



SAPIENZA
UNIVERSITÀ DI ROMA

Mathematical Analysis 1

Exercises, volume 6 — Series and integrals

The worked exercises on numerical series and, in the second part, on antiderivatives and integrals, one sheet per chapter, with the solution right after each statement. The sheets are by Fabio Furini and Gianluca Priori; many exercises are taken from M. Bramanti, *Esercitazioni di Analisi matematica 1*, Esculapio.

Fabio Furini

fabiofurini.github.io

Department of Computer, Control and Management Engineering — Sapienza University of Rome

Online version and downloadable material:
fabiofurini.github.io/mathematical-analysis-1/
github.com/fabiofurini/mathematical-analysis-1

Academic year 2026–2027

Contents

I	Series	2
1	Numerical series	3
II	Antiderivatives and integrals	16
2	Antiderivatives	17
3	Integrals	31



Part I

Series



Numerical series

Exercise 1.1

Determine, as $\alpha \in \mathbb{R}$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \frac{(e^{\alpha^2-3})^n}{\log(1+n)}$$

Solution

This is a series with positive terms. Applying the ratio test, we obtain

$$\lim_{n \rightarrow +\infty} \frac{(e^{\alpha^2-3})^{n+1}}{\log(1+n+1)} \frac{\log(1+n)}{(e^{\alpha^2-3})^n} = e^{\alpha^2-3}$$

The test guarantees that the given series converges for $-\sqrt{3} < \alpha < \sqrt{3}$ and diverges for $\alpha < -\sqrt{3}$ and for $\alpha > \sqrt{3}$, while it gives no information for $\alpha = \pm\sqrt{3}$. For $\alpha = \pm\sqrt{3}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{\log(1+n)}$$

which diverges by the comparison test, since

$$a_n = \frac{1}{\log(1+n)} > \frac{1}{n}$$

and the latter is the general term of the harmonic series, which diverges. Overall, the given series converges for $-\sqrt{3} < \alpha < \sqrt{3}$, while it diverges for $\alpha \leq -\sqrt{3}$ and for $\alpha \geq \sqrt{3}$.

Exercise 1.2

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \frac{\log n}{n^3}$$

Solution

First of all, we observe that the necessary condition for convergence is satisfied. Moreover, we have

$$a_n = \frac{\log n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$$

By comparison with the generalized harmonic series with exponent $p = 2$, the given series converges.

Exercise 1.3

Determine, as $\alpha \in \mathbb{R}$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \frac{n^{2-\alpha}}{\arctan \frac{1}{n^2} + \frac{1}{\sqrt{n}}}$$

Solution

This is a series with positive terms. Since

$$\arctan \frac{1}{n^2} + \frac{1}{\sqrt{n}} = \frac{1}{n^2} + o\left(\frac{1}{n^2}\right) + \frac{1}{\sqrt{n}} \sim \frac{1}{\sqrt{n}}$$

we have

$$a_n = \frac{n^{2-\alpha}}{\arctan \frac{1}{n^2} + \frac{1}{\sqrt{n}}} \sim \frac{n^{2-\alpha}}{\frac{1}{\sqrt{n}}} = \frac{1}{n^{\alpha-\frac{5}{2}}}$$

Therefore, by the limit comparison test, the series converges for $\alpha - \frac{5}{2} > 1$, that is, if and only if $\alpha > \frac{7}{2}$.

Exercise 1.4

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \sin\left(n\frac{\pi}{2}\right)}{2n+5}$$

Solution

Observing that

$$\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even, that is } n = 2k, \\ (-1)^k & \text{if } n \text{ is odd, that is } n = 2k+1. \end{cases}$$

the given series is actually summed only over the odd indices and can be rewritten in the

form

$$\sum_{n=0}^{\infty} \frac{(-1)^n \sin\left(n\frac{\pi}{2}\right)}{2n+5} = \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}(-1)^k}{2(2k+1)+5} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{4k+7}$$

The latter is an alternating series; it does not converge absolutely, since $\left|\frac{(-1)^{k+1}}{4k+7}\right| = \frac{1}{4k+7} \sim \frac{1}{4k}$ as $k \rightarrow +\infty$, where the last one is the general term of a harmonic series (which diverges). However, the given series converges (conditionally) by the Leibniz test. Indeed, it is immediate to prove that the sequence defined by $a_k = \frac{1}{4k+7}$ is positive, infinitesimal and monotonically decreasing.

Exercise 1.5

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=2}^{\infty} \frac{1 - \sqrt{1 - \frac{2}{n}}}{n}$$

Solution

Recalling that $\sqrt{1 - \frac{2}{n}} = 1 - \frac{1}{2}\frac{2}{n} + o\left(\frac{1}{n}\right)$, we have

$$a_n = \frac{1 - \sqrt{1 - \frac{2}{n}}}{n} = \frac{1 - 1 + \frac{1}{2}\frac{2}{n} + o\left(\frac{1}{n}\right)}{n} \sim \frac{\frac{1}{n}}{n} = \frac{1}{n^2}$$

which is the general term of the generalized harmonic series with exponent $2 > 1$. Therefore the original series converges by the limit comparison test.

Exercise 1.6

Determine, as $x \in \mathbb{R}$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \frac{e^{2/n} - 1 - \frac{x}{n}}{\sqrt{\frac{1}{n}}}$$

Solution

Left as an exercise. Use the second-order Maclaurin expansion of e^t , with $t = \frac{2}{n}$. By limit comparison with the generalized harmonic series, the given series will turn out to be divergent for $x \neq 2$ and convergent for $x = 2$.

Exercise 1.7

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \log \left(\frac{n^2 + 3\sqrt{n}}{n^2 + 4} \right)$$

Solution

This is a series with positive terms. Since

$$\log \left(\frac{n^2 + 3\sqrt{n}}{n^2 + 4} \right) = \log \left(1 + \frac{n^2 + 3\sqrt{n}}{n^2 + 4} - 1 \right) \sim \frac{n^2 + 3\sqrt{n}}{n^2 + 4} - 1$$

Then

$$a_n = \log \left(\frac{n^2 + 3\sqrt{n}}{n^2 + 4} \right) \sim \frac{n^2 + 3\sqrt{n}}{n^2 + 4} - 1 = \frac{3\sqrt{n} - 4}{n^2 + 4} \sim 3 \frac{1}{n^{\frac{3}{2}}}$$

Therefore, by comparison with the generalized harmonic series with exponent $\frac{3}{2} > 1$, it follows that the series converges by the limit comparison test.

Exercise 1.8

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}} - 1}{n + 1}$$

Solution

This is a series with positive terms. Since

$$e^{\frac{1}{n}} - 1 \sim \frac{1}{n}$$

we immediately have

$$a_n = \frac{e^{\frac{1}{n}} - 1}{n + 1} \sim \frac{\frac{1}{n}}{n} = \frac{1}{n^2}$$

Therefore, by comparison with the generalized harmonic series with exponent $2 > 1$, it follows that the series converges by the limit comparison test.

Exercise 1.9

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \left(e^{\frac{n^2+2n}{n^2+1}} - e \right)$$

Solution

This is a series with positive terms. We have

$$a_n = e^{\frac{n^2+2n}{n^2+1}} - e = e \left(e^{\frac{n^2+2n}{n^2+1} - 1} - 1 \right) = e \left(e^{\frac{2n-1}{n^2+1}} - 1 \right) \sim e \frac{2n-1}{n^2+1} \sim 2e \frac{1}{n}$$

which is the general term of the harmonic series. We used the fact that $\frac{2n-1}{n^2+1} \rightarrow 0$ and also $\frac{2n-1}{n^2+1} \sim \frac{2}{n}$ as $n \rightarrow +\infty$. Therefore, the original series diverges by the limit comparison test.

Exercise 1.10

Determine, as $\alpha > 0$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \left(\sin \frac{1}{n^{3\alpha}} \right) n^{\frac{3}{2}-2\alpha}$$

Solution

This is a series with positive terms. Since

$$\sin \frac{1}{n^{3\alpha}} \sim \frac{1}{n^{3\alpha}}$$

then we immediately have

$$a_n = \left(\sin \frac{1}{n^{3\alpha}} \right) n^{\frac{3}{2}-2\alpha} \sim \frac{n^{\frac{3}{2}-2\alpha}}{n^{3\alpha}} = \frac{1}{n^{5\alpha-\frac{3}{2}}}$$

Therefore, by the limit comparison test, the series converges for $5\alpha - \frac{3}{2} > 1$, that is, if and only if $\alpha > \frac{1}{2}$.

Exercise 1.11

Determine, as $\alpha > 0$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \left(\sqrt{1 + \frac{1}{n^\alpha}} - 1 \right) n^{2-\alpha}$$

Solution

This is a series with positive terms. Since

$$\sqrt{1 + \frac{1}{n^\alpha}} - 1 \sim \frac{1}{2} \frac{1}{n^\alpha}$$

then

$$a_n = \left(\sqrt{1 + \frac{1}{n^\alpha}} - 1 \right) n^{2-\alpha} \sim \frac{1}{2n^\alpha} n^{2-\alpha} = \frac{1}{2n^{2\alpha-2}}$$

Therefore, by the limit comparison test, the series converges for $2\alpha - 2 > 1$, that is, if and only if $\alpha > \frac{3}{2}$.

Exercise 1.12

Determine, as $\alpha > 0$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \left(\cos \frac{1}{n^\alpha} - 1 \right) n^{1-\alpha}$$

Solution

The given series has negative terms (think about the cosine of an angle between 0 and 1). Therefore, we can apply to it the tests for series with positive terms, factoring out the negative sign. Observing that, for $\alpha > 0$,

$$\left(\cos \frac{1}{n^\alpha} - 1 \right) n^{1-\alpha} \sim -\frac{n^{1-\alpha}}{2n^{2\alpha}} = -\frac{1}{2n^{3\alpha-1}}$$

we immediately obtain that, by the limit comparison test, the given series converges for $3\alpha - 1 > 1$, i.e., $\alpha > \frac{2}{3}$.

