



SAPIENZA
UNIVERSITÀ DI ROMA

Mathematical Analysis 1

Exercises, volume 5 — Derivatives

The worked exercises on differential calculus, one sheet per chapter, with the solution right after each statement: cusps, inflection points and tangents, computing derivatives, tangent lines, asymptotes, continuous extensions, differentiability, parametric equations, curve sketching, Maclaurin expansions, limits via asymptotic expansions, Newton's and bisection methods, finding extrema. The sheets are by Fabio Furini and Gianluca Priori; many exercises are taken from M. Bramanti, *Esercitazioni di Analisi matematica 1*, Esculapio.

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Online version and downloadable material:
fabiofurini.github.io/mathematical-analysis-1/
github.com/fabiofurini/mathematical-analysis-1

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Contents

1	Cusps, inflection points and tangents	2
2	Computing derivative functions	7
3	Tangent lines	22
4	Asymptotes	26
5	Continuous extensions	31
6	Differentiability	36
7	Parametric equations	43
8	Curve sketching	47
9	Maclaurin expansions	58
10	Limits via asymptotic expansions	62
11	Newton's method	67
12	Bisection method	70
13	Finding local and global extrema and extremum points	74



Cusps, inflection points and tangents

Exercise 1.1

Using the definition of derivative, determine the behavior at the origin of the following function:

$$f(x) = x^{\frac{4}{3}}$$

Solution

Let

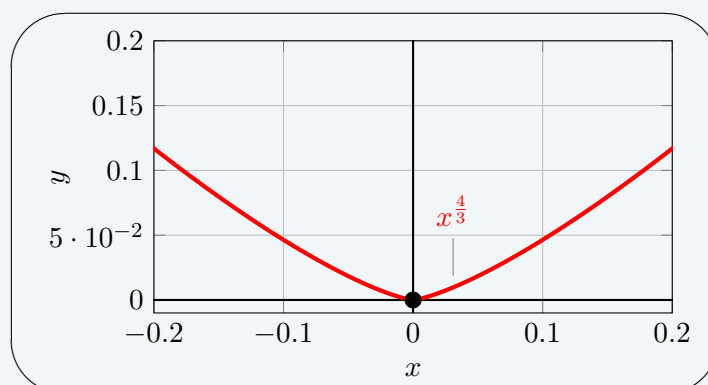
$$f(x) = x^{\frac{4}{3}}$$

then for $x_0 = 0$

$$\frac{f(h) - f(0)}{h} = \frac{h^{4/3}}{h} = h^{1/3}$$

and the limits are:

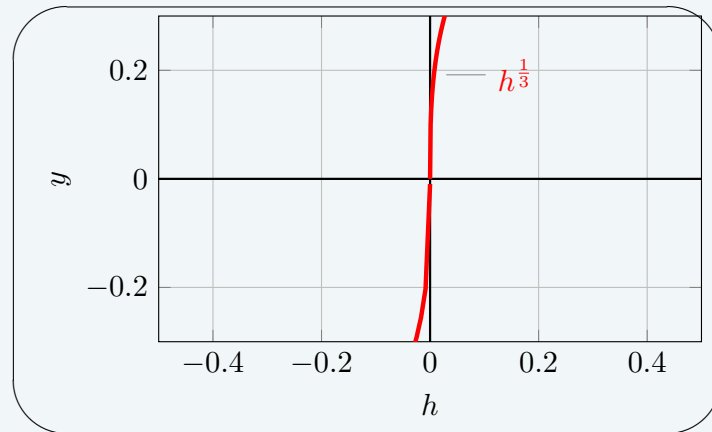
$$\lim_{h \rightarrow 0^-} h^{1/3} = 0 \quad \text{and} \quad \lim_{h \rightarrow 0^+} h^{1/3} = 0 \quad \text{hence} \quad \lim_{h \rightarrow 0} h^{1/3} = 0 \quad \text{and} \quad f'(0) = 0.$$



The function has a **point with horizontal tangent** at $x_0 = 0$. The function:

$$f(h) = h^{\frac{1}{3}}$$

is a power with positive rational exponent $\frac{m}{n}$ with n odd (hence defined on all of $\mathbb{R}$) and m odd (hence an odd function). Its graph is:



Exercise 1.2

Using the definition of derivative, determine the behavior at the origin of the following function:

$$f(x) = x^{\frac{2}{3}}$$

Solution

Let

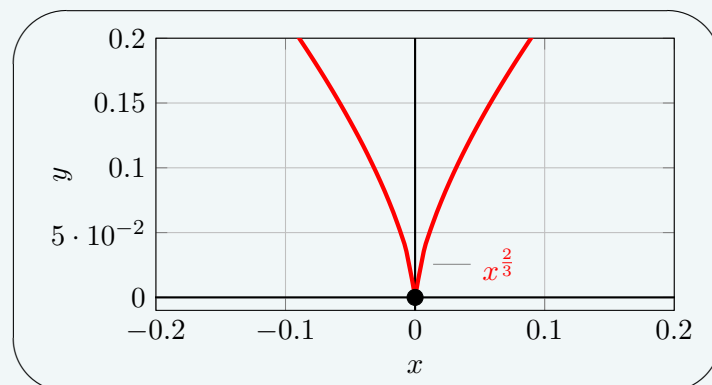
$$f(x) = x^{\frac{2}{3}}$$

then for $x_0 = 0$

$$\frac{f(h) - f(0)}{h} = \frac{h^{2/3} - 0}{h} = \frac{1}{h^{1/3}}$$

and the limits are:

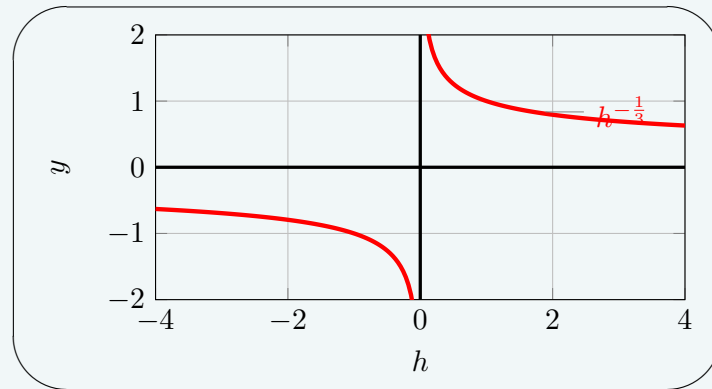
$$\lim_{h \rightarrow 0^-} \frac{1}{h^{1/3}} = -\infty \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{1}{h^{1/3}} = +\infty \quad \text{hence} \quad f'_-(0) = -\infty \quad \text{and} \quad f'_+(0) = +\infty$$



The function has a **cusp** at $x_0 = 0$. The function:

$$f(h) = h^{-\frac{1}{3}} = \frac{1}{h^{1/3}}$$

is a power with negative rational exponent $\frac{m}{n}$ with n odd (hence defined on all of $\mathbb{R}$) and m odd (hence an odd function). Its graph is:

**Exercise 1.3**

Using the definition of derivative, determine the behavior at the origin of the following function:

$$f(x) = x^{\frac{5}{2}}$$

Solution

Let

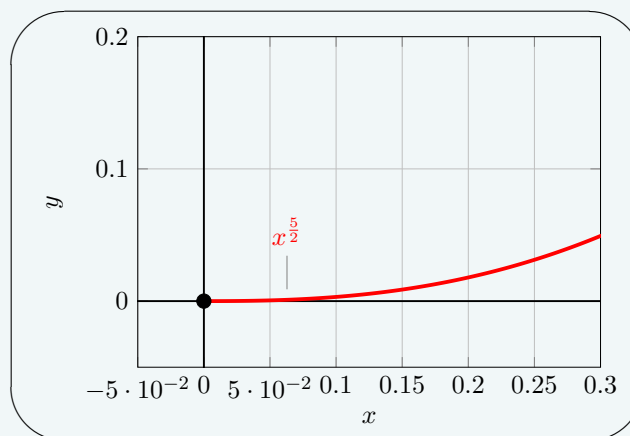
$$f(x) = x^{\frac{5}{2}}$$

then for $x_0 = 0$

$$\frac{f(h) - f(0)}{h} = \frac{h^{5/2}}{h} = h^{3/2}$$

and the limits are:

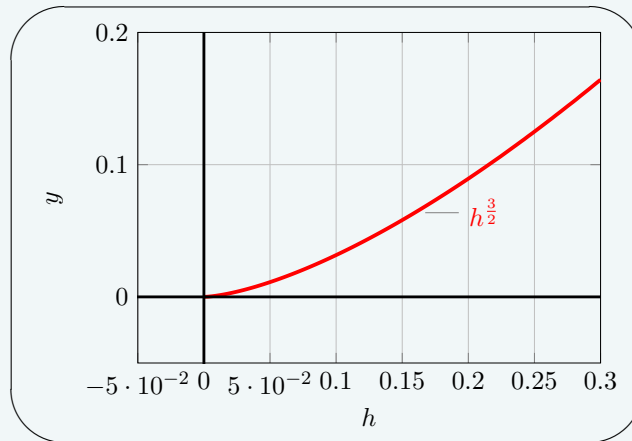
$$\lim_{h \rightarrow 0^+} h^{3/2} = 0.$$



The function has a **point with horizontal tangent** at $x_0 = 0$. The function:

$$f(h) = h^{\frac{3}{2}}$$

is a power with positive rational exponent $\frac{m}{n}$ with n even (hence defined only on $\mathbb{R}_+$). Its graph is:



Exercise 1.4

Using the definition of derivative, determine the behavior at the origin of the following function:

$$f(x) = x^{\frac{3}{2}}$$

Solution

Let

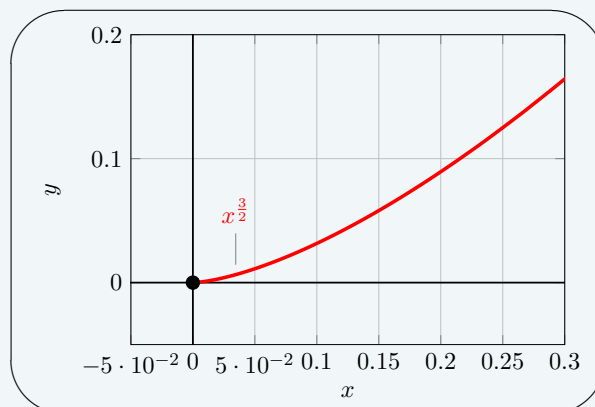
$$f(x) = x^{\frac{3}{2}}$$

then for $x_0 = 0$

$$\frac{f(h) - f(0)}{h} = \frac{h^{3/2}}{h} = h^{1/2}$$

and the limits are:

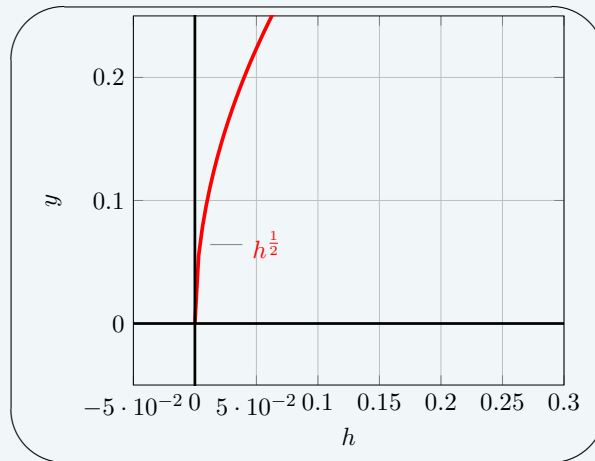
$$\lim_{h \rightarrow 0^+} h^{1/2} = 0.$$



The function has a **point with horizontal tangent** at $x_0 = 0$. The function:

$$f(h) = h^{\frac{1}{2}}$$

is a power with positive rational exponent $\frac{m}{n}$ with n even (hence defined only on $\mathbb{R}_+$). Its graph is:



Computing derivative functions

Table of the derivatives of the elementary functions:

	f	f'
1)	c (constant)	0
2)	x	1
3)	x^2	$2x$
4)	$\frac{1}{x}$	$-\frac{1}{x^2}$
5)	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$ (for $x > 0$)
6)	x^α ($\alpha \in \mathbb{R}, x > 0$)	$\alpha x^{\alpha-1}$
7)	$\sin x$	$\cos x$
8)	$\cos x$	$-\sin x$
9)	$\tan x$	$1 + \tan^2 x = \frac{1}{\cos^2 x}$
10)	$\cot x$	$-(1 + \cot^2 x) = -\frac{1}{\sin^2 x}$
11)	e^x	e^x
12)	$\log x$	$\frac{1}{x}$
13)	a^x ($a > 0$)	$a^x \log a$
14)	$\log_a x$ ($a > 0, a \neq 1$)	$\frac{1}{x \log a}$
15)	$\sinh x$	$\cosh x$
16)	$\cosh x$	$\sinh x$
17)	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
18)	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
19)	$\arctan x$	$\frac{1}{1+x^2}$

Exercise 2.1

Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$.

1. Write the difference quotient of f at the point $x = 1$.
2. Compute $f'(1)$ as the limit of the difference quotient.
3. Write the equation of the tangent line to the graph of f at the point $(1, 1)$.

Solution

1.

$$\frac{f(1+h) - f(1)}{h} = \frac{(1+h)^3 - 1}{h} = \frac{3h + 3h^2 + h^3}{h} = 3 + 3h + h^2$$

2.

$$f'(1) = \lim_{h \rightarrow 0} 3 + 3h + h^2 = 3$$

3. The tangent line has equation $y = f(1) + f'(1)(x - 1)$, hence

$$y = 1 + 3(x - 1)$$

from which

$$y = 3x - 2$$

Exercise 2.2Compute the derivative of the following function f , specifying the domain of f and of f' :

$$f(x) = x^2 \sin(x^3)$$

SolutionThe functions f and f' are defined on all of $\mathbb{R}$.

Using the rules for the derivative of a product and of composite functions, we have

$$f'(x) = 2x \cdot \sin(x^3) + x^2 \cdot \cos(x^3) \cdot 3x^2 = 2x \sin(x^3) + 3x^4 \cos(x^3)$$

Exercise 2.3Compute the derivative of the following function f , specifying the domain of f and of f' :

$$f(x) = x^3 \cos(e^{5x^2})$$

SolutionThe functions f and f' are defined on all of $\mathbb{R}$.

Using the rules for the derivative of a product and of composite functions, we have

$$\begin{aligned} f'(x) &= 3x^2 \cdot \cos(e^{5x^2}) - x^3 \cdot \sin(e^{5x^2}) \cdot e^{5x^2} \cdot 10x = \\ &= 3x^2 \cos(e^{5x^2}) - 10x^4 e^{5x^2} \sin(e^{5x^2}) \end{aligned}$$

Exercise 2.4

Compute the derivative of the following function f , specifying the domain of f and of f' :

$$f(x) = e^{1/x}$$

Solution

Using the rules for the derivative of composite functions, we have

$$f'(x) = -\frac{1}{x^2} \cdot e^{1/x}$$

The functions f and f' are defined on $\mathbb{R} \setminus \{0\}$.

Exercise 2.5

Compute the derivative of the following function:

$$f(x) = 3x^4 + 5x + x^{3/2} - 2x^{-3}$$

Solution

$$f'(x) = 12x^3 + 5 + \frac{3}{2}\sqrt{x} + \frac{6}{x^4}$$

Exercise 2.6

Compute the derivatives of the following functions:

$$(1) \ f(x) = \log(|x|), \quad (2) \ f(x) = \log(3x), \quad (3) \ f(x) = \log\left(\left|\frac{x+2}{3-x}\right|\right)$$

Solution

1.

$$(\log(|x|))' = \frac{1}{|x|} \cdot \operatorname{sgn}(x) = \frac{1}{x}$$

2.

$$(\log(3x))' = (\log 3 + \log x)' = \frac{1}{x}$$

3.

$$\begin{aligned}
 \left(\log \left(\left| \frac{x+2}{3-x} \right| \right) \right)' &= \frac{1}{\left| \frac{x+2}{3-x} \right|} \cdot \operatorname{sgn} \left(\frac{x+2}{3-x} \right) \cdot \left(\left(\frac{x+2}{3-x} \right) \right)' = \frac{\left(\left(\frac{x+2}{3-x} \right) \right)'}{\frac{x+2}{3-x}} \\
 &= \left(\left(\frac{x+2}{3-x} \right) \right)' \cdot \frac{3-x}{x+2} = \frac{3-x+x+2}{(3-x)^2} \cdot \frac{3-x}{x+2} \\
 &= \frac{5}{(3-x) \cdot (x+2)} = \frac{5}{-x^2+x+6}
 \end{aligned}$$

Exercise 2.7

Compute the derivative of the following function:

$$f(x) = e^{-3x} (x^2 + 2x - 1)$$

Solution

Using the rules for the derivative of a product and of composite functions, we have:

$$\begin{aligned}
 f'(x) &= -3e^{-3x} (x^2 + 2x - 1) + e^{-3x} (2x + 2) = \\
 &= e^{-3x} (-3x^2 - 6x + 3 + 2x + 2) = e^{-3x} (-3x^2 - 4x + 5)
 \end{aligned}$$

Exercise 2.8

Compute the derivative of the following function:

$$f(x) = \frac{1}{a} \arctan \left(\frac{x}{a} \right), \quad a > 0$$

Solution

$$f'(x) = \frac{1}{a} \cdot \frac{1}{1 + x^2/a^2} \cdot \frac{1}{a} = \frac{1}{a^2} \cdot \frac{1}{\frac{a^2+x^2}{a^2}} = \frac{1}{a^2 + x^2}$$

Exercise 2.9

Compute the derivative of the following function:

$$f(x) = x \log x$$

Solution

Using the rules for the derivative of a product, we have:

$$f'(x) = \log x + x \cdot \frac{1}{x} = \log x + 1.$$

Exercise 2.10

Compute the derivative of the following function:

$$f(x) = \arctan\left(\frac{1+x}{1-x}\right)$$

Solution

Using the rules for the derivative of a quotient and of composite functions, we have:

$$\begin{aligned} f'(x) &= \frac{1}{1 + \frac{(1+x)^2}{(1-x)^2}} \cdot \frac{(1-x) + (1+x)}{(1-x)^2} = \frac{(1-x)^2}{(1-x)^2 + (1+x)^2} \cdot \frac{2}{(1-x)^2} = \\ &= \frac{2}{1+x^2-2x+1+x^2+2x} = \frac{2}{2(x^2+1)} = \frac{1}{x^2+1} \end{aligned}$$

Exercise 2.11

Compute the derivative of the following function:

$$f(x) = e^{2x} (2 \sin 3x - 4 \cos 3x)$$

Solution

Using the rules for the derivative of a product and of composite functions, we have

$$\begin{aligned} f'(x) &= 2e^{2x} (2 \sin 3x - 4 \cos 3x) + e^{2x} (2 \cdot 3 \cos 3x + 4 \cdot 3 \sin 3x) = \\ &= 2e^{2x} (2 \sin 3x - 4 \cos 3x + 3 \cos 3x + 6 \sin 3x) = 2e^{2x} (8 \sin 3x - \cos 3x) \end{aligned}$$

Exercise 2.12

Compute the derivative of the following function:

$$f(x) = e^{\frac{x+2}{x-3}}$$

Solution

Using the rules for the derivative of composite functions, we have

$$f'(x) = e^{\frac{x+2}{x-3}} \left[\frac{(x-3) - (x+2)}{(x-3)^2} \right] = \frac{-5 e^{\frac{x+2}{x-3}}}{(x-3)^2}$$

Exercise 2.13

Compute the derivative of the following function:

$$f(x) = \tanh x$$

Solution

We compute the derivative of the hyperbolic tangent function

$$f(x) = \tanh x = \frac{\sinh x}{\cosh x}$$

Using the rules for the derivative of a quotient, we have

$$f'(x) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = 1 - \tanh^2 x$$

Moreover, since $\cosh^2 x - \sinh^2 x = 1$, we also have

$$f'(x) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x,$$

where $\operatorname{sech} x$ is the hyperbolic secant, defined as $\operatorname{sech} x = \frac{1}{\cosh x}$.

