersection of sets:

Observation 1.15: distributive properties (sets)

Given three sets A, B and C we have:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

- They can be rewritten in terms of predicates by observing that the intersection symbol $\cap$ corresponds to the conjunction $\wedge$ (“and”) and that the union symbol $\cup$ corresponds to the disjunction $\vee$ (“or”).

Observation 1.16: distributive properties (predicates)

Given three predicates $p(x), q(x)$ and $r(x)$ we have:

$$p(x) \wedge (q(x) \vee r(x)) = (p(x) \wedge q(x)) \vee (p(x) \wedge r(x))$$

$$p(x) \vee (q(x) \wedge r(x)) = (p(x) \vee q(x)) \wedge (p(x) \vee r(x)).$$

- De Morgan’s laws:

Proposition 1.17: De Morgan’s laws (sets)

Given the sets $B, C \subseteq U$, we have

$$\overline{B \cap C} = \overline{B} \cup \overline{C}$$

$$\overline{B \cup C} = \overline{B} \cap \overline{C}$$

- They can be rewritten in terms of predicates by observing that the complement operation corresponds to the negation $\neg$ (“not”).

Proposition 1.18: De Morgan’s laws (predicates)

Given two predicates $q(x)$ and $r(x)$ we have:

$$\neg(q(x) \wedge r(x)) = \neg q(x) \vee \neg r(x)$$

$$\neg(q(x) \vee r(x)) = \neg q(x) \wedge \neg r(x)$$

Principle of mathematical induction

2.1 The principle of mathematical induction

- We now present a proof technique called proof by induction. This procedure can be applied to theorems with the following structure:

“ for every $n \in \mathbb{N}$, $n \geq n_0$, property $p(n)$ holds ”

- The number n_0 is the smallest integer for which we want the property to be true; if $n_0 = 0$ the theorem simply states that the property is true for every $n \in \mathbb{N}$.

A proof by induction consists of the following two steps:

1. We prove that $p(n)$ is true for $n = n_0$ (**base case of the induction**).
2. We prove that, if n is an arbitrary natural number $\geq n_0$, the fact that $p(n)$ is true implies that $p(n + 1)$ is true (**inductive step**).

We can then conclude that for every $n \geq n_0$, $p(n)$ is true.

- The validity of this proof technique, *intuitively*, is based on the following fact:
 1. By point 1, we know that $p(n_0)$ is true. Suppose for example $n_0 = 1$: we therefore know that $p(1)$ is true (this must be proved explicitly).
 2. By point 2, since $p(1)$ is true, $p(2)$ will be true: indeed, we have proved that for any n , if $p(n)$ is true then $p(n + 1)$ is also true. But then, since $p(2)$ is true, $p(3)$ will be true; but then $p(4)$ is true, $\dots$ and so on, hence $p(n)$ is true for every $n \geq 1$.
- In practice, the proof consists of two phases.
 1. Prove $p(n_0)$ directly;
 2. Assume $p(n)$ as a hypothesis (**inductive hypothesis**) and prove $p(n + 1)$.
- This is the delicate point, often subject to misunderstandings. “Assuming as a hypothesis” $p(n)$ does not mean assuming the thesis as a hypothesis. What must be proved is that:

“ for every $n \geq n_0$, if $p(n)$ is true then $p(n+1)$ is also true ”

and not

“ if $p(n)$ is true for every n , then $p(n+1)$ is also true”

2.2 Proofs based on the principle of mathematical induction

2.2.1 Bernoulli's inequality

Observation 2.1: Bernoulli's inequality

For every integer $n \geq 0$, $x \in \mathbb{R}$, $x \geq -1$, we have:

$$(1+x)^n \geq 1+n x \quad (2.1)$$

Proof. By induction on n .

- **Base case of the induction**

Let $n = 0$. Then the statement becomes:

$$(1+x)^0 \geq 1+0 x \quad \text{i.e.} \quad 1 \geq 1$$

which is clearly true.