Exercise 1.13

Determine, as $\alpha \in \mathbb{R}$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \frac{|\alpha^2 - 5\alpha + 7|^n}{3n^2 + \log n}$$

Solution

First of all, we observe that this is a series with positive terms, since $\alpha^2 - 5\alpha + 7 > 0$ for every $\alpha \in \mathbb{R}$. Using the root test, we obtain

$$\lim_{n \rightarrow +\infty} \sqrt[n]{\frac{|\alpha^2 - 5\alpha + 7|^n}{3n^2 + \log n}} = \lim_{n \rightarrow +\infty} \frac{|\alpha^2 - 5\alpha + 7|}{\sqrt[n]{3n^2 + \log n}} = \lim_{n \rightarrow +\infty} \frac{|\alpha^2 - 5\alpha + 7|}{\sqrt[n]{3} \sqrt[n]{n^2}} = |\alpha^2 - 5\alpha + 7|$$

where we took into account the hierarchy of infinities and the limit $\sqrt[n]{n^2} \rightarrow 1$. Since $|\alpha^2 - 5\alpha + 7| < 1$ is satisfied if and only if $2 < \alpha < 3$, while $|\alpha^2 - 5\alpha + 7| > 1$ is satisfied

if and only if $\alpha < 2$ or $\alpha > 3$, and $|\alpha^2 - 5\alpha + 7| = 1$ if and only if $\alpha = 2$ or $\alpha = 3$, the test guarantees that the given series

- converges for $2 < \alpha < 3$;
- diverges for $\alpha < 2$ or $\alpha > 3$.

For $\alpha = 2$ or $\alpha = 3$, the given series has general term $a_n = \frac{1}{3n^2 + \log n} \sim \frac{1}{3n^2}$, which satisfies the necessary condition, and the series converges by limit comparison with the generalized harmonic series with exponent $2 > 1$.

Exercise 1.14

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \frac{1}{n \left(\sqrt{1 + \frac{3}{n^3}} - 1 \right)}$$

Solution

This is a series with positive terms. Since

$$\sqrt{1 + \frac{3}{n^3}} - 1 \sim \frac{1}{2} \frac{3}{n^3}$$

then

$$a_n = \frac{1}{n \left(\sqrt{1 + \frac{3}{n^3}} - 1 \right)} \sim \frac{1}{n \left(\frac{3}{2n^3} \right)} = \frac{2}{3} n^2$$

Therefore the given series diverges since its general term is not infinitesimal.

Exercise 1.15

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} e^{\sin n} \left(\sin \frac{1}{n} \right) \left(e^{\frac{1}{\sqrt{n}}} - 1 \right) \cos n$$

Solution

Left as an exercise. The given series converges absolutely and hence it also converges.

Exercise 1.16

Determine, as $\alpha \in \mathbb{R}$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=2}^{\infty} \frac{[2(\log n)^{\alpha-7}]^n}{n^2}$$

Solution

Left as an exercise. Apply the root test. The given series converges for $\alpha < 7$ and diverges for $\alpha \geq 7$.

Exercise 1.17

Determine, as $\alpha \in \mathbb{R}$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \frac{4^n}{n^3(7^{\alpha+2})^n}$$

Solution

Left as an exercise. Apply the root test. The given series converges for $\alpha \geq \frac{\log 4}{\log 7} - 2$.

Exercise 1.18

Determine, as $\alpha \in \mathbb{R}$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=0}^{\infty} \left[\cos \left(\frac{1}{n+2} \right)^{\alpha} - 1 \right] n^4$$

Solution

The given series has negative terms (think about the cosine of an angle between 0 and 1). Therefore, we can apply to it the tests for series with positive terms, factoring out the negative sign. Observing that

$$\left[\cos \left(\frac{1}{n+2} \right)^{\alpha} - 1 \right] n^4 \sim -\frac{n^4}{2(n+2)^{2\alpha}} \sim -\frac{n^4}{2n^{2\alpha}} = -\frac{1}{2n^{2\alpha-4}}$$

we immediately obtain that, by the limit comparison test, the given series converges for $2\alpha - 4 > 1$, i.e., $\alpha > \frac{5}{2}$.

Exercise 1.19

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \binom{n}{4} \frac{1}{3^n}$$

Solution

Work it out as an exercise using the ratio test. The given series converges.

Exercise 1.20

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \frac{\sqrt{n!}}{(\sqrt{n})^n}$$

Solution

This is a series with positive terms. Applying the ratio test, we obtain

$$\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow +\infty} \frac{\sqrt{(n+1)!}}{(\sqrt{n+1})^{n+1}} \frac{(\sqrt{n})^n}{\sqrt{n!}} = \lim_{n \rightarrow +\infty} \left(\sqrt{\frac{n}{n+1}} \right)^n$$

Recalling the fundamental limit

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n} \right)^n = e$$

we obtain

$$\lim_{n \rightarrow +\infty} \left(\sqrt{\frac{n}{n+1}} \right)^n = \lim_{n \rightarrow +\infty} \left(\sqrt{\frac{n+1}{n}} \right)^{-n} = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n} \right)^{-\frac{1}{2}n} = e^{-\frac{1}{2}} = l$$

Since $l = e^{-\frac{1}{2}} < 1$, the given series converges.

Exercise 1.21

Determine, as $\alpha > 0$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \left[\log \left(\frac{n+2}{n+1} \right) \right]^\alpha$$

Solution

Left as an exercise. Add and subtract 1 inside the argument of the logarithm and use a suitable asymptotic estimate. The given series converges for $\alpha > 1$.

Exercise 1.22

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} n \left(\frac{2}{\sqrt{n}} - \sin \frac{2}{\sqrt{n}} \right)^2$$

Solution

Left as an exercise. It suffices to expand the general term of the series up to the first nonzero term of the expansion. The given series converges.

Exercise 1.23

Determine, as $x \in \mathbb{R}$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=0}^{\infty} \frac{|x - 12|^{n+1}}{2n^2 e^{-n}}$$

Solution

Left as an exercise. Use the ratio test. The given series converges for $12 - \frac{1}{e} \leq x \leq 12 + \frac{1}{e}$ and diverges elsewhere.

Exercise 1.24

Determine, as $x \in \mathbb{R}$ varies, the behavior of the following series:

$$\sum_{n=0}^{\infty} \left(\frac{x}{|x-1|} \right)^n$$

Solution

This is a geometric series with ratio $q = \frac{x}{|x-1|}$, defined for $x \neq 1$. The series converges

absolutely (and hence also converges) if and only if

$$\left| \frac{x}{x-1} \right| < 1$$

that is, if and only if

$$-1 < \frac{x}{x-1} < 1$$

whose solution interval is $x < \frac{1}{2}$.

Exercise 1.25

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi) e^{-2n} \log n}{n}$$

Solution

Recall that $\cos(n\pi) = (-1)^n$; therefore the series is alternating. Since $\log n < n$ for every $n \in \mathbb{N}$, we have:

$$|a_n| = \left| (-1)^n \frac{e^{-2n} \log n}{n} \right| = e^{-2n} \left(\frac{\log n}{n} \right) < \left(\frac{1}{e^2} \right)^n$$

where the last one is the general term of the geometric series with ratio $q = \frac{1}{e^2} < 1$, which converges. By the comparison test we obtain that the given series converges absolutely, hence it also converges.

Exercise 1.26

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=0}^{\infty} \frac{1 + n \cos(n\pi)}{2n^2 + 1}$$

Solution

Left as an exercise. Observe that $\sum_{n=0}^{\infty} \frac{1}{2n^2+1}$ is convergent, hence it is possible to rewrite

$$\sum_{n=0}^{\infty} \frac{1 + n \cos(n\pi)}{2n^2 + 1} = \sum_{n=0}^{\infty} \frac{1}{2n^2 + 1} + \sum_{n=0}^{\infty} \frac{n \cos(n\pi)}{2n^2 + 1}$$

where $\cos(n\pi) = (-1)^n$. Determine the behavior of the second series; one will verify that it converges (conditionally) by the Leibniz test and that, in the end, the given series converges as a sum of convergent series.

Exercise 1.27

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos n}{3n^2 + 2\sqrt{n}}$$

Solution

The given series is alternating. Setting $a_n = (-1)^n \frac{\cos n}{3n^2 + 2\sqrt{n}}$, we have

$$|a_n| = \left| (-1)^n \frac{\cos n}{3n^2 + 2\sqrt{n}} \right| = \frac{|\cos n|}{3n^2 + 2\sqrt{n}} \leq \frac{1}{3n^2 + 2\sqrt{n}} \sim \frac{1}{3n^2}$$

where we used the fact that $|\cos n| \leq 1$ for every n and $3n^2 + 2\sqrt{n} \sim 3n^2$ as $n \rightarrow +\infty$. By the limit comparison test with the generalized harmonic series with exponent $2 > 1$ and by the comparison test, the given series converges absolutely, and hence it also converges.

Exercise 1.28

Determine the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=2}^{\infty} (-1)^n \frac{1 - \sin^2(n\pi/2)}{3n + \sin n}$$

Solution

This is an alternating series. Observing that

$$\sin^2(n\pi/2) = \begin{cases} 0 & \text{if } n \text{ is even, that is } n = 2k, \\ 1 & \text{if } n \text{ is odd, that is } n = 2k + 1. \end{cases}$$

and setting $a_n = \frac{1 - \sin^2(n\pi/2)}{3n + \sin n}$, we have

$$a_n = \begin{cases} \frac{1}{3n + \sin n} & \text{if } n = 2k, \\ 0 & \text{if } n = 2k + 1. \end{cases}$$

Hence the given series can be rewritten in the form

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{3n + \sin n} = \sum_{k=1}^{\infty} (-1)^{2k} \frac{1}{6k + \sin(2k)} = \sum_{k=1}^{\infty} \frac{1}{6k + \sin(2k)}$$

where, for the second equality, we used the fact that $n = 2k$. The last series obtained has positive terms. Since, as $k \rightarrow +\infty$, $\frac{1}{6k + \sin(2k)} \sim \frac{1}{6k}$, the limit comparison test with the harmonic series shows that the given series diverges.

Exercise 1.29

Determine, as $x \in \mathbb{R}$ varies, the behavior of the following series, making appropriate use of the tests studied:

$$\sum_{n=0}^{\infty} \left(\frac{2x}{1-x^2} \right)^n$$

Solution

Left as an exercise. Argue as in exercise 24.