Exercise 2.14

Compute the derivative of the following function:

$$f(x) = \coth x$$

Solution

We compute the derivative of the hyperbolic cotangent function

$$f(x) = \coth x = \frac{\cosh x}{\sinh x}$$

Using the rules for the derivative of a quotient, we have

$$f'(x) = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = 1 - \coth^2 x$$

Moreover, since $\cosh^2 x - \sinh^2 x = 1$, we also have

$$f'(x) = -\frac{\cosh^2 x - \sinh^2 x}{\sinh^2 x} = -\frac{1}{\sinh^2 x} = -\operatorname{csch}^2 x,$$

where $\operatorname{csch} x$ is the hyperbolic cosecant, defined as $\operatorname{csch} x = \frac{1}{\sinh x}$.

Exercise 2.15

Compute the derivative of the following function:

$$f(x) = e^{x^2+3x}$$

Solution

Using the rules for the derivative of composite functions, we have

$$f'(x) = e^{x^2+3x} \cdot (2x + 3)$$

Exercise 2.16

Compute the derivative of the following function:

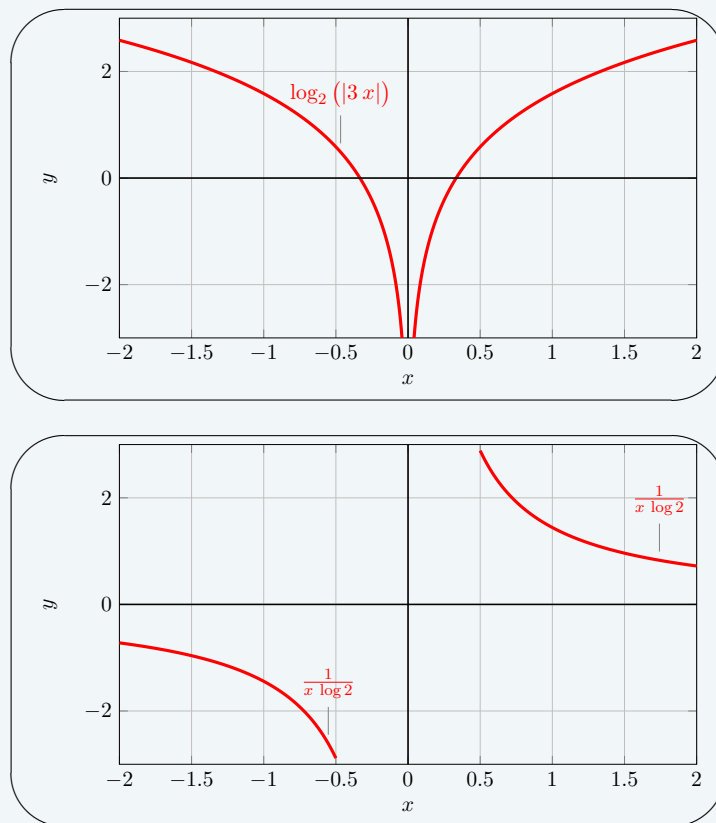
$$f(x) = \log_2(|3x|)$$

Solution

For $x \neq 0$, we have:

$$f(x) = \begin{cases} \log_2(3x), & x > 0 \\ \log_2(-3x), & x < 0 \end{cases} \quad \text{hence} \quad f'(x) = \begin{cases} \frac{3}{3x \log 2} = \frac{1}{x \log 2}, & x > 0 \\ \frac{-3}{-3x \log 2} = \frac{1}{x \log 2}, & x < 0 \end{cases}$$

Solution



The function is discontinuous at $x = 0$, hence it is not differentiable.

Exercise 2.17

Compute the derivative of the following function:

$$f(x) = \frac{x^2 + 3x - 2}{2x + 1}$$

Solution

Using the rules for the derivative of a quotient, we have

$$\begin{aligned} f'(x) &= \frac{(2x + 3)(2x + 1) - 2(x^2 + 3x - 2)}{(2x + 1)^2} = \frac{4x^2 + 2x + 6x + 3 - 2x^2 - 6x + 4}{(2x + 1)^2} \\ &= \frac{2x^2 + 2x + 7}{(2x + 1)^2} \end{aligned}$$

Exercise 2.18

Compute the derivative of the following function:

$$f(x) = x^2 \log(\cos x)$$

Solution

Using the rules for the derivative of composite functions, we have

$$f'(x) = 2x \cdot \log(\cos x) + x^2 \cdot \frac{1}{\cos x} \cdot (-\sin x) = 2x \log(\cos x) - x^2 \tan x$$

Exercise 2.19

Compute the derivative of the following function:

$$f(x) = \sqrt{\arctan(1+x^2)}$$

Solution

Using the rules for the derivative of composite functions, we have

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{\arctan(1+x^2)}} \cdot \frac{1}{1+(1+x^2)^2} \cdot 2x = \frac{x}{(1+(1+x^2)^2)\sqrt{\arctan(1+x^2)}} \\ &= \frac{x}{(x^4+2x^2+2)\sqrt{\arctan(1+x^2)}} \end{aligned}$$

Exercise 2.20

Determine whether the function

$$f(x) = e^{-x+1} - 2\pi x - 1$$

is invertible and compute the derivative of $f^{-1}(y)$ for $y = -2\pi$.

Solution

Since $f'(x) = -e^{-x+1} - 2\pi < 0$, for every $x \in \mathbb{R}$, we immediately obtain that f is strictly monotonically decreasing and hence invertible on $\mathbb{R}$. We also observe that

$$e^{-x+1} - 2\pi x - 1 = -2\pi \iff x = 1,$$

i.e., $f^{-1}(-2\pi) = 1$. From the theorem on the derivative of the inverse function we immedi-

ately obtain

$$(f^{-1})'(-2\pi) = \frac{1}{f'(1)} = -\frac{1}{1+2\pi}.$$

Remark: We would not have been able to write the analytic expression of f^{-1} , even though it exists since f is invertible; the theorem on the derivative of the inverse function allows us to overcome this obstacle, without going through the explicit expression of the inverse.

Exercise 2.21

Determine whether the function

$$f(x) = 4x + \pi \sin x$$

is invertible and compute the derivative of $f^{-1}(y)$ for $y = 4\pi$.

Solution

Since $f'(x) = 4 + \pi \cos x > 0$, for every $x \in \mathbb{R}$, we immediately obtain that f is strictly monotonically increasing and hence invertible on $\mathbb{R}$. We also observe that

$$4x + \pi \sin x = 4\pi \iff x = \pi,$$

i.e., $f^{-1}(4\pi) = \pi$. From the theorem on the derivative of the inverse function we immediately obtain

$$(f^{-1})'(4\pi) = \frac{1}{f'(\pi)} = \frac{1}{4 - \pi}.$$

Remark: We would not have been able to write the analytic expression of f^{-1} , even though it exists since f is invertible; the theorem on the derivative of the inverse function allows us to overcome this obstacle, without going through the explicit expression of the inverse.

Exercise 2.22

Let $f \in \mathcal{C}^1(\mathbb{R})$ be such that $f'(1) = 5e$. Setting $g(x) = f(\log x)$, compute $g'(e)$.

Solution

We observe that $g \in \mathcal{C}^1(0, +\infty)$, since it is a composition of functions of class $\mathcal{C}^1$ on their domains. Therefore, using the theorem on the derivative of composite functions, we obtain

$$g'(x) = f'(\log x) \frac{1}{x} \implies g'(e) = \frac{f'(1)}{e} = 5.$$

Exercise 2.23

Compute the derivative of the following function:

$$h(x) = x^{x \log x}$$

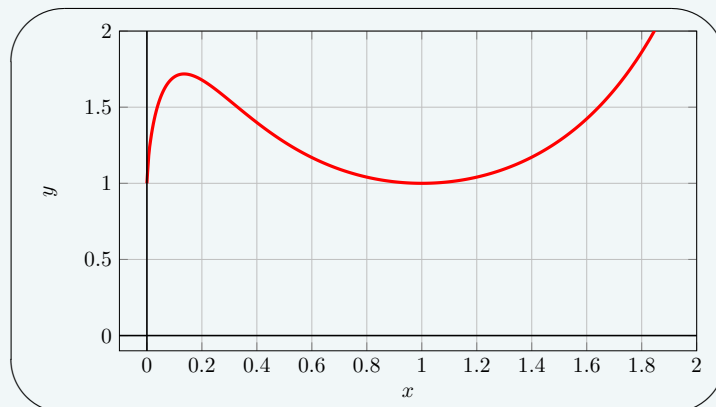
Solution

We use the rule for the derivative of a function raised to another function. The two functions are:

$$f(x) = x \quad \text{and} \quad g(x) = x \log x$$

Hence:

$$\begin{aligned} h'(x) &= \left(\exp(x \cdot \log^2 x) \right)' = \exp(x \cdot \log^2 x) \cdot (x \cdot \log^2 x)' = \\ &= x^{x \log x} \cdot \left(\log^2 x + x \cdot \left(\log^2 x \right)' \right) = x^{x \log x} \cdot \left(\log^2 x + x \cdot \frac{2 \cdot \log x}{x} \right) \\ &= x^{x \log x} \cdot \log x \cdot (\log x + 2) \end{aligned}$$

**Exercise 2.24**

Compute the derivative of the following function:

$$f(x) = \frac{ax + b}{cx + d}$$

Solution

Using the rules for the derivative of a quotient, we have

$$f'(x) = \frac{a(cx + d) - c(ax + b)}{(cx + d)^2} = \frac{acx + ad - acx - bc}{(cx + d)^2} = \frac{ad - bc}{(cx + d)^2}$$

Exercise 2.25

Compute the derivative of the following function:

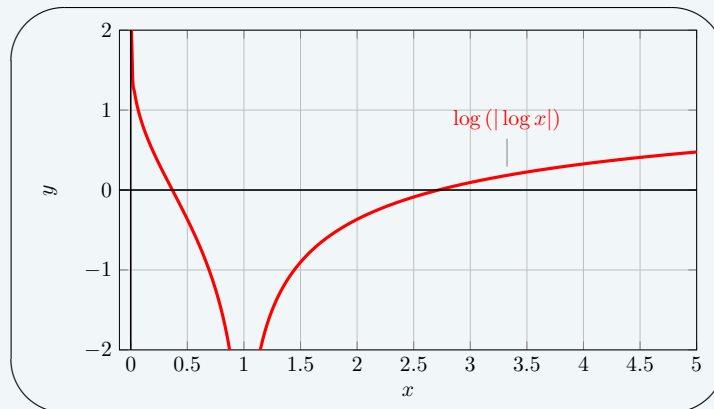
$$f(x) = \log(|\log x|)$$

Solution

For $x \neq 1$, we have:

$$f(x) = \begin{cases} \log(\log x), & x > 1 \\ \log(-\log x), & x < 1 \end{cases} \quad \text{hence} \quad f'(x) = \begin{cases} \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x \log x}, & x > 1 \\ \frac{1}{-\log x} \cdot -\frac{1}{x} = \frac{1}{x \log x}, & x < 1 \end{cases}$$

The function is discontinuous at $x = 1$, hence it is not differentiable.


Exercise 2.26

Compute the derivative of the following function f , specifying the domain of f and of f' :

$$f(x) = \sqrt[4]{x^2 \log(x^3)}$$

Solution

From $x^3 > 0$ and $x^2 \log(x^3) \geq 0$ it follows that f is defined for $x \in [1, +\infty)$, and for such x we have

$$f(x) = \sqrt[4]{3x^2 \log x}$$

By the usual differentiation rules, for $x > 1$ the function f is differentiable and we have:

$$f'(x) = \frac{\sqrt[4]{3}}{4\sqrt[4]{x^6 \log^3 x}} (2x \log x + x).$$

For $x = 1$, since the usual differentiation rules cannot be applied (the function $\sqrt[4]{y}$ is not differentiable at $y = 0$), we directly examine the limit of the difference quotient, where the

increment h makes sense only if $h > 0$:

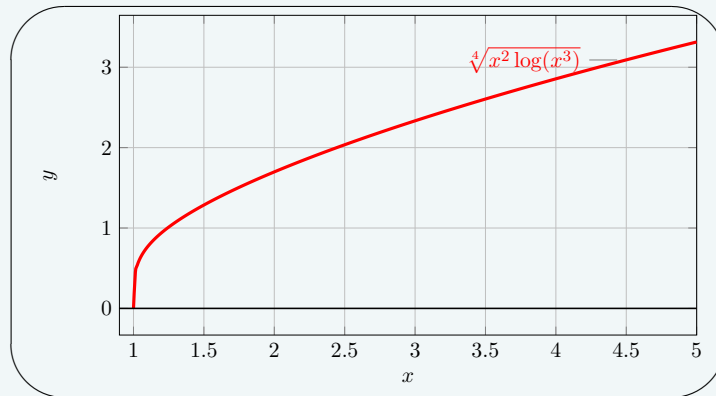
$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} &= \sqrt[4]{3} \lim_{h \rightarrow 0^+} \frac{\sqrt[4]{(1+h)^2 \log(1+h)}}{h} = \\ &= \sqrt[4]{3} \lim_{h \rightarrow 0^+} \frac{\log^{1/4}(1+h)}{h} = \sqrt[4]{3} \lim_{h \rightarrow 0^+} \frac{h^{1/4}}{h} = \sqrt[4]{3} \lim_{h \rightarrow 0^+} \frac{1}{h^{3/4}} = +\infty \end{aligned}$$

where we used $\log^{1/4}(1+h) \sim h^{1/4}$ for $h \rightarrow 0^+$. The function f is not differentiable at $x = 1$. At the point $(1, 0)$ the graph has a vertical tangent line.

Alternatively, we reach the same conclusion $f'(1) = +\infty$ by observing that f is continuous and that

$$\lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} f'(x) = \lim_{x \rightarrow 1^+} \frac{\sqrt[4]{3}}{4\sqrt[4]{x^6 \log^3 x}} (2x \log x + x) = +\infty$$

using Lagrange's mean value theorem or, equivalently, L'Hôpital's rule for the indeterminate forms $0/0$.



Exercise 2.27

Give an example of $f : \mathbb{R} \rightarrow \mathbb{R}$, continuous at $x = 1$, not differentiable at $x = 1$, and such that $f(0) = 2$.

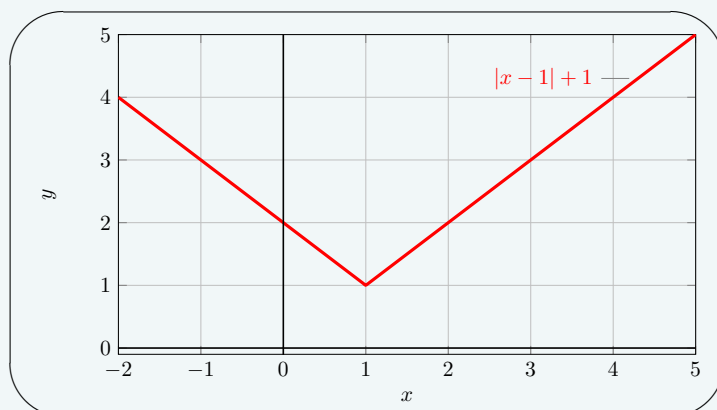
Solution

For example

$$f(x) = |x - 1| + 1.$$

The function satisfies $f(0) = 2$ and is continuous on all of $\mathbb{R}$.

At $x = 1$ we have $f'_+(1) = 1$, $f'_-(1) = -1$, hence the function is not differentiable.



Tangent lines

Exercise 3.1

Write the equation of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ for

$$f(x) = 3x^2 + 2x + 1, \quad x_0 = 2$$

Solution

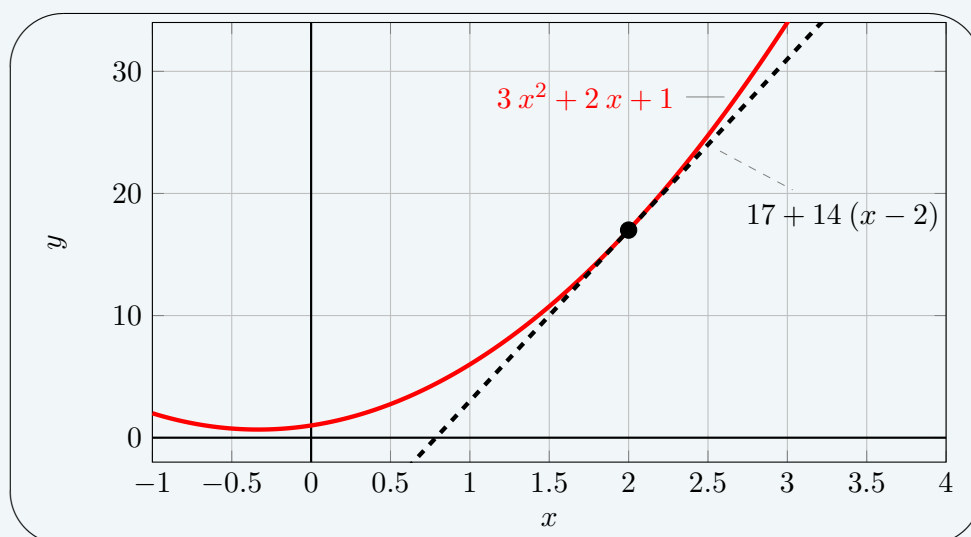
We compute the equation of the tangent line to the graph of the function

$f(x) = 3x^2 + 2x + 1$ at the point with abscissa $x_0 = 2$, we have :

$$f(2) = 17, \quad f'(x) = 6x + 2, \quad f'(2) = 14;$$

the tangent line is

$$y = f(2) + f'(2)(x - 2) = 17 + 14(x - 2)$$



Exercise 3.2

Write the equation of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ for

$$f(x) = \sin x, \quad x_0 = \frac{\pi}{3}$$

Solution

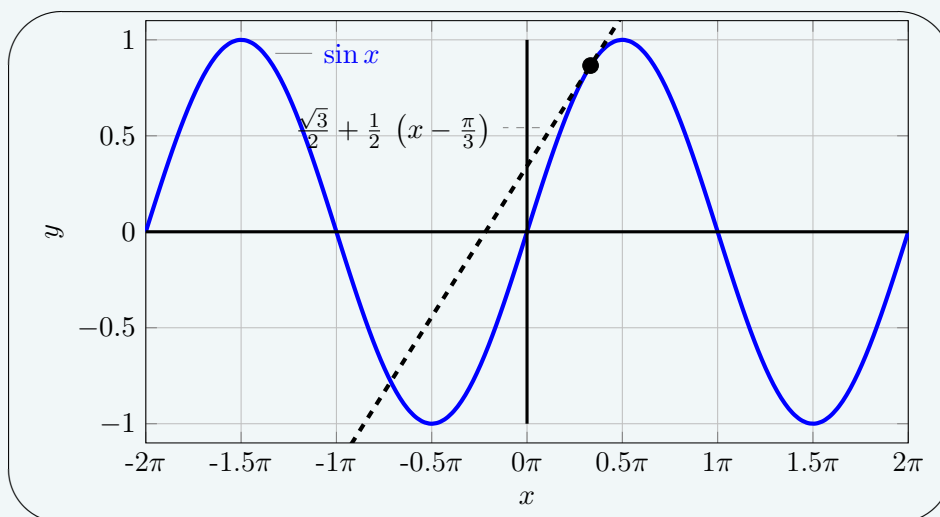
We compute the equation of the tangent line to the graph of the function

$f(x) = \sin x$ at the point with abscissa $x_0 = \frac{\pi}{3}$, we have :

$$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad f'(x) = \cos x, \quad f'\left(\frac{\pi}{3}\right) = \frac{1}{2};$$

the tangent line is

$$y = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

**Exercise 3.3**

Write the equation of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ for

$$f(x) = \log x, \quad x_0 = 1$$

Solution

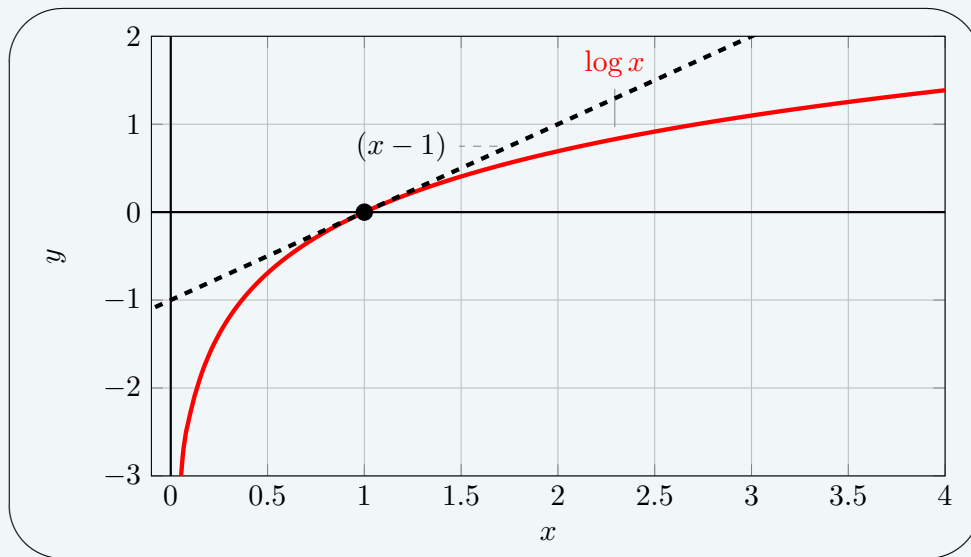
We compute the equation of the tangent line to the graph of the function

$f(x) = \log x$ at the point with abscissa $x_0 = 1$, we have :

$$f(1) = 0, \quad f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{\log(1 + h)}{h} = 1$$

the tangent line is

$$y = f(1) + f'(1)(x - 1) = 0 + (x - 1)$$



Exercise 3.4

Write the equation of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ for

$$f(x) = a^x, \quad x_0 = 2$$

Solution

Exercise 3.5

Write the equation of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ for

$$f(x) = (x \log |x|)^3, \quad x_0 = -1$$

Solution

Exercise 3.6

Write the equation of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ for

$$f(x) = \cos(\log x), \quad x_0 = e^{\frac{\pi}{2}}$$

Solution

Exercise 3.7

Write the equation of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ for

$$f(x) = e^{x^2}, \quad x_0 = \log 2$$

Solution

Exercise 3.8

Write the equation of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ for

$$f(x) = e^{-|x|}, \quad x_0 = -1$$

Solution

Asymptotes

Exercise 4.1

Determine the asymptotes (if any) of the following function on its natural domain

$$f(x) = \sqrt{\frac{x^3 - x}{x + 2}}$$

Solution

The function f is defined on

$$(-\infty, -2) \cup [-1, 0] \cup [1, +\infty).$$

Since

$$\lim_{x \rightarrow -2^-} f(x) = +\infty$$

the line $x = -2$ is a vertical asymptote.