- **Inductive step**

Suppose it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have:

$$(1+x)^n \geq 1+n x$$

and moreover we have

$$(1+x) \geq 0 \quad \text{since} \quad x \geq -1.$$

Then we can write:

$$\begin{aligned} (1+x)^{n+1} &= (1+x) \cdot (1+x)^n \\ &\geq (1+x) \cdot (1+n x) \\ &= 1+(n+1) x+n x^2 \\ &\geq 1+(n+1) x \end{aligned}$$

where in the last inequality we used the fact that $n x^2 \geq 0$.

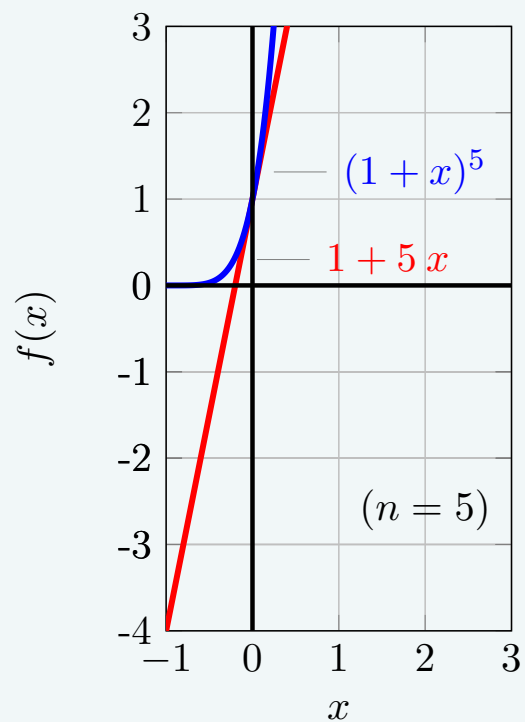
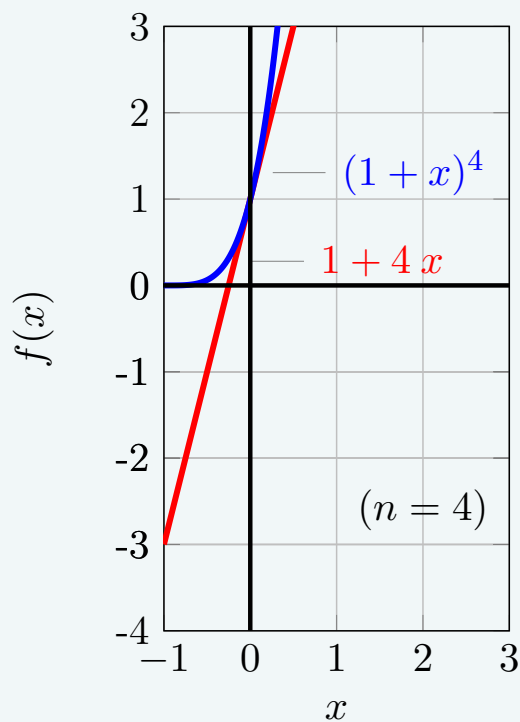
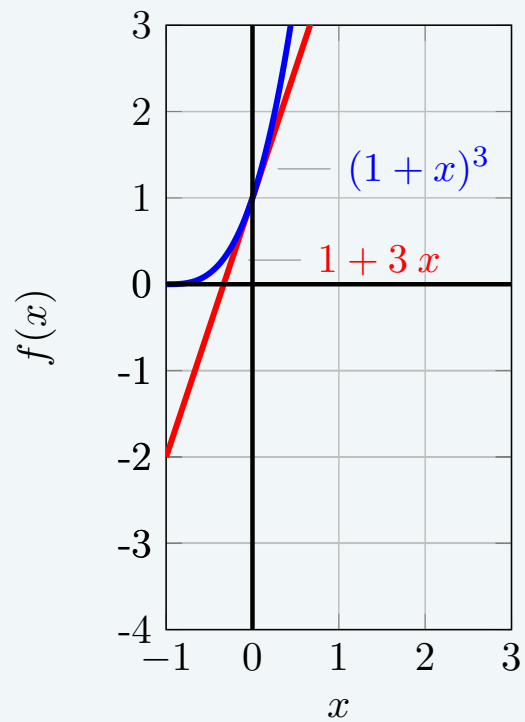
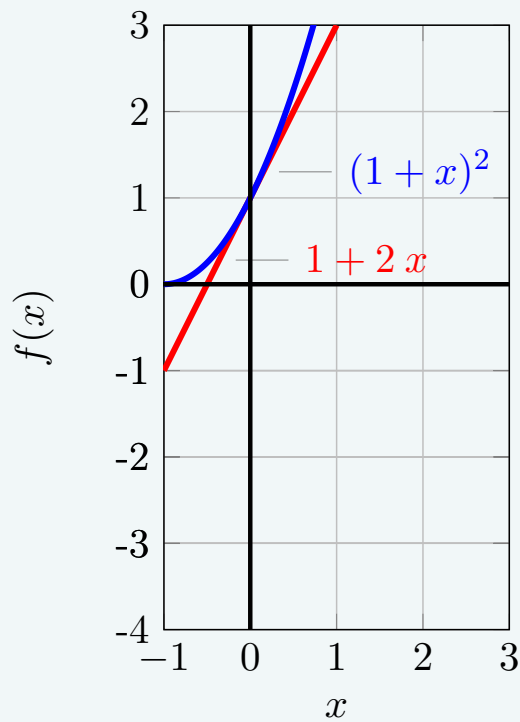
The chain of inequalities shows that, for $n+1$, we have