Part II

Antiderivatives and integrals



Antiderivatives

Exercise 2.1

Compute the following antiderivatives on the intervals where they are defined

$$(a) \int \left(2x + \sqrt[3]{x} - \frac{2}{\sqrt{x}} \right) dx, \quad (b) \int \frac{x - x^4}{\sqrt{x}} dx$$

$$(c) \int \tan^2 x \, dx, \quad (d) \int (3 - x)^5 dx$$

$$(e) \int \frac{1}{2 - 3x} dx, \quad (f) \int x^2 e^{x^3+1} dx$$

$$(g) \int x \sqrt{x^2 + 1} dx, \quad (h) \int \frac{\sin 2x}{1 + \sin^2 x} dx$$

$$(i) \int \frac{e^x}{e^x + 3} dx, \quad (j) \int \frac{1}{\tan x} dx$$

Solution

(a)

$$\begin{aligned} \int \left(2x + \sqrt[3]{x} - \frac{2}{\sqrt{x}} \right) dx &= \int (2x + x^{1/3} - 2x^{-1/2}) dx \\ &= x^2 + \frac{3}{4} x^{4/3} - 4x^{1/2} + c = x^2 + \frac{3}{4} \sqrt[3]{x^4} - 4\sqrt{x} + c \end{aligned}$$

on any interval I such that $I \subset (0, +\infty)$.

Solution

(b)

$$\int \frac{x - x^4}{\sqrt{x}} dx = \int (x^{1/2} - x^{7/2}) dx =$$

$$\frac{2}{3}x^{3/2} - \frac{2}{9}x^{9/2} + c = \frac{2}{3}x\sqrt{x} - \frac{2}{9}x^4\sqrt{x} + c$$

on any interval I such that $I \subset (0, +\infty)$.

Solution

(c)

$$\int \tan^2 x \, dx = \int [(1 + \tan^2 x) - 1] dx = \tan x - x + c$$

on any interval I that does not contain points of the form $\pi/2 + k\pi$ with $k \in \mathbb{Z}$.

Solution

(d)

$$\int (3 - x)^5 dx = - \int (x - 3)^5 dx = -\frac{1}{6}(x - 3)^6 + c$$

on any interval I of $\mathbb{R}$.

Solution

(e)

$$\int \frac{1}{2 - 3x} dx = - \int \frac{1}{3x - 2} dx = -\frac{1}{3} \log |3x - 2| + c$$

on any interval I such that $2/3 \notin I$.

Solution

(f)

$$\int x^2 e^{x^3+1} dx = \frac{1}{3} e^{x^3+1} + c$$

on any interval I of $\mathbb{R}$.

Solution

(g)

$$\int x\sqrt{x^2+1}dx = \frac{1}{3}(x^2+1)^{3/2} + c = \frac{1}{3}(x^2+1)\sqrt{x^2+1} + c$$

on any interval I of $\mathbb{R}$.**Solution**

(h)

$$\int \frac{\sin 2x}{1+\sin^2 x} dx = \int \frac{2\sin x \cos x}{1+\sin^2 x} dx = \log(1+\sin^2 x) + c$$

on any interval I of $\mathbb{R}$.**Solution**

(i)

$$\int \frac{e^x}{e^x+3} dx = \log(e^x+3) + c$$

on any interval I of $\mathbb{R}$.**Solution**

(j)

$$\int \frac{1}{\tan x} dx = \int \frac{\cos x}{\sin x} dx = \log |\sin x| + c$$

on any interval I that does not contain points of the form $k\pi$ with $k \in \mathbb{Z}$.**Exercise 2.2**

Compute the following antiderivatives using integration by parts

$$(a) \int x \sin x \, dx, \quad (b) \int x^2 \cos x \, dx$$

$$(c) \int \arcsin x \, dx, \quad (d) \int \sin^2 x \, dx$$

Solution

(a)

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + c$$

 on any interval I of $\mathbb{R}$.

Solution

(b) Integrating by parts twice in a row, we have

$$\begin{aligned} \int x^2 \cos x \, dx &= x^2 \sin x - 2 \int x \sin x \, dx = \\ &= x^2 \sin x + 2x \cos x - 2 \int \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + c \end{aligned}$$

 on any interval I of $\mathbb{R}$.

Solution

(c) Taking the constant function 1 as the factor to be integrated, we have

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = x \arcsin x + \sqrt{1-x^2} + c$$

 on any interval $I \subset (-1, 1)$.

Solution

 (d) Integrating by parts and using $\cos^2 x = 1 - \sin^2 x$, we obtain

$$\begin{aligned} \int \sin^2 x \, dx &= -\sin x \cos x + \int \cos^2 x \, dx \\ &= -\sin x \cos x + \int 1 \, dx - \int \sin^2 x \, dx \end{aligned}$$

from which

$$\int \sin^2 x \, dx = -\sin x \cos x + x - \int \sin^2 x \, dx + c$$

 hence, moving $\int \sin^2 x \, dx$ to the left-hand side,

$$2 \int \sin^2 x \, dx = -\sin x \cos x + x + c$$

 and finally, still denoting by c the arbitrary constant $c/2$,

$$\int \sin^2 x \, dx = -\frac{1}{2} \sin x \cos x + \frac{1}{2} x + c$$

on any interval I of $\mathbb{R}$.

Exercise 2.3

Compute the following antiderivatives using integration by substitution

$$(a) \int \frac{e^x - 1}{e^x + 1} dx, \quad (b) \int \frac{\sqrt{x}}{1+x} dx, \quad (c) \int x\sqrt{x+1} dx$$

Solution

(a) Setting $y = e^x$, $y > 0$, $x \in \mathbb{R}$, we have $x = \log y$ and we write $dx = \frac{1}{y} dy$, hence

$$\int \frac{e^x - 1}{e^x + 1} dx = \int \frac{y - 1}{y(y + 1)} dy.$$

Using partial fractions, we obtain

$$\frac{y - 1}{y(y + 1)} = -\frac{1}{y} + \frac{2}{y + 1}$$

from which, taking into account that $y > 0$,

$$\int \frac{y - 1}{y(y + 1)} dy = -\log y + 2 \log(y + 1) + c.$$

It follows, substituting back $y = e^x$,

$$\int \frac{e^x - 1}{e^x + 1} dx = -\log e^x + 2 \log(e^x + 1) + c = -x + 2 \log(e^x + 1) + c$$

on any interval I of $\mathbb{R}$.

Solution

(b) Setting $y = \sqrt{x}$, $x, y > 0$, we have $x = y^2$ and we write $dx = 2y dy$, hence

$$\int \frac{\sqrt{x}}{1+x} dx = 2 \int \frac{y^2}{1+y^2} dy.$$

From

$$\frac{y^2}{y^2 + 1} = \frac{y^2 + 1 - 1}{y^2 + 1} = 1 - \frac{1}{y^2 + 1}$$

we then have

$$2 \int \frac{y^2}{1+y^2} dy = 2y - 2 \arctan y + c.$$

It follows, substituting back $y = \sqrt{x}$,

$$\int \frac{\sqrt{x}}{1+x} dx = 2\sqrt{x} - 2 \arctan \sqrt{x} + c$$

on any interval $I \subset (0, +\infty)$.

Solution

(c) Setting $y = \sqrt{x+1}$, $x > -1$, $y > 0$, we have $x = y^2 - 1$ and we write $dx = 2y \, dy$, hence

$$\int x\sqrt{x+1} dx = 2 \int (y^2 - 1)y^2 \, dy = 2 \int (y^4 - y^2) dy = \frac{2}{5}y^5 - \frac{2}{3}y^3 + c.$$

It follows, substituting back $y = \sqrt{x+1}$,

$$\int x\sqrt{x+1} dx = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + c$$

on any interval $I \subset (-1, +\infty)$.

Exercise 2.4

Compute the following antiderivatives of rational functions

$$(a) \int \frac{5}{x^2 + 2x + 1} dx, \quad (b) \int \frac{1}{x^2 + x + 1} dx, \quad (c) \int \frac{x}{x^2 + x + 1} dx, \quad (d) \int \frac{x^3 + 1}{x^2 - 3x + 2} dx$$

$$(e) \int \frac{x}{x^3 + 3x^2 + 3x + 1} dx, \quad (f) \int \frac{1}{x^4 - 1} dx, \quad (g) \int \frac{x+1}{x^3 - x^2} dx, \quad (h) \int \frac{1}{x^3 + 1} dx$$

Solution

(a) The given integral is immediate:

$$\int \frac{5}{x^2 + 2x + 1} dx = \int \frac{5}{(x+1)^2} dx = -\frac{5}{x+1} + c$$

on any interval I such that $-1 \notin I$.

Solution

(b) Writing the (irreducible) denominator as a sum of squares:

$$\int \frac{1}{x^2 + x + 1} dx = \int \frac{1}{(x + 1/2)^2 + 3/4} dx = \frac{2}{\sqrt{3}} \arctan \left[\frac{2}{\sqrt{3}} \left(x + \frac{1}{2} \right) \right] + c$$

on any interval I of $\mathbb{R}$.

Solution

(c) Using also the previous exercise:

$$\begin{aligned} \int \frac{x}{x^2 + x + 1} dx &= \frac{1}{2} \int \frac{2x + 1}{x^2 + x + 1} dx - \frac{1}{2} \int \frac{1}{x^2 + x + 1} dx = \\ &= \frac{1}{2} \log(x^2 + x + 1) - \frac{1}{\sqrt{3}} \arctan \left[\frac{2}{\sqrt{3}} \left(x + \frac{1}{2} \right) \right] + c \end{aligned}$$

on any interval I of $\mathbb{R}$.

Solution

(d) We perform polynomial long division:

$$\frac{x^3 + 1}{x^2 - 3x + 2} = x + 3 + \frac{7x - 5}{x^2 - 3x + 2}.$$

Setting up the partial fractions

$$\frac{7x - 5}{x^2 - 3x + 2} = \frac{7x - 5}{(x - 1)(x - 2)} = \frac{A}{x - 1} + \frac{B}{x - 2},$$

we obtain

$$A = -2, \quad B = 9.$$

In conclusion:

$$\begin{aligned} \int \frac{x^3 + 1}{x^2 - 3x + 2} dx &= \int \left(x + 3 - \frac{2}{x - 1} + \frac{9}{x - 2} \right) dx = \\ &= \frac{1}{2} x^2 + 3x - 2 \log |x - 1| + 9 \log |x - 2| + c \end{aligned}$$

on any interval I such that $1 \notin I, 2 \notin I$.