For $x \rightarrow +\infty$ we have

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \sqrt{\frac{x^3 - x}{x^3 + 2x^2}} = 1$$

and

$$\lim_{x \rightarrow +\infty} f(x) - x = \lim_{x \rightarrow +\infty} \left(\sqrt{\frac{x^3 - x}{x + 2}} - x \right) \frac{\sqrt{\frac{x^3 - x}{x + 2}} + x}{\sqrt{\frac{x^3 - x}{x + 2}} + x} =$$

$$\lim_{x \rightarrow +\infty} \left(\frac{x^3 - x}{x + 2} - x^2 \right) \frac{1}{x \left(\sqrt{\frac{x^3 - x}{x^3 + 2x^2}} + 1 \right)} =$$

$$\lim_{x \rightarrow +\infty} \frac{-2x - 1}{x + 2} \frac{1}{\sqrt{\frac{x^3 - x}{x^3 + 2x^2}} + 1} = -1$$

hence we have the asymptote

$$y = x - 1, \quad x \rightarrow +\infty.$$

Solution

For $x \rightarrow -\infty$ we have

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{x} = \lim_{x \rightarrow -\infty} -\sqrt{\frac{x^3 - x}{x^3 + 2x^2}} = -1$$

and

$$\lim_{x \rightarrow -\infty} f(x) + x = \lim_{x \rightarrow -\infty} \left(\sqrt{\frac{x^3 - x}{x + 2}} + x \right) \frac{\sqrt{\frac{x^3 - x}{x + 2}} - x}{\sqrt{\frac{x^3 - x}{x + 2}} - x} =$$

$$\lim_{x \rightarrow -\infty} \left(\frac{x^3 - x}{x + 2} - x^2 \right) \frac{1}{-x \left(\sqrt{\frac{x^3 - x}{x^3 + 2x^2}} + 1 \right)} =$$

$$\lim_{x \rightarrow -\infty} \frac{2x + 1}{x + 2} \frac{1}{\sqrt{\frac{x^3 - x}{x^3 + 2x^2}} + 1} = 1$$

hence we have the asymptote

$$y = -x + 1, \quad x \rightarrow -\infty.$$

Solution

Using the asymptotic expansion

$$\sqrt{1 + y} = 1 + \frac{1}{2}y + o(y), \quad y \rightarrow 0$$

and taking into account the equivalence

$$\frac{2x + 1}{x^2 + 2x} \sim \frac{2}{x}, \quad x \rightarrow \pm\infty,$$

hence

$$\frac{2x + 1}{x^2 + 2x} \sim \frac{2}{x} + o\left(\frac{1}{x}\right), \quad x \rightarrow \pm\infty,$$

we can also proceed as follows:

$$f(x) = |x| \sqrt{1 - \frac{2x + 1}{x^2 + 2x}} =$$

$$|x| \sqrt{1 - (2/x) + o(1/x)} = |x|(1 - (1/x) + o(1/x)) =$$

$$|x| - \operatorname{sgn}(x) + o(1), \quad x \rightarrow \pm\infty$$

hence we recover the oblique asymptotes $y = -x + 1$ for $x \rightarrow -\infty$ and $y = x - 1$ for $x \rightarrow +\infty$.

Exercise 4.2

Determine the asymptotes (if any) of the following function on its natural domain

$$f(x) = \sqrt{9x^2 + 2x + 1}$$

Solution

The function f is defined on all of $\mathbb{R}$.

For $x \rightarrow +\infty$ we have

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \sqrt{9 + 2/x + 1/x^2} = 3$$

and

$$\lim_{x \rightarrow +\infty} f(x) - 3x = \lim_{x \rightarrow +\infty} \left(\sqrt{9x^2 + 2x + 1} - 3x \right) \frac{\sqrt{9x^2 + 2x + 1} + 3x}{\sqrt{9x^2 + 2x + 1} + 3x} =$$

$$\lim_{x \rightarrow +\infty} \frac{2x + 1}{3x \left(\sqrt{1 + 2/(9x) + 1/(9x^2)} + 1 \right)} = \frac{1}{3}$$

hence we have the asymptote

$$y = 3x + \frac{1}{3}, \quad x \rightarrow +\infty.$$

For $x \rightarrow -\infty$ we have

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{x} = \lim_{x \rightarrow -\infty} -\sqrt{9 + 2/x + 1/x^2} = -3$$

and

$$\lim_{x \rightarrow -\infty} f(x) + 3x = \lim_{x \rightarrow -\infty} \left(\sqrt{9x^2 + 2x + 1} + 3x \right) \frac{\sqrt{9x^2 + 2x + 1} - 3x}{\sqrt{9x^2 + 2x + 1} - 3x} =$$

$$\lim_{x \rightarrow -\infty} \frac{2x + 1}{-3x \left(\sqrt{1 + 2/(9x) + 1/(9x^2)} + 1 \right)} = -\frac{1}{3}$$

hence we have the asymptote

$$y = -3x - \frac{1}{3}, \quad x \rightarrow -\infty.$$

Solution

Alternatively, we can proceed as follows:

$$\begin{aligned} f(x) &= 3|x|\sqrt{1 + \frac{2}{9x} + \frac{1}{9x^2}} = \\ 3|x|\sqrt{1 + (2/9x) + o(1/x)} &= 3|x|(1 + (1/9x) + o(1/x)) = \\ 3|x| + (1/3)\operatorname{sgn}(x) + o(1), &\quad x \rightarrow \pm\infty \end{aligned}$$

hence we recover the oblique asymptotes $y = -3x - (1/3)$ for $x \rightarrow -\infty$ and $y = 3x + (1/3)$ for $x \rightarrow +\infty$.

Exercise 4.3

Determine the asymptotes (if any) of the following function on its natural domain

$$f(x) = x \left(e^{-\frac{1}{x}} + \sin \frac{1}{x} \right)$$

Solution

The function f is defined for $x \neq 0$. We have

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x e^{-\frac{1}{x}} + \lim_{x \rightarrow 0^-} x \sin \frac{1}{x} = -\infty + 0 = -\infty$$

where

$$\lim_{x \rightarrow 0^-} x e^{-\frac{1}{x}} = -\infty$$

follows from the fact that $e^{-1/x}$ is an infinite quantity of higher order than $1/x$ for $x \rightarrow 0^-$ and that $x e^{-1/x} < 0$ for $x < 0$. For the limit

$$\lim_{x \rightarrow 0^-} x \sin \frac{1}{x} = 0$$

we used the fact that the product of an infinitesimal (x) and a bounded function ($\sin(1/x)$) is an infinitesimal. The line $x = 0$ is therefore a vertical asymptote.

Solution

For $x \rightarrow \pm\infty$ we have

$$\lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \lim_{x \rightarrow \pm\infty} e^{-\frac{1}{x}} + \sin \frac{1}{x} = 1$$

and

$$\lim_{x \rightarrow \pm\infty} f(x) - x = \lim_{x \rightarrow \pm\infty} \frac{e^{-\frac{1}{x}} - 1}{\frac{1}{x}} + \frac{\sin \frac{1}{x}}{\frac{1}{x}} =$$

$$\lim_{y \rightarrow 0^\pm} \frac{e^{-y} - 1}{y} + \lim_{y \rightarrow 0^\pm} \frac{\sin y}{y} = -1 + 1 = 0$$

hence there is the oblique asymptote

$$y = x$$

both for $x \rightarrow -\infty$ and for $x \rightarrow +\infty$.

Solution

The same result can be obtained with the expansions $e^y = 1 + y + o(y)$, $\sin y = y + o(y)$ for $y \rightarrow 0$:

$$f(x) = x(1 - 1/x + o(1/x)) + x(1/x + o(1/x)) = x - 1 + 1 + o(1) = x + o(1)$$

for $x \rightarrow \pm\infty$, which means exactly that we have the oblique asymptote

$$y = x$$

both for $x \rightarrow -\infty$ and for $x \rightarrow +\infty$.

Continuous extensions

Exercise 5.1

For the following function, determine whether there exist continuous extensions at the endpoints of the domain. Then study the differentiability of such extensions.

$$f : [0, \pi/2) \rightarrow \mathbb{R}, \quad f(x) = (\cos x)^{(\pi/2)-x}$$

Solution

Setting $y = \pi/2 - x$ we have

$$\begin{aligned} \lim_{x \rightarrow \pi/2^-} f(x) &= \lim_{x \rightarrow \pi/2^-} e^{(\frac{\pi}{2}-x) \log \cos x} = \lim_{y \rightarrow 0^+} e^{y \log \sin y} = \lim_{y \rightarrow 0^+} e^{y \log(y+o(y))} = \\ &= \lim_{y \rightarrow 0^+} e^{y \log(y(1+o(1)))} = \lim_{y \rightarrow 0^+} e^{y \log y + y \log(1+o(1))} = e^0 = 1. \end{aligned}$$

We obtain a continuous extension by setting

$$f(\pi/2) = 1.$$

We examine the limit of the difference quotient:

$$\begin{aligned} \lim_{x \rightarrow \pi/2^-} \frac{f(x) - f(\pi/2)}{x - \pi/2} &= \lim_{x \rightarrow \pi/2^-} \frac{e^{(\frac{\pi}{2}-x) \log \cos x} - 1}{x - \pi/2} = \\ &= \lim_{x \rightarrow \pi/2^-} \frac{(\frac{\pi}{2}-x) \log \cos x}{x - \pi/2} = \lim_{x \rightarrow \pi/2^-} -\log \cos x = +\infty \end{aligned}$$

using the equivalence

$$e^t - 1 \sim t, \quad t \rightarrow 0$$

with

$$t = \left(\frac{\pi}{2} - x\right) \log \cos x, \quad x \rightarrow \pi/2^-.$$

It follows that f can be extended continuously at $x = \pi/2$, but this extension is not differentiable at $x = \pi/2$ (vertical tangent).

Exercise 5.2

For the following function, determine whether there exist continuous extensions at the endpoints of the domain. Then study the differentiability of such extensions.

$$f : (0, 1) \rightarrow \mathbb{R}, \quad f(x) = (\cos(\pi x/2))^{\log x}$$

Solution

We have

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} e^{\log x \log \cos(\pi x/2)} = e^0 = 1$$

since

$$\begin{aligned} \lim_{x \rightarrow 0} \log x \log \cos(\pi x/2) &= \lim_{x \rightarrow 0} \log x \log \left(1 - \frac{\pi^2}{8} x^2 + o(x^2) \right) = \\ &= \lim_{x \rightarrow 0} \left(-\frac{\pi^2}{8} x^2 + o(x^2) \right) \log x = \lim_{x \rightarrow 0} -\frac{\pi^2}{8} x^2 \log x = 0. \end{aligned}$$

We obtain a continuous extension by setting

$$f(0) = 1.$$

Solution

We examine the limit of the difference quotient:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} &= \lim_{x \rightarrow 0} \frac{e^{\log x \log \cos(\pi x/2)} - 1}{x} = \\ &= \lim_{x \rightarrow 0} \frac{\log x \log \cos(\pi x/2)}{x} = \lim_{x \rightarrow 0} -\frac{\pi^2}{8} x \log x = 0 \end{aligned}$$

using the equivalences

$$e^t - 1 \sim t, \quad t \rightarrow 0$$

with

$$t = \log x \log \cos(\pi x/2), \quad x \rightarrow 0$$

and

$$\log \cos(\pi x/2) \sim -\frac{\pi^2}{8} x^2, \quad x \rightarrow 0.$$

It follows that f can be extended continuously at $x = 0$, and this extension is differentiable at $x = 0$ with

$$f'(0) = 0.$$

Solution

We now examine the behavior for $x \rightarrow 1$ by setting $y = x - 1$:

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} e^{\log x \log \cos(\pi x/2)} = \lim_{y \rightarrow 0} e^{\log(1+y) \log(-\sin(\pi y/2))} = e^0 = 1$$

since

$$\begin{aligned}\lim_{y \rightarrow 0} \log(1+y) \log(-\sin(\pi y/2)) &= \lim_{y \rightarrow 0} y \log(-\pi y/2 + o(y)) = \\ &= \lim_{y \rightarrow 0} y \log(-\pi y/2(1 + o(1))) = \\ &= \lim_{y \rightarrow 0} y \log(-\pi y/2) + \lim_{y \rightarrow 0} y \log(1 + o(1)) = 0 + 0 = 0.\end{aligned}$$

We obtain a continuous extension by setting

$$f(1) = 1.$$

We examine the limit of the difference quotient:

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1} \frac{e^{\log x \log \cos(\pi x/2)} - 1}{x - 1} = \lim_{y \rightarrow 0} \frac{e^{\log(1+y) \log(-\sin(\pi y/2))} - 1}{y} = \\ &= \lim_{y \rightarrow 0} \frac{\log(1+y) \log(-\sin(\pi y/2))}{y} = \lim_{y \rightarrow 0} \log(-\sin(\pi y/2)) = -\infty\end{aligned}$$

using the equivalences

$$e^t - 1 \sim t, \quad t \rightarrow 0$$

with

$$t = \log(1+y) \log(-\sin(\pi y/2)), \quad y \rightarrow 0$$

and

$$\log(1+y) \sim y, \quad y \rightarrow 0.$$

It follows that f can be extended continuously at $x = 1$, but this extension is not differentiable at $x = 1$ (vertical tangent).

Exercise 5.3

For the following function, determine whether there exist continuous extensions at the endpoints of the domain. Then study the differentiability of such extensions.

$$f : (0, 1] \rightarrow \mathbb{R}, \quad f(x) = \frac{\sqrt{\cos x} - 1}{x^2}$$

Solution

We expand the function $\sqrt{\cos x}$ to order 3 with initial point $x = 0$, using

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^3)$$

and

$$\sqrt{1+y} = 1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 + o(y^3)$$

with $y = -(1/2)x^2 + o(x^2)$:

$$\sqrt{\cos x} = 1 - \frac{1}{4}x^2 + o(x^3).$$

It follows that

$$f(x) = \frac{\sqrt{\cos x} - 1}{x^2} = -\frac{1}{4} + o(x)$$

hence f can be extended continuously at $x = 0$ by setting

$$f(0) = -\frac{1}{4}$$

and this extension turns out to be differentiable with

$$f'(0) = 0.$$

Indeed

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} -\frac{1}{4} + o(x) = -\frac{1}{4}$$

and

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{-\frac{1}{4} + o(x) + \frac{1}{4}}{x} = \lim_{x \rightarrow 0} \frac{o(x)}{x} = 0.$$

Exercise 5.4

For the following function, determine whether there exist continuous extensions at the endpoints of the domain. Then study the differentiability of such extensions.

$$f : (0, \pi/2] \rightarrow \mathbb{R}, \quad f(x) = (1 - \cos x)^{\log(1+x)}$$

Solution

We have

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} e^{\log(1+x) \log(1-\cos x)} = e^0 = 1$$

since

$$\lim_{x \rightarrow 0} \log(1+x) \log(1-\cos x) = \lim_{x \rightarrow 0} x \log \left(\frac{1}{2}x^2 + o(x^2) \right) =$$

$$= \lim_{x \rightarrow 0} x \log \left(\frac{1}{2}x^2 (1 + o(1)) \right) =$$

$$= \lim_{x \rightarrow 0} x \log \left(\frac{1}{2}x^2 \right) + \lim_{x \rightarrow 0} x \log(1 + o(1)) = 0 + 0 = 0.$$

We obtain a continuous extension by setting

$$f(0) = 1.$$

We examine the limit of the difference quotient:

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{e^{\log(1+x) \log(1-\cos x)} - 1}{x} =$$

$$= \lim_{x \rightarrow 0} \frac{\log(1+x) \log(1-\cos x)}{x} = \lim_{x \rightarrow 0} \log(1-\cos x) = -\infty$$

using the equivalence

$$e^t - 1 \sim t, \quad t \rightarrow 0$$

with

$$t = \log(1+x) \log(1-\cos x), \quad x \rightarrow 0.$$

It follows that f can be extended continuously at $x = 0$, but this extension is not differentiable at $x = 0$ (vertical tangent).

Differentiability

Exercise 6.1

Study the differentiability of the function $f(x) = x^2 \cdot \log |x|$ defined for $x \neq 0$.