$$(1+x)^{n+1} \geq 1+(n+1) x$$

which is exactly the desired statement, for $n+1$.

□

Example 2.1: Bernoulli's inequality



2.2.2 Some important summations

Observation 2.2: sum of the first n natural numbers

For every integer $n \geq 1$, we have:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

Proof. By induction on n .

- **Base case of the induction**

Let $n = 1$. Then the statement becomes:

$$\sum_{k=1}^1 k = \frac{1(1+1)}{2} \quad \text{i.e.} \quad 1 = 1$$

which is clearly true.

- **Inductive step**

Suppose it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

Hence we can write:

$$\begin{aligned} \sum_{k=1}^{n+1} k &= \sum_{k=1}^n k + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2} \end{aligned}$$

which is exactly the desired statement, for $n+1$.

□

Observation 2.3: sum of the first n odd numbers

For every integer $n \geq 1$, we have:

$$\sum_{k=0}^{n-1} (2k+1) = n^2 \quad \text{or equivalently} \quad \sum_{k=1}^n (2k-1) = n^2$$

Proof. By induction on n .

- **Base case of the induction**

Let $n = 1$. Then the statement becomes:

$$\sum_{k=0}^0 (2k+1) = 1^2 \quad \text{i.e.} \quad 1 = 1$$

which is clearly true.

- **Passo induttivo**

Suppose it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have:

$$\sum_{k=0}^{n-1} (2k+1) = n^2$$

Hence we can write:

$$\sum_{k=0}^{(n+1)-1} (2k+1) = \sum_{k=0}^{n-1} (2k+1) + 2n+1 = n^2 + 2n+1 = (n+1)^2$$

which is exactly the desired statement, for $n+1$.

The second sum coincides with the first one, by an index shift ($k \rightarrow k+1$):

$$\sum_{k=1}^n (2k-1) = \sum_{k=0}^{n-1} (2(k+1)-1) = \sum_{k=0}^{n-1} (2k+1)$$

□

Observation 2.4: sum of the first n even numbers

For every integer $n \geq 1$, we have:

$$\sum_{k=1}^n 2k = n(n+1)$$

Proof. By induction on n .

- **Base case of the induction**

Let $n = 1$. Then the statement becomes:

$$\sum_{k=1}^1 2k = 1 \cdot 2 \quad \text{i.e.} \quad 2 = 2$$

which is clearly true.

- **Passo induttivo**

Suppose it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have:

$$\sum_{k=1}^n 2k = n(n+1)$$

Hence we can write:

$$\begin{aligned} \sum_{k=1}^{n+1} 2k &= \sum_{k=1}^n 2k + 2(n+1) = n^2 + n + 2(n+1) \\ &= n^2 + 2n + 1 + n + 1 = (n+1)^2 + (n+1) = (n+1)((n+1)+1) \end{aligned}$$

which is exactly the desired statement, for $n+1$.

