

SAPIENZA
UNIVERSITÀ DI ROMA

Mathematical Analysis 1

Exercises, volume 7 — Integrals

The worked exercises on integral calculus, with the solution right after each statement: antiderivatives, definite and improper integrals, Riemann sums and properties of the integral, integral functions and the fundamental theorem, definite integrals and areas, convergence of improper integrals. The sheets are by Fabio Furini and Gianluca Priori; many exercises are taken from M. Bramanti, *Esercitazioni di Analisi matematica 1*, Esculapio.

Fabio Furini
fabiofurini.github.io
Department of Computer, Control and Management Engineering — Sapienza University of Rome

Online version and downloadable material:
fabiofurini.github.io/mathematical-analysis-1/
github.com/fabiofurini/mathematical-analysis-1

Academic year 2026–2027

Contents

1 Antiderivatives 2

2 Integrals 16

3 Riemann integral and properties 34

4 Integral functions and the fundamental theorem 38

5 Definite integrals, areas and mean value 42

6 Improper integrals 45

Chapter 1

Antiderivatives

Exercise 1.1

Compute the following antiderivatives on the intervals where they are defined

$$\text{(a)} \int \left(2x + \sqrt[3]{x} - \frac{2}{\sqrt{x}}\right) dx, \quad \text{(b)} \int \frac{x - x^4}{\sqrt{x}} dx$$

$$\text{(c)} \int \tan^2 x \, dx, \quad \text{(d)} \int (3 - x)^5 dx$$

$$\text{(e)} \int \frac{1}{2 - 3x} dx, \quad \text{(f)} \int x^2 e^{x^3+1} dx$$

$$\text{(g)} \int x\sqrt{x^2 + 1} dx, \quad \text{(h)} \int \frac{\sin 2x}{1 + \sin^2 x} dx$$

$$\text{(i)} \int \frac{e^x}{e^x + 3} dx, \quad \text{(j)} \int \frac{1}{\tan x} dx$$

Solution

(a)

$$\int \left(2x + \sqrt[3]{x} - \frac{2}{\sqrt{x}}\right) dx = \int (2x + x^{1/3} - 2x^{-1/2}) dx$$

$$= x^2 + \frac{3}{4}x^{4/3} - 4x^{1/2} + c = x^2 + \frac{3}{4}\sqrt[3]{x^4} - 4\sqrt{x} + c$$

on any interval I such that $I \subset (0, +\infty)$.

Solution

(b)

$$\int \frac{x - x^4}{\sqrt{x}} dx = \int (x^{1/2} - x^{7/2}) dx =$$

$$\frac{2}{3}x^{3/2} - \frac{2}{9}x^{9/2} + c = \frac{2}{3}x\sqrt{x} - \frac{2}{9}x^4\sqrt{x} + c$$

on any interval I such that $I \subset (0, +\infty)$.

Solution

(c)

$$\int \tan^2 x \, dx = \int [(1 + \tan^2 x) - 1] dx = \tan x - x + c$$

on any interval I that does not contain points of the form $\pi/2 + k\pi$ with $k \in \mathbb{Z}$.

Solution

(d)

$$\int (3 - x)^5 dx = -\int (x - 3)^5 dx = -\frac{1}{6}(x - 3)^6 + c$$

on any interval I of $\mathbb{R}$.

Solution

(e)

$$\int \frac{1}{2 - 3x} dx = -\int \frac{1}{3x - 2} dx = -\frac{1}{3} \log |3x - 2| + c$$

on any interval I such that $2/3 \notin I$.

Solution

(f)

$$\int x^2 e^{x^3+1} dx = \frac{1}{3} e^{x^3+1} + c$$

on any interval I of $\mathbb{R}$.

Solution

(g)

$$\int x\sqrt{x^2+1}dx = \frac{1}{3}(x^2+1)^{3/2} + c = \frac{1}{3}(x^2+1)\sqrt{x^2+1} + c$$

on any interval I of $\mathbb{R}$.

Solution

(h)

$$\int \frac{\sin 2x}{1+\sin^2 x}dx = \int \frac{2\sin x\cos x}{1+\sin^2 x}dx = \log(1+\sin^2 x) + c$$

on any interval I of $\mathbb{R}$.

Solution

(i)

$$\int \frac{e^x}{e^x+3}dx = \log(e^x+3) + c$$

on any interval I of $\mathbb{R}$.

Solution

(j)

$$\int \frac{1}{\tan x}dx = \int \frac{\cos x}{\sin x}dx = \log|\sin x| + c$$

on any interval I that does not contain points of the form $k\pi$ with $k \in \mathbb{Z}$.

Exercise 1.2

Compute the following antiderivatives using integration by parts

$$\text{(a)} \int x\sin x\ dx, \quad \text{(b)} \int x^2\cos x\ dx$$

$$\text{(c)} \int \arcsin x\ dx, \quad \text{(d)} \int \sin^2 x\ dx$$

Solution

(a)

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + c$$

on any interval I of $\mathbb{R}$.

Solution

(b) Integrating by parts twice in a row, we have

$$\int x^2 \cos x \, dx = x^2 \sin x - 2 \int x \sin x \, dx =$$

$$x^2 \sin x + 2x \cos x - 2 \int \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + c$$

on any interval I of $\mathbb{R}$.

Solution

(c) Taking the constant function 1 as the factor to be integrated, we have

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = x \arcsin x + \sqrt{1-x^2} + c$$

on any interval $I \subset (-1, 1)$.

Solution

(d) Integrating by parts and using $\cos^2 x = 1 - \sin^2 x$, we obtain

$$\int \sin^2 x \, dx = -\sin x \cos x + \int \cos^2 x \, dx$$

$$= -\sin x \cos x + \int 1 \, dx - \int \sin^2 x \, dx$$

from which

$$\int \sin^2 x \, dx = -\sin x \cos x + x - \int \sin^2 x \, dx + c$$

hence, moving $\int \sin^2 x \, dx$ to the left-hand side,

$$2 \int \sin^2 x \, dx = -\sin x \cos x + x + c$$

and finally, still denoting by c the arbitrary constant $c/2$,

$$\int \sin^2 x \, dx = -\frac{1}{2} \sin x \cos x + \frac{1}{2} x + c$$

on any interval I of $\mathbb{R}$.

Exercise 1.3

Compute the following antiderivatives using integration by substitution

$$\text{(a)} \int \frac{e^x - 1}{e^x + 1} dx, \quad \text{(b)} \int \frac{\sqrt{x}}{1 + x} dx, \quad \text{(c)} \int x\sqrt{x + 1} dx$$

Solution

(a) Setting $y = e^x$, $y > 0, x \in \mathbb{R}$, we have $x = \log y$ and we write $dx = \frac{1}{y} dy$, hence

$$\int \frac{e^x - 1}{e^x + 1} dx = \int \frac{y - 1}{y(y + 1)} dy.$$

Using partial fractions, we obtain

$$\frac{y - 1}{y(y + 1)} = -\frac{1}{y} + \frac{2}{y + 1}$$

from which, taking into account that $y > 0$,

$$\int \frac{y - 1}{y(y + 1)} dy = -\log y + 2\log(y + 1) + c.$$

It follows, substituting back $y = e^x$,

$$\int \frac{e^x - 1}{e^x + 1} dx = -\log e^x + 2\log(e^x + 1) + c = -x + 2\log(e^x + 1) + c$$

on any interval I of $\mathbb{R}$.

Solution

(b) Setting $y = \sqrt{x}$, $x, y > 0$, we have $x = y^2$ and we write $dx = 2y\, dy$, hence

$$\int \frac{\sqrt{x}}{1 + x} dx = 2 \int \frac{y^2}{1 + y^2} dy.$$

From

$$\frac{y^2}{y^2 + 1} = \frac{y^2 + 1 - 1}{y^2 + 1} = 1 - \frac{1}{y^2 + 1}$$

we then have

$$2 \int \frac{y^2}{1 + y^2} dy = 2y - 2\arctan y + c.$$

It follows, substituting back $y = \sqrt{x}$,

$$\int \frac{\sqrt{x}}{1+x}dx = 2\sqrt{x} - 2\arctan\sqrt{x} + c$$

on any interval $I \subset (0, +\infty)$.

Solution

(c) Setting $y = \sqrt{x+1}$, $x > -1, y > 0$, we have $x = y^2 - 1$ and we write $dx = 2y\ dy$, hence

$$\int x\sqrt{x+1}dx = 2\int (y^2-1)y^2\ dy = 2\int (y^4 - y^2)dy = \frac{2}{5}y^5 - \frac{2}{3}y^3 + c.$$

It follows, substituting back $y = \sqrt{x+1}$,

$$\int x\sqrt{x+1}dx = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + c$$

on any interval $I \subset (-1, +\infty)$.

Exercise 1.4

Compute the following antiderivatives of rational functions

(a) $\displaystyle\int \frac{5}{x^2+2x+1}dx$, (b) $\displaystyle\int \frac{1}{x^2+x+1}dx$, (c) $\displaystyle\int \frac{x}{x^2+x+1}dx$, (d) $\displaystyle\int \frac{x^3+1}{x^2-3x+2}dx$

(e) $\displaystyle\int \frac{x}{x^3+3x^2+3x+1}dx$, (f) $\displaystyle\int \frac{1}{x^4-1}dx$, (g) $\displaystyle\int \frac{x+1}{x^3-x^2}dx$, (h) $\displaystyle\int \frac{1}{x^3+1}dx$

Solution

(a) The given integral is immediate:

$$\int \frac{5}{x^2+2x+1}dx = \int \frac{5}{(x+1)^2}dx = -\frac{5}{x+1} + c$$

on any interval I such that $-1 \notin I$.

Solution

(b) Writing the (irreducible) denominator as a sum of squares:

$$\int \frac{1}{x^2+x+1}dx = \int \frac{1}{(x+1/2)^2+3/4}dx = \frac{2}{\sqrt{3}}\arctan\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]+c$$

on any interval I of $\mathbb{R}$.

Solution

(c) Using also the previous exercise:

$$\int \frac{x}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{1}{x^2+x+1}dx =$$

$$\frac{1}{2}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\arctan\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]+c$$

on any interval I of $\mathbb{R}$.

Solution

(d) We perform polynomial long division:

$$\frac{x^3+1}{x^2-3x+2} = x+3+\frac{7x-5}{x^2-3x+2}.$$

Setting up the partial fractions

$$\frac{7x-5}{x^2-3x+2} = \frac{7x-5}{(x-1)(x-2)} = \frac{A}{x-1}+\frac{B}{x-2},$$

we obtain

$$A=-2,\ B=9.$$

In conclusion:

$$\int \frac{x^3+1}{x^2-3x+2}dx = \int \left(x+3-\frac{2}{x-1}+\frac{9}{x-2}\right)dx =$$

$$\frac{1}{2}x^2+3x-2\log|x-1|+9\log|x-2|+c$$

on any interval I such that $1\notin I$, $2\notin I$.