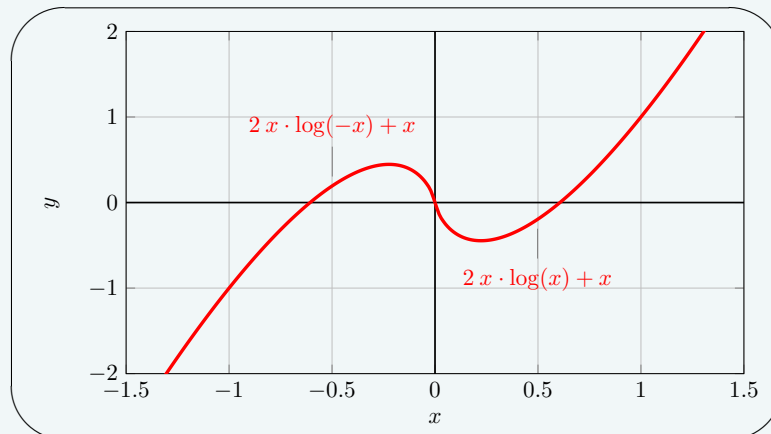
Solution

We know that the following limit exists

$$\lim_{x \rightarrow 0} x^2 \cdot \log |x| = 0$$

and we can extend f by continuity by defining $f(0) = 0$, so that f is continuous at 0 as well. For $x \neq 0$, we have:

$$f(x) = \begin{cases} x^2 \cdot \log x, & x > 0 \\ x^2 \cdot \log(-x), & x < 0 \end{cases} \quad \text{hence} \quad f'(x) = \begin{cases} 2x \cdot \log(x) + x, & x > 0 \\ 2x \cdot \log(-x) + x, & x < 0 \end{cases}$$

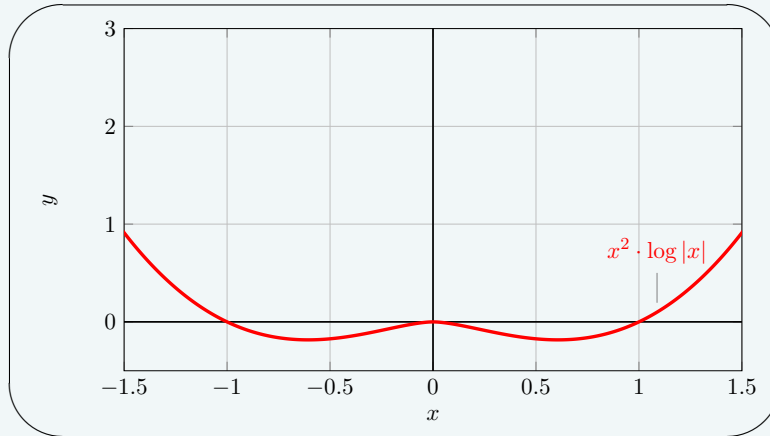


At $x = 0$ the function is continuous, and we have:

$$\lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^+} 2x \cdot \log x + x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^-} f'(x) = \lim_{x \rightarrow 0^-} 2x \cdot \log(-x) + x = 0$$

hence the derivative limit theorem can be applied and we have:

$$f'_+(0) = 0, f'_-(0) = 0, f'(0) = 0 \quad \text{so that at } x = 0 \text{ the function is differentiable}$$

**Exercise 6.2**

Determine the values of α and $\beta \in \mathbb{R}$ for which the function

$$f(x) = \begin{cases} \frac{(1+x)^\alpha - 1}{x} & \text{if } x > 0 \\ \beta x + 2 & \text{if } x \leq 0 \end{cases}$$

is continuous and differentiable at $x = 0$.

Solution

We start with continuity at the junction point:

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \frac{(1+x)^\alpha - 1}{x} = \alpha, \\ \lim_{x \rightarrow 0^-} f(x) &= f(0) = 2; \end{aligned}$$

hence the function is continuous at $x = 0$ if and only if $\alpha = 2$ and $\beta \in \mathbb{R}$.

For $\alpha = 2$ and $x > 0$, we have:

$$f(x) = \frac{(1+x)^2 - 1}{x} = x + 2.$$

Therefore it is clear that, in order to smoothly join two lines passing through the same point (in our case, the y -intercept $(0, 2)$), their equations must coincide, from which $\beta = 1$. Alternatively, one can write the analytic expression of $f'(x)$ and impose the condition

$$\lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^-} f'(x)$$

and observe that the same conclusion is reached.

Exercise 6.3

Determine the values of α and $\beta \in \mathbb{R}$ for which the function

$$f(x) = \begin{cases} \frac{e^{x-1} - 1}{x \sin(x^2 - 1)} & \text{if } 0 < x < 1 \\ \alpha x + \beta & \text{if } 1 \leq x \leq 2 \\ (x-2)^2 \log^2(x-2) & \text{if } x > 2 \end{cases}$$

is continuous and differentiable on $(0, +\infty)$.

Solution

The given function is continuous on the intervals $(0, 1)$, $(1, 2)$ and $(2, +\infty)$, since each branch is a composition of continuous functions.

In order to ensure continuity at $x = 1$, we impose the equality

$$\lim_{x \rightarrow 1^+} (\alpha x + \beta) = \lim_{x \rightarrow 1^-} \frac{e^{x-1} - 1}{x \sin(x^2 - 1)}$$

and, observing that, for $x \rightarrow 1^-$,

$$f(x) \sim \frac{x-1}{x(x^2-1)} = \frac{1}{x(x+1)} \rightarrow \frac{1}{2},$$

we obtain $\alpha + \beta = \frac{1}{2}$.

We now study continuity at the junction point $x = 2$; we impose

$$\lim_{x \rightarrow 2^+} (x-2)^2 \log^2(x-2) = \lim_{x \rightarrow 2^-} (\alpha x + \beta),$$

and since

$$\lim_{x \rightarrow 2^+} (x-2)^2 \log^2(x-2) = 0$$

we obtain $2\alpha + \beta = 0$. Solving the system of the two conditions we get $\alpha = -\frac{1}{2}$ and $\beta = 1$, which guarantee the continuity of f on $(0, +\infty)$.

To check differentiability, we observe that computing $f'(x)$ on $(0, 1)$ is much more complicated than on $(2, +\infty)$. It is therefore convenient to first check the differentiability of f at $x = 2$. Since

$$f'(x) = 2(x-2) \log^2(x-2) + 2(x-2) \log(x-2) \quad \forall x > 2,$$

we have that

$$\lim_{x \rightarrow 2^+} f'(x) = 0 \neq -\frac{1}{2} = \lim_{x \rightarrow 2^-} f'(x)$$

Therefore, for $\alpha = -\frac{1}{2}$ and $\beta = 1$, the function is **not** differentiable on $(0, +\infty)$, due to the presence of a corner point at $x = 2$. We might ask what would happen for other values of α and β , but the answer is immediate: the function would not be continuous on $(0, +\infty)$, and hence not differentiable either.

Exercise 6.4

Given the function

$$f(x) = \begin{cases} \frac{\pi}{\arctan \frac{1}{x}} & \text{if } x < 0 \\ \lambda & \text{if } x = 0 \\ \frac{\log(1 - 2x^4)}{x^\alpha} & \text{if } x > 0 \end{cases}$$

determine for which values of the real parameters $\alpha, \lambda \in \mathbb{R}$ the function is continuous and differentiable at $x = 0$.

Solution

The given function is continuous on the intervals $(-\infty, 0)$ and $(0, +\infty)$ since it is a composition of continuous functions. It remains to ensure continuity at the junction point $x = 0$, where the following must hold:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0) = \lambda$$

We have

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\pi}{\arctan \frac{1}{x}} = -2$$

and

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \frac{\log(1 - 2x^4)}{x^\alpha} = \lim_{x \rightarrow 0^+} \frac{-2x^4}{x^\alpha} = -2 \lim_{x \rightarrow 0^+} x^{4-\alpha} \\ &= \begin{cases} 0 & \text{if } \alpha < 4 \\ -2 & \text{if } \alpha = 4 \\ -\infty & \text{if } \alpha > 4 \end{cases} \end{aligned}$$

It is clear that the function is continuous at $x = 0$ if and only if $\alpha = 4, \lambda = -2$. Let us check whether, for the values of α, λ obtained, the function is also differentiable at $x = 0$. We rewrite the function substituting the values of α, λ obtained:

$$f(x) = \begin{cases} \frac{\pi}{\arctan \frac{1}{x}} & \text{if } x < 0 \\ -2 & \text{if } x = 0 \\ \frac{\log(1 - 2x^4)}{x^4} & \text{if } x > 0 \end{cases}$$

and we write the analytic expression of its first derivative:

$$f'(x) = \begin{cases} \frac{\pi}{(x^2 + 1) \left(\arctan \frac{1}{x}\right)^2} & \text{if } x < 0 \\ \frac{-8x^7 - 4x^3(1 - 2x^4) \log(1 - 2x^4)}{x^8(1 - 2x^4)} & \text{if } x > 0 \end{cases}$$

We have

$$\lim_{x \rightarrow 0^-} f'(x) = \lim_{x \rightarrow 0^-} \frac{\pi}{(x^2 + 1) \left(\arctan \frac{1}{x}\right)^2} = \frac{4}{\pi}$$

while

$$\lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^+} \frac{-8x^7 - 4x^3(1 - 2x^4) \log(1 - 2x^4)}{x^8(1 - 2x^4)} = 0$$

since

$$\lim_{x \rightarrow 0^+} \frac{-8x^7 - 4x^3(1 - 2x^4) \log(1 - 2x^4)}{x^8(1 - 2x^4)} = \lim_{x \rightarrow 0^+} \frac{-8x^{11} + o(x^{11})}{x^8} = \lim_{x \rightarrow 0^+} \frac{-8x^{11}}{x^8} = 0$$

We conclude that $\nexists \alpha, \lambda \in \mathbb{R}$ for which the function is also differentiable at $x = 0$.

Exercise 6.5

Prove that the following inequality holds:

$$\log(1 + x) \leq x \quad \forall x \geq 0$$

Solution

The function $f(x) = \log(1 + x)$ is continuous and differentiable on $(-1, +\infty)$. We can apply Lagrange's theorem on every interval of the form $[0, x]$, with $x > 0$. Therefore, for a suitable $\eta \in (0, x)$, $x > 0$, we obtain

$$\log(1 + x) = \log(1 + x) - \log(1 + 0) = \frac{1}{1 + \eta} x < x$$

where we took into account that $\frac{1}{1 + \eta} < 1$ for every $\eta > 0$.

Exercise 6.6

Determine the minimum and maximum points (if any), relative and absolute, of the function

$$g(x) = x + |\cos x|$$

on $[0, 2\pi]$. At how many points does $g(x)$ vanish? Why?

Solution

The function is continuous on the closed and bounded interval $[0, 2\pi]$, hence by the Weierstrass theorem the absolute maximum and minimum points exist. We have

$$g(x) = \begin{cases} x + \cos x & \text{if } x \in [0, \pi/2] \cup [3\pi/2, 2\pi] \\ x - \cos x & \text{if } x \in (\pi/2, 3\pi/2) \end{cases}$$

and

$$g'(x) = \begin{cases} 1 - \sin x & \text{if } x \in (0, \pi/2) \cup (3\pi/2, 2\pi) \\ 1 + \sin x & \text{if } x \in (\pi/2, 3\pi/2) \end{cases}$$

and verify as an exercise that the function is not differentiable at the junction points. The first derivative is always positive where it is defined, hence the function is always increasing

and attains its absolute maximum and minimum values at the endpoints of the interval. More precisely, $x = 0$ is the absolute minimum point with $g(0) = 1$ and $x = 2\pi$ is the absolute maximum point with $g(2\pi) = 2\pi + 1$. The function never vanishes since it is always increasing and $g(0) = 1$.

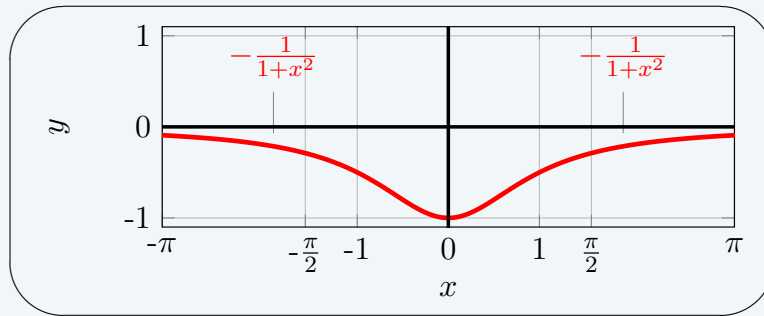
Exercise 6.7

Study the differentiability of the function $f(x) = \arctan \frac{1}{x}$ defined for $x \neq 0$.

Solution

For $x \neq 0$, we have:

$$f'(x) = -\frac{1}{1+x^2}$$



But in this case the hypothesis of continuity of f at 0 fails, since:

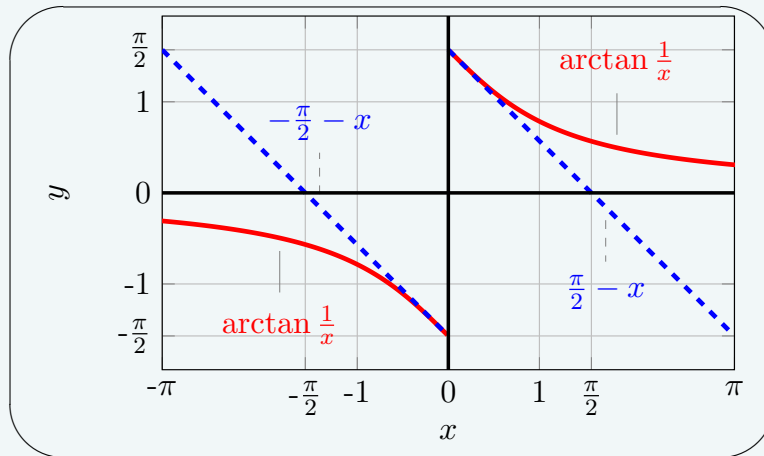
$$\lim_{x \rightarrow 0^+} f(x) = \frac{\pi}{2} \quad \text{and} \quad \lim_{x \rightarrow 0^-} f(x) = -\frac{\pi}{2}$$

and it is not possible to use the derivative limit theorem to compute the derivative at $x = 0$. Since it is not continuous at $x = 0$, the function is not differentiable. If instead we consider only the left limit of the derivative, the hypotheses of the derivative limit theorem are satisfied and since:

$$\lim_{x \rightarrow 0^-} f'(x) = \lim_{x \rightarrow 0^-} -\frac{1}{1+x^2} = -1 \quad \text{then} \quad f'_-(0) = -1$$

moreover, since:

$$\lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^+} -\frac{1}{1+x^2} = -1 \quad \text{then} \quad f'_+(0) = -1$$



Parametric equations

Exercise 7.1

Determine the number of solutions of the equation $f(x) = k$ as the real parameter k varies, in the following cases:

$$f(x) = \frac{2x^2 - 1}{x}$$

$$f(x) = x^2 e^{-x^2}$$

$$f(x) = \frac{e^x - 1}{\log(e^x - 1)}$$

$$f(x) = \log^2 x + 2 \log x$$

$$f(x) = e - x \log^2 x$$

$$f(x) = \frac{x}{x+1} - \arctan x$$

Solution

All the functions are (infinitely) differentiable on their domains. To answer, it is enough to determine the limits at the endpoints of the domain, the monotonicity behavior and the extremum points (if any).

Solution

The function $f(x) = \frac{2x^2-1}{x} = 2x - \frac{1}{x}$ is defined for $x \neq 0$, is odd, with

$$\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty, \quad \lim_{x \rightarrow 0^\pm} f(x) = \mp\infty$$

(the graph is a hyperbola with asymptotes $x = 0$, $y = 2x$).

Moreover, we have

$$f'(x) = 2 + \frac{1}{x^2},$$

which is positive for every $x \neq 0$. The function is strictly increasing both on $(-\infty, 0)$ and on $(0, +\infty)$, and on both intervals it takes all real values between the limits $-\infty$ and $+\infty$. In conclusion, for every value of $k \in \mathbb{R}$ the equation $f(x) = k$ has two solutions: one positive, the other negative. In particular, the equation $f(x) = 0$ has the two solutions $x = \pm\sqrt{1/2}$.

Solution

The function $f(x) = x^2 e^{-x^2}$ is defined for $x \in \mathbb{R}$, is even, with

$$\lim_{x \rightarrow \pm\infty} f(x) = 0.$$

The function takes only non-negative values; the point $x = 0$, where $f(0) = 0$, is the (unique) absolute minimum point.

Moreover, we have

$$f'(x) = 2x(1 - x^2)e^{-x^2},$$

from which, for $x > 0$, f is strictly increasing on $(0, 1)$ and strictly decreasing on $(1, +\infty)$. The point $x = 1$ is an absolute maximum point with value $f(1) = 1/e$. By even symmetry we obtain the behavior for $x < 0$.

In conclusion, for every value $k < 0$ or $k > 1/e$, the equation $f(x) = k$ has no solutions: the set of values of f is the interval $[0, 1/e]$.

For $k = 0$, the equation $f(x) = 0$ has the unique solution $x = 0$.

For $0 < k < 1/e$, the equation $f(x) = k$ has four solutions: one in the interval $(0, 1)$, one in the interval $(1, +\infty)$, and the other two opposite to these by even symmetry.

For $k = 1/e$, the equation $f(x) = 1/e$ has the two solutions $x = \pm 1$.

Solution

The function $f(x) = \frac{e^x - 1}{\log(e^x - 1)}$ is defined for $e^x - 1 > 0$, $e^x - 1 \neq 1$, hence for

$$x \in (0, \log 2) \cup (\log 2, +\infty).$$

We have

$$\lim_{x \rightarrow 0} f(x) = \lim_{y \rightarrow 0} \frac{y}{\log y} = 0, \quad \lim_{x \rightarrow \log 2^\pm} f(x) = \lim_{y \rightarrow 1^\pm} \frac{y}{\log y} = \pm\infty,$$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{y \rightarrow +\infty} \frac{y}{\log y} = +\infty.$$

Moreover, we have

$$f'(x) = \frac{e^x [\log(e^x - 1) - 1]}{\log^2(e^x - 1)},$$

from which f is strictly decreasing on $(0, \log 2)$, where it takes all values between the limits $-\infty$ and 0 ; strictly decreasing on $(\log 2, \log(e + 1))$, where it takes all values $k > e$; the

point $x = \log(e + 1)$ is a relative minimum point with value $f(\log(1 + e)) = e$; f is again strictly increasing on the interval $(\log(1 + e), +\infty)$, where it again takes all values $k > e$. In conclusion, for every value $k < 0$ the equation $f(x) = k$ has one solution, which lies in the interval $(0, \log 2)$.

For $0 \leq k < e$, the equation $f(x) = k$ has no solutions.

For $k = e$, the equation $f(x) = e$ has the unique solution $x = \log(e + 1)$.

For $k > e$, the equation $f(x) = k$ has two solutions, one in $(\log 2, \log(1 + e))$, the other in $(\log(e + 1), +\infty)$.

Solution

The function $f(x) = \log^2 x + 2 \log x$ is defined for $x > 0$. We have

$$\lim_{x \rightarrow 0} f(x) = \lim_{y \rightarrow -\infty} y^2 + 2y = +\infty, \quad \lim_{x \rightarrow +\infty} f(x) = \lim_{y \rightarrow +\infty} y^2 + 2y = +\infty.$$

Moreover, we have

$$f'(x) = \frac{2(\log x + 1)}{x},$$

from which f is strictly decreasing on $(0, 1/e)$, where it takes all values $k > -1$; the point $x = 1/e$ is the absolute minimum point with value $f(1/e) = -1$; f is strictly increasing on the interval $(1/e, +\infty)$, where it again takes all values $k > -1$.

In conclusion, for every value $k < -1$ the equation $f(x) = k$ has no solutions.

For $k = -1$, the equation $f(x) = -1$ has the unique solution $x = 1/e$.

For $k > -1$, the equation $f(x) = k$ has two solutions, one in $(0, 1/e)$, the other in $(1/e, +\infty)$.

In particular, the equation $f(x) = 0$ has the two solutions $x = 1/e^2$, $x = 1$.

Solution

The function $f(x) = e - x \log^2 x$ is defined for $x > 0$. We have

$$\lim_{x \rightarrow 0} f(x) = e, \quad \lim_{x \rightarrow +\infty} f(x) = -\infty.$$

Moreover, we have

$$f'(x) = -\log x(\log x + 2),$$

from which f is strictly decreasing on $(0, 1/e^2)$, where it takes all values $k \in (e - \frac{4}{e^2}, e)$; the point $x = 1/e^2$ is a relative minimum point with value $f(1/e^2) = e - \frac{4}{e^2}$; f is strictly increasing on the interval $(1/e^2, 1)$, where it again takes all values $k \in (e - \frac{4}{e^2}, e)$; the point $x = 1$ is the absolute maximum point with value $f(1) = e$; f is again strictly decreasing on $(1, +\infty)$, where it takes all values $k \in (-\infty, e)$.

In conclusion, for every value $k < e - \frac{4}{e^2}$ the equation $f(x) = k$ has only one solution, which lies in the interval $(1, +\infty)$. In particular, the equation $f(x) = 0$ has the unique solution $x = e$.