□

2.2.3 Sum of the terms of the geometric progression

Observation 2.5: sum of the first n terms of the geometric progression ($a = 1$)

Given $q \in \mathbb{R}$, for every integer $n \geq 1$ we have (with the convention $0^0 = 1$ for the case $q = 0$):

$$\sum_{k=1}^n q^{k-1} = \begin{cases} \frac{q^n - 1}{q - 1} & \text{if } q \neq 1 \\ n & \text{otherwise} \end{cases} \quad (2.2)$$

Proof. By induction on n .

- **Base case of the induction**

Let $n = 1$. Then the statement becomes:

$$\sum_{k=1}^1 q^{k-1} = \frac{q^1 - 1}{q - 1} \quad \text{i.e.} \quad 1 = 1$$

which is clearly true.

- **Inductive step**

Suppose it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have:

$$\sum_{k=1}^n q^{k-1} = \frac{q^n - 1}{q - 1}$$

Hence we can write

$$\begin{aligned} \sum_{k=1}^{n+1} q^{k-1} &= \sum_{k=1}^n q^{k-1} + q^n = \frac{q^n - 1}{q - 1} + q^n \\ &= \frac{q^n - 1 + q^{n+1} - q^n}{q - 1} = \frac{q^{n+1} - 1}{q - 1} \end{aligned}$$

which is exactly the desired statement, for $n+1$.

□

2.2.4 Inequality of arithmetic and geometric means

Observation 2.6: inequality of arithmetic and geometric means

For every integer $n \geq 2$ and every vector $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}_{\geq 0}^n$, we have:

$$\frac{\sum_{i=1}^n a_i}{n} \geq \sqrt[n]{\prod_{i=1}^n a_i} \quad (2.3)$$

- The inequality (2.3) states that the *arithmetic mean* is greater than or equal to the *geometric mean*.
- If $a_i = a$ for every $i \in \{1, 2, \dots, n\}$, we have:

$$\frac{\sum_{i=1}^n a_i}{n} = \frac{n a}{n} = a \quad \text{and} \quad \sqrt[n]{\prod_{i=1}^n a_i} = \sqrt[n]{a^n} = a$$

and the arithmetic mean and the geometric mean are equal (this is the only case in which equality holds).

Proof. By induction on n .

- **Base case of the induction**

Let $n = 2$. Since both sides are non-negative, we can square them:

$$\begin{aligned} \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} &\iff \left(\frac{a_1 + a_2}{2} \right)^2 \geq a_1 a_2 \\ &\iff \frac{a_1^2 + 2 a_1 a_2 + a_2^2}{4} \geq a_1 a_2 \\ &\iff a_1^2 - 2 a_1 a_2 + a_2^2 \geq 0 \\ &\iff (a_1 - a_2)^2 \geq 0 \end{aligned}$$

which is clearly true.

• **Inductive step**

Suppose it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, the arithmetic mean of n non-negative real numbers is greater than or equal to their geometric mean. We need to prove that:

$$\frac{\sum_{i=1}^{n+1} a_i}{n+1} \geq \sqrt[n+1]{\prod_{i=1}^{n+1} a_i} \quad (\text{A})$$

Let α be the arithmetic mean of the $n+1$ non-negative real numbers:

$$\alpha = \frac{\sum_{i=1}^{n+1} a_i}{n+1} \quad (\text{B})$$

If $a_i = \alpha$ for every $i \in \{1, 2, \dots, n+1\}$, then (A) holds with equality. Otherwise, there is at least one value greater than α and at least one value smaller than α . Without loss of generality, we reorder the values so that:

$$a_n > \alpha \quad \text{and} \quad a_{n+1} < \alpha$$

Then we have:

$$a_n - \alpha > 0 \quad \text{and} \quad \alpha - a_{n+1} > 0 \implies (a_n - \alpha)(\alpha - a_{n+1}) > 0 \quad (\text{C})$$