Solution

(e) We set up the partial fractions:

$$\frac{x}{x^3 + 3x^2 + 3x + 1} = \frac{x}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

We obtain

$$A = 0, \quad B = 1, \quad C = -1.$$

In conclusion

$$\begin{aligned} \int \frac{x}{x^3 + 3x^2 + 3x + 1} dx &= \int \left(\frac{1}{(x+1)^2} - \frac{1}{(x+1)^3} \right) dx \\ &= -\frac{1}{x+1} + \frac{1}{2} \frac{1}{(x+1)^2} + c \end{aligned}$$

on any interval I such that $-1 \notin I$.

Solution

(f) We set up the partial fractions:

$$\frac{1}{x^4 - 1} = \frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}.$$

We obtain

$$A = \frac{1}{4}, \quad B = -\frac{1}{4}, \quad C = 0, \quad D = -\frac{1}{2}.$$

In conclusion

$$\begin{aligned} \int \frac{1}{x^4 - 1} dx &= \int \left(\frac{1}{4} \frac{1}{x-1} - \frac{1}{4} \frac{1}{x+1} - \frac{1}{2} \frac{1}{x^2+1} \right) dx \\ &= \frac{1}{4} \log |x-1| - \frac{1}{4} \log |x+1| - \frac{1}{2} \arctan x + c \end{aligned}$$

on any interval I such that $\pm 1 \notin I$.

Solution

(g) We set up the partial fractions:

$$\frac{x+1}{x^3 - x^2} = \frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

We obtain

$$A = -2, \quad B = -1, \quad C = 2.$$

In conclusion

$$\begin{aligned}\int \frac{x+1}{x^3-x^2} dx &= \int \left(-\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1} \right) dx \\ &= -2 \log |x| + \frac{1}{x} + 2 \log |x-1| + c\end{aligned}$$

on any interval I such that $0, 1 \notin I$.

Solution

(h) We set up the partial fractions:

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

We obtain

$$A = \frac{1}{3}, \quad B = -\frac{1}{3}, \quad C = \frac{2}{3}.$$

Hence

$$\begin{aligned}\int \frac{1}{x^3+1} dx &= \int \left(\frac{1}{3} \frac{1}{x+1} + \frac{1}{3} \frac{-x+2}{x^2-x+1} \right) dx \\ &= \frac{1}{3} \log |x+1| - \frac{1}{6} \int \frac{2x-1}{x^2-x+1} dx + \frac{1}{2} \int \frac{1}{x^2-x+1} dx \\ &= \frac{1}{3} \log |x+1| - \frac{1}{6} \log(x^2-x+1) + \frac{1}{2} \int \frac{1}{(x-1/2)^2+3/4} dx \\ &= \frac{1}{3} \log |x+1| - \frac{1}{6} \log(x^2-x+1) + \frac{1}{\sqrt{3}} \arctan \left[\frac{2}{\sqrt{3}} \left(x - \frac{1}{2} \right) \right] + c\end{aligned}$$

on any interval I such that $-1 \notin I$.

Exercise 2.5

Compute the following antiderivatives

$$\begin{aligned}\text{(a)} \quad & \int \frac{e^x+1}{e^{2x}+1} dx, \quad \text{(b)} \quad \int x^2 \log(x+1) dx, \quad \text{(c)} \quad \int \frac{\tan x}{\sin^2 x - \cos^2 x} dx \\ \text{(d)} \quad & \int \frac{x+\sqrt{x}}{2+\sqrt{x}} dx, \quad \text{(e)} \quad \int \frac{1}{\cos x} dx, \quad \text{(f)} \quad \int \sqrt{4-x^2} dx\end{aligned}$$

Solution

(a) With the substitution $y = e^x$, $y > 0$, $x \in \mathbb{R}$, we have $x = \log y$, we write $dx = \frac{1}{y} dy$, and

we obtain

$$\int \frac{e^x + 1}{e^{2x} + 1} dx = \int \frac{y + 1}{y(y^2 + 1)} dy.$$

We set up the partial fractions:

$$\frac{y + 1}{y(y^2 + 1)} = \frac{A}{y} + \frac{By + C}{y^2 + 1}.$$

We have

$$A = 1, B = -1, C = 1,$$

from which, taking into account also that $y > 0$,

$$\begin{aligned} \int \frac{y + 1}{y(y^2 + 1)} dy &= \int \left(\frac{1}{y} - \frac{y}{y^2 + 1} + \frac{1}{y^2 + 1} \right) dy \\ &= \log y - \frac{1}{2} \log(y^2 + 1) + \arctan y + c. \end{aligned}$$

Then, substituting back $y = e^x$, we have

$$\int \frac{e^x + 1}{e^{2x} + 1} dx = x - \frac{1}{2} \log(e^{2x} + 1) + \arctan e^x + c$$

on any interval I of $\mathbb{R}$.

Solution

(b) Integrating by parts and then computing the integral of a rational function, we have

$$\begin{aligned} \int x^2 \log(x + 1) dx &= \frac{1}{3} x^3 \log(x + 1) - \frac{1}{3} \int \frac{x^3}{x + 1} dx \\ &= \frac{1}{3} x^3 \log(x + 1) - \frac{1}{3} \int (x^2 - x + 1) dx + \frac{1}{3} \int \frac{1}{x + 1} dx \\ &= \frac{1}{3} x^3 \log(x + 1) - \frac{1}{9} x^3 + \frac{1}{6} x^2 - \frac{1}{3} x + \frac{1}{3} \log(x + 1) + c \end{aligned}$$

on any interval $I \subset (-1, +\infty)$.

Solution

(c) From $\tan x = \frac{\sin x}{\cos x}$ and $\sin^2 x = 1 - \cos^2 x$, we have

$$\int \frac{\tan x}{\sin^2 x - \cos^2 x} dx = \int \frac{\sin x}{\cos x(1 - 2\cos^2 x)} dx.$$

Now setting $y = \cos x$, we write $dy = -\sin x dx$ and we obtain

$$\int \frac{\sin x}{\cos x(1 - 2\cos^2 x)} dx = \frac{1}{2} \int \frac{1}{y(y - \sqrt{1/2})(y + \sqrt{1/2})} dy.$$

Using partial fractions, we then have

$$\frac{1}{y(y - \sqrt{1/2})(y + \sqrt{1/2})} = -\frac{2}{y} + \frac{1}{y - \sqrt{1/2}} + \frac{1}{y + \sqrt{1/2}},$$

from which

$$\frac{1}{2} \int \frac{1}{y(y - \sqrt{1/2})(y + \sqrt{1/2})} = -\log |y| + \frac{1}{2} \log |y^2 - 1/2| + c.$$

In conclusion,

$$\int \frac{\tan x}{\sin^2 x - \cos^2 x} dx = -\log |\cos x| + \frac{1}{2} \log |\cos^2 x - 1/2| + c$$

on any interval I that does not contain points of the form $x = \pi/2 + k\pi$ or of the form $x = \pi/4 + k\pi/2$ with k an integer.

Solution

(d) With the substitution $y = \sqrt{x}$, $x, y > 0$, we have $x = y^2$, we write $dx = 2y dy$ and we obtain

$$\int \frac{x + \sqrt{x}}{2 + \sqrt{x}} dx = 2 \int \frac{y^3 + y^2}{y + 2} dy.$$

Integrating the rational function, we have

$$\begin{aligned} 2 \int \frac{y^3 + y^2}{y + 2} dy &= 2 \int (y^2 - y + 2) dy - 8 \int \frac{1}{y + 2} dy \\ &= \frac{2}{3} y^3 - y^2 + 4y - 8 \log(y + 2) + c. \end{aligned}$$

Substituting back $y = \sqrt{x}$, we conclude

$$\int \frac{x + \sqrt{x}}{2 + \sqrt{x}} dx = \frac{2}{3} x \sqrt{x} - x + 4\sqrt{x} - 8 \log(\sqrt{x} + 2) + c$$

on any interval $I \subset (0, +\infty)$.

Solution

(e) Using the rationalizing substitution $y = \tan \frac{x}{2}$, $dx = \frac{2}{1+y^2} dy$ and taking into account that $\cos x = \frac{1-y^2}{1+y^2}$, we have

$$\int \frac{1}{\cos x} dx = -2 \int \frac{1}{y^2 - 1} dy.$$

Using partial fractions, we then have

$$-2 \int \frac{1}{y^2 - 1} dy = \int \left(\frac{1}{y+1} - \frac{1}{y-1} \right) dy = \log \left| \frac{y+1}{y-1} \right| + c.$$

Going back to $y = \tan \frac{x}{2}$, we conclude

$$\int \frac{1}{\cos x} dx = \log \left| \frac{\tan(x/2) + 1}{\tan(x/2) - 1} \right| + c$$

on any interval I that does not contain points of the form $x = \pi/2 + k\pi$ with k an integer.

Solution

(f) Using the rationalizing substitution $x = 2 \sin y$, $-\pi/2 < y < \pi/2$, $-2 < x < 2$, $dx = 2 \cos y \, dy$, and taking into account also that $\cos y > 0$, we obtain

$$\int \sqrt{4 - x^2} dx = 4 \int \cos^2 y \, dy.$$

From $4 \cos^2 y = 2 + 2 \cos(2y)$, we then have

$$4 \int \cos^2 y \, dy = 2y + \sin(2y) = 2y + 2 \sin y \cos y + c.$$

Now using $\sin y = x/2$, $y = \arcsin(x/2)$, $\cos y = \sqrt{1 - \sin^2 y} = \frac{1}{2} \sqrt{4 - x^2}$, we conclude

$$\int \sqrt{4 - x^2} dx = 2 \arcsin(x/2) + (x/2) \sqrt{4 - x^2} + c$$

on any interval $I \subset (-2, 2)$.