For $k = e - \frac{4}{e^2}$, the equation $f(x) = e - \frac{4}{e^2}$ has two solutions: one is $x = 1/e^2$, the other lies in the interval $(1, e)$.

For $e - \frac{4}{e^2} < k < e$, the equation $f(x) = k$ has three solutions, one in $(0, 1/e^2)$, one in $(1/e^2, 1)$, one in $(1, e)$.

For $k = e$, the equation $f(x) = e$ has the unique solution $x = 1$.
 For $k > e$, the equation $f(x) = k$ has no solutions.

Solution

The function $f(x) = \frac{x}{x+1} - \arctan x$ is defined for $x \neq -1$. We have

$$\lim_{x \rightarrow -\infty} f(x) = 1 + \frac{\pi}{2}, \quad \lim_{x \rightarrow -1^\pm} f(x) = \mp\infty, \quad \lim_{x \rightarrow +\infty} f(x) = 1 - \frac{\pi}{2}.$$

Moreover, we have

$$f'(x) = -\frac{2x}{(x+1)^2(x^2+1)}$$

from which f is strictly increasing on $(-\infty, -1)$, where it takes all values $k \in (1 + \frac{\pi}{2}, +\infty)$; f is also strictly increasing on the interval $(-1, 0)$, where it takes all values $k \in (-\infty, 0)$; the point $x = 0$ is a relative maximum point with value $f(0) = 0$; f is strictly decreasing on $(0, +\infty)$, where it takes all values $k \in (1 - \frac{\pi}{2}, 0)$.

In conclusion, for every value $k \leq 1 - \frac{\pi}{2}$ the equation $f(x) = k$ has only one solution, which lies in the interval $(-1, 0)$.

For $1 - \frac{\pi}{2} < k < 0$, the equation $f(x) = k$ has two solutions: one is in $(-1, 0)$, the other lies in the interval $(0, +\infty)$.

For $k = 0$, the equation $f(x) = 0$ has the unique solution $x = 0$.

For $0 < k \leq 1 + \frac{\pi}{2}$, the equation $f(x) = k$ has no solutions.

For $k > 1 + \frac{\pi}{2}$, the equation $f(x) = k$ has a unique solution, which lies in the interval $(-\infty, -1)$.

Curve sketching

Exercise 8.1

For each of the following functions of the real variable x , determine:

1. the domain and the limits at its endpoints, pointing out any continuous extensions;
2. any local extrema and the intervals of monotonicity;
3. any inflection points and the intervals of convexity;
4. any asymptotes;
5. a qualitative graph that takes into account all the previous elements.

Exercise 8.2

$$f(x) = x - \frac{1}{x}$$

Solution

The function $f(x) = x - \frac{1}{x}$ is defined for $x \neq 0$, hence its natural domain D is given by

$$D = (-\infty, 0) \cup (0, +\infty).$$

We have

$$\lim_{x \rightarrow -\infty} f(x) = -\infty, \quad \lim_{x \rightarrow 0^-} f(x) = +\infty, \quad \lim_{x \rightarrow 0^+} f(x) = -\infty, \quad \lim_{x \rightarrow +\infty} f(x) = +\infty,$$

in particular the line $x = 0$ (the y -axis) is a vertical asymptote. From

$$\lim_{x \rightarrow \pm\infty} f(x) - x = \lim_{x \rightarrow \pm\infty} -\frac{1}{x} = 0$$

it then follows that the line $y = x$ is an oblique asymptote as $x \rightarrow \pm\infty$. The presence of these asymptotes can also be obtained from analytic geometry: the curve with equation $y = x - \frac{1}{x}$, in implicit form $x^2 - xy - 1 = 0$, is a hyperbola whose asymptotes are precisely

the lines $x = 0$ and $y = x$ (the asymptotes of a hyperbola are obtained by setting equal to 0 the homogeneous part of degree 2 of the equation).

The function $f(x)$ is infinitely differentiable on its domain D . The first derivative is

$$f'(x) = 1 + \frac{1}{x^2}$$

and $f'(x) > 0$ for every $x \in D$. Hence the function $f(x)$ is strictly increasing on the interval $(-\infty, 0)$ and also strictly increasing on the interval $(0, +\infty)$.

The second derivative is

$$f''(x) = -\frac{2}{x^3}$$

and it has the opposite sign of x . Hence the function $f(x)$ is strictly convex on the interval $(-\infty, 0)$ and strictly concave on the interval $(0, +\infty)$.

The position of the graph with respect to the oblique asymptote is easily deduced from $f(x) - x = -1/x$: we have $f(x) > x$ for $x < 0$ while $f(x) < x$ for $x > 0$.

Now collect all the previous elements in a qualitative graph.

Exercise 8.3

$$f(x) = \frac{\sqrt{x}}{\sqrt{x}-1}$$

Solution

The function $f(x) = \frac{\sqrt{x}}{\sqrt{x}-1}$ is defined for $x \geq 0$, $x \neq 1$, hence its natural domain D is given by

$$D = [0, 1) \cup (1, +\infty).$$

We have

$$\lim_{x \rightarrow 0} f(x) = f(0) = 0, \quad \lim_{x \rightarrow 1^-} f(x) = -\infty, \quad \lim_{x \rightarrow 1^+} f(x) = +\infty, \quad \lim_{x \rightarrow +\infty} f(x) = +1,$$

in particular the line $x = 1$ is a vertical asymptote and the line $y = 1$ is a horizontal asymptote as $x \rightarrow +\infty$.

The function $f(x)$ is continuous on D and infinitely differentiable on $D \setminus \{0\}$. At the point $x = 0$ the difference quotient has limit

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{x}}{x(\sqrt{x}-1)} = -\lim_{x \rightarrow 0} \frac{\sqrt{x}}{x} = -\lim_{x \rightarrow 0} \frac{1}{\sqrt{x}} = -\infty$$

hence the function $f(x)$ is not differentiable at $x = 0$ and at the point $(0, 0)$ the graph has the y -axis as a vertical tangent.

For $x \in D \setminus \{0\}$ the first derivative is

$$f'(x) = \frac{-1}{2\sqrt{x}(\sqrt{x}-1)^2}$$

and $f'(x) < 0$ for every x . Hence the function $f(x)$ is strictly decreasing on the interval $[0, 1)$ and also strictly decreasing on the interval $(1, +\infty)$.

The second derivative is

$$f''(x) = \frac{1}{4} \frac{3\sqrt{x} - 1}{x\sqrt{x}(\sqrt{x} - 1)^3}$$

which vanishes at $x = 1/9$, is positive for $0 < x < 1/9$, negative for $1/9 < x < 1$, and positive again for $x > 1$. Hence the function $f(x)$ is strictly convex on the interval $(0, 1/9)$, strictly concave on the interval $(1/9, 1)$, and strictly convex again on the interval $(1, +\infty)$. At $x = 1/9$ there is an inflection point: at the point $(1/9, -1/2)$ the graph crosses from above to below the tangent line.

Now collect all the previous elements in a qualitative graph.

Exercise 8.4

$$f(x) = \sqrt{x^2 - 1} - \sqrt{x^2 + 1}$$

Solution

The function $f(x) = \sqrt{x^2 - 1} - \sqrt{x^2 + 1}$ is defined for $x \leq -1$, $x \geq 1$, hence its natural domain D is given by

$$D = (-\infty, -1] \cup [1, +\infty).$$

The function under consideration is even: the following study could be restricted to the interval $[1, +\infty)$. We have

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \frac{-2}{\sqrt{x^2 - 1} + \sqrt{x^2 + 1}} = 0$$

in particular the line $y = 0$ (the x -axis) is a horizontal asymptote as $x \rightarrow \pm\infty$. We then have

$$\lim_{x \rightarrow -1} f(x) = f(-1) = -\sqrt{2}, \quad \lim_{x \rightarrow 1} f(x) = f(1) = -\sqrt{2}.$$

The function $f(x)$ is continuous on D and infinitely differentiable on $D \setminus \{-1, 1\}$.

For $x \in D \setminus \{-1, 1\}$ the first derivative is

$$f'(x) = \frac{x}{\sqrt{x^2 - 1}} - \frac{x}{\sqrt{x^2 + 1}} = \frac{x(\sqrt{x^2 + 1} - \sqrt{x^2 - 1})}{\sqrt{x^4 - 1}}$$

and it has the same sign as x . Hence the function $f(x)$ is strictly decreasing on the interval $(-\infty, -1)$ and strictly increasing on the interval $(1, +\infty)$.

At the points $x = \pm 1$ we have

$$\lim_{x \rightarrow \pm 1} f'(x) = \pm\infty$$

hence the function $f(x)$ is not differentiable at $x = \pm 1$ and at the points $(\pm 1, -\sqrt{2})$ the graph has a vertical tangent.

The second derivative is

$$f''(x) = -\frac{1}{(x^2 - 1)\sqrt{x^2 - 1}} - \frac{1}{(x^2 + 1)\sqrt{x^2 + 1}}$$

and it is negative for every x . Hence the function $f(x)$ is strictly concave on the interval $(-\infty, -1)$ and on the interval $(1, +\infty)$.

Now collect all the previous elements in a qualitative graph, respecting in particular the symmetry of the graph with respect to the y -axis (even symmetry).

Exercise 8.5

$$f(x) = \frac{\sqrt{2x-1}}{\log(2x-1)}$$

Solution

The function $f(x) = \frac{\sqrt{2x-1}}{\log(2x-1)}$ is defined for $2x-1 > 0$, $2x-1 \neq 1$, hence its natural domain D is given by

$$D = (1/2, 1) \cup (1, +\infty).$$

We have

$$\lim_{x \rightarrow 1/2} f(x) = 0, \quad \lim_{x \rightarrow 1^\pm} f(x) = \pm\infty, \quad \lim_{x \rightarrow +\infty} f(x) = \lim_{y \rightarrow +\infty} \frac{y^{1/2}}{\log y} = +\infty$$

in particular the line $x = 1$ is a vertical asymptote. The order of infinity of $f(x)$ as $x \rightarrow +\infty$ is lower than that of $\sqrt{x}$, which rules out an oblique asymptote (the order of infinity of functions with an oblique asymptote is that of x).

The function $f(x)$ is continuous on D ; the limit $\lim_{x \rightarrow 1/2} f(x) = 0$ allows us to extend the function continuously also to $x = 1/2$ by setting $f(1/2) = 0$.

The function is infinitely differentiable on D . The first derivative is

$$f'(x) = \frac{\log(2x-1) - 2}{\sqrt{2x-1} \log^2(2x-1)}.$$

At the point $x = 1/2$, the difference quotient of the continuous extension has limit

$$\begin{aligned} \lim_{x \rightarrow 1/2} \frac{f(x) - f(1/2)}{x - 1/2} &= \lim_{x \rightarrow 1/2} \frac{\sqrt{2x-1}}{(x - 1/2) \log(2x-1)} = \\ &= \sqrt{2} \lim_{x \rightarrow 1/2} \frac{1}{\sqrt{x - 1/2} \log(2x-1)} = \sqrt{2} \lim_{y \rightarrow 0} \frac{1}{y^{1/2} \log y} = -\infty. \end{aligned}$$

The continuous extension of f is not differentiable at $x = 1/2$. At the point $(1/2, 0)$ its graph has a vertical tangent.

Going back to $f'(x)$ at the points of D , we have $f'(x) > 0$ for $x > (e^2 + 1)/2$, $f'(x) < 0$ for $x < (e^2 + 1)/2$, $f'(x) = 0$ for $x = (e^2 + 1)/2$. Hence the function $f(x)$ is strictly decreasing on the interval $(1/2, 1)$ and on the interval $(1, (e^2 + 1)/2)$, and strictly increasing on the interval $((e^2 + 1)/2, +\infty)$. The point $x = (e^2 + 1)/2$ is a local minimum point with value $f((e^2 + 1)/2) = e/2$.

The second derivative is

$$f''(x) = -\frac{\log^2(2x-1) - 8}{(2x-1)\sqrt{2x-1} \log^3(2x-1)}$$

and it is positive for $x \in (1/2, (e^{-\sqrt{8}} + 1)/2)$, negative for $x \in ((e^{-\sqrt{8}} + 1)/2, 1)$, positive for $x \in (1, (e^{\sqrt{8}} + 1)/2)$, negative for $x \in ((e^{\sqrt{8}} + 1)/2, +\infty)$. It follows that f is strictly convex for $x \in (1/2, (e^{-\sqrt{8}} + 1)/2)$, strictly concave for $x \in ((e^{-\sqrt{8}} + 1)/2, 1)$, strictly convex for $x \in (1, (e^{\sqrt{8}} + 1)/2)$, strictly concave for $x \in ((e^{\sqrt{8}} + 1)/2, +\infty)$. The points $x = (e^{\pm\sqrt{8}} + 1)/2$ are inflection points with respective values $f((e^{\pm\sqrt{8}} + 1)/2) = \pm e^{\pm\sqrt{2}}/\sqrt{8}$.

Now collect all the previous elements in a qualitative graph.

Exercise 8.6

$$f(x) = xe^{\frac{1}{x}}$$

Solution

The function $f(x) = xe^{\frac{1}{x}}$ is defined for $x \neq 0$, hence its natural domain D is given by

$$D = (-\infty, 0) \cup (0, +\infty).$$

We have

$$\lim_{x \rightarrow -\infty} f(x) = -\infty, \quad \lim_{x \rightarrow 0^-} f(x) = 0,$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{y \rightarrow +\infty} \frac{e^y}{y} = +\infty, \quad \lim_{x \rightarrow +\infty} f(x) = +\infty$$

in particular the line $x = 0$ (the y -axis) is a vertical asymptote. As $x \rightarrow \pm\infty$, from $e^y = 1 + y + o(y)$ as $y \rightarrow 0$, we obtain

$$f(x) = xe^{\frac{1}{x}} = x \left(1 + \frac{1}{x} + o\left(\frac{1}{x}\right) \right) = x + 1 + o(1)$$

hence the line $y = x + 1$ is an oblique asymptote as $x \rightarrow \pm\infty$.

The function $f(x)$ is continuous on D ; the limit $\lim_{x \rightarrow 0^-} f(x) = 0$ allows us to extend the function continuously from the left to $x = 0$ by setting $f(0) = 0$.

The function is infinitely differentiable on D . The first derivative is

$$f'(x) = \left(1 - \frac{1}{x}\right) e^{\frac{1}{x}} = \frac{x-1}{x} e^{\frac{1}{x}}.$$

At the point $x = 0$, the left difference quotient of the extension has limit

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^-} e^{\frac{1}{x}} = 0$$

The left continuous extension of f is left-differentiable at $x = 0$ with $f'_-(0) = 0$. The negative x half-axis is a tangent half-line to the graph at the point $(0, 0)$.

Going back to $f'(x)$ at the points of D , we have $f'(x) > 0$ for $x < 0$, $f'(x) < 0$ for $0 < x < 1$, again $f'(x) > 0$ for $x > 1$, and $f'(x) = 0$ for $x = 1$. Hence the function $f(x)$ is strictly increasing on the interval $(-\infty, 0)$, strictly decreasing on the interval $(0, 1)$, and strictly increasing again on the interval $(1, +\infty)$. The point $x = 1$ is a local minimum point with value $f(1) = e$.

The second derivative is

$$f''(x) = \frac{1}{x^3} e^{\frac{1}{x}}$$

and it has the same sign as x . It follows that f is strictly convex for $x \in (0, +\infty)$ and strictly concave for $x \in (-\infty, 0)$.

Now collect all the previous elements in a qualitative graph.

Exercise 8.7

$$f(x) = e^{\frac{1-|x|}{1+x}}$$

Solution

The function $f(x) = e^{\frac{1-|x|}{1+x}}$ is defined for $x \neq -1$, hence its natural domain D is given by

$$D = (-\infty, -1) \cup (-1, +\infty).$$

On the set $(-\infty, -1) \cup (-1, 0]$ the function is constant: $f(x) = e^{\frac{1+x}{1+x}} = e$ for every $x \leq 0$, $x \neq -1$. Obviously we can extend the function continuously to $x = -1$ by defining $f(-1) = e$. We then have

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} e^{\frac{1-x}{1+x}} = e^{-1} = \frac{1}{e}$$

in particular the line $y = 1/e$ is a horizontal asymptote as $x \rightarrow +\infty$.

The function $f(x)$ is continuous on D .

The function is infinitely differentiable on $D \setminus \{0\}$. The first derivative is obviously 0 for $x < 0$, while it is

$$f'(x) = -\frac{2}{(x+1)^2} e^{\frac{1-x}{1+x}} \quad \text{for } x > 0.$$

At the point $x = 0$, the left derivative is 0 while

$$\lim_{x \rightarrow 0^+} f'(x) = -2e.$$

The function is not differentiable at $x = 0$ since $f'_-(0) \neq f'_+(0)$. The graph has a corner point at $(0, e)$: on the left it consists of the half-line $y = e$, $x \leq 0$, while the right branch has tangent half-line $y = -2ex + e$, $x \geq 0$.

We then have $f'(x) < 0$ for every $x > 0$, hence f is strictly decreasing on the interval $(0, +\infty)$.

Since the function is constant for $x \leq 0$, we only need to compute the second derivative for $x > 0$, where it is

$$f''(x) = \frac{4x+8}{(x+1)^4} e^{\frac{1-x}{1+x}}$$

and it is positive. It follows that f is strictly convex for $x \in (0, +\infty)$.

Now collect all the previous elements in a qualitative graph.

Exercise 8.8

$$f(x) = \left| \frac{\log x}{x} \right|$$

Solution

The function $f(x) = \left| \frac{\log x}{x} \right|$ is defined for $x > 0$, hence its natural domain D is given by

$$D = (0, +\infty)$$

where its expression can be simplified to

$$f(x) = \frac{|\log x|}{x}, \quad x \in D.$$

Keep in mind, now and in what follows, the sign of $\log x$, which gives $|\log x| = -\log x$ for $0 < x < 1$ and $|\log x| = \log x$ for $x > 1$. We have

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{-\log x}{x} = +\infty, \quad \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{\log x}{x} = 0$$

in particular the line $x = 0$ (the y -axis) is a vertical asymptote while the line $y = 0$ (the x -axis) is a horizontal asymptote as $x \rightarrow +\infty$.

The function $f(x)$ is continuous on D , it clearly takes only non-negative values, and the point $x = 1$, where $f(1) = 0$, is an absolute minimum point.

The function is infinitely differentiable on $D \setminus \{1\}$. The first derivative is

$$f'(x) = -\frac{1 - \log x}{x^2} \quad \text{for } 0 < x < 1; \quad f'(x) = \frac{1 - \log x}{x^2} \quad \text{for } x > 1.$$

At the point $x = 1$, the left derivative is

$$\lim_{x \rightarrow 1^-} f'(x) = \lim_{x \rightarrow 1^-} -\frac{1 - \log x}{x^2} = -1,$$

while the right derivative is

$$\lim_{x \rightarrow 1^+} f'(x) = \lim_{x \rightarrow 1^+} \frac{1 - \log x}{x^2} = 1$$

The function is not differentiable at $x = 1$ since $f'_-(1) \neq f'_+(1)$. The graph has a corner point at $(1, 0)$: on the left the tangent half-line has equation $y = -x + 1$, on the right $y = x - 1$.

Solution

We then have $f'(x) < 0$ for every $x \in (0, 1)$, hence f is strictly decreasing on the interval $(0, 1)$. We have $f'(x) > 0$ for every $x \in (1, e)$ and $f'(x) < 0$ for every $x \in (e, +\infty)$, $f'(e) = 0$, hence f is strictly increasing on the interval $(1, e)$ and strictly decreasing on the interval $(e, +\infty)$; the point $x = e$ is a relative maximum point with value $f(e) = 1/e$.

On $D \setminus \{1\}$ the second derivative is

$$f''(x) = \frac{3 - 2 \log x}{x^3} \quad \text{for } 0 < x < 1; \quad f''(x) = \frac{2 \log x - 3}{x^3} \quad \text{for } x > 1.$$

It follows that f is strictly convex for $x \in (0, 1)$ and on $(e^{3/2}, +\infty)$; strictly concave on $(1, e^{3/2})$. The point $x = e^{3/2}$ is an inflection point with value $f(e^{3/2}) = 3/2e^{3/2}$.