From (B) we have:

$$(n+1) \alpha = \sum_{i=1}^{n+1} a_i \iff n \alpha = \sum_{i=1}^{n-1} a_i + \underbrace{a_n + a_{n+1} - \alpha}_{=y} \iff \alpha = \frac{\sum_{i=1}^{n-1} a_i + y}{n}$$

where $y = a_n + a_{n+1} - \alpha \geq a_n - \alpha > 0$, since $a_{n+1} \geq 0$. Hence α is also the arithmetic mean of the n non-negative numbers $a_1, a_2, \dots, a_{n-1}$ and y . By the inductive hypothesis, $\alpha^n \geq \left(\prod_{i=1}^{n-1} a_i \right) y$, and therefore:

$$\alpha^{n+1} = \alpha^n \alpha \geq \left(\prod_{i=1}^{n-1} a_i \right) y \alpha \quad (\text{D})$$

From (C) it follows that:

$$(a_n - \alpha)(\alpha - a_{n+1}) = \underbrace{(a_n + a_{n+1} - \alpha)}_{=y} \alpha - a_n a_{n+1} > 0$$

and therefore:

$$y \alpha > a_n a_{n+1} \quad (\text{E})$$

Substituting (E) into (D) (the product $\prod_{i=1}^{n-1} a_i$ is non-negative), we obtain:

$$\alpha^{n+1} \geq \prod_{i=1}^{n+1} a_i \iff \frac{\sum_{i=1}^{n+1} a_i}{n+1} \geq \sqrt[n+1]{\prod_{i=1}^{n+1} a_i}$$

that is, (A), which is exactly the desired statement, for $n+1$.

□

2.2.5 Fibonacci sequence

- The **Fibonacci sequence** is defined by the following two initial values and recurrence relation:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for every integer } n \geq 2. \quad (2.4)$$

- The first terms are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...: each term is the sum of the two preceding ones.
- The two solutions of the equation $x^2 - x - 1 = 0$ are called the **golden ratio** ϕ and the **conjugate of the golden ratio** $\hat{\phi}$:

$$\phi = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \hat{\phi} = \frac{1 - \sqrt{5}}{2}$$

Since they are solutions of $x^2 = x + 1$, the following two equalities hold:

$$\phi^2 = \phi + 1 \quad \text{and} \quad \hat{\phi}^2 = \hat{\phi} + 1 \quad (2.5)$$

- The following proposition gives a closed formula for F_n : the n -th term can be computed directly, without going through all the preceding terms.

Observation 2.7: closed formula of the Fibonacci sequence

For every integer $n \geq 0$, we have:

$$F_n = \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}} \quad (2.6)$$

Proof. By induction on n . Since the recurrence relation links F_{n+1} to the **two** preceding terms, the base case of the induction checks the statement for $n = 0$ and for $n = 1$, and the inductive step assumes it true for n and for $n - 1$.

- **Base case of the induction**

Let $n = 0$ and $n = 1$. Then the statement becomes:

$$F_0 = \frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = \frac{1 - 1}{\sqrt{5}} = 0$$

$$\text{and} \quad F_1 = \frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

which is clearly true.

- **Inductive step**

Suppose it is true for n and for $n - 1$, with $n \geq 1$, and let us prove it for $(n + 1)$. By the definition of the Fibonacci sequence we have $F_{n+1} = F_n + F_{n-1}$, while by the inductive hypothesis we have:

$$F_n = \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}} \quad \text{and} \quad F_{n-1} = \frac{\phi^{n-1} - \hat{\phi}^{n-1}}{\sqrt{5}}$$

Hence we can write:

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} = \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}} + \frac{\phi^{n-1} - \hat{\phi}^{n-1}}{\sqrt{5}} \\ &= \frac{(\phi^n + \phi^{n-1}) - (\hat{\phi}^n + \hat{\phi}^{n-1})}{\sqrt{5}} = \frac{\phi^{n-1}(\phi + 1) - \hat{\phi}^{n-1}(\hat{\phi} + 1)}{\sqrt{5}} \\ &= \frac{\phi^{n-1} \phi^2 - \hat{\phi}^{n-1} \hat{\phi}^2}{\sqrt{5}} = \frac{\phi^{n+1} - \hat{\phi}^{n+1}}{\sqrt{5}} \end{aligned}$$

where in the second-to-last equality we used the relations (2.5). This is exactly the desired statement, for $n + 1$.

□

- The Fibonacci sequence is studied in more detail in the chapter “[Fibonacci sequence](#)” of Part 2.