Solution

(e) We set up the partial fractions:

$$\frac{x}{x^3+3x^2+3x+1} = \frac{x}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

We obtain

$$A = 0,\ B = 1,\ C = -1.$$

In conclusion

$$\int \frac{x}{x^3+3x^2+3x+1}dx = \int \left(\frac{1}{(x+1)^2} - \frac{1}{(x+1)^3}\right) dx$$

$$= -\frac{1}{x+1} + \frac{1}{2}\frac{1}{(x+1)^2} + c$$

on any interval I such that $-1 \notin I$.

Solution

(f) We set up the partial fractions:

$$\frac{1}{x^4-1} = \frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}.$$

We obtain

$$A = \frac{1}{4},\ B = -\frac{1}{4},\ C = 0,\ D = -\frac{1}{2}.$$

In conclusion

$$\int \frac{1}{x^4-1}dx = \int \left(\frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x^2+1}\right) dx$$

$$= \frac{1}{4}\log|x-1| - \frac{1}{4}\log|x+1| - \frac{1}{2}\arctan x + c$$

on any interval I such that $\pm 1 \notin I$.

Solution

(g) We set up the partial fractions:

$$\frac{x+1}{x^3-x^2} = \frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

We obtain

$$A = -2,\ B = -1,\ C = 2.$$

In conclusion

$$\int \frac{x+1}{x^3-x^2}dx = \int \left(-\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1}\right) dx$$

$$= -2\log|x| + \frac{1}{x} + 2\log|x-1| + c$$

on any interval I such that $0, 1 \notin I$.

Solution

(h) We set up the partial fractions:

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

We obtain

$$A = \frac{1}{3},\ B = -\frac{1}{3},\ C = \frac{2}{3}.$$

Hence

$$\int \frac{1}{x^3+1}dx = \int \left(\frac{1}{3}\frac{1}{x+1} + \frac{1}{3}\frac{-x+2}{x^2-x+1}\right) dx$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{1}{x^2-x+1}dx$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log(x^2-x+1) + \frac{1}{2}\int \frac{1}{(x-1/2)^2+3/4}dx$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\arctan\left[\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right] + c$$

on any interval I such that $-1 \notin I$.

Exercise 1.5

Compute the following antiderivatives

$$\text{(a)} \int \frac{e^x+1}{e^{2x}+1}dx, \quad \text{(b)} \int x^2\log(x+1)dx, \quad \text{(c)} \int \frac{\tan x}{\sin^2 x - \cos^2 x}dx$$

$$\text{(d)} \int \frac{x+\sqrt{x}}{2+\sqrt{x}}dx, \quad \text{(e)} \int \frac{1}{\cos x}dx, \quad \text{(f)} \int \sqrt{4-x^2}dx$$

Solution

(a) With the substitution $y = e^x$, $y > 0$, $x \in \mathbb{R}$, we have $x = \log y$, we write $dx = \frac{1}{y}dy$, and

we obtain

$$\int \frac{e^x+1}{e^{2x}+1}dx = \int \frac{y+1}{y(y^2+1)}dy.$$

We set up the partial fractions:

$$\frac{y+1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1}.$$

We have

$$A = 1,\ B = -1,\ C = 1,$$

from which, taking into account also that $y > 0$,

$$\int \frac{y+1}{y(y^2+1)}dy = \int \left(\frac{1}{y} - \frac{y}{y^2+1} + \frac{1}{y^2+1}\right) dy$$

$$= \log y - \frac{1}{2}\log(y^2+1) + \arctan y + c.$$

Then, substituting back $y = e^x$, we have

$$\int \frac{e^x+1}{e^{2x}+1}dx = x - \frac{1}{2}\log(e^{2x}+1) + \arctan e^x + c$$

on any interval I of $\mathbb{R}$.

Solution

(b) Integrating by parts and then computing the integral of a rational function, we have

$$\int x^2\log(x+1)dx = \frac{1}{3}x^3\log(x+1) - \frac{1}{3}\int \frac{x^3}{x+1}dx$$

$$= \frac{1}{3}x^3\log(x+1) - \frac{1}{3}\int (x^2-x+1)dx + \frac{1}{3}\int \frac{1}{x+1}dx$$

$$= \frac{1}{3}x^3\log(x+1) - \frac{1}{9}x^3 + \frac{1}{6}x^2 - \frac{1}{3}x + \frac{1}{3}\log(x+1) + c$$

on any interval $I \subset (-1, +\infty)$.

Solution

(c) From $\tan x = \frac{\sin x}{\cos x}$ and $\sin^2 x = 1 - \cos^2 x$, we have

$$\int \frac{\tan x}{\sin^2 x - \cos^2 x}dx = \int \frac{\sin x}{\cos x(1-2\cos^2 x)}dx.$$

Now setting $y = \cos x$, we write $dy = -\sin x\, dx$ and we obtain

$$\int \frac{\sin x}{\cos x(1-2\cos^2 x)}dx = \frac{1}{2}\int \frac{1}{y(y-\sqrt{1/2})(y+\sqrt{1/2})}dy.$$

Using partial fractions, we then have

$$\frac{1}{y(y-\sqrt{1/2})(y+\sqrt{1/2})} = -\frac{2}{y} + \frac{1}{y-\sqrt{1/2}} + \frac{1}{y+\sqrt{1/2}},$$

from which

$$\frac{1}{2}\int \frac{1}{y(y-\sqrt{1/2})(y+\sqrt{1/2})} = -\log|y| + \frac{1}{2}\log|y^2-1/2| + c.$$

In conclusion,

$$\int \frac{\tan x}{\sin^2 x - \cos^2 x}dx = -\log|\cos x| + \frac{1}{2}\log|\cos^2 x - 1/2| + c$$

on any interval I that does not contain points of the form $x = \pi/2 + k\pi$ or of the form $x = \pi/4 + k\pi/2$ with k an integer.

Solution

(d) With the substitution $y = \sqrt{x}$, $x, y > 0$, we have $x = y^2$, we write $dx = 2y\,dy$ and we obtain

$$\int \frac{x+\sqrt{x}}{2+\sqrt{x}}dx = 2\int \frac{y^3+y^2}{y+2}dy.$$

Integrating the rational function, we have

$$2\int \frac{y^3+y^2}{y+2}dy = 2\int (y^2-y+2)dy - 8\int \frac{1}{y+2}dy$$

$$= \frac{2}{3}y^3 - y^2 + 4y - 8\log(y+2) + c.$$

Substituting back $y = \sqrt{x}$, we conclude

$$\int \frac{x+\sqrt{x}}{2+\sqrt{x}}dx = \frac{2}{3}x\sqrt{x} - x + 4\sqrt{x} - 8\log(\sqrt{x}+2) + c$$

on any interval $I \subset (0, +\infty)$.

Solution

(e) Using the rationalizing substitution $y = \tan\frac{x}{2}$, $dx = \frac{2}{1+y^2}dy$ and taking into account that $\cos x = \frac{1-y^2}{1+y^2}$, we have

$$\int \frac{1}{\cos x}dx = -2\int \frac{1}{y^2-1}dy.$$

Using partial fractions, we then have

$$-2\int \frac{1}{y^2-1}dy = \int \left(\frac{1}{y+1} - \frac{1}{y-1}\right) dy = \log\left|\frac{y+1}{y-1}\right| + c.$$

Going back to $y = \tan\frac{x}{2}$, we conclude

$$\int \frac{1}{\cos x}dx = \log\left|\frac{\tan(x/2)+1}{\tan(x/2)-1}\right| + c$$

on any interval I that does not contain points of the form $x = \pi/2 + k\pi$ with k an integer.

Solution

(f) Using the rationalizing substitution $x = 2\sin y$, $-\pi/2 < y < \pi/2$, $-2 < x < 2$, $dx = 2\cos y\, dy$, and taking into account also that $\cos y > 0$, we obtain

$$\int \sqrt{4-x^2}dx = 4\int \cos^2 y\, dy.$$

From $4\cos^2 y = 2 + 2\cos(2y)$, we then have

$$4\int \cos^2 y\, dy = 2y + \sin(2y) = 2y + 2\sin y\cos y + c.$$

Now using $\sin y = x/2$, $y = \arcsin(x/2)$, $\cos y = \sqrt{1-\sin^2 y} = \frac{1}{2}\sqrt{4-x^2}$, we conclude

$$\int \sqrt{4-x^2}dx = 2\arcsin(x/2) + (x/2)\sqrt{4-x^2} + c$$

on any interval $I \subset (-2, 2)$.

Exercise 1.6

Compute the following limits

$$\text{(a)} \lim_{x\to 0^+} \frac{1}{x^3}\int_0^{x^2} \log(1+\sqrt{t})dt, \quad \text{(b)} \lim_{x\to 0^+} \frac{1}{x^4}\int_0^{x^2} (1-\cos\sqrt{t})dt$$

$$\text{(c)} \lim_{k\to+\infty} \left(2\int_4^{4e} \frac{\log(kx)}{x}dx - \log(k(k+1))\right),$$

$$\text{(d)} \lim_{k\to+\infty} \left(\int_1^3 \log(k(x+2))dx - 3\log(k+1)\right)$$

Solution

(a) Setting $f(x) = \int_0^{x^2} \log(1+\sqrt{t})dt$, the limit takes the indeterminate form $\frac{0}{0}$ given by $\lim_{x\to 0^+} \frac{f(x)}{x^3}$. The hypotheses of L'Hôpital's rule are satisfied, so we can move on to computing the limit

$$\lim_{x\to 0^+} \frac{f'(x)}{3x^2}.$$

We compute the derivative $f'(x)$ using the Fundamental theorem of calculus and the chain rule (derivative of a composite function):

$$f'(x) = 2x\log(1+\sqrt{x^2}) = 2x\log(1+x), \quad x > 0.$$

In conclusion

$$\lim_{x\to 0^+} \frac{1}{x^3}\int_0^{x^2} \log(1+\sqrt{t})dt = \lim_{x\to 0^+} \frac{2x\log(1+x)}{3x^2}$$

$$= \frac{2}{3}\lim_{x\to 0^+} \frac{\log(1+x)}{x} = \frac{2}{3}.$$

Solution

(b) Setting $f(x) = \int_0^{x^2} (1-\cos\sqrt{t})dt$, the limit takes the indeterminate form $\frac{0}{0}$ given by $\lim_{x\to 0^+} \frac{f(x)}{x^4}$. The hypotheses of L'Hôpital's rule are satisfied, so we can move on to computing the limit

$$\lim_{x\to 0^+} \frac{f'(x)}{4x^3}.$$

We compute the derivative $f'(x)$ using the Fundamental theorem of calculus and the chain rule (derivative of a composite function):