Now collect all the previous elements in a qualitative graph.

Exercise 8.9

$$f(x) = \sqrt{1 - |e^{2x} - 1|}$$

Solution

The function $f(x) = \sqrt{1 - |e^{2x} - 1|}$ is defined for $|e^{2x} - 1| \leq 1$, hence for $-1 \leq e^{2x} - 1 \leq 1$, from which $0 \leq e^{2x} \leq 2$ and finally $x \leq (\log 2)/2$, since $e^{2x} > 0$ for every x . The natural domain D is given by

$$D = (-\infty, (\log 2)/2].$$

Keep in mind, now and in what follows, the sign of $e^{2x} - 1$, which gives $|e^{2x} - 1| = 1 - e^{2x}$ for $x < 0$ and $|e^{2x} - 1| = e^{2x} - 1$ for $x > 0$. In particular

$$f(x) = \sqrt{e^{2x}} = e^x \text{ for } x < 0; \quad f(x) = \sqrt{2 - e^{2x}} \text{ for } 0 \leq x \leq (\log 2)/2.$$

The behavior of e^x for $x < 0$ is well known; in particular it follows that

$$\lim_{x \rightarrow -\infty} f(x) = 0$$

(the x -axis is a horizontal asymptote as $x \rightarrow -\infty$) and that f is strictly increasing on $(-\infty, 0)$. We then have

$$\lim_{x \rightarrow (\log 2)/2} f(x) = f((\log 2)/2) = 0.$$

The function $f(x)$ is continuous on D , it clearly takes only non-negative values, and the point $x = (\log 2)/2$, where $f((\log 2)/2) = 0$, is an absolute minimum point.

The function is infinitely differentiable on $D \setminus \{0, (\log 2)/2\}$. The first derivative is

$$f'(x) = e^x \text{ for } x < 0; \quad f'(x) = -\frac{e^{2x}}{\sqrt{2 - e^{2x}}} \text{ for } 0 < x < (\log 2)/2.$$

At the point $x = 0$, the left derivative is

$$\lim_{x \rightarrow 0^-} f'(x) = \lim_{x \rightarrow 0^-} e^x = 1,$$

while the right derivative is

$$\lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^+} -\frac{e^{2x}}{\sqrt{2 - e^{2x}}} = -1$$

The function is not differentiable at $x = 0$ since $f'_-(0) \neq f'_+(0)$. The graph has a corner point at $(0, 1)$: on the left the tangent half-line has equation $y = x + 1$, on the right $y = -x + 1$.

Solution

At the point $x = (\log 2)/2$ we have

$$\lim_{x \rightarrow (\log 2)/2} f'(x) = \lim_{x \rightarrow (\log 2)/2} -\frac{e^{2x}}{\sqrt{2 - e^{2x}}} = -\infty.$$

The function is not differentiable at $x = (\log 2)/2$. At the point $((\log 2)/2, 0)$ the graph has a vertical tangent.

We then have $f'(x) < 0$ for every $x \in (0, (\log 2)/2)$, hence f is strictly decreasing on the interval $(0, (\log 2)/2)$. The point $x = 0$ is an (absolute) maximum point with value $f(0) = 1$. On $(-\infty, 0)$ the function coincides with e^x , and it is well known that the exponential function is strictly convex. We then have

$$f''(x) = -\frac{e^{2x}(4 - e^{2x})}{(2 - e^{2x})^{3/2}} \text{ for } 0 < x < (\log 2)/2$$

which is negative on that interval. It follows that f is strictly concave for $x \in (0, (\log 2)/2)$. Now collect all the previous elements in a qualitative graph.

(i) The function $f(x) = x - \arctan x$ has natural domain $D = \mathbb{R}$ with

$$\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty.$$

It is an odd function, which would allow us to study it only for $x \geq 0$. From

$$\lim_{x \rightarrow \pm\infty} f(x) - x = \lim_{x \rightarrow \pm\infty} -\arctan x = \mp \frac{\pi}{2}$$

it then follows that the function has the oblique asymptotes

$$y = x + \pi/2, \quad x \rightarrow -\infty; \quad y = x - \pi/2, \quad x \rightarrow +\infty.$$

The function is infinitely differentiable on all of $\mathbb{R}$. The first derivative is

$$f'(x) = 1 - \frac{1}{x^2 + 1} = \frac{x^2}{x^2 + 1}$$

with $f'(0) = 0$ and $f'(x) > 0$ for every $x \neq 0$. It follows that f is strictly increasing on all of $\mathbb{R}$ and that the point $x = 0$ is an inflection point with horizontal tangent, with corresponding value $f(0) = 0$.

The second derivative is

$$f''(x) = \frac{2x}{(x^2 + 1)^2}$$

and it has the same sign as x . In particular f is strictly concave on $(-\infty, 0)$ and strictly convex on $(0, +\infty)$.

Now collect all the previous elements in a qualitative graph, respecting the central symmetry of the graph with respect to the origin (odd symmetry).

Exercise 8.10

$$f(x) = x - \arctan x$$

Solution

The function $f(x) = x - \arctan x$ has natural domain $D = \mathbb{R}$ with

$$\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty.$$

It is an odd function, which would allow us to study it only for $x \geq 0$.

From

$$\lim_{x \rightarrow \pm\infty} f(x) - x = \lim_{x \rightarrow \pm\infty} -\arctan x = \mp \frac{\pi}{2}$$

it then follows that the function has the oblique asymptotes

$$y = x + \pi/2, \quad x \rightarrow -\infty; \quad y = x - \pi/2, \quad x \rightarrow +\infty.$$

The function is infinitely differentiable on all of $\mathbb{R}$. The first derivative is

$$f'(x) = 1 - \frac{1}{x^2 + 1} = \frac{x^2}{x^2 + 1}$$

with $f'(0) = 0$ and $f'(x) > 0$ for every $x \neq 0$. It follows that f is strictly increasing on all of $\mathbb{R}$ and that the point $x = 0$ is an inflection point with horizontal tangent, with corresponding value $f(0) = 0$.

The second derivative is

$$f''(x) = \frac{2x}{(x^2 + 1)^2}$$

and it has the same sign as x . In particular f is strictly concave on $(-\infty, 0)$ and strictly convex on $(0, +\infty)$.

Now collect all the previous elements in a qualitative graph, respecting the central symmetry of the graph with respect to the origin (odd symmetry).

Exercise 8.11

$$f(x) = 1 - x^2 + \log |x|$$

Solution

The function $f(x) = 1 - x^2 + \log |x|$ is defined for $x \neq 0$, hence its natural domain D is given by

$$D = (-\infty, 0) \cup (0, +\infty).$$

It is an even function, which would allow us to study it only for $x > 0$.

From $f(x) \sim -x^2$ as $x \rightarrow \pm\infty$ we have

$$\lim_{x \rightarrow \pm\infty} f(x) = -\infty$$

and we deduce that the function has no oblique asymptotes. We then have

$$\lim_{x \rightarrow 0^\pm} f(x) = -\infty,$$

in particular the line $x = 0$ (the y -axis) is a vertical asymptote.

The function is infinitely differentiable on all of D . The first derivative is

$$f'(x) = -2x + \frac{1}{x} = \frac{1 - 2x^2}{x}.$$

Analyzing its sign for $x > 0$, we have $f'(\sqrt{1/2}) = 0$, $f'(x) > 0$ for $0 < x < \sqrt{1/2}$, $f'(x) < 0$ for $x > \sqrt{1/2}$. It follows that f is strictly increasing on $(0, \sqrt{1/2})$, strictly decreasing on $(\sqrt{1/2}, +\infty)$, and that the point $x = \sqrt{1/2}$ is an (absolute) maximum point with corresponding value $f(\sqrt{1/2}) = 1/2 - (\log 2)/2$. In particular the graph meets the x -axis at two points with positive abscissa. One of these points is $x = 1$; the other has abscissa in the interval $(0, \sqrt{1/2})$ and can be approximated, if needed, with one of the algorithms studied for finding the zeros of smooth functions.

The study of monotonicity and extrema of f for $x < 0$ follows by even symmetry: the point $x = -\sqrt{1/2}$ is an (absolute) maximum point with corresponding value $f(-\sqrt{1/2}) = 1/2 - (\log 2)/2$, etc., etc.

The second derivative is

$$f''(x) = -2 - \frac{1}{x^2}$$

and it is negative for every $x \in D$. In particular f is strictly concave both on $(-\infty, 0)$ and on $(0, +\infty)$.

Now collect all the previous elements in a qualitative graph, respecting the axial symmetry of the graph with respect to the y -axis (even symmetry).

Maclaurin expansions

Exercise 9.1

Using the Maclaurin expansion of the following functions $f(x)$, determine the equation of the tangent line to the graph at the point $(0, f(0))$ and the local position of the graph with respect to this tangent

$$f(x) = 1 + \sin(\sqrt{1+x^3} - 1)$$

$$f(x) = 2\sqrt{1 + \sinh x} - x$$

$$f(x) = 8\sqrt{1 + \sin x} + x^2$$

$$f(x) = 8\sqrt{1 + \log(1+x)} + 3x^2$$

$$f(x) = 8\sqrt{1 - 2x - x^2} - 8 + 8x$$

Solution

From

$$\sqrt{1+x^3} - 1 = \frac{1}{2}x^3 + o(x^3)$$

e

$$\sin y = y - \frac{1}{6}y^3 + o(y^3)$$

we have

$$f(x) = 1 + \sin(\sqrt{1+x^3} - 1) = 1 + \frac{1}{2}x^3 + o(x^3).$$

The equation of the tangent line is

$$y = 1.$$

The term $\frac{1}{2}x^3 + o(x^3)$ says that there exists a neighborhood $(-\delta, \delta)$, $\delta > 0$, of $x = 0$ where the graph crosses the tangent line from below for $-\delta < x < 0$ to above for $0 < x < \delta$. The point $x = 0$ is an inflection point with horizontal tangent.

Solution

From

$$\sinh x = x + o(x^2)$$

e

$$2\sqrt{1+y} = 2 + y - \frac{1}{4}y^2 + o(y^2)$$

we have

$$f(x) = 2\sqrt{1+\sinh x} - x = 2 + x - \frac{1}{4}x^2 + o(x^2) - x = 2 - \frac{1}{4}x^2 + o(x^2).$$

The equation of the tangent line is

$$y = 2.$$

The term $-\frac{1}{4}x^2 + o(x^2)$ says that there exists a neighborhood $(-\delta, \delta)$, $\delta > 0$, of $x = 0$ where the graph lies below the tangent line for $-\delta < x < \delta$, $x \neq 0$. In particular, since the tangent line is horizontal, the point $x = 0$ is a local maximum point.

Solution

From

$$\sin x = x - \frac{1}{6}x^3 + o(x^3)$$

e

$$8\sqrt{1+y} = 8 + 4y - y^2 + \frac{1}{2}y^3 + o(y^3)$$

we have

$$f(x) = 8\sqrt{1+\sin x} + x^2 =$$

$$8 + 4x - \frac{2}{3}x^3 - \left(x - \frac{1}{6}x^3\right)^2 + \frac{1}{2}\left(x - \frac{1}{6}x^3\right)^3 + o(x^3) + x^2 =$$

$$8 + 4x - \frac{2}{3}x^3 - x^2 + \frac{1}{2}x^3 + x^2 + o(x^3) = 8 + 4x - \frac{1}{6}x^3 + o(x^3).$$

The equation of the tangent line is

$$y = 8 + 4x.$$

The term $-\frac{1}{6}x^3 + o(x^3)$ says that there exists a neighborhood $(-\delta, \delta)$, $\delta > 0$, of $x = 0$ where the graph crosses the tangent line from above for $-\delta < x < 0$ to below for $0 < x < \delta$. The point $x = 0$ is an inflection point.

Solution

From

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)$$

e

$$8\sqrt{1+y} = 8 + 4y - y^2 + \frac{1}{2}y^3 + o(y^3)$$

we have

$$\begin{aligned}
 f(x) &= 8\sqrt{1 + \log(1+x)} + 3x^2 = \\
 &= 8 + 4\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right) - \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)^2 + \\
 &\quad \frac{1}{2}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)^3 + o(x^3) + 3x^2 = \\
 &= 8 + 4x - 2x^2 + \frac{4}{3}x^3 - x^2 + x^3 + \frac{1}{2}x^3 + 3x^2 + o(x^3) = \\
 &= 8 + 4x + \frac{17}{6}x^3 + o(x^3).
 \end{aligned}$$

The equation of the tangent line is

$$y = 8 + 4x.$$

The term $\frac{17}{6}x^3 + o(x^3)$ says that there exists a neighborhood $(-\delta, \delta)$, $\delta > 0$, of $x = 0$ where the graph crosses the tangent line from below for $-\delta < x < 0$ to above for $0 < x < \delta$. The point $x = 0$ is an inflection point.

Exercise 9.2

Write the Maclaurin expansion of order 3 of the function

$$f(x) = e^x - e^{-x^2} - \sin x$$

and determine its order of infinitesimal. Based only on the expansion obtained, say whether the function has at $x = 0$ a relative minimum point, a relative maximum point, an inflection point, or none of these. Justify your answer.

Solution

Since we have to expand $f(x)$ to the third order, we can stop at the second order in the expansion of e^{-x^2} . That is:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$$

$$e^{-x^2} = 1 - x^2 + o(x^3)$$

$$\sin x = x - \frac{1}{6}x^3 + o(x^3)$$

We then have

$$\begin{aligned}
 f(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3) - (1 - x^2 + o(x^3)) - \left(x - \frac{1}{6}x^3 + o(x^3)\right) \\
 &= \frac{3}{2}x^2 + \frac{1}{3}x^3 + o(x^3)
 \end{aligned}$$

In particular, we have

$$f(x) \sim \frac{3}{2}x^2 \quad \text{as } x \rightarrow 0$$

from which we can immediately state that the order of infinitesimal of $f(x)$ is $\alpha = 2$ and the principal part is $\frac{3}{2}x^2$. The point $x = 0$ is a local minimum point; indeed, the term $\frac{3}{2}x^2 + o(x^2)$ says that there exists a neighborhood $(-\delta, \delta)$, $\delta > 0$, of $x = 0$ where the graph lies entirely above the horizontal tangent line $y = 0$.

Exercise 9.3

Determine the Taylor polynomial of order 3, centered at the point $x_0 = \frac{\pi}{3}$, of the function $f(x) = \cos x$.

Solution

In general, the Taylor polynomial of order n , centered at x_0 , of a function $f(x)$ is

$$T_{n,f,x_0}(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

with $f^{(0)}(x_0) = f(x_0)$. In this case, we need the first, second and third derivatives, evaluated at $x_0 = \frac{\pi}{3}$. We have

$$f' \left(\frac{\pi}{3} \right) = -\sin \left(\frac{\pi}{3} \right) = -\frac{\sqrt{3}}{2}$$

$$f'' \left(\frac{\pi}{3} \right) = -\cos \left(\frac{\pi}{3} \right) = -\frac{1}{2}$$

$$f''' \left(\frac{\pi}{3} \right) = \sin \left(\frac{\pi}{3} \right) = \frac{\sqrt{3}}{2}$$

and the Taylor polynomial of order 3, centered at the point $x_0 = \frac{\pi}{3}$, of the function $f(x) = \cos x$ is

$$T_{3,f,\frac{\pi}{3}}(x) = \frac{1}{2} - \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{3} \right) - \frac{1}{4} \left(x - \frac{\pi}{3} \right)^2 + \frac{\sqrt{3}}{12} \left(x - \frac{\pi}{3} \right)^3$$

Limits via asymptotic expansions

Exercise 10.1

Compute the following limits:

$$\lim_{x \rightarrow 0} \frac{6(x - \sin x) - x^3}{\sin x^5}$$

$$\lim_{x \rightarrow 0} \frac{\sinh^2 x + 2(1 - \cosh x)}{(1 - \cos x)^2}$$

$$\lim_{x \rightarrow 0} \frac{4(1 - \cos x)^2 - x^2 \sin^2 x}{\sin^2 x \log(1 + x^4)}$$

$$\lim_{x \rightarrow 0} \frac{1 + x \sin x - e^{x^2}}{x \sin(x^3)}$$

$$\lim_{x \rightarrow 0} \frac{x^2 \cos x - \sinh x^2 + \frac{1}{2}x^4}{x^2 - \arctan x^2}$$

$$\lim_{x \rightarrow 0} \frac{x \sinh x - 2 \cosh x + 2}{(e^{\sin x} - 1)^2}$$

$$\lim_{x \rightarrow 0} \frac{8\sqrt{1 + \sin x} - 8 - 4x + x^2}{(2e^x - 2 - 2x - x^2) \cosh^2 x}$$

$$\lim_{x \rightarrow 0} \frac{2 \log(1 + \sin x) - 2x + x^2}{(2e^x - 2 - 2x - x^2) \cos^2 x}$$

$$\lim_{x \rightarrow +\infty} x \left(\sqrt{4 + \frac{5}{x}} - 2 \cos \frac{1}{x} \right)$$

Solution

We determine the principal part of the Maclaurin expansion of the numerator:

$$6(x - \sin x) - x^3 = 6\left(\frac{x^3}{6} - \frac{x^5}{120} + o(x^5)\right) - x^3 = -\frac{x^5}{20} + o(x^5).$$

The denominator is equivalent to x^5 :

$$\sin x^5 \sim x^5.$$

Hence the given limit is

$$\lim_{x \rightarrow 0} \frac{6(x - \sin x) - x^3}{\sin x^5} = \lim_{x \rightarrow 0} \frac{-\frac{x^5}{20}}{x^5} = -\frac{1}{20}$$

Solution

We determine the principal part of the Maclaurin expansion of the numerator:

$$\sinh^2 x + 2(1 - \cosh x) = \left(x + \frac{x^3}{6} + o(x^3)\right)^2 - x^2 - \frac{x^4}{12} + o(x^4) = \frac{x^4}{4} + o(x^4).$$

The denominator is equivalent to $x^4/4$:

$$(1 - \cos x)^2 \sim \frac{x^4}{4}.$$

Hence the given limit is

$$\lim_{x \rightarrow 0} \frac{\sinh^2 x + 2(1 - \cosh x)}{(1 - \cos x)^2} = \lim_{x \rightarrow 0} \frac{\frac{x^4}{4}}{\frac{x^4}{4}} = 1.$$

Solution

We determine the principal part of the Maclaurin expansion of the numerator:

$$\begin{aligned} 4(1 - \cos x)^2 - x^2 \sin^2 x &= 4\left(\frac{x^2}{2} - \frac{x^4}{24} + o(x^4)\right)^2 - x^2\left(x - \frac{x^3}{6} + o(x^3)\right)^2 \\ &= \frac{x^6}{6} + o(x^6). \end{aligned}$$

The denominator is equivalent to x^6 :

$$\sin^2 x \log(1 + x^4) \sim x^2 \cdot x^4 = x^6.$$

Hence the given limit is

$$\lim_{x \rightarrow 0} \frac{4(1 - \cos x)^2 - x^2 \sin^2 x}{\sin^2 x \log(1 + x^4)} = \lim_{x \rightarrow 0} \frac{\frac{x^6}{6}}{x^6} = \frac{1}{6}.$$

Solution

We determine the principal part of the Maclaurin expansion of the numerator:

$$\begin{aligned} 1 + x \sin x - e^{x^2} &= 1 + x \left(x - \frac{x^3}{6} + o(x^3) \right) - \left(1 + x^2 + \frac{x^4}{2} + o(x^4) \right) \\ &= -\frac{2}{3}x^4 + o(x^4). \end{aligned}$$

The denominator is equivalent to x^4 :

$$x \sin(x^3) \sim x \cdot x^3 = x^4.$$

Hence the given limit is

$$\lim_{x \rightarrow 0} \frac{1 + x \sin x - e^{x^2}}{x \sin(x^3)} = \lim_{x \rightarrow 0} \frac{-\frac{2}{3}x^4}{x^4} = -\frac{2}{3}.$$