Exercise 2.6

Compute the following limits

$$(a) \lim_{x \rightarrow 0^+} \frac{1}{x^3} \int_0^{x^2} \log(1 + \sqrt{t}) dt, \quad (b) \lim_{x \rightarrow 0^+} \frac{1}{x^4} \int_0^{x^2} (1 - \cos \sqrt{t}) dt$$

$$(c) \lim_{k \rightarrow +\infty} \left(2 \int_4^{4e} \frac{\log(kx)}{x} dx - \log(k(k+1)) \right),$$

$$(d) \lim_{k \rightarrow +\infty} \left(\int_1^3 \log(k(x+2)) dx - 3 \log(k+1) \right)$$

Solution

(a) Setting $f(x) = \int_0^{x^2} \log(1 + \sqrt{t}) dt$, the limit takes the indeterminate form $\frac{0}{0}$ given by $\lim_{x \rightarrow 0^+} \frac{f(x)}{x^3}$. The hypotheses of L'Hôpital's rule are satisfied, so we can move on to computing the limit

$$\lim_{x \rightarrow 0^+} \frac{f'(x)}{3x^2}.$$

We compute the derivative $f'(x)$ using the Fundamental theorem of calculus and the chain rule (derivative of a composite function):

$$f'(x) = 2x \log(1 + \sqrt{x^2}) = 2x \log(1 + x), \quad x > 0.$$

In conclusion

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{1}{x^3} \int_0^{x^2} \log(1 + \sqrt{t}) dt &= \lim_{x \rightarrow 0^+} \frac{2x \log(1 + x)}{3x^2} \\ &= \frac{2}{3} \lim_{x \rightarrow 0^+} \frac{\log(1 + x)}{x} = \frac{2}{3}. \end{aligned}$$

Solution

(b) Setting $f(x) = \int_0^{x^2} (1 - \cos \sqrt{t}) dt$, the limit takes the indeterminate form $\frac{0}{0}$ given by $\lim_{x \rightarrow 0^+} \frac{f(x)}{x^4}$. The hypotheses of L'Hôpital's rule are satisfied, so we can move on to computing the limit

$$\lim_{x \rightarrow 0^+} \frac{f'(x)}{4x^3}.$$

We compute the derivative $f'(x)$ using the Fundamental theorem of calculus and the chain rule (derivative of a composite function):

$$f'(x) = 2x(1 - \cos \sqrt{x^2}) = 2x(1 - \cos x), \quad x > 0.$$

In conclusion

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{1}{x^4} \int_0^{x^2} (1 - \cos \sqrt{t}) dt &= \lim_{x \rightarrow 0^+} \frac{2x(1 - \cos x)}{4x^3} \\ &= \frac{1}{2} \lim_{x \rightarrow 0^+} \frac{1 - \cos x}{x^2} = \frac{1}{4}. \end{aligned}$$

Solution

(c) We compute

$$\begin{aligned}
 2 \int_4^{4e} \frac{\log(kx)}{x} dx &= [\log^2(kx)]_4^{4e} = \log^2(4ke) - \log^2(4k) \\
 &= (\log(4ke) - \log(4k))(\log(4ke) + \log(4k)) \\
 &= \log \frac{4ke}{4k} \log(16k^2e) = \log(16k^2e).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \lim_{k \rightarrow +\infty} \left(2 \int_4^{4e} \frac{\log(kx)}{x} dx - \log(k(k+1)) \right) &= \lim_{k \rightarrow +\infty} \log(16k^2e) - \log(k(k+1)) \\
 &= \lim_{k \rightarrow +\infty} \log \frac{16k^2e}{k^2+k} = \log(16e) = 1 + \log 16.
 \end{aligned}$$

Solution

(d) We compute

$$\begin{aligned}
 \int_1^3 \log(k(x+2)) dx &= [x \log(kx+2k)]_1^3 - \int_1^3 \frac{x}{x+2} dx \\
 &= 3 \log(5k) - \log(3k) - \int_1^3 \left(1 - \frac{2}{x+2} \right) dx \\
 &= 3 \log(5k) - \log(3k) - 2 + 2 \log \frac{5}{3}.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \lim_{k \rightarrow +\infty} \left(\int_1^3 \log(k(x+2)) dx - 3 \log(k+1) \right) \\
 &= \lim_{k \rightarrow +\infty} 3 \log(5k) - \log(3k) - 2 + 2 \log \frac{5}{3} - 3 \log(k+1) \\
 &= \lim_{k \rightarrow +\infty} 3 \log \frac{5k}{k+1} - \log(3k) - 2 + 2 \log \frac{5}{3} = -\infty.
 \end{aligned}$$

Integrals

Exercise 3.1

Determine the measure (area) of the set $A \subset \mathbb{R}^2$ defined as follows:

$$A = \left\{ (x, y) \in \mathbb{R}^2; -1 < x < 1, x^2 - 1 \leq y \leq \frac{x+1}{x+2} \right\}$$

Solution

We need to compute

$$\int_{-1}^1 \left(\frac{x+1}{x+2} - (x^2 - 1) \right) dx.$$

We have

$$\begin{aligned} \int_{-1}^1 \left(\frac{x+1}{x+2} + 1 - x^2 \right) dx &= \int_{-1}^1 \left(2 - \frac{1}{x+2} - x^2 \right) dx \\ &= \left[2x - \log(x+2) - \frac{x^3}{3} \right]_{-1}^1 = \frac{10}{3} - \log 3. \end{aligned}$$

Exercise 3.2

Compute

$$\int_0^1 \frac{e^x + e^{x/2}}{1 + e^x} dx$$

Solution

With the substitution $y = e^{x/2}$, $x = 2 \log y$, $dx = \frac{2}{y} dy$, we obtain

$$\begin{aligned} \int_0^1 \frac{e^x + e^{x/2}}{1 + e^x} dx &= 2 \int_1^{\sqrt{e}} \frac{y^2 + y}{(1 + y^2)y} dy \\ &= 2 \int_1^{\sqrt{e}} \frac{y + 1}{1 + y^2} dy = [\log(y^2 + 1) + 2 \arctan y]_1^{\sqrt{e}} \\ &= \log \frac{e + 1}{2} + 2 \arctan \sqrt{e} - \frac{\pi}{2}. \end{aligned}$$

Exercise 3.3

Compute

$$\int_{-\pi}^{\pi} e^{-|x|} \cos x \, dx$$

Solution

We observe that the function is even, since $f(-x) = f(x)$ for every $x \in [-\pi, \pi]$. Hence, by symmetry

$$\int_{-\pi}^{\pi} e^{-|x|} \cos x \, dx = 2 \int_0^{\pi} e^{-|x|} \cos x \, dx = 2 \int_0^{\pi} e^{-x} \cos x \, dx$$

where in the last equality we used the fact that $|x| = x$ for every $x \geq 0$. Integrating by parts twice, we have

$$\begin{aligned} I &= \int e^{-x} \cos x \, dx = e^{-x} \sin x + \int e^{-x} \sin x \, dx \\ &= e^{-x} \sin x - e^{-x} \cos x - \int e^{-x} \cos x \, dx = e^{-x} \sin x - e^{-x} \cos x - I \end{aligned}$$

from which

$$2I = e^{-x} \sin x - e^{-x} \cos x$$

and finally

$$I = \int e^{-x} \cos x \, dx = \frac{e^{-x} \sin x - e^{-x} \cos x}{2} + c$$

To compute the definite integral, we evaluate the antiderivative between 0 and π , not forgetting the factor 2 in front of the integral:

$$2[I]_0^{\pi} = e^{-\pi} + 1$$

Exercise 3.4

Compute

$$\int_0^2 \frac{e^x \log(1 + e^x)}{1 + e^x} dx$$

Solution

With the substitution $1 + e^x = t$, $e^x dx = dt$, we obtain

$$\begin{aligned} \int_0^2 \frac{e^x \log(1 + e^x)}{1 + e^x} dx &= \int_2^{e^2+1} \frac{\log t}{t} dt \\ &= \left[\frac{\log^2 t}{2} \right]_2^{e^2+1} = \frac{1}{2} (\log^2(e^2 + 1) - \log^2 2) \end{aligned}$$

Exercise 3.5

Determine whether the following integral

$$\int_0^1 \frac{\sin^3 \sqrt{x}}{e^x(1 - \cos x)} dx$$

exists and is finite.

Solution

Since the integrand is continuous on $(0, 1]$, to determine whether the given integral converges it is enough to study the behavior of f in a right neighborhood of $x = 0$. As $x \rightarrow 0^+$ we have

$$f(x) \sim \frac{(\sqrt{x})^3}{\frac{1}{2}x^2} = \frac{x^{3/2}}{\frac{1}{2}x^2} = 2 \frac{1}{x^{1/2}}$$

Since $1/2 < 1$, by the limit comparison test the given integral exists and is finite; more formally, the function $f(x)$ is improperly integrable in a right neighborhood of the origin.

Exercise 3.6

Determine whether the following integral

$$\int_0^{+\infty} \frac{x^2}{(3 + 5x^5) \arctan x^{3/2}} dx$$

exists and is finite.

Solution

The integrand is continuous on $(0, +\infty)$. To determine whether the given integral converges we need to study the behavior of the integrand in a right neighborhood of $x = 0$, where the denominator is not defined, and in a neighborhood of $+\infty$. It is convenient to rewrite this integral of the third kind as

$$\int_0^1 \frac{x^2}{(3+5x^5) \arctan x^{3/2}} dx + \int_1^{+\infty} \frac{x^2}{(3+5x^5) \arctan x^{3/2}} dx$$

in order to highlight the two separate analyses to be carried out. As $x \rightarrow 0^+$ we have

$$f(x) \sim \frac{x^2}{3x^{3/2}} = \frac{1}{3x^{-1/2}}$$

Since $-1/2 < 1$, by the limit comparison test the function $f(x)$ is improperly integrable in a right neighborhood of the origin. Let us now verify that it is also improperly integrable in a neighborhood of $+\infty$. As $x \rightarrow +\infty$ we have

$$f(x) \sim \frac{x^2}{5x^{5\pi/2}} = \frac{2}{5\pi} \frac{1}{x^3}$$

Since $3 > 1$, by the limit comparison test the function $f(x)$ is improperly integrable also in a neighborhood of $+\infty$. Overall, therefore, the improper integral exists and is finite. We observe that, in fact, since $f(x) \sim \frac{\sqrt{x}}{3}$ as $x \rightarrow 0^+$, the function can be extended by continuity at $x = 0$ and we can set, by definition, $f(0) = 0$.

Exercise 3.7

Determine for which values of the parameter $\alpha \in \mathbb{R}$ the following improper integral

$$\int_0^1 \frac{\log(1+x^2)}{x^\alpha(1+x^3)} dx$$

exists and is finite.

Solution

The integrand is continuous on $(0, 1)$. To determine for which values of $\alpha \in \mathbb{R}$ the given integral converges it is enough to study the behavior of the integrand in a right neighborhood of $x = 0$, where the denominator is not defined. As $x \rightarrow 0^+$ we have

$$f(x) \sim \frac{x^2}{x^\alpha} = \frac{1}{x^{\alpha-2}}$$

which is improperly integrable for $\alpha - 2 < 1$, i.e., for $\alpha < 3$.

Exercise 3.8

Given the following improper integral

$$\int_1^{+\infty} \frac{x \arctan(x^2) \log^\alpha(1 + \frac{1}{x})}{x^4 + 1} dx$$

study its convergence as $\alpha \in \mathbb{R}$ varies. Then, for $\alpha = 0$, compute it using the definition.

Solution

The integrand is continuous on $[1, +\infty)$. To determine for which values of $\alpha \in \mathbb{R}$ the given integral converges it is enough to study the behavior of the integrand in a neighborhood of $+\infty$. As $x \rightarrow +\infty$ we have

$$f(x) \sim \frac{x}{x^4} \frac{\pi}{2} \left(\frac{1}{x}\right)^\alpha = \frac{\pi}{2} \frac{1}{x^{\alpha+3}}$$

where we needed the asymptotic estimates $x^4 + 1 \sim x^4$, $\log^\alpha(1 + \frac{1}{x}) \sim (\frac{1}{x})^\alpha$, $x \rightarrow +\infty$. By the limit comparison test, the improper integral exists and is finite if $\alpha + 3 > 1$, that is, $\alpha > -2$.

For $\alpha = 0$, the integral becomes

$$\int_1^{+\infty} \frac{x \arctan(x^2)}{x^4 + 1} dx$$

We can proceed by substitution, setting $\arctan(x^2) = t$, $\frac{x}{1+x^4} dx = \frac{1}{2} dt$, and we obtain

$$\int_1^{+\infty} \frac{x \arctan(x^2)}{x^4 + 1} dx = \int_{\pi/4}^{\pi/2} \frac{t}{2} dt = \frac{3}{64} \pi^2$$

Exercise 3.9

After determining whether or not the function is summable, compute the following integrals

$$(a) \int_{-1}^0 \frac{1}{(1-x)\sqrt{1+x}} dx, \quad (b) \int_{-1}^1 \frac{1}{(1-x)\sqrt{1+x}} dx, \quad (c) \int_0^{+\infty} x e^{-x^2} dx$$

$$(d) \int_0^{+\infty} x^3 e^{-x^2} dx, \quad (e) \int_4^{+\infty} \frac{1}{x(x-3)} dx$$

These are all continuous and positive functions on their respective intervals of integration, hence they are all integrable functions. Each integral represents the measure of the region under the graph. In all cases, the region under the graph is an unbounded subset of the plane, so the integral is either a positive number or $+\infty$. Recall that a function is called summable when it is integrable with finite integral.

Solution

(a) The function

$$f(x) = \frac{1}{(1-x)\sqrt{1+x}}$$

is continuous on the interval $(-1, 0]$. Hence, we need to study its behavior in a right neighborhood of $x = -1$. As $x \rightarrow -1^+$ we have

$$f(x) \sim \frac{1}{\sqrt{1+x}} = \frac{1}{(1+x)^{\frac{1}{2}}}, \quad x \rightarrow -1^+$$

hence the function is summable on $(-1, 0]$ by the comparison test, since $\frac{1}{2} < 1$. This means that

$$\int_{-1}^0 f(x)dx = \lim_{\varepsilon \rightarrow 0^+} \int_{-1+\varepsilon}^0 f(x)dx$$

and that this limit exists, finite and positive.

Setting $y = \sqrt{1+x}$, $x = y^2 - 1$, $dx = 2y dy$, we obtain

$$\int_{-1+\varepsilon}^0 f(x)dx = \int_{\sqrt{\varepsilon}}^1 \frac{2}{2-y^2} dy,$$

hence, taking into account that $\sqrt{\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$, we have

$$\int_{-1}^0 f(x)dx = \lim_{\delta \rightarrow 0^+} \int_{\delta}^1 \frac{2}{2-y^2} dy = \int_0^1 \frac{2}{2-y^2} dy.$$

Note that, in this case, after the change of variable, we obtained the integral of a bounded function starting from the integral of an unbounded one.

Using partial fractions, and taking into account also that $|y - \sqrt{2}| = \sqrt{2} - y$ on the interval of integration, we have

$$\frac{2}{2-y^2} = -\frac{2}{(y-\sqrt{2})(y+\sqrt{2})} = \frac{1}{\sqrt{2}} \left(\frac{1}{y+\sqrt{2}} - \frac{1}{y-\sqrt{2}} \right),$$

hence

$$\begin{aligned} \int_{-1}^0 f(x)dx &= \frac{1}{\sqrt{2}} \int_0^1 \left(\frac{1}{y+\sqrt{2}} - \frac{1}{y-\sqrt{2}} \right) dy \\ &= \frac{1}{\sqrt{2}} \left[\log \frac{y+\sqrt{2}}{\sqrt{2}-y} \right]_0^1 = \frac{1}{\sqrt{2}} \log \frac{1+\sqrt{2}}{\sqrt{2}-1}. \end{aligned}$$

Solution

(b) We have

$$\int_{-1}^1 \frac{1}{(1-x)\sqrt{1+x}} dx = \int_{-1}^0 \frac{1}{(1-x)\sqrt{1+x}} dx + \int_0^1 \frac{1}{(1-x)\sqrt{1+x}} dx$$

where the first integral is finite and was computed in the previous item. The integral now

given is therefore finite if and only if so is

$$\int_0^1 \frac{1}{(1-x)\sqrt{1+x}} dx.$$

The function

$$f(x) = \frac{1}{(1-x)\sqrt{1+x}}$$

is continuous and bounded on every interval $[0, 1 - \varepsilon]$, $\varepsilon > 0$. Moreover, we have

$$f(x) \sim \frac{1}{\sqrt{2}} \frac{1}{1-x}, \quad x \rightarrow 1^-$$

hence the function is not summable on $[0, 1)$ by the comparison test. This means that

$$\int_0^1 f(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_0^{1-\varepsilon} f(x) dx = +\infty,$$

hence the given integral is also

$$\int_{-1}^1 \frac{1}{(1-x)\sqrt{1+x}} dx = +\infty.$$

Solution

(c) The function $f(x) = xe^{-x^2}$ is continuous and bounded on every interval $[0, a]$ with $a > 0$. Moreover, we have

$$f(x) = O\left(\frac{1}{x^\alpha}\right), \quad x \rightarrow +\infty,$$

for any $\alpha > 0$; in particular, choosing $\alpha > 1$, the function is summable on $[0, +\infty)$. This means that

$$\int_0^{+\infty} f(x) dx = \lim_{a \rightarrow +\infty} \int_0^a f(x) dx$$

and that this limit exists, finite and positive. Indeed

$$\lim_{a \rightarrow +\infty} \int_0^a xe^{-x^2} dx = -\frac{1}{2} \lim_{a \rightarrow +\infty} [e^{-x^2}]_0^a = -\frac{1}{2} \lim_{a \rightarrow +\infty} (e^{-a^2} - 1) = \frac{1}{2}.$$

The given integral is equal to $1/2$.

Solution

(d) The function $f(x) = x^3e^{-x^2}$ is continuous and bounded on every interval $[0, a]$ with $a > 0$. Moreover, we have

$$f(x) = O\left(\frac{1}{x^\alpha}\right), \quad x \rightarrow +\infty,$$

for any $\alpha > 0$; in particular, choosing $\alpha > 1$, the function is summable on $[0, +\infty)$. This

means that

$$\int_0^{+\infty} f(x)dx = \lim_{a \rightarrow +\infty} \int_0^a f(x)dx$$

and that this limit exists, finite and positive. Indeed, integrating by parts, and taking into account the integral computed in the previous item,

$$\begin{aligned} \lim_{a \rightarrow +\infty} \int_0^a x^3 e^{-x^2} dx &= \lim_{a \rightarrow +\infty} \left\{ -\frac{1}{2} [x^2 e^{-x^2}]_0^a + \int_0^a x e^{-x^2} dx \right\} \\ &= \lim_{a \rightarrow +\infty} \left(-\frac{1}{2} a^2 e^{-a^2} + \int_0^a x e^{-x^2} dx \right) = \frac{1}{2}. \end{aligned}$$

The given integral is equal to $1/2$.

Solution

(e) The function $f(x) = \frac{1}{x(x-3)}$ is continuous and bounded on every interval $[4, a]$ with $a > 4$. Moreover, we have

$$f(x) \sim \frac{1}{x^2}, \quad x \rightarrow +\infty,$$

hence the function is summable on $[4, +\infty)$ by comparison. This means that

$$\int_4^{+\infty} f(x)dx = \lim_{a \rightarrow +\infty} \int_4^a f(x)dx$$

and that this limit exists, finite and positive. Indeed, using partial fractions,

$$\begin{aligned} \lim_{a \rightarrow +\infty} \int_4^a \frac{1}{x(x-3)} dx &= \frac{1}{3} \lim_{a \rightarrow +\infty} \int_4^a \left(\frac{1}{x-3} - \frac{1}{x} \right) dx = \frac{1}{3} \lim_{a \rightarrow +\infty} \left[\log \frac{x-3}{x} \right]_4^a \\ &= \frac{1}{3} \lim_{a \rightarrow +\infty} \left(\log \frac{a-3}{a} - \log \frac{1}{4} \right) = -\frac{1}{3} \log \frac{1}{4} = \frac{1}{3} \log 4. \end{aligned}$$

The given integral is equal to $\frac{1}{3} \log 4$.

Exercise 3.10

Determine all $\alpha \geq 0$ for which the function

$$f(x) = (\sqrt{1+x} - \sqrt{x})^\alpha$$

is summable on $[0, +\infty)$

Solution

The function $f(x)$ is continuous and positive on $[0, +\infty)$, so it is enough to analyze its behavior in a neighborhood of $+\infty$, that is, as $x \rightarrow +\infty$. Multiplying and dividing by

$(\sqrt{1+x} + \sqrt{x})^\alpha$, we have

$$f(x) = \frac{1}{(\sqrt{1+x} + \sqrt{x})^\alpha} \sim \frac{1}{(2\sqrt{x})^\alpha} = \frac{1}{2^\alpha} \frac{1}{x^{\alpha/2}}, \quad x \rightarrow +\infty.$$

By comparison, it follows that f is summable on $[0, +\infty)$ if and only if $\alpha/2 > 1$, hence if and only if

$$\alpha > 2.$$

Exercise 3.11

Given

$$\int_0^{+\infty} \frac{\log(1 + e^{x^2})}{2 + 3x^{5/2} + x^{3\alpha} + 12x} dx$$

determine for which values of $\alpha > 0$ it converges, that is, it exists and is finite.

Solution

The function $f(x)$ is continuous and positive on $[0, +\infty)$, so it is enough to analyze its behavior in a neighborhood of $+\infty$, that is, as $x \rightarrow +\infty$. We have

$$f(x) = \frac{\log(1 + e^{x^2})}{2 + 3x^{5/2} + x^{3\alpha} + 12x} \sim \frac{\log(e^{x^2})}{3x^{5/2} + x^{3\alpha}} = \frac{x^2}{3x^{5/2} + x^{3\alpha}} \sim$$

$$\sim \begin{cases} \frac{x^2}{3x^{5/2}} = \frac{1}{3} \frac{1}{x^{1/2}} & \text{if } 3\alpha < 5/2, \text{ i.e., if and only if } \alpha < 5/6 \\ \frac{x^2}{4x^{5/2}} = \frac{1}{4} \frac{1}{x^{1/2}} & \text{if } 3\alpha = 5/2, \text{ i.e., if and only if } \alpha = 5/6 \\ \frac{x^2}{x^{3\alpha}} = \frac{1}{x^{3\alpha-2}} & \text{if } 3\alpha > 5/2, \text{ i.e., if and only if } \alpha > 5/6 \end{cases}$$

In the first two cases, the improper integral diverges by limit comparison, since $1/2 < 1$. In the third case, instead, it exists and is finite if and only if $3\alpha - 2 > 1$, that is, if and only if $\alpha > 1$. In conclusion, the improper integral exists and is finite (converges) for $\alpha > 1$, and diverges for $\alpha \leq 1$.

Exercise 3.12

Given

$$\int_0^{+\infty} \frac{3 + 2x^{3/2} + 5x^{4\alpha} + x^{1/2}}{\log(1 + e^{x^5})} dx$$

determine for which values of $\alpha > 0$ it converges, that is, it exists and is finite.

Solution

The function $f(x)$ is continuous and positive on $[0, +\infty)$, so it is enough to analyze its behavior in a neighborhood of $+\infty$, that is, as $x \rightarrow +\infty$. We have

$$f(x) = \frac{3 + 2x^{3/2} + 5x^{4\alpha} + x^{1/2}}{\log(1 + e^{x^5})} \sim \frac{2x^{3/2} + 5x^{4\alpha}}{\log(e^{x^5})} = \frac{2x^{3/2} + 5x^{4\alpha}}{x^5} \sim$$

$$\sim \begin{cases} \frac{2x^{3/2}}{x^5} = 2\frac{1}{x^{7/2}} & \text{if } 4\alpha < 3/2, \text{ i.e., if and only if } \alpha < 3/8 \\ \frac{7x^{3/2}}{x^5} = 7\frac{1}{x^{7/2}} & \text{if } 4\alpha = 3/2, \text{ i.e., if and only if } \alpha = 3/8 \\ \frac{5x^{4\alpha}}{x^5} = 5\frac{1}{x^{5-4\alpha}} & \text{if } 4\alpha > 3/2, \text{ i.e., if and only if } \alpha > 3/8 \end{cases}$$

In the first two cases, the improper integral converges by limit comparison, since $7/2 > 1$. In the third case, instead, it exists and is finite if and only if $5 - 4\alpha > 1$, that is, if and only if $\alpha < 1$. In conclusion, the improper integral exists and is finite (converges) for $\alpha < 1$, and diverges for $\alpha \geq 1$.

Exercise 3.13

Given

$$\int_1^{+\infty} \frac{\arctan x}{\sqrt{x}(1 + x^\alpha)} dx$$

determine for which values of $\alpha \in \mathbb{R}$ it converges, that is, it exists and is finite.

Solution

The function $f(x)$ is continuous and positive on $[1, +\infty)$, so it is enough to analyze its behavior in a neighborhood of $+\infty$, that is, as $x \rightarrow +\infty$. We have

$$f(x) \sim \begin{cases} \frac{\pi/2}{x^{1/2} + x^{1/2+\alpha}} \sim \frac{\pi}{2x^{1/2}} & \text{if } \alpha < 0 \\ \frac{\pi/2}{2x^{1/2}} = \frac{\pi}{4x^{1/2}} & \text{if } \alpha = 0 \\ \frac{\pi/2}{x^{1/2} + x^{1/2+\alpha}} \sim \frac{\pi}{2x^{1/2+\alpha}} & \text{if } \alpha > 0 \end{cases}$$

In the first two cases, the improper integral diverges by limit comparison, since $1/2 < 1$. In the third case, instead, it exists and is finite if and only if $\frac{1}{2} + \alpha > 1$, that is, if and only if $\alpha > \frac{1}{2}$. In conclusion, the improper integral exists and is finite (converges) in a neighborhood $U(+\infty)$ for $\alpha > \frac{1}{2}$, and diverges for $\alpha \leq \frac{1}{2}$.

Exercise 3.14

Perform the change of variable $y = \sqrt{x}$ in

$$\int_4^{+\infty} \frac{1}{\sqrt{x}(\sqrt{x}+2)(\sqrt{x}+6)} dx$$

and write the corresponding integral in dy . Then determine the value of the integral.

Solution

From $y = \sqrt{x}$, $x = y^2$, $dx = 2y dy$, we obtain

$$\int_4^{+\infty} \frac{1}{\sqrt{x}(\sqrt{x}+2)(\sqrt{x}+6)} dx = \int_2^{+\infty} \frac{2}{(y+2)(y+6)} dy.$$

Using partial fractions,

$$\begin{aligned} \int_2^{+\infty} \frac{2}{(y+2)(y+6)} dy &= \lim_{a \rightarrow +\infty} \int_2^a \frac{2}{(y+2)(y+6)} dy \\ &= \frac{1}{2} \lim_{a \rightarrow +\infty} \int_2^a \left(\frac{1}{y+2} - \frac{1}{y+6} \right) dy = \frac{1}{2} \lim_{a \rightarrow +\infty} \left[\log \frac{y+2}{y+6} \right]_2^a \\ &= \frac{1}{2} \lim_{a \rightarrow +\infty} \left(\log \frac{a+2}{a+6} - \log \frac{1}{2} \right) = -\frac{1}{2} \log \frac{1}{2} = \frac{1}{2} \log 2. \end{aligned}$$

Exercise 3.15

Given the function

$$f : [1, +\infty) \rightarrow \mathbb{R}, \quad f(x) = \int_1^x \frac{(t^3 - 27)(t-1)^3}{t^7 + 1} dt$$

- Write $f'(x)$.
- Find the possible relative maximum or minimum points in the interior of the domain, specifying the sign of the values taken at those points.
- State whether $\lim_{x \rightarrow +\infty} f(x)$ exists and whether it is finite or not, justifying your answer.
- Determine for which value of α the limit

$$\lim_{x \rightarrow 1} (x-1)^\alpha f(x)$$

is finite and different from 0, justifying your answer.

Solution

By the Fundamental theorem of calculus

$$f'(x) = \frac{(x^3 - 27)(x - 1)^3}{x^7 + 1}.$$

Solution

The first derivative vanishes at the endpoint $x = 1$ of the domain and at the interior point $x = 3$. It is negative for $1 < x < 3$ and positive for $x > 3$. The point $x = 3$ is an interior (absolute) minimum point. The corresponding value

$$f(3) = \int_1^3 \frac{(t^3 - 27)(t - 1)^3}{t^7 + 1} dt$$

is negative, as follows from the fact that $f(1) = 0$ and that f is strictly decreasing on $[1, 3]$. The same conclusion is reached directly by observing that the integrand is negative on the interval $[1, 3]$.

Solution

The function f is strictly increasing on $(3, +\infty)$, hence the limit exists, finite or $+\infty$. By definition,

$$\lim_{x \rightarrow +\infty} f(x) = \int_1^{+\infty} \frac{(t^3 - 27)(t - 1)^3}{t^7 + 1} dt,$$

hence the question is equivalent to asking whether or not the function

$$g(t) = \frac{(t^3 - 27)(t - 1)^3}{t^7 + 1}$$

is summable on $[1, +\infty)$.

The function $g(t)$ is continuous on $[1, +\infty)$, so it is enough to examine its behavior as $t \rightarrow +\infty$. We have

$$g(t) \sim \frac{1}{t}, \quad t \rightarrow +\infty,$$

from which the function is not summable on $[1, +\infty)$.

In conclusion, $\lim_{x \rightarrow +\infty} f(x) = +\infty$.

Solution

We have $\lim_{x \rightarrow 1} f(x) = f(1) = 0$, hence for $\alpha \geq 0$ the given limit is zero. For $\alpha < 0$, say $\alpha = -\beta$ with $\beta > 0$, the given limit

$$\lim_{x \rightarrow 1^+} \frac{f(x)}{(x - 1)^\beta}$$

has the indeterminate form $\frac{0}{0}$. Applying L'Hôpital's rule, we have

$$\begin{aligned}\lim_{x \rightarrow 1^+} \frac{f(x)}{(x-1)^\beta} &= \lim_{x \rightarrow 1^+} \frac{f'(x)}{\beta(x-1)^{\beta-1}} \\ &= \lim_{x \rightarrow 1^+} \frac{(x^3 - 27)(x-1)^3}{\beta(x-1)^{\beta-1}(x^7 + 1)} = -\frac{13}{\beta} \lim_{x \rightarrow 1^+} \frac{(x-1)^3}{(x-1)^{\beta-1}}.\end{aligned}$$

This limit exists for every $\beta > 0$, but it is finite and different from zero, as required, if and only if $\beta = 4$ (and in this case it equals $-13/4$).

In conclusion, the given limit exists, finite and different from zero, if and only if $\alpha = -4$.

Exercise 3.16

Compute

$$\int_0^{+\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+4)} dx$$

Then explain how one can anticipate, without using antiderivatives, the summability of the function on the given interval.

Solution

The function

$$f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+4)}$$

is continuous and positive on $(0, +\infty)$, unbounded as $x \rightarrow 0$, and the interval of integration is unbounded. Writing

$$\int_0^{+\infty} f(x) dx = \int_0^1 f(x) dx + \int_1^{+\infty} f(x) dx,$$

it is necessary and sufficient to examine the asymptotic behavior of $f(x)$ as $x \rightarrow 0$ and as $x \rightarrow +\infty$. We have

$$f(x) \sim \frac{1}{4\sqrt{x}}, \quad x \rightarrow 0; \quad f(x) \sim \frac{1}{x^{3/2}}, \quad x \rightarrow +\infty.$$

Therefore, $f(x)$ is summable on $(0, +\infty)$ because, by comparison, it is summable both on $(0, 1]$ and on $[1, +\infty)$.

We can compute the value with the substitution $y = \sqrt{x}$, $x = y^2$, $dx = 2y \, dy$:

$$\begin{aligned} \int_0^{+\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+4)} dx &= \int_0^{+\infty} \frac{2}{(y+1)(y+4)} dy \\ &= \frac{2}{3} \int_0^{+\infty} \left(\frac{1}{y+1} - \frac{1}{y+4} \right) dy = \frac{2}{3} \lim_{a \rightarrow +\infty} \left[\log \frac{y+1}{y+4} \right]_0^a \\ &= \frac{2}{3} \lim_{a \rightarrow +\infty} \left(\log \frac{a+1}{a+4} - \log \frac{1}{4} \right) = -\frac{2}{3} \log \frac{1}{4} = \frac{2}{3} \log 4. \end{aligned}$$

Exercise 3.17

Let

$$f : [1/2, +\infty[\rightarrow \mathbb{R}, \quad f(x) = \int_1^x \frac{\log t}{t^2} dt$$

- Determine the possible local extremum points and the monotonicity of f .
- Justify the existence of a horizontal asymptote for f .
- Compute $\lim_{x \rightarrow 1} \frac{f(x)}{(x-1)^2}$.

Solution

By the Fundamental theorem of calculus,

$$f'(x) = \frac{\log x}{x^2}$$

for every x in the given domain $[1/2, +\infty)$. The derivative vanishes at $x = 1$, is negative for $1/2 \leq x < 1$ and positive for $x > 1$. The function f is therefore strictly decreasing on $[1/2, 1)$ and strictly increasing on $(1, +\infty)$. The point $x = 1$ is an absolute minimum point with value $f(1) = 0$; in particular the function takes only non-negative values (positive for every $x \neq 1$ in the given domain).

Solution

We have to justify that $\lim_{x \rightarrow +\infty} f(x)$ exists and is finite. By definition,

$$\lim_{x \rightarrow +\infty} f(x) = \int_1^{+\infty} \frac{\log t}{t^2} dt,$$

hence we have to justify the fact that the function

$$g(t) = \frac{\log t}{t^2}$$

is summable on $[1, +\infty)$.

The function $g(t)$ is continuous on $[1, +\infty)$, so it is enough to examine its behavior as $t \rightarrow +\infty$. Since $\log t$ is an infinity of lower order than any power, we have

$$g(t) = O\left(\frac{1}{t^{2-\varepsilon}}\right), \quad t \rightarrow +\infty,$$

for every $\varepsilon > 0$, from which, choosing ε such that $2 - \varepsilon > 1$, the function g is summable on $[1, +\infty)$.

Solution

We have $\lim_{x \rightarrow 1} f(x) = f(1) = 0$, hence the given limit has the indeterminate form $\frac{0}{0}$. Applying L'Hôpital's rule, we have

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{f(x)}{(x-1)^2} &= \lim_{x \rightarrow 1} \frac{f'(x)}{2(x-1)} \\ &= \lim_{x \rightarrow 1} \frac{\log x}{2(x-1)x^2} = \frac{1}{2} \lim_{x \rightarrow 1} \frac{\log x}{x-1} = \frac{1}{2} \lim_{y \rightarrow 0} \frac{\log(1+y)}{y} = \frac{1}{2}. \end{aligned}$$

Exercise 3.18

Consider

$$\int_2^{+\infty} \frac{x+2}{x(x+1)(x-1)} dx.$$

Before computing it, justify the fact that this integral exists and is finite.

Solution

The function

$$f(x) = \frac{x+2}{x(x+1)(x-1)}$$

is continuous and positive on $[2, +\infty)$ and has the asymptotic behavior

$$f(x) \sim \frac{1}{x^2}, \quad x \rightarrow +\infty,$$

hence it is improperly integrable in a neighborhood of $+\infty$ by comparison.

From the partial fractions

$$\frac{x+2}{x(x+1)(x-1)} = -\frac{2}{x} + \frac{1}{2} \frac{1}{x+1} + \frac{3}{2} \frac{1}{x-1},$$

we obtain the antiderivative on $[2, +\infty)$

$$\int \frac{x+2}{x(x+1)(x-1)} dx = \log \frac{(x+1)^{1/2}(x-1)^{3/2}}{x^2}.$$

Hence

$$\begin{aligned} \int_2^{+\infty} \frac{x+2}{x(x+1)(x-1)} dx &= \lim_{a \rightarrow +\infty} \left[\log \frac{(x+1)^{1/2}(x-1)^{3/2}}{x^2} \right]_2^a \\ &= \lim_{a \rightarrow +\infty} \left(\log \frac{(a+1)^{1/2}(a-1)^{3/2}}{a^2} - \log \frac{3^{1/2}}{4} \right) = \log \frac{4}{\sqrt{3}}. \end{aligned}$$

Exercise 3.19

Using the change of variable $y = e^x$, compute

$$\int_0^{+\infty} \frac{e^x}{(e^x+2)(e^x+3)} dx.$$

Solution

From $y = e^x$, $x = \log y$, $dx = \frac{1}{y} dy$ and using partial fractions, it follows that

$$\begin{aligned} \int_0^{+\infty} \frac{e^x}{(e^x+2)(e^x+3)} dx &= \int_1^{+\infty} \frac{1}{(y+2)(y+3)} dy \\ &= \int_1^{+\infty} \left(\frac{1}{y+2} - \frac{1}{y+3} \right) dy = \lim_{a \rightarrow +\infty} \left[\log \frac{y+2}{y+3} \right]_1^a \\ &= \lim_{a \rightarrow +\infty} \left(\log \frac{a+2}{a+3} - \log \frac{3}{4} \right) = \log \frac{4}{3} \end{aligned}$$

Exercise 3.20

Given the function

$$f : [1, +\infty) \rightarrow \mathbb{R}, \quad f(x) = \int_1^x \frac{t^3 - 27}{t^5 + 1} dt$$

- Write $f'(x)$.
- Determine the possible relative maximum or minimum points.
- Write $f''(x)$.
- To justify the presence or absence of inflection points, carry out separately a brief study of the polynomial function $y = -2x^5 + 135x^2 + 3$, sketching a qualitative graph. This polynomial is one of the factors of $f''(x)$.
- Based on the previous item, state whether f has inflection points. (You are not asked to find these points explicitly, but only to state whether there are any, how many there are, and to locate them with respect to other notable points such as relative maximum or minimum points).
- State whether $\lim_{x \rightarrow +\infty} f(x)$ exists and whether it is finite or not, justifying your answer (if it is finite, you are not asked to compute it).

Solution

By the Fundamental theorem of calculus,

$$f'(x) = \frac{x^3 - 27}{x^5 + 1}$$

for all x in the given domain $[1, +\infty)$.

Solution

The derivative vanishes at $x = 3$, is negative on $[1, 3)$ and positive on $(3, +\infty)$. The function is strictly decreasing on $[1, 3)$ and strictly increasing on $(3, +\infty)$. The point $x = 3$ is an absolute minimum point, with value

$$f(3) = \int_1^3 \frac{t^3 - 27}{t^5 + 1} dt,$$

which can be computed using partial fractions if necessary, and which is negative since $f(1) = 0$ and f is strictly decreasing on $[1, 3)$. The sign of $f(3)$ can also be determined directly by observing that the integrand is strictly negative for $t \in [1, 3)$.

Solution

$$f''(x) = \frac{x^2(-2x^5 + 135x^2 + 3)}{(x^5 + 1)^2}.$$

Solution

We have $\lim_{x \rightarrow \pm\infty} y(x) = \mp\infty$. Then $y'(x) = -10x(x^3 - 27)$, from which $y(x)$ is strictly decreasing on $(-\infty, 0)$, the point $x = 0$ is a relative minimum point with value $y(0) = 3$, $y(x)$ is strictly increasing on $(0, 3)$, the point $x = 3$ is a relative maximum point with value $y(3) = 732$, and $y(x)$ is strictly decreasing on $(3, +\infty)$. In particular, $y(x)$ has a unique zero at $x = \alpha$ with $\alpha \in (3, +\infty)$, $y(x) > 0$ for $x < \alpha$, $y(x) < 0$ for $x > \alpha$.

Solution

On the given domain, the second derivative has the same sign as $-2x^5 + 135x^2 + 3$. By what we saw in the previous item, the function f has an inflection point at $x = \alpha$, with $\alpha \in (3, +\infty)$, is strictly convex on $(1, \alpha)$ and strictly concave on $(\alpha, +\infty)$.

Solution

The function is strictly increasing on $(3, +\infty)$, hence the limit exists, finite or $+\infty$.

By definition,

$$\lim_{x \rightarrow +\infty} f(x) = \int_1^{+\infty} \frac{t^3 - 27}{t^5 + 1} dt,$$

hence the question is equivalent to asking whether or not the function

$$g(t) = \frac{t^3 - 27}{t^5 + 1}$$

is summable on $[1, +\infty)$.

The function $g(t)$ is continuous on $[1, +\infty)$, so it is enough to examine its behavior as $t \rightarrow +\infty$. We have

$$g(t) \sim \frac{1}{t^2}, \quad t \rightarrow +\infty,$$

from which the function is summable on $[1, +\infty)$.

In conclusion, $\lim_{x \rightarrow +\infty} f(x)$ exists and is finite. If necessary, it can be computed using partial fractions.