$$f'(x) = 2x(1-\cos\sqrt{x^2}) = 2x(1-\cos x), \quad x > 0.$$

In conclusion

$$\lim_{x\to 0^+} \frac{1}{x^4}\int_0^{x^2} (1-\cos\sqrt{t})dt = \lim_{x\to 0^+} \frac{2x(1-\cos x)}{4x^3}$$

$$= \frac{1}{2}\lim_{x\to 0^+} \frac{1-\cos x}{x^2} = \frac{1}{4}.$$

Solution

(c) We compute

$$2\int_4^{4e}\frac{\log(kx)}{x}dx = [\log^2(kx)]_4^{4e} = \log^2(4ke) - \log^2(4k)$$

$$= (\log(4ke) - \log(4k))(\log(4ke) + \log(4k))$$

$$= \log\frac{4ke}{4k}\log(16k^2e) = \log(16k^2e).$$

It follows that

$$\lim_{k\to+\infty}\left(2\int_4^{4e}\frac{\log(kx)}{x}dx - \log(k(k+1))\right) = \lim_{k\to+\infty}\log(16k^2e) - \log(k(k+1))$$

$$= \lim_{k\to+\infty}\log\frac{16k^2e}{k^2+k} = \log(16e) = 1 + \log 16.$$

Solution

(d) We compute

$$\int_1^3\log(k(x+2))dx = [x\log(kx+2k)]_1^3 - \int_1^3\frac{x}{x+2}dx$$

$$= 3\log(5k) - \log(3k) - \int_1^3\left(1 - \frac{2}{x+2}\right)dx$$

$$= 3\log(5k) - \log(3k) - 2 + 2\log\frac{5}{3}.$$

It follows that

$$\lim_{k\to+\infty}\left(\int_1^3\log(k(x+2))dx - 3\log(k+1)\right)$$

$$= \lim_{k\to+\infty}3\log(5k) - \log(3k) - 2 + 2\log\frac{5}{3} - 3\log(k+1)$$

$$= \lim_{k\to+\infty}3\log\frac{5k}{k+1} - \log(3k) - 2 + 2\log\frac{5}{3} = -\infty.$$

Integrals

Exercise 2.1

Determine the measure (area) of the set $A \subset \mathbb{R}^2$ defined as follows:

$$A = \left\{(x, y) \in \mathbb{R}^2; -1 < x < 1, x^2 - 1 \leq y \leq \frac{x+1}{x+2}\right\}$$

Solution

We need to compute

$$\int_{-1}^{1} \left(\frac{x+1}{x+2} - (x^2 - 1)\right) dx.$$

We have

$$\int_{-1}^{1} \left(\frac{x+1}{x+2} + 1 - x^2\right) dx = \int_{-1}^{1} \left(2 - \frac{1}{x+2} - x^2\right) dx$$

$$= \left[2x - \log(x+2) - \frac{x^3}{3}\right]_{-1}^{1} = \frac{10}{3} - \log 3.$$

Exercise 2.2

Compute

$$\int_0^1 \frac{e^x + e^{x/2}}{1 + e^x} dx$$

Solution

With the substitution $y = e^{x/2}$, $x = 2\log y$, $dx = \frac{2}{y}dy$, we obtain

$$\int_0^1 \frac{e^x + e^{x/2}}{1+e^x}dx = 2\int_1^{\sqrt{e}} \frac{y^2+y}{(1+y^2)y}dy$$

$$= 2\int_1^{\sqrt{e}} \frac{y+1}{1+y^2}dy = [\log(y^2+1) + 2\arctan y]_1^{\sqrt{e}}$$

$$= \log\frac{e+1}{2} + 2\arctan\sqrt{e} - \frac{\pi}{2}.$$

Exercise 2.3

Compute

$$\int_{-\pi}^{\pi} e^{-|x|}\cos x\, dx$$

Solution

We observe that the function is even, since $f(-x) = f(x)$ for every $x \in [-\pi, \pi]$. Hence, by symmetry

$$\int_{-\pi}^{\pi} e^{-|x|}\cos x\, dx = 2\int_0^{\pi} e^{-|x|}\cos x\, dx = 2\int_0^{\pi} e^{-x}\cos x\, dx$$

where in the last equality we used the fact that $|x| = x$ for every $x \geq 0$. Integrating by parts twice, we have

$$I = \int e^{-x}\cos x\, dx = e^{-x}\sin x + \int e^{-x}\sin x\, dx$$

$$= e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x\, dx = e^{-x}\sin x - e^{-x}\cos x - I$$

from which

$$2I = e^{-x}\sin x - e^{-x}\cos x$$

and finally

$$I = \int e^{-x}\cos x\, dx = \frac{e^{-x}\sin x - e^{-x}\cos x}{2} + c$$

To compute the definite integral, we evaluate the antiderivative between 0 and π, not forgetting the factor 2 in front of the integral:

$$2[I]_0^{\pi} = e^{-\pi} + 1$$

Exercise 2.4

Compute

$$\int_0^2 \frac{e^x \log(1+e^x)}{1+e^x} dx$$

Solution

With the substitution $1 + e^x = t$, $e^x\, dx = dt$, we obtain

$$\int_0^2 \frac{e^x \log(1+e^x)}{1+e^x} dx = \int_2^{e^2+1} \frac{\log t}{t} dt$$

$$= \left[\frac{\log^2 t}{2}\right]_2^{e^2+1} = \frac{1}{2}(\log^2(e^2+1) - \log^2 2)$$

Exercise 2.5

Determine whether the following integral

$$\int_0^1 \frac{\sin^3 \sqrt{x}}{e^x(1-\cos x)} dx$$

exists and is finite.

Solution

Since the integrand is continuous on $(0, 1]$, to determine whether the given integral converges it is enough to study the behavior of f in a right neighborhood of $x = 0$. As $x \to 0^+$ we have

$$f(x) \sim \frac{(\sqrt{x})^3}{\frac{1}{2}x^2} = \frac{x^{3/2}}{\frac{1}{2}x^2} = 2\frac{1}{x^{1/2}}$$

Since $1/2 < 1$, by the limit comparison test the given integral exists and is finite; more formally, the function $f(x)$ is improperly integrable in a right neighborhood of the origin.

Exercise 2.6

Determine whether the following integral

$$\int_0^{+\infty} \frac{x^2}{(3+5x^5)\arctan x^{3/2}} dx$$

exists and is finite.

Solution

The integrand is continuous on $(0, +\infty)$. To determine whether the given integral converges we need to study the behavior of the integrand in a right neighborhood of $x = 0$, where the denominator is not defined, and in a neighborhood of $+\infty$. It is convenient to rewrite this integral of the third kind as

$$\int_0^1 \frac{x^2}{(3+5x^5)\arctan x^{3/2}}dx + \int_1^{+\infty} \frac{x^2}{(3+5x^5)\arctan x^{3/2}}dx$$

in order to highlight the two separate analyses to be carried out. As $x \to 0^+$ we have

$$f(x) \sim \frac{x^2}{3x^{3/2}} = \frac{1}{3x^{-1/2}}$$

Since $-1/2 < 1$, by the limit comparison test the function $f(x)$ is improperly integrable in a right neighborhood of the origin. Let us now verify that it is also improperly integrable in a neighborhood of $+\infty$. As $x \to +\infty$ we have

$$f(x) \sim \frac{x^2}{5x^5\frac{\pi}{2}} = \frac{2}{5\pi}\frac{1}{x^3}$$

Since $3 > 1$, by the limit comparison test the function $f(x)$ is improperly integrable also in a neighborhood of $+\infty$. Overall, therefore, the improper integral exists and is finite. We observe that, in fact, since $f(x) \sim \frac{\sqrt{x}}{3}$ as $x \to 0^+$, the function can be extended by continuity at $x = 0$ and we can set, by definition, $f(0) = 0$.

Exercise 2.7

Determine for which values of the parameter $\alpha \in \mathbb{R}$ the following improper integral

$$\int_0^1 \frac{\log(1+x^2)}{x^\alpha(1+x^3)}dx$$

exists and is finite.

Solution

The integrand is continuous on $(0, 1)$. To determine for which values of $\alpha \in \mathbb{R}$ the given integral converges it is enough to study the behavior of the integrand in a right neighborhood of $x = 0$, where the denominator is not defined. As $x \to 0^+$ we have

$$f(x) \sim \frac{x^2}{x^\alpha} = \frac{1}{x^{\alpha-2}}$$

which is improperly integrable for $\alpha - 2 < 1$, i.e., for $\alpha < 3$.

Exercise 2.8

Given the following improper integral

$$\int_1^{+\infty} \frac{x \arctan\left(x^2\right) \log^\alpha\left(1+\frac{1}{x}\right)}{x^4+1} dx$$

study its convergence as $\alpha \in \mathbb{R}$ varies. Then, for $\alpha = 0$, compute it using the definition.

Solution

The integrand is continuous on $[1, +\infty)$. To determine for which values of $\alpha \in \mathbb{R}$ the given integral converges it is enough to study the behavior of the integrand in a neighborhood of $+\infty$. As $x \to +\infty$ we have

$$f(x) \sim \frac{x}{x^4} \frac{\pi}{2} \left(\frac{1}{x}\right)^\alpha = \frac{\pi}{2} \frac{1}{x^{\alpha+3}}$$

where we needed the asymptotic estimates $x^4+1 \sim x^4$, $\log^\alpha\left(1+\frac{1}{x}\right) \sim \left(\frac{1}{x}\right)^\alpha$, $x \to +\infty$. By the limit comparison test, the improper integral exists and is finite if $\alpha + 3 > 1$, that is, $\alpha > -2$.
For $\alpha = 0$, the integral becomes

$$\int_1^{+\infty} \frac{x \arctan(x^2)}{x^4+1} dx$$

We can proceed by substitution, setting $\arctan(x^2) = t$, $\frac{x}{1+x^4}\, dx = \frac{1}{2}\, dt$, and we obtain

$$\int_1^{+\infty} \frac{x \arctan\left(x^2\right)}{x^4+1}\, dx = \int_{\pi/4}^{\pi/2} \frac{t}{2}\, dt = \frac{3}{64}\pi^2$$

Exercise 2.9

After determining whether or not the function is summable, compute the following integrals

$$\text{(a)} \int_{-1}^{0} \frac{1}{(1-x)\sqrt{1+x}} dx, \quad \text{(b)} \int_{-1}^{1} \frac{1}{(1-x)\sqrt{1+x}} dx, \quad \text{(c)} \int_0^{+\infty} x e^{-x^2} dx$$

$$\text{(d)} \int_0^{+\infty} x^3 e^{-x^2} dx, \quad \text{(e)} \int_4^{+\infty} \frac{1}{x(x-3)} dx$$

These are all continuous and positive functions on their respective intervals of integration, hence they are all integrable functions. Each integral represents the measure of the region under the graph. In all cases, the region under the graph is an unbounded subset of the plane, so the integral is either a positive number or $+\infty$. Recall that a function is called summable when it is integrable with finite integral.