Solution

We determine the principal part of the Maclaurin expansion of the numerator:

$$\begin{aligned} x^2 \cos x - \sinh x^2 + \frac{1}{2}x^4 &= x^2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4) \right) - \left(x^2 + \frac{x^6}{6} + o(x^6) \right) + \frac{1}{2}x^4 = -\frac{x^6}{8} + o(x^6). \end{aligned}$$

For the denominator, from the expansion $\arctan x = x - \frac{x^3}{3} + o(x^3)$, it follows that

$$x^2 - \arctan x^2 = \frac{x^6}{3} + o(x^6).$$

Hence the given limit is:

$$\lim_{x \rightarrow 0} \frac{x^2 \cos x - \sinh x^2 + \frac{1}{2}x^4}{x^2 - \arctan x^2} = \lim_{x \rightarrow 0} \frac{-\frac{x^6}{8}}{\frac{x^6}{3}} = -\frac{3}{8}.$$

Solution

We determine the principal part of the Maclaurin expansion of the numerator:

$$\begin{aligned} x \sinh x - 2 \cosh x + 2 &= x \left(x + \frac{x^3}{6} + o(x^3) \right) - 2 \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + o(x^4) \right) + 2 = \frac{x^4}{12} + o(x^4). \end{aligned}$$

The denominator is equivalent to x^2 :

$$(e^{\sin x} - 1)^2 \sim (\sin x)^2 \sim x^2.$$

Hence the given limit is

$$\lim_{x \rightarrow 0} \frac{x \sinh x - 2 \cosh x + 2}{(e^{\sin x} - 1)^2} = \lim_{x \rightarrow 0} \frac{\frac{x^4}{12}}{x^2} = \lim_{x \rightarrow 0} \frac{x^2}{12} = 0.$$

Solution

We expand to order 3 the composite function $8\sqrt{1 + \sin x}$ with initial point $x = 0$. From $\sin x \sim x - x^3/6$ and $8\sqrt{1 + y} \sim 8 + 4y - y^2 + y^3/2$, by composition, we have

$$\begin{aligned} 8\sqrt{1 + \sin x} &= 8 + 4 \left(x - \frac{x^3}{6} \right) - \left(x - \frac{x^3}{6} \right)^2 + \frac{1}{2} \left(x - \frac{x^3}{6} \right)^3 + o(x^3) \\ &= 8 + 4x - x^2 - \frac{1}{6}x^3 + o(x^3). \end{aligned}$$

It follows that the numerator, in the given limit, is equivalent to $-x^3/6$. In the denominator we have the factor $\cosh^2 x$ which converges to 1, while

$$2e^x - 2 - 2x - x^2 \sim x^3/3.$$

The given limit is

$$\lim_{x \rightarrow 0} \frac{8\sqrt{1 + \sin x} - 8 - 4x + x^2}{(2e^x - 2 - 2x - x^2) \cosh^2 x} = \lim_{x \rightarrow 0} \frac{-\frac{x^3}{6}}{\frac{x^3}{3}} = -\frac{1}{2}.$$

Solution

We expand to order 3 the composite function $2 \log(1 + \sin x)$ with initial point $x = 0$. From $\sin x \sim x - x^3/6$ and $2 \log(1 + y) \sim 2y - y^2 + 2y^3/3$, by composition, we have

$$\begin{aligned} 2 \log(1 + \sin x) &= 2 \left(x - \frac{x^3}{6} \right) - \left(x - \frac{x^3}{6} \right)^2 + \frac{2}{3} \left(x - \frac{x^3}{6} \right)^3 + o(x^3) \\ &= 2x - x^2 + \frac{1}{3}x^3 + o(x^3). \end{aligned}$$

It follows that the numerator, in the given limit, is equivalent to $x^3/3$. In the denominator we have the factor $\cos^2 x$ which converges to 1, while

$$2e^x - 2 - 2x - x^2 \sim x^3/3.$$

The given limit is

$$\lim_{x \rightarrow 0} \frac{2 \log(1 + \sin x) - 2x + x^2}{(2e^x - 2 - 2x - x^2) \cos^2 x} = \lim_{x \rightarrow 0} \frac{\frac{1}{3}x^3}{\frac{1}{3}x^3} = 1.$$

Solution

$$\lim_{x \rightarrow +\infty} x \left(\sqrt{4 + \frac{5}{x}} - 2 \cos \frac{1}{x} \right) = \lim_{x \rightarrow +\infty} 2x \left(\sqrt{1 + \frac{5}{4x}} - \cos \frac{1}{x} \right)$$

From $\cos y = 1 + o(y)$ and $\sqrt{1+y} = 1 + \frac{1}{2}y + o(y)$, by composition, we have

$$\begin{aligned} \lim_{x \rightarrow +\infty} 2x \left(\sqrt{1 + \frac{5}{4x}} - \cos \frac{1}{x} \right) &= \lim_{x \rightarrow +\infty} 2x \left(1 + \frac{1}{2} \cdot \frac{5}{4x} + o\left(\frac{1}{x}\right) - 1 + o\left(\frac{1}{x}\right) \right) \\ &= \lim_{x \rightarrow +\infty} 2x \left(\frac{5}{8x} + o\left(\frac{1}{x}\right) \right) \\ &= \frac{5}{4} + o(1) \\ &= \frac{5}{4} \end{aligned}$$

Newton's method

Exercise 11.1

Consider

$$f(x) = e^{-x} - x$$

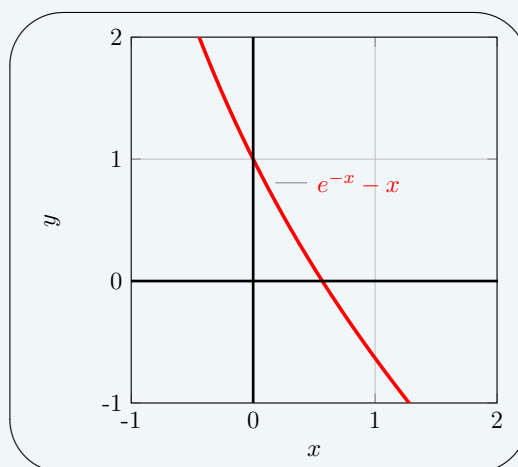
and verify whether Newton's method can be applied on the interval $[0, 1]$. If so, find an estimate of $x \in [0, 1]$ such that:

$$e^{-x} - x = 0$$

Solution

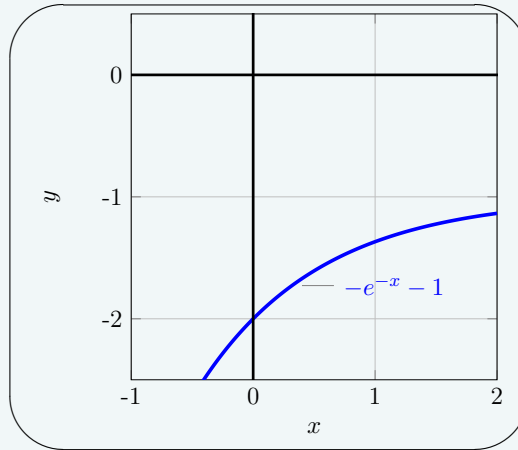
We consider the function:

$$f(x) = e^{-x} - x, \quad f'(x) = -e^{-x} - 1, \quad f''(x) = e^{-x}$$



We have:

$$f(0) = 1 > 0 \quad \text{and} \quad f(1) < 0$$



We consider the interval $[0, 1]$; for $x \in [0, 1]$ we have:

$$f'(x) < 0 \quad \text{and} \quad f''(x) > 0$$

Solution

Hence on the interval $[0, 1]$ the function $f(x) = e^{-x} - x$ satisfies the hypotheses of the theorem.

We are in the case of a function that is strictly decreasing on $[0, 1]$, since $f'(x) < 0, x \in [0, 1]$, and convex on $[0, 1]$, since $f''(x) > 0, x \in [0, 1]$. The endpoints of the chosen interval are: $a = 0$ and $b = 1$.

Hypothesis 3 holds, that is $f(0) \cdot f''(0) > 0$, and the sequence becomes:

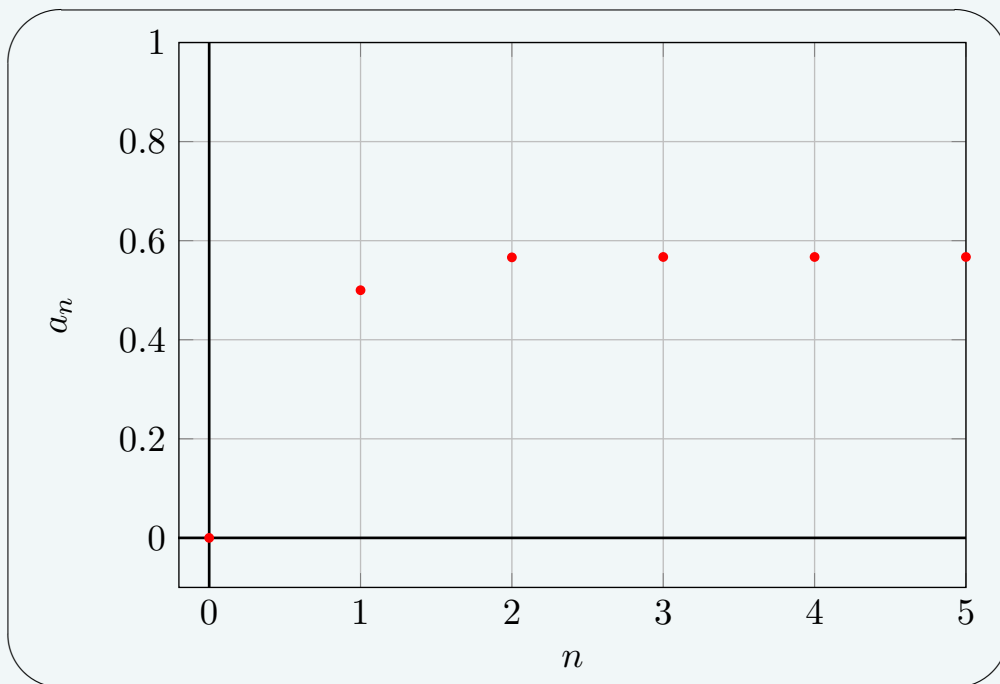
$$x_0 = 0, \quad x_{n+1} = x_n - \frac{e^{-x_n} - x_n}{-e^{-x_n} - 1} = \frac{x_n + 1}{1 + e^{x_n}} \quad \text{with } n \in \mathbb{N}$$

Since

$$\begin{aligned} x_n - \frac{e^{-x_n} - x_n}{-e^{-x_n} - 1} &= x_n + \frac{e^{-x_n} - x_n}{e^{-x_n} + 1} = x_n + \frac{e^{-x_n}}{e^{-x_n}} \frac{1 - e^{x_n} x_n}{1 + e^{x_n}} \\ &= \frac{x_n(1 + e^{x_n}) + 1 - e^{x_n} x_n}{1 + e^{x_n}} = \frac{x_n + 1}{1 + e^{x_n}} \end{aligned}$$

we therefore have:

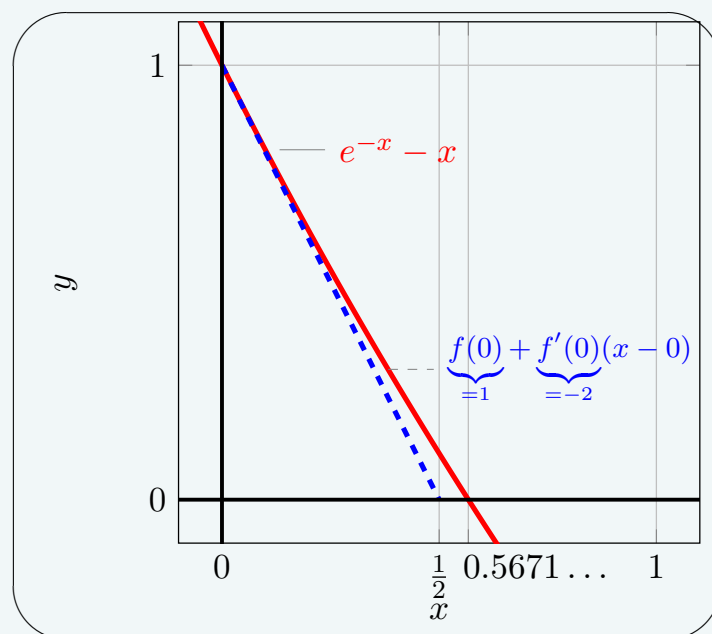
$$x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 0.5663 \dots, \quad x_3 = 0.5671 \dots, \quad x_4 = 0.5671 \dots, \quad x_5 = 0.5671 \dots$$



Solution

At the first iteration the tangent line is:

$$y = 1 - 2x \quad \text{and} \quad x_1 = 0 - \frac{1}{-2} = \frac{1}{2}$$



Bisection method

Exercise 12.1

Using the bisection method, find the solutions of the equation

$$x^3 + 2x - 1 = 0$$

correct to two decimal places.

(First of all, one must determine how many solutions there are.)

Solution

Written in the form

$$x^3 = 1 - 2x,$$

the equation suggests first comparing the graphs of the functions $y = x^3$ and $y = 1 - 2x$, in order to determine beforehand the number of solutions of the equation. Comparing the two graphs, we deduce that there exists a unique solution $\alpha \in (0, 1)$, i.e., a unique zero of the function $f(x) = x^3 + 2x - 1$ in that interval. More formally:

$$\exists! \alpha \in (0, 1) : f(\alpha) = 0$$

We therefore proceed by applying the bisection method to the interval $[a, b] = [0, 1]$, with $f(0) = -1 < 0$ and $f(1) = 2 > 0$.

Iteration 1 ($n = 0$)

$$\begin{aligned} [a_0, b_0] &= [0, 1] \\ c_0 &= \frac{a_0 + b_0}{2} = \frac{1}{2} \end{aligned}$$

with $f(c_0) = \frac{1}{8} > 0$. Then, in the next iteration, $b_1 = c_0$.

Solution

Iteration 2 ($n = 1$)

$$[a_1, b_1] = \left[0, \frac{1}{2}\right], f(0) < 0, f\left(\frac{1}{2}\right) > 0$$

$$c_1 = \frac{a_1 + b_1}{2} = \frac{1}{4}$$

with $f(c_1) < 0$. Then, in the next iteration, $a_2 = c_1$.

Iteration 3 ($n = 2$)

$$[a_2, b_2] = \left[\frac{1}{4}, \frac{1}{2} \right], f\left(\frac{1}{4}\right) < 0, f\left(\frac{1}{2}\right) > 0$$

$$c_2 = \frac{a_2 + b_2}{2} = \frac{3}{8}$$

with $f(c_2) < 0$. Then, in the next iteration, $a_3 = c_2$. To reach the desired level of approximation, we will have to wait until iteration 9 ($n = 8$), where we will obtain

$$a_8 = \frac{29}{64} \simeq 0.453 \quad b_8 = \frac{117}{256} \simeq 0.457$$

We can finally state that $\alpha \in \left(\frac{29}{64}, \frac{117}{256}\right)$ correct to two decimal places.

Exercise 12.2

Using the bisection method, find the solutions of the equation

$$x + \log x = 0$$

correct to two decimal places.

(First of all, one must determine how many solutions there are.)

Solution

Written in the form

$$\log x = -x,$$

the equation suggests first comparing the graphs of the functions $y = \log x$ and $y = -x$, in order to determine beforehand the number of solutions of the equation. Comparing the two graphs, we deduce that there exists a unique solution $\alpha \in (0, 1)$, i.e., a unique zero of the function $f(x) = x + \log x$ in that interval. More formally:

$$\exists! \alpha \in (0, 1) : f(\alpha) = 0$$

We therefore proceed by applying the bisection method to the interval $[a, b] = [0 + \varepsilon, 1]$, $\varepsilon > 0$, with $f(1) = 1 > 0$ and $f < 0$ in a right neighborhood of the point $x_0 = 0$ of arbitrarily small radius ε (note that $y = \log x$ is not defined at $x = 0$).

Iteration 1 ($n = 0$)

$$[a_0, b_0] = [0 + \varepsilon, 1]$$

$$c_0 = \frac{a_0 + b_0}{2} = \frac{1 + \varepsilon}{2} \simeq \frac{1}{2}$$

with $f(c_0) = \frac{1}{2} - \log 2 < 0$. Then, in the next iteration, $a_1 = c_0$.

Iteration 2 ($n = 1$)



$$[a_1, b_1] = \left[\frac{1}{2}, 1 \right], f\left(\frac{1}{2}\right) < 0, f(1) > 0$$

$$c_1 = \frac{a_1 + b_1}{2} = \frac{3}{4}$$

with $f(c_1) > 0$. Then, in the next iteration, $b_2 = c_1$.

Iteration 3 ($n = 2$)

$$[a_2, b_2] = \left[\frac{1}{2}, \frac{3}{4} \right], f\left(\frac{1}{2}\right) < 0, f\left(\frac{3}{4}\right) > 0$$

$$c_2 = \frac{a_2 + b_2}{2} = \frac{5}{8}$$

with $f(c_2) > 0$. Then, in the next iteration, $b_3 = c_2$.

Solution

Iteration 4 ($n = 3$)

$$[a_3, b_3] = \left[\frac{1}{2}, \frac{5}{8} \right], f\left(\frac{1}{2}\right) < 0, f\left(\frac{5}{8}\right) > 0$$

$$c_3 = \frac{a_3 + b_3}{2} = \frac{9}{16}$$

with $f(c_3) < 0$. Then, in the next iteration, $a_4 = c_3$. Observe that

$$a_3 = 0.5, b_3 = 0.625$$

and we are therefore still far from having two correct decimal places. To reach the desired level of approximation, we will have to wait until iteration 10 ($n = 9$), where we will obtain

$$a_9 = \frac{145}{256} \simeq 0.566 \quad b_9 = \frac{291}{512} \simeq 0.568$$

We can finally state that $\alpha \in \left(\frac{145}{256}, \frac{291}{512}\right)$ correct to two decimal places.

Exercise 12.3

Using the bisection method, find the solutions of the equation

$$2 \sin x = x$$

correct to one decimal place.

(First of all, one must determine how many solutions there are.)

Exercise 12.4

Using the bisection method, find the solutions of the equation

$$x^2 - 2 - \log x = 0$$

correct to one decimal place.

(First of all, one must determine how many solutions there are.)