Solution

(a) The function

$$f(x) = \frac{1}{(1-x)\sqrt{1+x}}$$

is continuous on the interval $(-1, 0]$. Hence, we need to study its behavior in a right neighborhood of $x = -1$. As $x \to -1^+$ we have

$$f(x) \sim \frac{1}{\sqrt{1+x}} = \frac{1}{(1+x)^{\frac{1}{2}}}, \quad x \to -1^+$$

hence the function is summable on $(-1, 0]$ by the comparison test, since $\frac{1}{2} < 1$. This means that

$$\int_{-1}^{0} f(x)dx = \lim_{\varepsilon \to 0^+} \int_{-1+\varepsilon}^{0} f(x)dx$$

and that this limit exists, finite and positive.
Setting $y = \sqrt{1+x}$, $x = y^2 - 1$, $dx = 2y\, dy$, we obtain

$$\int_{-1+\varepsilon}^{0} f(x)dx = \int_{\sqrt{\varepsilon}}^{1} \frac{2}{2-y^2}dy,$$

hence, taking into account that $\sqrt{\varepsilon} \to 0$ as $\varepsilon \to 0$, we have

$$\int_{-1}^{0} f(x)dx = \lim_{\delta \to 0^+} \int_{\delta}^{1} \frac{2}{2-y^2}dy = \int_{0}^{1} \frac{2}{2-y^2}dy.$$

Note that, in this case, after the change of variable, we obtained the integral of a bounded function starting from the integral of an unbounded one.
Using partial fractions, and taking into account also that $|y - \sqrt{2}| = \sqrt{2} - y$ on the interval of integration, we have

$$\frac{2}{2-y^2} = -\frac{2}{(y-\sqrt{2})(y+\sqrt{2})} = \frac{1}{\sqrt{2}}\left(\frac{1}{y+\sqrt{2}} - \frac{1}{y-\sqrt{2}}\right),$$

hence

$$\int_{-1}^{0} f(x)dx = \frac{1}{\sqrt{2}} \int_{0}^{1} \left(\frac{1}{y+\sqrt{2}} - \frac{1}{y-\sqrt{2}}\right) dy$$

$$= \frac{1}{\sqrt{2}} \left[\log \frac{y+\sqrt{2}}{\sqrt{2}-y}\right]_0^1 = \frac{1}{\sqrt{2}} \log \frac{1+\sqrt{2}}{\sqrt{2}-1}.$$

Solution

(b) We have

$$\int_{-1}^{1} \frac{1}{(1-x)\sqrt{1+x}}dx = \int_{-1}^{0} \frac{1}{(1-x)\sqrt{1+x}}dx + \int_{0}^{1} \frac{1}{(1-x)\sqrt{1+x}}dx$$

where the first integral is finite and was computed in the previous item. The integral now

given is therefore finite if and only if so is

$$\int_0^1 \frac{1}{(1-x)\sqrt{1+x}}dx.$$

The function

$$f(x) = \frac{1}{(1-x)\sqrt{1+x}}$$

is continuous and bounded on every interval $[0, 1-\varepsilon]$, $\varepsilon > 0$. Moreover, we have

$$f(x) \sim \frac{1}{\sqrt{2}}\frac{1}{1-x}, \quad x \to 1^-$$

hence the function is not summable on $[0, 1)$ by the comparison test. This means that

$$\int_0^1 f(x)dx = \lim_{\varepsilon \to 0^+} \int_0^{1-\varepsilon} f(x)dx = +\infty,$$

hence the given integral is also

$$\int_{-1}^1 \frac{1}{(1-x)\sqrt{1+x}}dx = +\infty.$$

Solution

(c) The function $f(x) = xe^{-x^2}$ is continuous and bounded on every interval $[0, a]$ with $a > 0$. Moreover, we have

$$f(x) = O\left(\frac{1}{x^\alpha}\right), \quad x \to +\infty,$$

for any $\alpha > 0$; in particular, choosing $\alpha > 1$, the function is summable on $[0, +\infty)$. This means that

$$\int_0^{+\infty} f(x)dx = \lim_{a \to +\infty} \int_0^a f(x)dx$$

and that this limit exists, finite and positive. Indeed

$$\lim_{a \to +\infty} \int_0^a xe^{-x^2}dx = -\frac{1}{2}\lim_{a \to +\infty}[e^{-x^2}]_0^a = -\frac{1}{2}\lim_{a \to +\infty}(e^{-a^2} - 1) = \frac{1}{2}.$$

The given integral is equal to $1/2$.

Solution

(d) The function $f(x) = x^3e^{-x^2}$ is continuous and bounded on every interval $[0, a]$ with $a > 0$. Moreover, we have

$$f(x) = O\left(\frac{1}{x^\alpha}\right), \quad x \to +\infty,$$

for any $\alpha > 0$; in particular, choosing $\alpha > 1$, the function is summable on $[0, +\infty)$. This

means that

$$\int_0^{+\infty} f(x)dx = \lim_{a\to+\infty} \int_0^a f(x)dx$$

and that this limit exists, finite and positive. Indeed, integrating by parts, and taking into account the integral computed in the previous item,

$$\lim_{a\to+\infty} \int_0^a x^3 e^{-x^2} dx = \lim_{a\to+\infty} \left\{ -\frac{1}{2}[x^2 e^{-x^2}]_0^a + \int_0^a x e^{-x^2} dx \right\}$$

$$= \lim_{a\to+\infty} \left(-\frac{1}{2} a^2 e^{-a^2} + \int_0^a x e^{-x^2} dx \right) = \frac{1}{2}.$$

The given integral is equal to $1/2$.

Solution

(e) The function $f(x) = \frac{1}{x(x-3)}$ is continuous and bounded on every interval $[4, a]$ with $a > 4$. Moreover, we have

$$f(x) \sim \frac{1}{x^2}, \quad x \to +\infty,$$

hence the function is summable on $[4, +\infty)$ by comparison. This means that

$$\int_4^{+\infty} f(x)dx = \lim_{a\to+\infty} \int_4^a f(x)dx$$

and that this limit exists, finite and positive. Indeed, using partial fractions,

$$\lim_{a\to+\infty} \int_4^a \frac{1}{x(x-3)} dx = \frac{1}{3} \lim_{a\to+\infty} \int_4^a \left(\frac{1}{x-3} - \frac{1}{x} \right) dx = \frac{1}{3} \lim_{a\to+\infty} \left[\log \frac{x-3}{x} \right]_4^a$$

$$= \frac{1}{3} \lim_{a\to+\infty} \left(\log \frac{a-3}{a} - \log \frac{1}{4} \right) = -\frac{1}{3} \log \frac{1}{4} = \frac{1}{3} \log 4.$$

The given integral is equal to $\frac{1}{3} \log 4$.

Exercise 2.10

Determine all $\alpha \geq 0$ for which the function

$$f(x) = \left(\sqrt{1+x} - \sqrt{x} \right)^\alpha$$

is summable on $[0, +\infty)$

Solution

The function $f(x)$ is continuous and positive on $[0, +\infty)$, so it is enough to analyze its behavior in a neighborhood of $+\infty$, that is, as $x \to +\infty$. Multiplying and dividing by

$\left(\sqrt{1+x}+\sqrt{x}\right)^{\alpha}$, we have

$$f(x)=\frac{1}{\left(\sqrt{1+x}+\sqrt{x}\right)^{\alpha}}\sim\frac{1}{(2\sqrt{x})^{\alpha}}=\frac{1}{2^{\alpha}}\frac{1}{x^{\alpha/2}},\ x\to+\infty.$$

By comparison, it follows that f is summable on $[0,+\infty)$ if and only if $\alpha/2>1$, hence if and only if

$$\alpha>2.$$

Exercise 2.11

Given

$$\int_0^{+\infty}\frac{\log\left(1+e^{x^2}\right)}{2+3x^{5/2}+x^{3\alpha}+12x}\,dx$$

determine for which values of $\alpha>0$ it converges, that is, it exists and is finite.

Solution

The function $f(x)$ is continuous and positive on $[0,+\infty)$, so it is enough to analyze its behavior in a neighborhood of $+\infty$, that is, as $x\to+\infty$. We have

$$f(x)=\frac{\log\left(1+e^{x^2}\right)}{2+3x^{5/2}+x^{3\alpha}+12x}\sim\frac{\log\left(e^{x^2}\right)}{3x^{5/2}+x^{3\alpha}}=\frac{x^2}{3x^{5/2}+x^{3\alpha}}\sim$$

$$\sim\begin{cases}\frac{x^2}{3x^{5/2}}=\frac{1}{3}\frac{1}{x^{1/2}} & \text{if } 3\alpha<5/2, \text{ i.e., if and only if } \alpha<5/6\\ \frac{x^2}{4x^{5/2}}=\frac{1}{4}\frac{1}{x^{1/2}} & \text{if } 3\alpha=5/2, \text{ i.e., if and only if } \alpha=5/6\\ \frac{x^2}{x^{3\alpha}}=\frac{1}{x^{3\alpha-2}} & \text{if } 3\alpha>5/2, \text{ i.e., if and only if } \alpha>5/6\end{cases}$$

In the first two cases, the improper integral diverges by limit comparison, since $1/2<1$. In the third case, instead, it exists and is finite if and only if $3\alpha-2>1$, that is, if and only if $\alpha>1$. In conclusion, the improper integral exists and is finite (converges) for $\alpha>1$, and diverges for $\alpha\leq 1$.

Exercise 2.12

Given

$$\int_0^{+\infty}\frac{3+2x^{3/2}+5x^{4\alpha}+x^{1/2}}{\log\left(1+e^{x^5}\right)}\,dx$$

determine for which values of $\alpha>0$ it converges, that is, it exists and is finite.

Solution

The function $f(x)$ is continuous and positive on $[0, +\infty)$, so it is enough to analyze its behavior in a neighborhood of $+\infty$, that is, as $x \to +\infty$. We have

$$f(x) = \frac{3 + 2x^{3/2} + 5x^{4\alpha} + x^{1/2}}{\log\left(1 + e^{x^5}\right)} \sim \frac{2x^{3/2} + 5x^{4\alpha}}{\log\left(e^{x^5}\right)} = \frac{2x^{3/2} + 5x^{4\alpha}}{x^5} \sim$$

$$\sim \begin{cases} \frac{2x^{3/2}}{x^5} = 2\frac{1}{x^{7/2}} & \text{if } 4\alpha < 3/2, \text{ i.e., if and only if } \alpha < 3/8 \\ \frac{7x^{3/2}}{x^5} = 7\frac{1}{x^{7/2}} & \text{if } 4\alpha = 3/2, \text{ i.e., if and only if } \alpha = 3/8 \\ \frac{5x^{4\alpha}}{x^5} = 5\frac{1}{x^{5-4\alpha}} & \text{if } 4\alpha > 3/2, \text{ i.e., if and only if } \alpha > 3/8 \end{cases}$$

In the first two cases, the improper integral converges by limit comparison, since $7/2 > 1$. In the third case, instead, it exists and is finite if and only if $5 - 4\alpha > 1$, that is, if and only if $\alpha < 1$. In conclusion, the improper integral exists and is finite (converges) for $\alpha < 1$, and diverges for $\alpha \geq 1$.

Exercise 2.13

Given

$$\int_1^{+\infty} \frac{\arctan x}{\sqrt{x}(1 + x^\alpha)}\, dx$$

determine for which values of $\alpha \in \mathbb{R}$ it converges, that is, it exists and is finite.

Solution

The function $f(x)$ is continuous and positive on $[1, +\infty)$, so it is enough to analyze its behavior in a neighborhood of $+\infty$, that is, as $x \to +\infty$. We have

$$f(x) \sim \begin{cases} \frac{\pi/2}{x^{1/2} + x^{1/2+\alpha}} \sim \frac{\pi}{2x^{1/2}} & \text{if } \alpha < 0 \\ \frac{\pi/2}{2x^{1/2}} = \frac{\pi}{4x^{1/2}} & \text{if } \alpha = 0 \\ \frac{\pi/2}{x^{1/2} + x^{1/2+\alpha}} \sim \frac{\pi}{2x^{1/2+\alpha}} & \text{if } \alpha > 0 \end{cases}$$

In the first two cases, the improper integral diverges by limit comparison, since $1/2 < 1$. In the third case, instead, it exists and is finite if and only if $\frac{1}{2} + \alpha > 1$, that is, if and only if $\alpha > \frac{1}{2}$. In conclusion, the improper integral exists and is finite (converges) in a neighborhood $U(+\infty)$ for $\alpha > \frac{1}{2}$, and diverges for $\alpha \leq \frac{1}{2}$.

Exercise 2.14

Perform the change of variable $y = \sqrt{x}$ in

$$\int_4^{+\infty} \frac{1}{\sqrt{x}(\sqrt{x}+2)(\sqrt{x}+6)}dx$$

and write the corresponding integral in dy. Then determine the value of the integral.

Solution

From $y = \sqrt{x}$, $x = y^2$, $dx = 2y\,dy$, we obtain

$$\int_4^{+\infty} \frac{1}{\sqrt{x}(\sqrt{x}+2)(\sqrt{x}+6)}dx = \int_2^{+\infty} \frac{2}{(y+2)(y+6)}dy.$$

Using partial fractions,

$$\int_2^{+\infty} \frac{2}{(y+2)(y+6)}dy = \lim_{a\to+\infty} \int_2^a \frac{2}{(y+2)(y+6)}dy$$

$$= \frac{1}{2}\lim_{a\to+\infty} \int_2^a \left(\frac{1}{y+2} - \frac{1}{y+6}\right)dy = \frac{1}{2}\lim_{a\to+\infty} \left[\log\frac{y+2}{y+6}\right]_2^a$$

$$= \frac{1}{2}\lim_{a\to+\infty} \left(\log\frac{a+2}{a+6} - \log\frac{1}{2}\right) = -\frac{1}{2}\log\frac{1}{2} = \frac{1}{2}\log 2.$$

Exercise 2.15

Given the function

$$f : [1, +\infty) \to \mathbb{R}, \quad f(x) = \int_1^x \frac{(t^3-27)(t-1)^3}{t^7+1}dt$$

- Write $f'(x)$.
- Find the possible relative maximum or minimum points in the interior of the domain, specifying the sign of the values taken at those points.
- State whether $\lim_{x\to+\infty} f(x)$ exists and whether it is finite or not, justifying your answer.
- Determine for which value of α the limit

$$\lim_{x\to 1}(x-1)^\alpha f(x)$$

is finite and different from 0, justifying your answer.

Solution

By the Fundamental theorem of calculus

$$f'(x) = \frac{(x^3 - 27)(x-1)^3}{x^7 + 1}.$$

Solution

The first derivative vanishes at the endpoint $x = 1$ of the domain and at the interior point $x = 3$. It is negative for $1 < x < 3$ and positive for $x > 3$. The point $x = 3$ is an interior (absolute) minimum point. The corresponding value

$$f(3) = \int_1^3 \frac{(t^3 - 27)(t-1)^3}{t^7 + 1} dt$$

is negative, as follows from the fact that $f(1) = 0$ and that f is strictly decreasing on $[1, 3]$. The same conclusion is reached directly by observing that the integrand is negative on the interval $[1, 3)$.

Solution

The function f is strictly increasing on $(3, +\infty)$, hence the limit exists, finite or $+\infty$.
By definition,

$$\lim_{x \to +\infty} f(x) = \int_1^{+\infty} \frac{(t^3 - 27)(t-1)^3}{t^7 + 1} dt,$$

hence the question is equivalent to asking whether or not the function

$$g(t) = \frac{(t^3 - 27)(t-1)^3}{t^7 + 1}$$

is summable on $[1, +\infty)$.
The function $g(t)$ is continuous on $[1, +\infty)$, so it is enough to examine its behavior as $t \to +\infty$. We have

$$g(t) \sim \frac{1}{t}, \quad t \to +\infty,$$

from which the function is not summable on $[1, +\infty)$.

In conclusion, $\lim_{x \to +\infty} f(x) = +\infty$.

Solution

We have $\lim_{x \to 1} f(x) = f(1) = 0$, hence for $\alpha \geq 0$ the given limit is zero. For $\alpha < 0$, say $\alpha = -\beta$ with $\beta > 0$, the given limit

$$\lim_{x \to 1^+} \frac{f(x)}{(x-1)^\beta}$$

has the indeterminate form $\frac{0}{0}$. Applying L'Hôpital's rule, we have

$$\lim_{x\to 1^+} \frac{f(x)}{(x-1)^\beta} = \lim_{x\to 1^+} \frac{f'(x)}{\beta(x-1)^{\beta-1}}$$

$$= \lim_{x\to 1^+} \frac{(x^3-27)(x-1)^3}{\beta(x-1)^{\beta-1}(x^7+1)} = -\frac{13}{\beta} \lim_{x\to 1^+} \frac{(x-1)^3}{(x-1)^{\beta-1}}.$$

This limit exists for every $\beta > 0$, but it is finite and different from zero, as required, if and only if $\beta = 4$ (and in this case it equals $-13/4$).

In conclusion, the given limit exists, finite and different from zero, if and only if $\alpha = -4$.

Exercise 2.16

Compute

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+4)} dx$$

Then explain how one can anticipate, without using antiderivatives, the summability of the function on the given interval.

Solution

The function

$$f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+4)}$$

is continuous and positive on $(0, +\infty)$, unbounded as $x \to 0$, and the interval of integration is unbounded. Writing

$$\int_0^{+\infty} f(x)dx = \int_0^1 f(x)dx + \int_1^{+\infty} f(x)dx,$$

it is necessary and sufficient to examine the asymptotic behavior of $f(x)$ as $x \to 0$ and as $x \to +\infty$. We have

$$f(x) \sim \frac{1}{4}\frac{1}{\sqrt{x}}, \quad x \to 0; \qquad f(x) \sim \frac{1}{x^{3/2}}, \quad x \to +\infty.$$

Therefore, $f(x)$ is summable on $(0, +\infty)$ because, by comparison, it is summable both on $(0, 1]$ and on $[1, +\infty)$.

We can compute the value with the substitution $y = \sqrt{x}$, $x = y^2$, $dx = 2y\,dy$:

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+4)}dx = \int_0^{+\infty} \frac{2}{(y+1)(y+4)}dy$$

$$= \frac{2}{3}\int_0^{+\infty} \left(\frac{1}{y+1} - \frac{1}{y+4}\right) dy = \frac{2}{3}\lim_{a\to+\infty}\left[\log\frac{y+1}{y+4}\right]_0^a$$

$$= \frac{2}{3}\lim_{a\to+\infty}\left(\log\frac{a+1}{a+4} - \log\frac{1}{4}\right) = -\frac{2}{3}\log\frac{1}{4} = \frac{2}{3}\log 4.$$

Exercise 2.17

Let

$$f : [1/2, +\infty[\longrightarrow \mathbb{R},\ f(x) = \int_1^x \frac{\log t}{t^2}dt$$

- Determine the possible local extremum points and the monotonicity of f.
- Justify the existence of a horizontal asymptote for f.
- Compute $\lim_{x\to 1} \frac{f(x)}{(x-1)^2}$.

Solution

By the Fundamental theorem of calculus,

$$f'(x) = \frac{\log x}{x^2}$$

for every x in the given domain $[1/2, +\infty)$. The derivative vanishes at $x = 1$, is negative for $1/2 \le x < 1$ and positive for $x > 1$. The function f is therefore strictly decreasing on $[1/2, 1)$ and strictly increasing on $(1, +\infty)$. The point $x = 1$ is an absolute minimum point with value $f(1) = 0$; in particular the function takes only non-negative values (positive for every $x \neq 1$ in the given domain).

Solution

We have to justify that $\lim_{x\to+\infty} f(x)$ exists and is finite. By definition,

$$\lim_{x\to+\infty} f(x) = \int_1^{+\infty} \frac{\log t}{t^2}dt,$$

hence we have to justify the fact that the function

$$g(t) = \frac{\log t}{t^2}$$

is summable on $[1, +\infty)$.

The function $g(t)$ is continuous on $[1, +\infty)$, so it is enough to examine its behavior as $t \to +\infty$. Since $\log t$ is an infinity of lower order than any power, we have

$$g(t) = O\left(\frac{1}{t^{2-\varepsilon}}\right), \quad t \to +\infty,$$

for every $\varepsilon > 0$, from which, choosing ε such that $2 - \varepsilon > 1$, the function g is summable on $[1, +\infty)$.

Solution

We have $\lim_{x\to 1} f(x) = f(1) = 0$, hence the given limit has the indeterminate form $\frac{0}{0}$. Applying L'Hôpital's rule, we have

$$\lim_{x\to 1} \frac{f(x)}{(x-1)^2} = \lim_{x\to 1} \frac{f'(x)}{2(x-1)}$$

$$= \lim_{x\to 1} \frac{\log x}{2(x-1)x^2} = \frac{1}{2} \lim_{x\to 1} \frac{\log x}{x-1} = \frac{1}{2} \lim_{y\to 0} \frac{\log(1+y)}{y} = \frac{1}{2}.$$

Exercise 2.18

Consider

$$\int_2^{+\infty} \frac{x+2}{x(x+1)(x-1)} dx.$$

Before computing it, justify the fact that this integral exists and is finite.

Solution

The function

$$f(x) = \frac{x+2}{x(x+1)(x-1)}$$

is continuous and positive on $[2, +\infty)$ and has the asymptotic behavior

$$f(x) \sim \frac{1}{x^2}, \quad x \to +\infty,$$

hence it is improperly integrable in a neighborhood of $+\infty$ by comparison.

From the partial fractions

$$\frac{x+2}{x(x+1)(x-1)} = -\frac{2}{x} + \frac{1}{2}\frac{1}{x+1} + \frac{3}{2}\frac{1}{x-1},$$

we obtain the antiderivative on $[2, +\infty)$

$$\int \frac{x+2}{x(x+1)(x-1)} dx = \log \frac{(x+1)^{1/2}(x-1)^{3/2}}{x^2}.$$

Hence

$$\int_2^{+\infty} \frac{x+2}{x(x+1)(x-1)}dx = \lim_{a\to+\infty}\left[\log\frac{(x+1)^{1/2}(x-1)^{3/2}}{x^2}\right]_2^a$$

$$= \lim_{a\to+\infty}\left(\log\frac{(a+1)^{1/2}(a-1)^{3/2}}{a^2} - \log\frac{3^{1/2}}{4}\right) = \log\frac{4}{\sqrt{3}}.$$

Exercise 2.19

Using the change of variable $y = e^x$, compute

$$\int_0^{+\infty} \frac{e^x}{(e^x+2)(e^x+3)}dx.$$

Solution

From $y = e^x$, $x = \log y$, $dx = \frac{1}{y}dy$ and using partial fractions, it follows that

$$\int_0^{+\infty} \frac{e^x}{(e^x+2)(e^x+3)}dx = \int_1^{+\infty} \frac{1}{(y+2)(y+3)}dy$$

$$= \int_1^{+\infty}\left(\frac{1}{y+2} - \frac{1}{y+3}\right)dy = \lim_{a\to+\infty}\left[\log\frac{y+2}{y+3}\right]_1^a$$

$$= \lim_{a\to+\infty}\left(\log\frac{a+2}{a+3} - \log\frac{3}{4}\right) = \log\frac{4}{3}$$

Exercise 2.20

Given the function

$$f : [1,+\infty) \to \mathbb{R}, \quad f(x) = \int_1^x \frac{t^3-27}{t^5+1}dt$$

- Write $f'(x)$.
- Determine the possible relative maximum or minimum points.
- Write $f''(x)$.
- To justify the presence or absence of inflection points, carry out separately a brief study of the polynomial function $y = -2x^5 + 135x^2 + 3$, sketching a qualitative graph. This polynomial is one of the factors of $f''(x)$.
- Based on the previous item, state whether f has inflection points. (You are not asked to find these points explicitly, but only to state whether there are any, how many there are, and to locate them with respect to other notable points such as relative maximum or minimum points).
- State whether $\lim_{x\to+\infty} f(x)$ exists and whether it is finite or not, justifying your answer (if it is finite, you are not asked to compute it).

Solution

By the Fundamental theorem of calculus,

$$f'(x) = \frac{x^3 - 27}{x^5 + 1}$$

for all x in the given domain $[1, +\infty)$.

Solution

The derivative vanishes at $x = 3$, is negative on $[1, 3)$ and positive on $(3, +\infty)$. The function is strictly decreasing on $[1, 3)$ and strictly increasing on $(3, +\infty)$. The point $x = 3$ is an absolute minimum point, with value

$$f(3) = \int_1^3 \frac{t^3 - 27}{t^5 + 1} dt,$$

which can be computed using partial fractions if necessary, and which is negative since $f(1) = 0$ and f is strictly decreasing on $[1, 3)$. The sign of $f(3)$ can also be determined directly by observing that the integrand is strictly negative for $t \in [1, 3)$.

Solution

$$f''(x) = \frac{x^2(-2x^5 + 135x^2 + 3)}{(x^5 + 1)^2}.$$

Solution

We have $\lim_{x \to \pm\infty} y(x) = \mp\infty$. Then $y'(x) = -10x(x^3 - 27)$, from which $y(x)$ is strictly decreasing on $(-\infty, 0)$, the point $x = 0$ is a relative minimum point with value $y(0) = 3$, $y(x)$ is strictly increasing on $(0, 3)$, the point $x = 3$ is a relative maximum point with value $y(3) = 732$, and $y(x)$ is strictly decreasing on $(3, +\infty)$. In particular, $y(x)$ has a unique zero at $x = \alpha$ with $\alpha \in (3, +\infty)$, $y(x) > 0$ for $x < \alpha$, $y(x) < 0$ for $x > \alpha$.

Solution

On the given domain, the second derivative has the same sign as $-2x^5 + 135x^2 + 3$. By what we saw in the previous item, the function f has an inflection point at $x = \alpha$, with $\alpha \in (3, +\infty)$, is strictly convex on $(1, \alpha)$ and strictly concave on $(\alpha, +\infty)$.

Solution

The function is strictly increasing on $(3, +\infty)$, hence the limit exists, finite or $+\infty$.

By definition,

$$\lim_{x\to+\infty} f(x) = \int_1^{+\infty} \frac{t^3-27}{t^5+1}dt,$$

hence the question is equivalent to asking whether or not the function

$$g(t) = \frac{t^3-27}{t^5+1}$$

is summable on $[1,+\infty)$.

The function $g(t)$ is continuous on $[1,+\infty)$, so it is enough to examine its behavior as $t\to+\infty$. We have

$$g(t) \sim \frac{1}{t^2}, \quad t\to+\infty,$$

from which the function is summable on $[1,+\infty)$.

In conclusion, $\lim_{x\to+\infty} f(x)$ exists and is finite. If necessary, it can be computed using partial fractions.

Chapter 3

Riemann integral and properties

Exercise 3.1

Let $f(x) = 2x + 1$ on $[0, 2]$ and let P_n be the uniform partition of $[0, 2]$ into n intervals.

- Compute $s(f, P_n)$ and $S(f, P_n)$.
- Deduce that f is integrable and compute $\int_0^2 (2x + 1)\, dx$.

Solution

The points of the partition are $x_k = \frac{2k}{n}$ and $\Delta x_k = \frac{2}{n}$. The function is increasing, so on $I_k = [x_{k-1}, x_k]$

$$m_k = f(x_{k-1}) = \frac{4(k-1)}{n} + 1, \qquad M_k = f(x_k) = \frac{4k}{n} + 1$$

With $\sum_{k=1}^n k = \frac{n(n+1)}{2}$:

$$S(f, P_n) = \sum_{k=1}^{n} \left(\frac{4k}{n} + 1 \right) \frac{2}{n} = \frac{8}{n^2} \cdot \frac{n(n+1)}{2} + 2 = 6 + \frac{4}{n}$$

$$s(f, P_n) = \sum_{k=1}^{n} \left(\frac{4(k-1)}{n} + 1 \right) \frac{2}{n} = \frac{8}{n^2} \cdot \frac{(n-1)n}{2} + 2 = 6 - \frac{4}{n}$$

For every n we have $6 - \frac{4}{n} \le \sup_P s(f, P) \le \inf_P S(f, P) \le 6 + \frac{4}{n}$; letting $n \to +\infty$ we get that f is integrable and

$$\int_0^2 (2x + 1)\, dx = 6$$

This is the area of the trapezoid with bases $f(0) = 1$, $f(2) = 5$ and height 2: $\frac{(1+5)\cdot 2}{2} = 6$.

Exercise 3.2

Using the formula $\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2$, compute the lower and upper sums of $f(x) = x^3$

on $[0,1]$ with respect to the uniform partition P_n, and deduce the value of $\int_0^1 x^3\,dx$.

Solution

With $x_k = \frac{k}{n}$ and f increasing, $m_k = \frac{(k-1)^3}{n^3}$ and $M_k = \frac{k^3}{n^3}$. Hence

$$S(f,P_n) = \frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{(n+1)^2}{4n^2}, \qquad s(f,P_n) = \frac{1}{n^4}\sum_{k=1}^{n-1} k^3 = \frac{(n-1)^2}{4n^2}$$

Both tend to $\frac{1}{4}$ as $n \to +\infty$ and the difference $S - s = \frac{1}{n}$ tends to 0. Therefore

$$\int_0^1 x^3\,dx = \frac{1}{4}$$

Exercise 3.3

Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = -1$ if $x \notin \mathbb{Q}$.

- Prove that f is not integrable on $[0,1]$.
- Prove that $|f|$ is integrable and compute its integral.
- What does this say about the theorem on the integral of the absolute value?

Solution

Every interval I_k of a partition contains rational and irrational numbers, so $m_k = -1$ and $M_k = 1$. For every partition

$$s(f,P) = -\sum_{k=1}^{n} \Delta x_k = -1, \qquad S(f,P) = 1$$

The supremum of the lower sums (-1) differs from the infimum of the upper sums (1): f is not integrable.

On the other hand $|f(x)| = 1$ for every x: it is constant, hence integrable, and $\int_0^1 |f(x)|\,dx = 1$.

The theorem says that if f is integrable then $|f|$ is integrable; the example shows that the converse is false.

Exercise 3.4

Without computing the integrals, prove the estimates

$$\text{(a)}\ \ \frac{1}{2} \le \int_0^1 \frac{dx}{1+x^4} \le 1, \qquad \text{(b)}\ \ 1 \le \int_0^1 \sqrt{1+x^3}\,dx \le \sqrt{2}$$

and decide, again without computing them, which of the two integrals $\int_0^1 x^2\,dx$ and $\int_0^1 x^3\,dx$ is larger, and which of the two integrals $\int_1^2 x^2\,dx$ and $\int_1^2 x^3\,dx$.

Solution

We use the Corollary on the estimates of the integral, with $b-a=1$.

- (a) On $[0,1]$ we have $0 \le x^4 \le 1$, hence $\frac{1}{2} \le \frac{1}{1+x^4} \le 1$.
- (b) On $[0,1]$ we have $0 \le x^3 \le 1$, hence $1 \le \sqrt{1+x^3} \le \sqrt{2}$.

For $x \in [0,1]$ we have $x^3 \le x^2$, so by the monotonicity of the integral $\int_0^1 x^3\,dx \le \int_0^1 x^2\,dx$. For $x \in [1,2]$ we have $x^3 \ge x^2$, hence $\int_1^2 x^3\,dx \ge \int_1^2 x^2\,dx$. (Indeed $\frac{1}{4} < \frac{1}{3}$ and $\frac{15}{4} > \frac{7}{3}$.)

Exercise 3.5

Let $f(x) = |x-1|$ on $[0,3]$. Compute $\int_0^3 f(x)\,dx$ with the geometric interpretation, the integral mean of f on $[0,3]$ and all the points $c \in [0,3]$ given by the integral mean value theorem.

Solution

On $[0,3]$ the graph of f forms two triangles with the x-axis: on $[0,1]$ with base 1 and height 1, on $[1,3]$ with base 2 and height 2. By additivity

$$\int_0^3 |x-1|\,dx = \frac{1}{2} + 2 = \frac{5}{2}$$

The integral mean is $\frac{1}{3}\cdot\frac{5}{2} = \frac{5}{6}$. The required points solve $|c-1| = \frac{5}{6}$: $c = \frac{1}{6}$ or $c = \frac{11}{6}$, both in $[0,3]$. The point of the theorem exists but it is not necessarily unique.

Exercise 3.6

Let $f : [a,b] \to \mathbb{R}$ be continuous and non-negative, with $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for every $x \in [a,b]$. Is the conclusion still true if f is only integrable?

Solution

By contradiction, let $f(x_0) > 0$ for some $x_0 \in [a,b]$. By the sign-preserving property (continuity of f at x_0) there is an interval $[c,d] \subseteq [a,b]$, with $c < d$ and $x_0 \in [c,d]$, on which $f(x) \ge \frac{f(x_0)}{2}$. By additivity, by $f \ge 0$ and by the Corollary on the estimates of the integral

$$\int_a^b f(x)\,dx \ge \int_c^d f(x)\,dx \ge \frac{f(x_0)}{2}\,(d-c) > 0$$

which contradicts the assumption.

Without continuity the conclusion is false: the function equal to 1 at $x = a$ and to 0 elsewhere is integrable, non-negative, with integral 0 (Remark on changes at finitely many points), but it is not identically zero.

Exercise 3.7

Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$, both with the Heine–Cantor theorem and directly, finding an explicit δ for every $\varepsilon > 0$. (Hint: $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$.)

Solution

The function is continuous on the closed and bounded interval $[0, 1]$: by the Heine–Cantor theorem it is uniformly continuous.

Directly: for $0 \leq y \leq x$ we have $\sqrt{x} \leq \sqrt{y} + \sqrt{x - y}$ (squaring: $x \leq y + (x - y) + 2\sqrt{y(x - y)}$), hence $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$. Given $\varepsilon > 0$, with $\delta = \varepsilon^2$:

$$|x - y| < \varepsilon^2 \quad \Longrightarrow \quad |\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|} < \varepsilon$$

The δ does not depend on the points: this is exactly uniform continuity (even on $[0, +\infty)$).

Chapter 4

Integral functions and the fundamental theorem

Exercise 4.1

Compute the derivative of the functions

(a) $F(x) = \displaystyle\int_0^{x^3} \cos(t^2)\,dt$, (b) $G(x) = \displaystyle\int_{\sin x}^{1} e^{t^2}\,dt$, (c) $H(x) = \displaystyle\int_x^{2x} \log(1+t^2)\,dt$

Solution

The integrands are continuous on $\mathbb{R}$; we apply the theorem on the differentiation of integral functions, $\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(t)\,dt = f(\beta(x))\,\beta'(x) - f(\alpha(x))\,\alpha'(x)$.

- (a) $\beta(x) = x^3$, α constant: $F'(x) = \cos(x^6)\cdot 3x^2$.
- (b) β constant, $\alpha(x) = \sin x$: $G'(x) = -e^{\sin^2 x}\cos x$.
- (c) $\beta(x) = 2x$, $\alpha(x) = x$: $H'(x) = 2\log(1+4x^2) - \log(1+x^2)$.

Exercise 4.2

Compute the limits

(a) $\displaystyle\lim_{x\to 0}\frac{1}{x^2}\int_0^x (e^t - 1)\,dt$, (b) $\displaystyle\lim_{x\to 0^+}\frac{1}{x^3}\int_0^{x^2} \sin\sqrt{t}\,dt$, (c) $\displaystyle\lim_{x\to+\infty}\frac{1}{x}\int_0^x \arctan t\,dt$

Solution

In all cases we have an indeterminate form and we use L'Hôpital's rule, differentiating the numerator with the fundamental theorem of calculus.

- (a) Form $\frac{0}{0}$: $\displaystyle\lim_{x\to 0}\frac{e^x - 1}{2x} = \frac{1}{2}$.

- (b) Form $\frac{0}{0}$; the derivative of the numerator is $\sin\sqrt{x^2}\cdot 2x = 2x\sin x$ (for $x>0$): $\lim_{x\to 0^+}\dfrac{2x\sin x}{3x^2}=\dfrac{2}{3}$.
- (c) The numerator tends to $+\infty$, because $\arctan t \geq \frac{\pi}{4}$ for $t\geq 1$. Form $\frac{\infty}{\infty}$: $\lim_{x\to+\infty}\dfrac{\arctan x}{1}=\dfrac{\pi}{2}$.

Exercise 4.3

Compute, with the fundamental formula of integral calculus,

$$\text{(a)}\ \int_0^{\pi/4}\tan^2 x\,dx, \qquad \text{(b)}\ \int_1^4\frac{x-1}{\sqrt{x}}\,dx, \qquad \text{(c)}\ \int_{-1}^{2}|x|\,dx$$

Solution

- (a) $\tan^2 x = (1+\tan^2 x)-1$ has antiderivative $\tan x - x$: $\left[\tan x - x\right]_0^{\pi/4} = 1-\dfrac{\pi}{4}$.
- (b) $\frac{x-1}{\sqrt{x}} = x^{1/2}-x^{-1/2}$: $\left[\dfrac{2}{3}x^{3/2}-2x^{1/2}\right]_1^4 = \left(\dfrac{16}{3}-4\right)-\left(\dfrac{2}{3}-2\right)=\dfrac{8}{3}$.
- (c) By additivity: $\displaystyle\int_{-1}^{0}(-x)\,dx+\int_0^2 x\,dx = \left[-\frac{x^2}{2}\right]_{-1}^{0}+\left[\frac{x^2}{2}\right]_0^2=\frac{1}{2}+2=\frac{5}{2}$.

Exercise 4.4

Given the function

$$F(x)=\int_0^x (t^2-1)\,e^{-t^2}\,dt, \quad x\in\mathbb{R}$$

- find the intervals of monotonicity and the local extremum points of F;
- determine the sign of $F(1)$ and of $F(-1)$;
- prove that F is odd.

Solution

- The integrand is continuous, hence $F'(x)=(x^2-1)\,e^{-x^2}$, which has the sign of x^2-1: F is increasing on $(-\infty,-1]$ and on $[1,+\infty)$, decreasing on $[-1,1]$. The point $x=-1$ is a local maximum point, $x=1$ a local minimum point.
- $F(0)=0$ and F is decreasing on $[-1,1]$: hence $F(1)<0<F(-1)$.
- The function $H(x)=F(x)+F(-x)$ has derivative $H'(x)=F'(x)-F'(-x)=0$, because F' is even. Hence H is constant and $H(x)=H(0)=0$: $F(-x)=-F(x)$.

Exercise 4.5

Let $G(x) = \int_2^x \frac{dt}{\log t}$ for $x > 1$. Determine the sign, the monotonicity and the concavity of G, and write the equation of the tangent line to the graph at the point with abscissa $x = e$, knowing that $G(e) = \gamma$ (you are not asked to compute γ).

Solution

The integrand $\frac{1}{\log t}$ is continuous on $(1, +\infty)$, which is an interval containing 2. By the fundamental theorem of calculus

$$G'(x) = \frac{1}{\log x} > 0, \qquad G''(x) = -\frac{1}{x \log^2 x} < 0, \quad \forall x > 1$$

Hence G is increasing and concave on $(1, +\infty)$. Since $G(2) = 0$ and G is increasing, $G(x) < 0$ for $1 < x < 2$ and $G(x) > 0$ for $x > 2$ (in particular $\gamma > 0$). The tangent line at $x = e$ has slope $G'(e) = 1$:

$$y = \gamma + (x - e)$$

Exercise 4.6

- Find a continuous function f such that $\int_0^x f(t)\,dt = x^2 + \sin x$ for every $x \in \mathbb{R}$.
- Is there a continuous function f such that $\int_0^x f(t)\,dt = x^2 + 1$ for every $x \in \mathbb{R}$?

Solution

- If f is continuous, by the first part of the fundamental theorem the left-hand side is differentiable with derivative $f(x)$. Differentiating: $f(x) = 2x + \cos x$. Check with the second part: $\int_0^x (2t + \cos t)\,dt = \left[t^2 + \sin t\right]_0^x = x^2 + \sin x$.
- No: for $x = 0$ the left-hand side equals $\int_0^0 f = 0$, the right-hand side 1. Differentiating is not enough: the integral function of $2x$ is x^2, not $x^2 + 1$.

Exercise 4.7

Let $F(x) = \int_0^x e^{\sin t}\,dt$. Write the equation of the tangent line to the graph of F at the point with abscissa 0 and decide whether F is invertible on $\mathbb{R}$.

Solution

$F(0) = 0$ and $F'(x) = e^{\sin x}$, hence $F'(0) = 1$: the tangent line is $y = x$.

Since $F'(x) = e^{\sin x} > 0$ for every x, F is increasing on $\mathbb{R}$, hence injective and invertible

onto its image. Moreover $F'(x) \geq e^{-1}$: for $x > 0$ we have $F(x) \geq \frac{x}{e} \to +\infty$, and by the same argument $F(x) \leq \frac{x}{e} \to -\infty$ as $x \to -\infty$. Being continuous, F is a bijection of $\mathbb{R}$ onto $\mathbb{R}$.

Chapter 5

Definite integrals, areas and mean value

Exercise 5.1

Compute by parts

$$\text{(a)} \quad \int_0^1 x\,e^{2x}\,dx, \qquad \text{(b)} \quad \int_1^e x^2 \log x\,dx$$

Solution

- (a) With $f = x$, $g' = e^{2x}$, $g = \frac{e^{2x}}{2}$:

$$\int_0^1 x\,e^{2x}\,dx = \left[\frac{x\,e^{2x}}{2}\right]_0^1 - \frac{1}{2}\int_0^1 e^{2x}\,dx = \frac{e^2}{2} - \left[\frac{e^{2x}}{4}\right]_0^1 = \frac{e^2}{2} - \frac{e^2-1}{4} = \frac{e^2+1}{4}$$

- (b) With $f = \log x$, $g' = x^2$, $g = \frac{x^3}{3}$:

$$\int_1^e x^2 \log x\,dx = \left[\frac{x^3}{3}\log x\right]_1^e - \int_1^e \frac{x^2}{3}\,dx = \frac{e^3}{3} - \frac{e^3-1}{9} = \frac{2e^3+1}{9}$$

Exercise 5.2

Compute by substitution

$$\text{(a)} \quad \int_0^3 x\sqrt{x+1}\,dx, \qquad \text{(b)} \quad \int_0^{\log 3} \frac{e^x}{(e^x+1)^2}\,dx, \qquad \text{(c)} \quad \int_0^{\pi/2} \sin^3 x\,dx$$

Solution

- (a) With $t = x+1$, $x = t-1$, $dx = dt$, limits 1 and 4:

$$\int_1^4 (t-1)\sqrt{t}\,dt = \left[\frac{2}{5}t^{5/2} - \frac{2}{3}t^{3/2}\right]_1^4 = \left(\frac{64}{5} - \frac{16}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right) = \frac{116}{15}$$

- (b) With $t = e^x$, $dt = e^x\,dx$, limits 1 and 3:

$$\int_1^3 \frac{dt}{(t+1)^2} = \left[-\frac{1}{t+1}\right]_1^3 = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

- (c) $\sin^3 x = (1-\cos^2 x)\sin x$; with $t = \cos x$, $dt = -\sin x\,dx$, limits 1 and 0:

$$\int_0^{\pi/2} \sin^3 x\,dx = \int_0^1 (1-t^2)\,dt = 1 - \frac{1}{3} = \frac{2}{3}$$

Exercise 5.3

Compute

$$\int_0^1 \frac{x}{x^2+3x+2}\,dx$$

Solution

The denominator has roots -1 and -2: $x^2+3x+2 = (x+1)(x+2)$. From $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$ we get $x = A(x+2) + B(x+1)$; with $x=-1$: $A=-1$; with $x=-2$: $B=2$. Hence

$$\int_0^1 \left(-\frac{1}{x+1} + \frac{2}{x+2}\right) dx = \big[-\log(x+1)+2\log(x+2)\big]_0^1 = -\log 2 + 2\log 3 - 2\log 2 = \log\frac{9}{8}$$

Exercise 5.4

Compute the area of the bounded region between the graphs of $y = x^3$ and $y = x$.

Solution

The intersections solve $x^3 = x$: $x \in \{-1, 0, 1\}$. On $[-1,0]$ we have $x^3 \geq x$, on $[0,1]$ we have $x \geq x^3$. Hence

$$\text{Area} = \int_{-1}^0 (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

(the two parts are equal because both functions are odd).

Exercise 5.5

Find the measure of the set

$$A = \left\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2,\ 0 \leq y \leq x\,e^{-x}\right\}$$

Solution

$x\,e^{-x} \geq 0$ on $[0,2]$, hence the measure is $\int_0^2 x\,e^{-x}\,dx$. By parts, with $f = x$, $g' = e^{-x}$:

$$\int x\,e^{-x}\,dx = -x\,e^{-x} + \int e^{-x}\,dx = -(x+1)\,e^{-x} + c$$

$$\int_0^2 x\,e^{-x}\,dx = \big[-(x+1)\,e^{-x}\big]_0^2 = 1 - \frac{3}{e^2}$$

Exercise 5.6

Compute the area of the region between the graph of $y = \log x$, the x-axis and the lines $x = \frac{1}{e}$ and $x = e$.

Solution

The logarithm is negative on $\left[\frac{1}{e}, 1\right)$ and positive on $(1, e]$: the area is $\int_{1/e}^{e} |\log x|\,dx$. With the antiderivative $x\log x - x$:

$$\int_{1/e}^{1} (-\log x)\,dx = \big[x - x\log x\big]_{1/e}^{1} = 1 - \frac{2}{e}, \qquad \int_1^e \log x\,dx = \big[x\log x - x\big]_1^e = 1$$

The area is $2 - \frac{2}{e}$.

Exercise 5.7

Compute the mean value of $f(x) = \frac{1}{1+x^2}$ on $[-1,1]$ and the points $c \in [-1,1]$ at which $f(c)$ equals the mean value.

Solution

$$\frac{1}{2}\int_{-1}^{1} \frac{dx}{1+x^2} = \frac{1}{2}\big[\arctan x\big]_{-1}^{1} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

From $\frac{1}{1+c^2} = \frac{\pi}{4}$ we get $c^2 = \frac{4}{\pi} - 1$, that is, $c = \pm\sqrt{\frac{4}{\pi} - 1} \approx \pm 0.52$, both in $[-1,1]$.

Chapter 6

Improper integrals

Exercise 6.1

Compute, by means of the definition, the improper integrals

$$\text{(a)}\ \int_1^{+\infty} \frac{dx}{x(x+1)}, \qquad \text{(b)}\ \int_0^{+\infty} x\,e^{-x}\,dx, \qquad \text{(c)}\ \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

Solution

- (a) $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$, hence

$$\int_1^{\omega} \frac{dx}{x(x+1)} = \left[\log\frac{x}{x+1}\right]_1^{\omega} = \log\frac{\omega}{\omega+1} + \log 2 \to \log 2 \quad \text{as} \quad \omega \to +\infty$$

- (b) By parts, $\int x\,e^{-x}\,dx = -(x+1)\,e^{-x} + c$; hence $\int_0^{\omega} x\,e^{-x}\,dx = 1 - (\omega+1)\,e^{-\omega} \to 1$, by the hierarchy of infinities.

- (c) The integrand is unbounded near 1:

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon\to 0^+} \big[\arcsin x\big]_0^{1-\varepsilon} = \arcsin 1 = \frac{\pi}{2}$$

Exercise 6.2

- Compute by parts the indefinite integral $\displaystyle\int \frac{\log(x+2)}{(x+1)^2}\,dx$.

- Then compute $\displaystyle\int_0^{+\infty} \frac{\log(x+2)}{(x+1)^2}\,dx$.

Solution

With $f = \log(x+2)$ and $g' = \frac{1}{(x+1)^2}$, $g = -\frac{1}{x+1}$:

$$\int \frac{\log(x+2)}{(x+1)^2}\,dx = -\frac{\log(x+2)}{x+1} + \int \frac{dx}{(x+1)(x+2)}$$

and, with the partial fractions $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$,

$$\int \frac{\log(x+2)}{(x+1)^2}\,dx = -\frac{\log(x+2)}{x+1} + \log\frac{x+1}{x+2} + c$$

As $\omega \to +\infty$ we have $\frac{\log(\omega+2)}{\omega+1} \to 0$ and $\log\frac{\omega+1}{\omega+2} \to 0$; at $x = 0$ the antiderivative equals $-\log 2 + \log\frac{1}{2} = -2\log 2$. Hence

$$\int_0^{+\infty} \frac{\log(x+2)}{(x+1)^2}\,dx = 0 - (-2\log 2) = 2\log 2$$

Exercise 6.3

Determine the behaviour of the improper integrals

(a) $\displaystyle\int_1^{+\infty} \frac{\sqrt{x}+1}{x^2+1}\,dx$, (b) $\displaystyle\int_0^1 \frac{dx}{x+\sqrt{x}}$, (c) $\displaystyle\int_0^{\pi/2} \frac{dx}{\sin x}$, (d) $\displaystyle\int_1^{+\infty} \frac{\log x}{x^2}\,dx$

Solution

The integrands are continuous and positive on the intervals considered (in (d) positive for $x > 1$).

- (a) As $x \to +\infty$: $\frac{\sqrt{x}+1}{x^2+1} \sim \frac{1}{x^{3/2}}$; since $\frac{3}{2} > 1$, it converges.
- (b) As $x \to 0^+$: $x + \sqrt{x} \sim \sqrt{x}$, so the integrand is asymptotic to $\frac{1}{x^{1/2}}$; since $\frac{1}{2} < 1$, it converges.
- (c) As $x \to 0^+$: $\sin x \sim x$, hence $\frac{1}{\sin x} \sim \frac{1}{x}$: it diverges.
- (d) Eventually $\log x \le \sqrt{x}$ (hierarchy of infinities), hence $0 \le \frac{\log x}{x^2} \le \frac{1}{x^{3/2}}$: by comparison it converges. (By parts one also finds the value: 1.)

Exercise 6.4

Determine for which values of the parameter $\alpha \in \mathbb{R}$ the following integrals converge:

(a) $\displaystyle\int_0^{+\infty} \frac{dx}{x^\alpha\,(1+x)}$, (b) $\displaystyle\int_0^1 \frac{1-\cos x}{x^\alpha}\,dx$

Solution

- (a) We split at $x = 1$. As $x \to 0^+$ the integrand is asymptotic to $\frac{1}{x^\alpha}$: it converges if and only if $\alpha < 1$. As $x \to +\infty$ it is asymptotic to $\frac{1}{x^{\alpha+1}}$: it converges if and only if $\alpha + 1 > 1$, that is, $\alpha > 0$. The integral converges if and only if $0 < \alpha < 1$.
- (b) The integrand is continuous and non-negative on $(0, 1]$. As $x \to 0^+$, $1 - \cos x \sim \frac{x^2}{2}$, hence $\frac{1-\cos x}{x^\alpha} \sim \frac{1}{2} \cdot \frac{1}{x^{\alpha-2}}$: it converges if and only if $\alpha - 2 < 1$, that is, $\alpha < 3$. (For $\alpha \leq 2$ the integrand is bounded and the integral is a Riemann integral.)

Exercise 6.5

Compute

$$\text{(a)} \int_e^{+\infty} \frac{dx}{x \log^3 x}, \qquad \text{(b)} \int_0^{+\infty} e^{-\sqrt{x}}\, dx$$

Solution

- (a) With $t = \log x$, $dt = \frac{dx}{x}$, an antiderivative is $-\frac{1}{2\log^2 x}$:

$$\int_e^{\omega} \frac{dx}{x \log^3 x} = \frac{1}{2} - \frac{1}{2\log^2 \omega} \to \frac{1}{2}$$

- (b) With $x = t^2$, $t \geq 0$, $dx = 2t\, dt$: $\displaystyle\int_0^{\omega} e^{-\sqrt{x}}\, dx = \int_0^{\sqrt{\omega}} 2t\, e^{-t}\, dt = 2\big[-(t+1)\, e^{-t}\big]_0^{\sqrt{\omega}} \to 2$.

Exercise 6.6

Prove that $\displaystyle\int_1^{+\infty} \frac{\cos x}{x^{3/2}}\, dx$ converges. Can the comparison tests be applied directly to the integrand?

Solution

The integrand changes sign infinitely many times, so the comparison tests (which require non-negative functions) do not apply directly. However

$$\left|\frac{\cos x}{x^{3/2}}\right| \leq \frac{1}{x^{3/2}}$$

and $\int_1^{+\infty} \frac{dx}{x^{3/2}}$ converges ($\frac{3}{2} > 1$). By comparison $\int_1^{+\infty} \left|\frac{\cos x}{x^{3/2}}\right| dx$ converges and, by the theorem on absolute convergence, the given integral converges.

Exercise 6.7

Using the integral test, determine for which $p > 0$ the series

$$\sum_{k=2}^{\infty} \frac{1}{k\,(\log k)^p}$$

converges.

Solution

The function $f(x) = \frac{1}{x(\log x)^p}$ is continuous, positive and decreasing on $[2, +\infty)$ (increasing and positive denominator). By the integral test (starting from $k_0 = 2$) the series has the same behaviour as $\int_2^{+\infty} f(x)\,dx$. With $t = \log x$:

$$\int_2^{\omega} \frac{dx}{x(\log x)^p} = \int_{\log 2}^{\log \omega} \frac{dt}{t^p}$$

Since $\log 2 > 0$, the integral on $[\log 2, 1]$ is finite; on $[1, +\infty)$, by the theorem on the standard improper integral at infinity, it converges if and only if $p > 1$. The series converges if and only if $p > 1$.