Finding local and global extrema and extremum points

Exercise 13.1

Compute all the maximum and minimum points (global and local) of the function:

$$f(x) = 5 + 54x - 2x^3 \quad \text{on the interval } [0, 4]$$

Solution

1. The values of the function at the endpoints of the interval are: $f(0) = 5$ and $f(4) = 93$.
2. The derivative is:

$$f'(x) = 54 - 6x^2 = -6(x^2 - 9) = -6(x + 3)(x - 3)$$

We solve the equation:

$$f'(x) = 0 \iff -6(x + 3)(x - 3) = 0 \iff x = \pm 3$$

$$x_1 = 3 \in [0, 4] \quad \text{stationary point}$$

3. We have $f'(x) \geq 0$ for $x \in [-3, 3]$. Studying the sign of f' near $x = 3$, we deduce that $x = 3$ is a local maximum point and $f(3) = 113$ is a local maximum.
4. We have:

$$f(3) = 113 > f(0) = 5, \quad f(3) = 113 > f(4) = 93$$

We therefore conclude that:

- $f(0) = 5$ is the **global minimum** and $x = 0$ is a **global minimum point**
- $f(3) = 113$ is the **global maximum** and $x = 3$ is a **global maximum point**
- $f(4) = 93$ is a **local minimum** and $x = 4$ is a **local minimum point**

Exercise 13.2

Compute all the maximum and minimum points (global and local) of the function:

$$f(x) = 2x^3 - 3x^2 - 12x + 1 \quad \text{on the interval } [-2, 3]$$

Solution

1. The values of the function at the endpoints of the interval are: $f(-2) = -3$ and $f(3) = -8$.
2. The derivative is:

$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x+1)(x-2)$$

We solve the equation:

$$f'(x) = 0 \iff 6(x^2 - x - 2) = 0 \iff x = \frac{1 \pm 3}{2}$$

$$x_1 = -1 \in [-2, 3] \quad \text{and} \quad x_2 = 2 \in [-2, 3] \quad \text{stationary points}$$

3. We have $f'(x) \geq 0$ for $x \in (-\infty, -1] \cup [2, +\infty)$. Studying the sign of f' near $x = -1$, we deduce that $x = -1$ is a local maximum point and $f(-1) = 8$ is a local maximum. Studying the sign of f' near $x = 2$, we deduce that $x = 2$ is a local minimum point and $f(2) = -19$ is a local minimum.
4. We have:

$$f(-1) = 8 > f(-2) = -3, \quad f(-1) = 8 > f(3) = -8$$

$$f(2) = -19 < f(-2) = -3, \quad f(2) = -19 < f(3) = -8$$

We therefore conclude that:

- $f(-2) = -3$ is a **local minimum** and $x = -2$ is a **local minimum point**
- $f(-1) = 8$ is the **global maximum** and $x = -1$ is a **global maximum point**
- $f(2) = -19$ is the **global minimum** and $x = 2$ is a **global minimum point**
- $f(3) = -8$ is a **local maximum** and $x = 3$ is a **local maximum point**

Exercise 13.3

Compute all the maximum and minimum points (global and local) of the function:

$$f(x) = x^3 - 6x^2 + 9x + 2 \quad \text{on the interval } [-1, 4]$$

Solution

1. The values of the function at the endpoints of the interval are: $f(-1) = -14$ and $f(4) = 6$.

2. The derivative is:

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 3)(x - 1)$$

We solve the equation:

$$f'(x) = 0 \iff 3(x^2 - 4x + 3) = 0 \iff x = \frac{4 \pm 2}{2}$$

$$x_1 = 1 \in [-1, 4] \text{ and } x_2 = 3 \in [-1, 4] \text{ stationary points}$$

3. We have $f'(x) \geq 0$ for $x \in (-\infty, 1] \cup [3, +\infty)$. Studying the sign of f' near $x = 1$, we deduce that $x = 1$ is a local maximum point and $f(1) = 6$ is a local maximum. Studying the sign of f' near $x = 3$, we deduce that $x = 3$ is a local minimum point and $f(3) = 2$ is a local minimum.

4. We have:

$$f(1) = 6 > f(-1) = -14, \quad f(1) = 6 \geq f(4) = 6$$

$$f(3) = 2 > f(-1) = -14, \quad f(3) = 2 < f(4) = 6$$

We therefore conclude that:

- $f(-1) = -14$ is the **global minimum** and $x = -1$ is a **global minimum point**
- $f(1) = 6$ is the **global maximum** and $x = 1$ is a **global maximum point**
- $f(3) = 2$ is a **local minimum** and $x = 3$ is a **local minimum point**
- $f(4) = 6$ is the **global maximum** and $x = 4$ is a **global maximum point**

Exercise 13.4

Compute all the maximum and minimum points (global and local) of the function:

$$f(x) = x^4 - 2x^2 + 3 \quad \text{on the interval } [-2, 3]$$

Solution

1. The values of the function at the endpoints of the interval are: $f(-2) = 11$ and $f(3) = 66$.

2. The derivative is:

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$$

We solve the equation:

$$f'(x) = 0 \iff 4x(x^2 - 1) = 0 \iff x = \pm 1 \text{ and } x = 0$$

$$x_1 = -1 \in [-2, 3], \quad x_2 = 0 \in [-2, 3] \text{ and } x_3 = 1 \in [-2, 3] \text{ stationary points}$$

3. We have $f'(x) \geq 0$ for $x \in [-1, 0] \cup [1, +\infty)$. Studying the sign of f' near $x = -1$, we deduce that $x = -1$ is a local minimum point and $f(-1) = 2$ is a local minimum. Studying the sign of f' near $x = 0$, we deduce that $x = 0$ is a local maximum point

and $f(0) = 3$ is a local maximum. Studying the sign of f' near $x = 1$, we deduce that $x = 1$ is a local minimum point and $f(1) = 2$ is a local minimum.

4. We have:

$$f(-1) = 2 < f(-2) = 11, \quad f(-1) = 2 < f(3) = 66$$

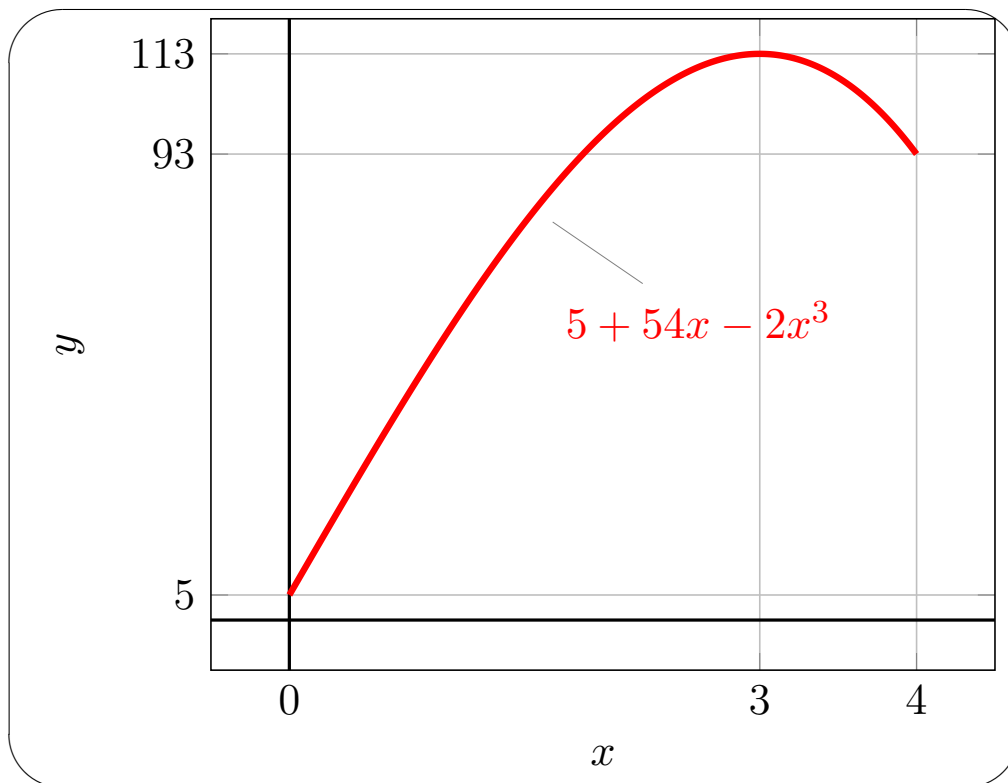
$$f(0) = 3 < f(-2) = 11, \quad f(0) = 3 < f(3) = 66$$

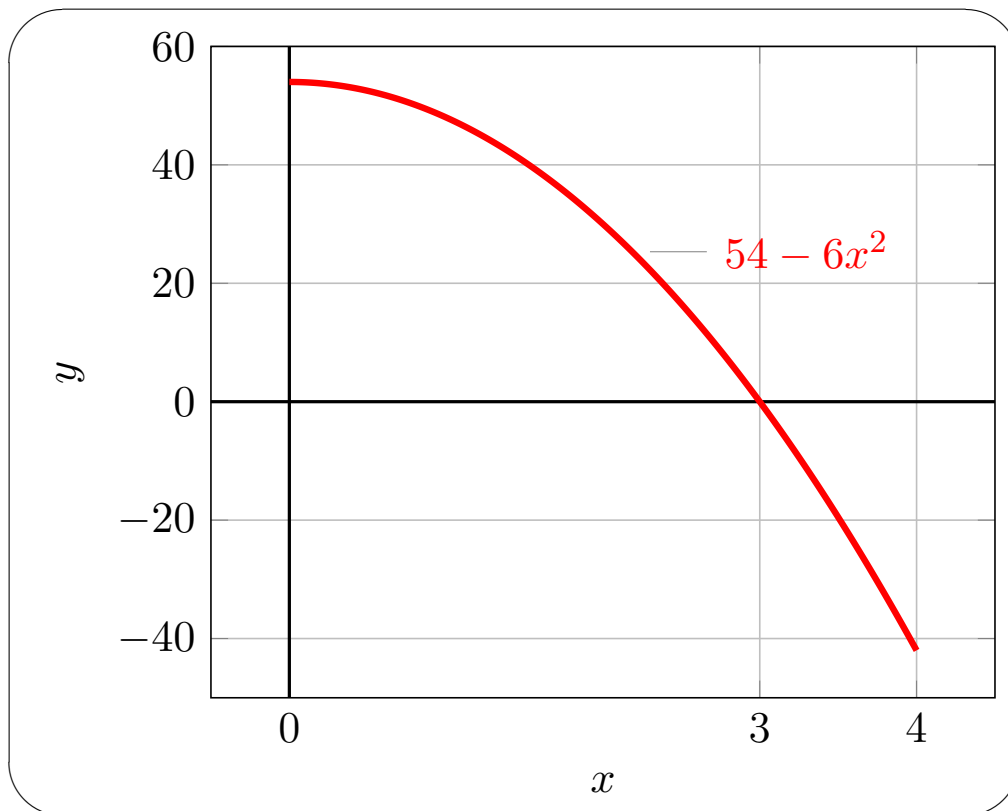
$$f(1) = 2 < f(-2) = 11, \quad f(1) = 2 < f(3) = 66$$

We therefore conclude that:

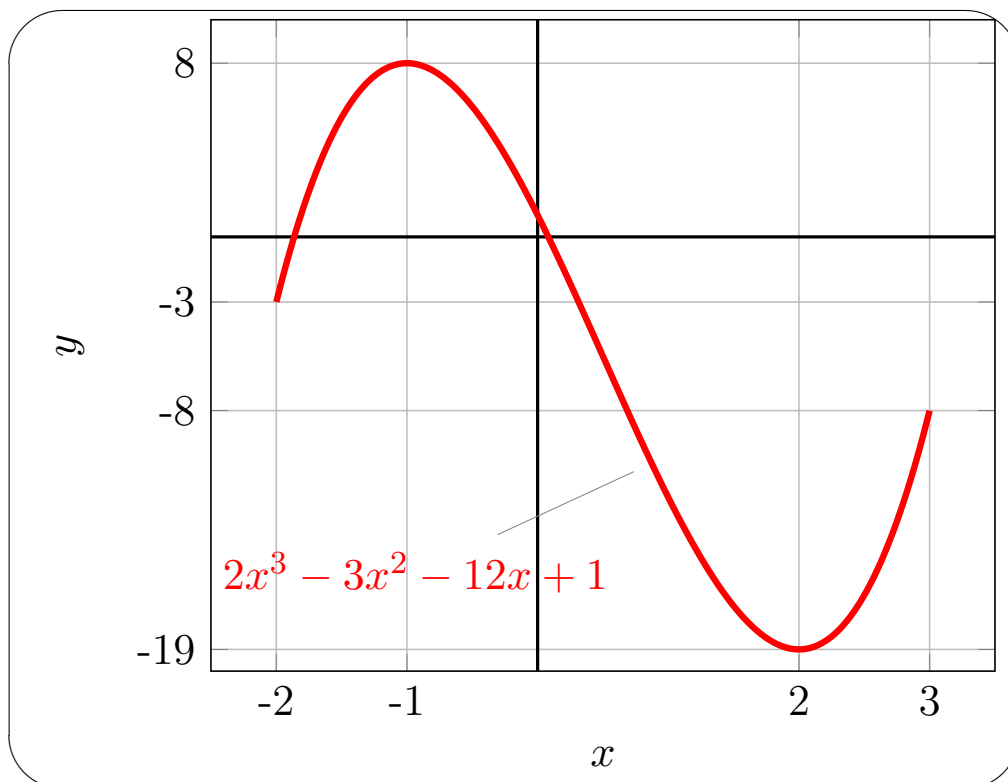
- $f(-2) = 11$ is a **local maximum** and $x = -2$ is a **local maximum point**
- $f(-1) = 2$ is the **global minimum** and $x = -1$ is a **global minimum point**
- $f(0) = 3$ is a **local maximum** and $x = 0$ is a **local maximum point**
- $f(1) = 2$ is the **global minimum** and $x = 1$ is a **global minimum point**
- $f(3) = 66$ is the **global maximum** and $x = 3$ is a **global maximum point**

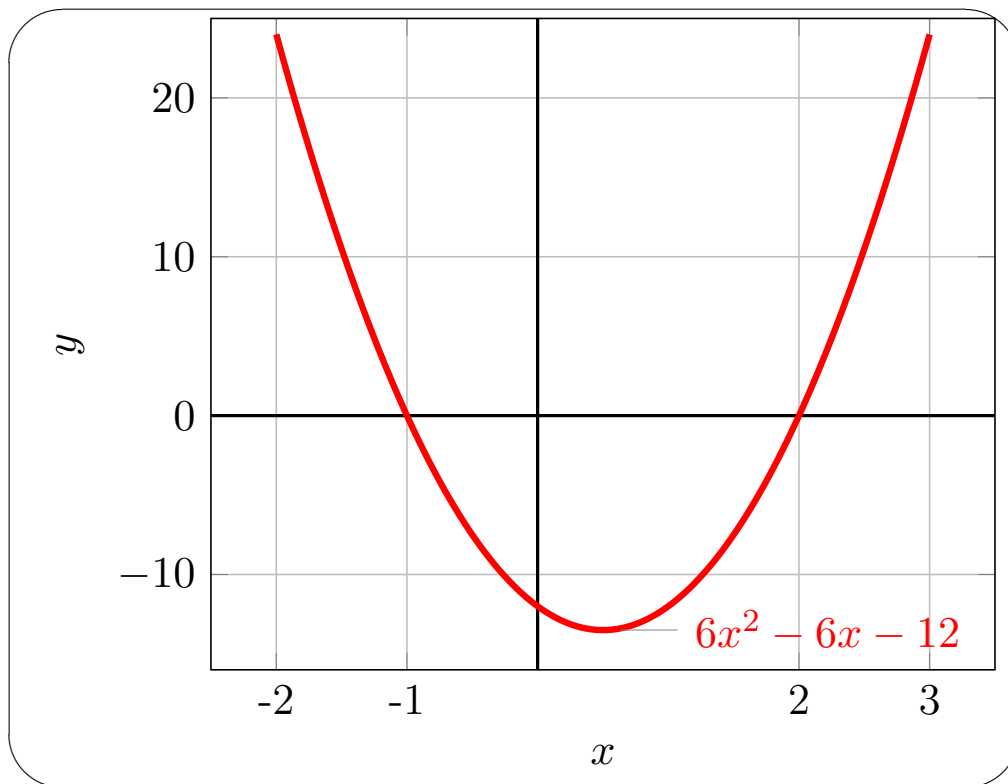
These are the graphs of the function $f(x) = 5 + 54x - 2x^3$ and of its derivative on the interval $[0, 4]$:



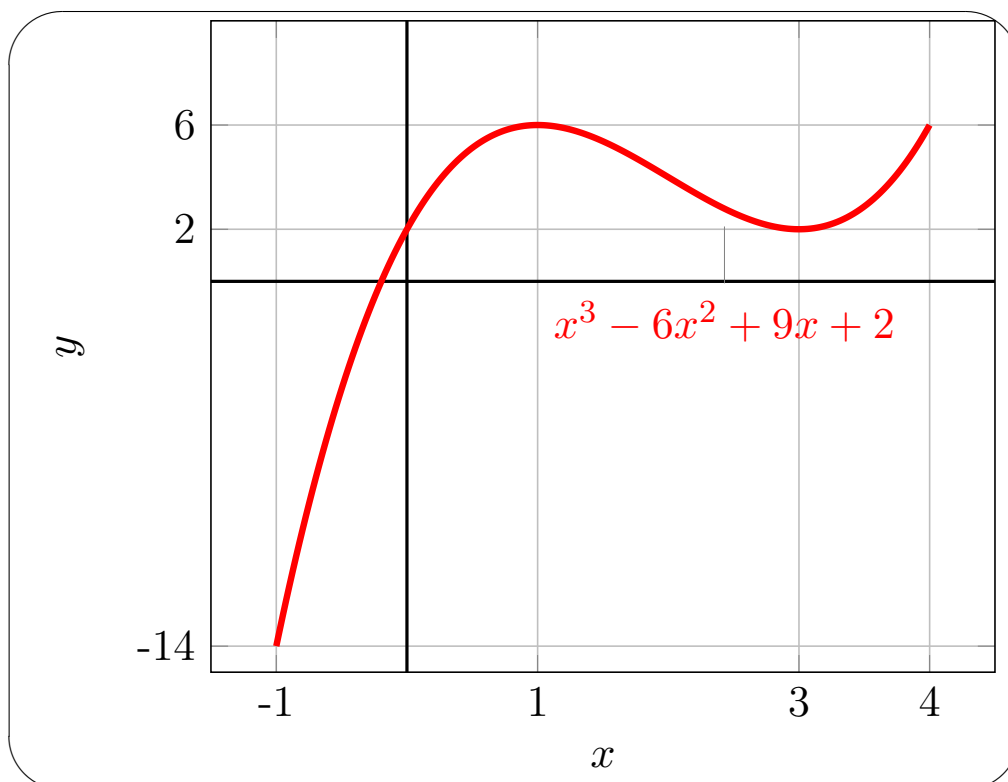


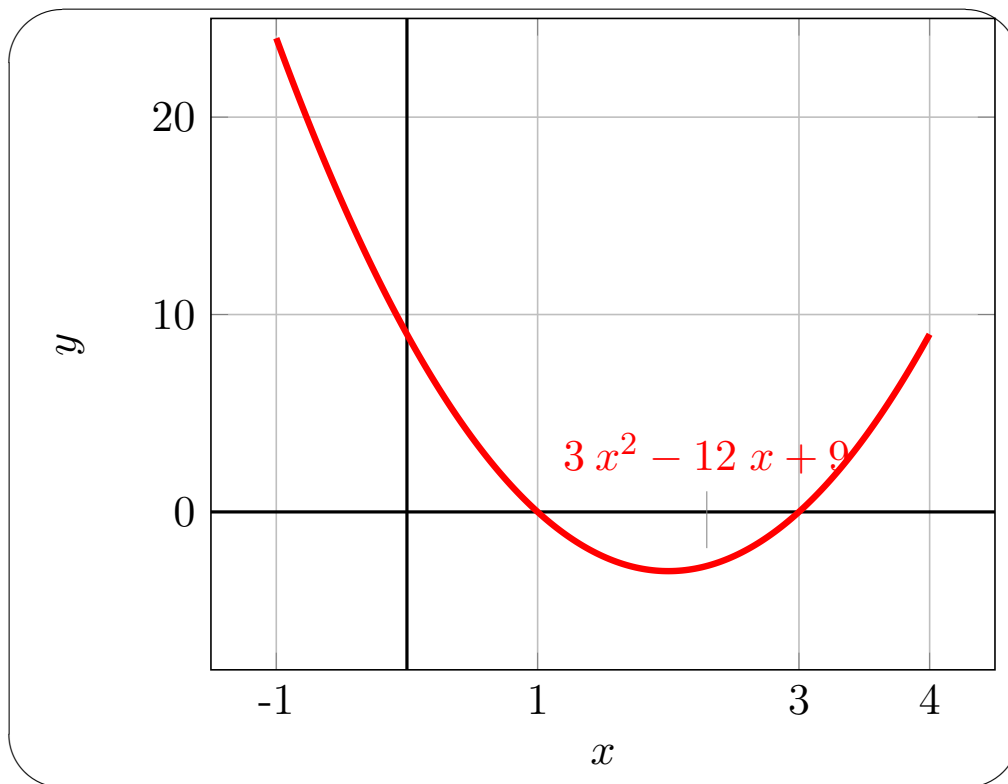
These are the graphs of the function $f(x) = 2x^3 - 3x^2 - 12x + 1$ and of its derivative on the interval $[-2, 3]$:





These are the graphs of the function $f(x) = x^3 - 6x^2 + 9x + 2$ and of its derivative on the interval $[-1, 4]$:





These are the graphs of the function $f(x) = x^4 - 2x^2 + 3$ and of its derivative on the interval $[-2, 3]$:

