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UNIVERSITÀ DI ROMA

Mathematical Analysis 1

Exercises, volume 4 — Limits

The worked exercises on limits, in two parts as the volume of lecture notes, one sheet per chapter, with the solution right after each statement. Limits of sequences: properties of sequences, verifying and computing limits, additional exercises. Limits of functions: limits of real functions of a real variable. The sheets are by Fabio Furini and Gianluca Priori; many exercises are taken from M. Bramanti, *Esercitazioni di Analisi matematica 1*, Esculapio.

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Online version and downloadable material:
fabiofurini.github.io/mathematical-analysis-1/
github.com/fabiofurini/mathematical-analysis-1

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Part I

Limits of sequences



Properties of sequences

Exercise 1.1

Consider the sequence

$$a_n = \log \left(1 + (-1)^n \frac{n}{n+1} \right), \quad \text{for } n = 1, 2, 3, \dots$$

Questions:

1. Is the sequence bounded above? If so, determine $\sup \{a_n\}$.
2. Does the sequence have a maximum? If so, determine $\max \{a_n\}$.
3. Is the sequence bounded below? If so, determine $\inf \{a_n\}$.
4. Does the sequence have a minimum? If so, determine $\min \{a_n\}$.

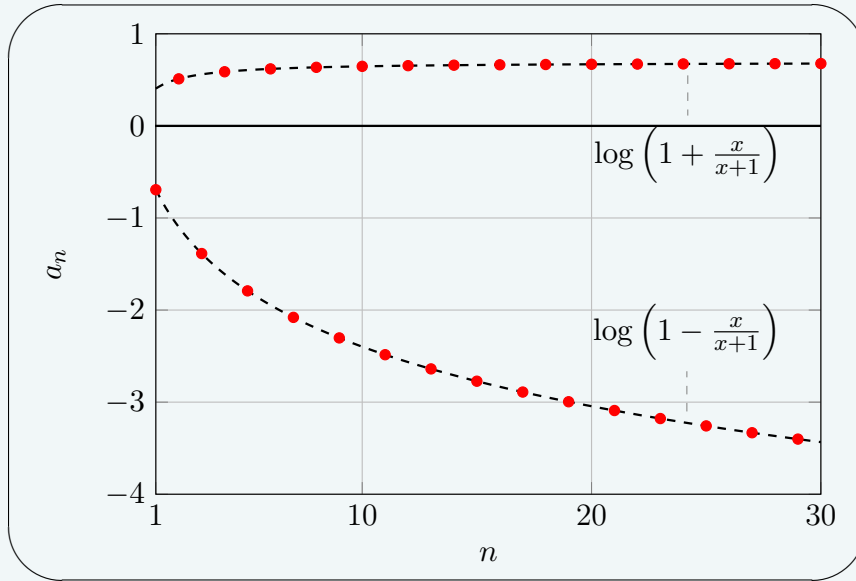
Solution

For $n = 1$ we have

$$\log \left(1 - 1 \frac{1}{2} \right) = \log \frac{1}{2} \approx -0.69314.$$

For $n = 2$ we have

$$\log \left(1 + 1 \frac{2}{3} \right) = \log \frac{5}{3} \approx 0.510826.$$



The presence of the alternating sign $(-1)^n$ suggests studying the behavior of the sequence by distinguishing what happens for even n and odd n .

Solution

- If n is even:

$$a_n = \log \left(1 + \frac{n}{n+1} \right), \quad \text{for } n = 2, 4, 6 \dots$$

hence the argument of the logarithm is

$$1 + \frac{n}{n+1} = 1 + \frac{n+1-1}{n+1} = 2 - \frac{1}{n+1}$$

which varies in

$$\left[\frac{5}{3}, 2 \right) \text{ and the logarithm in } \left[\log \frac{5}{3}, \log 2 \right),$$

i.e., it is positive and bounded above. Hence

$$\sup \{ a_n : n \text{ is even} \} = \log 2 \approx 0.69314$$

but $\log 2$ is not the maximum of these values.

- If n is odd:

$$a_n = \log \left(1 - \frac{n}{n+1} \right), \quad \text{for } n = 1, 3, 5 \dots$$

hence the argument of the logarithm is

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

which varies in

$$\left(0, \frac{1}{2} \right] \text{ and the logarithm in } \left(-\infty, \log \frac{1}{2} \right],$$

i.e., it is negative and, as n increases, it is unbounded below.

The sequence as a whole is therefore bounded above but not bounded below; it has no maximum (although its sup is finite) and no minimum (because it is unbounded below).

Exercise 1.2

Consider the sequence

$$a_n = e^{-\frac{1}{n}} \sin n, \quad \text{for } n = 1, 2, 3, \dots$$

Questions:

1. Is the sequence bounded above?
2. Is the sequence bounded below?
3. Is it eventually positive?
4. Does it never vanish?
5. Does it have a limit (finite or infinite)?

Solution

For $n = 1$ we have

$$e^{-1} \sin 1 \approx 0.3679 \cdot 0.8415 \approx 0.309560.$$

For $n = 2$ we have

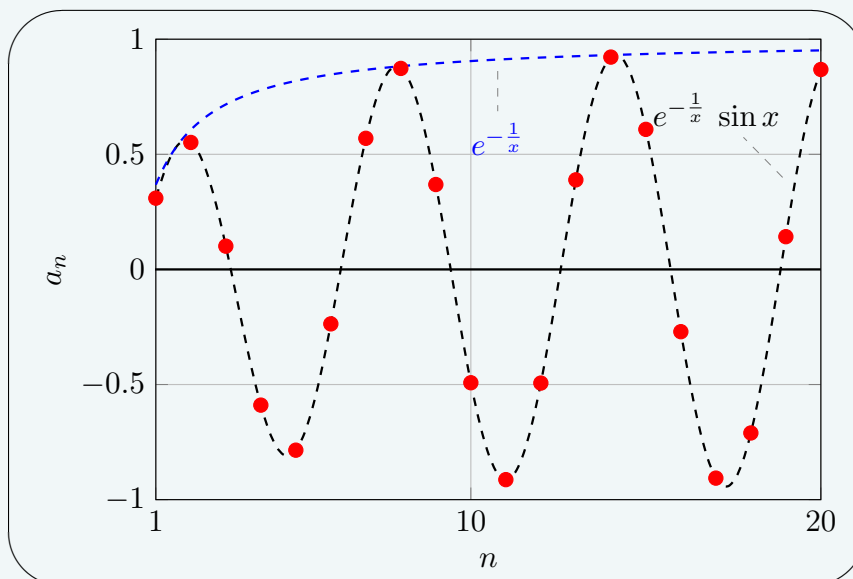
$$e^{-\frac{1}{2}} \sin 2 \approx 0.6065 \cdot 0.9093 \approx 0.551517.$$

For $n = 10$ we have

$$e^{-\frac{1}{10}} \sin 10 \approx 0.9048 \cdot -0.5440 \approx -0.492251.$$

For $n = 20$ we have

$$e^{-\frac{1}{20}} \sin 20 \approx 0.9512 \cdot 0.9129 \approx 0.8684.$$



Solution

The sequence is the product of the sequence

$$n \mapsto e^{-\frac{1}{n}} \quad \text{with} \quad \lim_{n \rightarrow +\infty} e^{-\frac{1}{n}} = 1,$$

which is bounded, always nonzero, convergent, and of the sequence

$$n \mapsto \sin n,$$

which is bounded but irregular, and never zero (since n starts from 1, by the irrationality of π , the angle n is never an integer multiple of π).

Hence, the sequence given by the product of these two is bounded, never zero, and irregular.

Since

$$e^{-\frac{1}{n}} \rightarrow 1$$

and the sign of $\sin n$ is not eventually constant, the sequence given by the product of the two is not eventually positive.

Exercise 1.3

Consider the sequence

$$a_n = e^n \sin n, \quad \text{for } n = 1, 2, 3, \dots$$

Questions:

1. Is the sequence bounded above?
2. Is the sequence bounded below?
3. Is it eventually positive?
4. Does it never vanish?
5. Does it have a limit (finite or infinite)?

Solution

For $n = 1$ we have

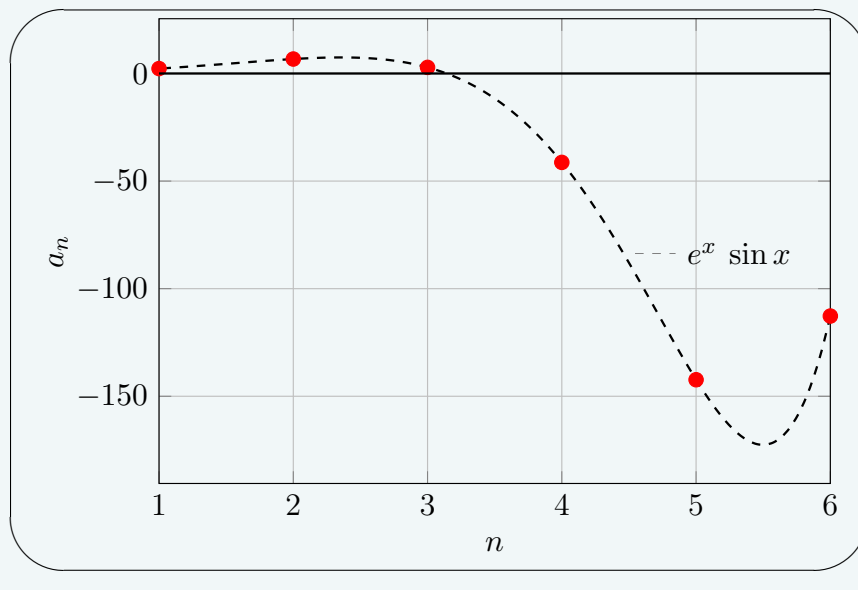
$$e^1 \sin 1 \approx 2.7183 \cdot 0.8415 \approx 2.2874.$$

For $n = 2$ we have

$$e^2 \sin 2 \approx 7.3891 \cdot 0.9093 \approx 6.7188.$$

For $n = 5$ we have

$$e^5 \sin 5 \approx 148.4132 \cdot -0.9589 \approx -142.3170.$$



Solution

The sequence is the product of the sequence

$$n \mapsto e^n \quad \text{with} \quad \lim_{n \rightarrow +\infty} e^n = +\infty,$$

which is unbounded above, bounded below, always nonzero, divergent, and of the sequence

$$n \mapsto \sin n,$$

which is bounded but irregular, and never zero (since n starts from 1, by the irrationality of π , the angle n is never an integer multiple of π).

Hence, the sequence given by the product of these two is unbounded above, unbounded below, never zero, and irregular.

Since

$$e^n \rightarrow +\infty$$

and the sign of $\sin n$ is not eventually constant, the sequence given by the product of the two is not eventually positive.

Exercise 1.4

Consider the sequence

$$a_n = \frac{n^{(-1)^n}}{n+1}, \quad \text{for } n = 1, 2, 3, \dots$$

Questions:

1. Is the sequence bounded above? If so, determine $\sup \{a_n\}$.
2. Does the sequence have a maximum? If so, determine $\max \{a_n\}$.

3. Is the sequence bounded below? If so, determine $\inf \{a_n\}$.
4. Does the sequence have a minimum? If so, determine $\min \{a_n\}$.

Solution

For $n = 1$ we have

$$1^{(-1)^1} \frac{1}{2} = 1 \frac{1}{2}$$

For $n = 2$ we have

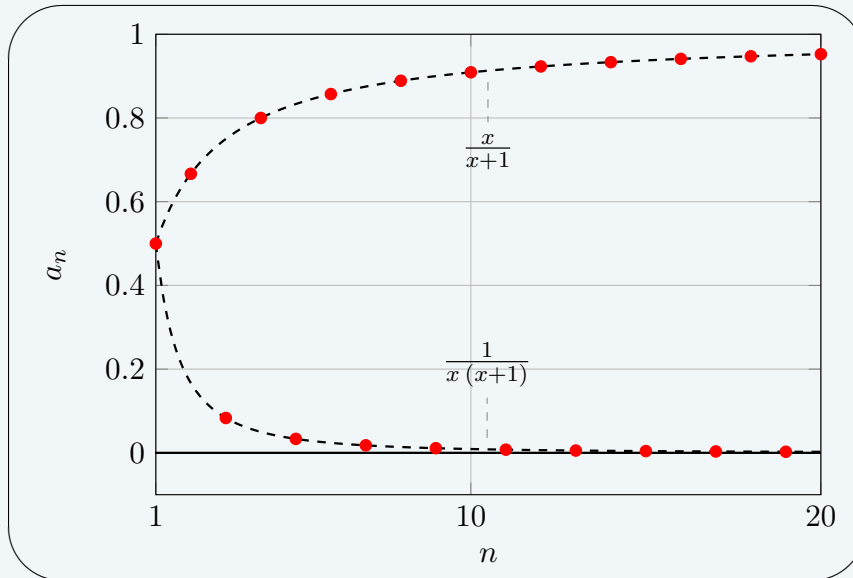
$$2^{(-1)^2} \frac{1}{3} = 2 \frac{1}{3}.$$

For $n = 3$ we have

$$3^{(-1)^3} \frac{1}{4} = \frac{1}{3} \frac{1}{4}.$$

For $n = 4$ we have

$$4^{(-1)^4} \frac{1}{5} = 4 \frac{1}{5}.$$



The presence of the alternating sign $(-1)^n$ suggests studying the behavior of the sequence by distinguishing what happens for even n and odd n .

Solution

- If n is even:

$$a_n = \frac{n}{n+1}, \quad \text{for } n = 2, 4, 6 \dots$$

Adding $+1$ and -1 to the numerator, we obtain

$$a_n = \frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1}$$

which varies in

$$\left[\frac{2}{3}, 1\right),$$

i.e., it is positive and bounded above. Hence

$$\sup \{a_n : n \text{ is even} \} = 1$$

but 1 is not the maximum of these values.

- If n is odd:

$$a_n = \frac{\frac{1}{n}}{n+1} = \frac{1}{n} \cdot \frac{1}{n+1} = \frac{1}{n(n+1)}, \quad \text{for } n = 1, 3, 5 \dots$$

which varies in

$$\left(0, \frac{1}{2}\right],$$

i.e., it is positive and bounded below. Hence

$$\inf \{a_n : n \text{ is odd} \} = 0$$

but 0 is not the minimum of these values.

The sequence as a whole is bounded; it has no maximum (although its sup is finite) and no minimum (although its inf is finite).

Verifying limits of sequences

Exercise 2.1

Using the definition of limit, verify that:

$$\lim_{n \rightarrow +\infty} \frac{n}{2n+5} = \frac{1}{2}$$

Solution

We have

$$a_n = \frac{n}{2n+5}$$

We must verify that for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$n > n(\varepsilon) \Rightarrow \frac{1}{2} - \varepsilon < a_n < \frac{1}{2} + \varepsilon$$

We have

$$a_n - \frac{1}{2} = -\frac{5}{4n+10},$$

hence we must verify that

$$n > n(\varepsilon) \Rightarrow -\varepsilon < -\frac{5}{4n+10} < \varepsilon$$

The second inequality is always true, while the first is equivalent to

$$4n+10 > \frac{5}{\varepsilon} \quad \text{hence it is satisfied for } n > \frac{5}{4\varepsilon} - \frac{5}{2}$$

For a fixed $\varepsilon > 0$, it suffices to choose the first integer

$$n(\varepsilon) > \frac{5}{4\varepsilon} - \frac{5}{2}$$

to satisfy the condition required by the definition of limit.

Exercise 2.2

Using the definition of limit, verify that:

$$\lim_{n \rightarrow +\infty} \frac{1}{\sqrt{n+1}} = 0$$

Solution

We have

$$a_n = \frac{1}{\sqrt{n+1}}$$

We must verify that for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$n > n(\varepsilon) \Rightarrow -\varepsilon < a_n < \varepsilon$$

The first inequality is always true, while the second is equivalent to

$$\sqrt{n+1} > \frac{1}{\varepsilon} \quad \text{hence it is satisfied for } n > \frac{1}{\varepsilon^2} - 1$$

For a fixed $\varepsilon > 0$, it suffices to choose the first integer

$$n(\varepsilon) > \frac{1}{\varepsilon^2} - 1$$

to satisfy the condition required by the definition of limit.

Exercise 2.3

Using the definition of limit, verify that:

$$\lim_{n \rightarrow +\infty} \log \left(1 + \frac{1}{n} \right) = 0$$

Solution

We have

$$a_n = \log \left(1 + \frac{1}{n} \right)$$

We must verify that for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$n > n(\varepsilon) \Rightarrow -\varepsilon < a_n < \varepsilon$$

The first inequality is always true, while the second is equivalent to

$$1 + \frac{1}{n} < e^\varepsilon \quad \text{hence it is satisfied for } n > \frac{1}{e^\varepsilon - 1}$$

For a fixed $\varepsilon > 0$, it suffices to choose the first integer

$$n(\varepsilon) > \frac{1}{e^\varepsilon - 1}$$

to satisfy the condition required by the definition of limit.

Exercise 2.4

Using the definition of limit, verify that:

$$\lim_{n \rightarrow +\infty} \sqrt{4 + \frac{1}{n}} = 2$$

Solution

We have

$$a_n = \sqrt{4 + \frac{1}{n}}$$

We must verify that for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$n > n(\varepsilon) \Rightarrow 2 - \varepsilon < a_n < 2 + \varepsilon$$

The first inequality is always true since $a_n > 2$ for every n , while the second is equivalent to

$$4 + \frac{1}{n} < 4 + \varepsilon^2 + 4\varepsilon \quad \text{hence it is satisfied for } n > \frac{1}{\varepsilon^2 + 4\varepsilon}$$

For a fixed $\varepsilon > 0$, it suffices to choose the first integer

$$n(\varepsilon) > \frac{1}{\varepsilon^2 + 4\varepsilon}$$

to satisfy the condition required by the definition of limit.

Exercise 2.5

Using the definition of limit, verify that:

$$\lim_{n \rightarrow +\infty} (n^2 - 1) = +\infty$$

Solution

We have

$$a_n = n^2 - 1$$

We must verify that for every $M > 0$ there exists $n(M) \in \mathbb{N}$ such that

$$n > n(M) \Rightarrow a_n > M$$

The inequality

$$n^2 - 1 > M \quad \text{is satisfied for } n > \sqrt{M + 1}$$

Hence, for a fixed $M > 0$, it suffices to choose the first integer

$$n(M) > \sqrt{M+1}$$

to satisfy the required divergence condition.

Exercise 2.6

Using the definition of limit, verify that:

$$\lim_{n \rightarrow +\infty} \log(\sqrt{n} + 1) = +\infty$$

Solution

We have

$$a_n = \log(\sqrt{n} + 1)$$

We must verify that for every $M > 0$ there exists $n(M) \in \mathbb{N}$ such that

$$n > n(M) \Rightarrow a_n > M$$

The inequality

$$\log(\sqrt{n} + 1) > M \quad \text{is satisfied for } n > (e^M - 1)^2$$

Hence, for a fixed $M > 0$, it suffices to choose the first integer

$$n(M) > (e^M - 1)^2$$

to satisfy the required divergence condition.

Computing limits of sequences

3.1 Computing limits with basic techniques

If:

$$a_n \rightarrow \ell_a \in \mathbb{R} \quad \text{and} \quad b_n \rightarrow \ell_b \in \mathbb{R},$$

we have:

$$\begin{aligned} a_n \pm b_n &\rightarrow \ell_a \pm \ell_b, & \frac{a_n}{b_n} &\rightarrow \frac{\ell_a}{\ell_b} \quad (b_n, \ell_b \neq 0, \text{ eventually}), \\ a_n b_n &\rightarrow \ell_a \ell_b, & a_n^{b_n} &\rightarrow \ell_a^{\ell_b} \quad (a_n, \ell_a > 0, \text{ eventually}). \end{aligned}$$

If:

$$a_n \rightarrow \ell_a \in \mathbb{R}, \quad b_n \rightarrow +\infty \quad \text{and} \quad c_n \rightarrow +\infty,$$

we have:

$$\begin{aligned} a_n + b_n &\rightarrow \ell_a + \infty = +\infty, & a_n - b_n &\rightarrow \ell_a - \infty = -\infty, \\ b_n + c_n &\rightarrow +\infty + \infty = +\infty, & -b_n - c_n &\rightarrow -\infty - \infty = -\infty. \end{aligned}$$

If:

$$a_n \rightarrow \ell_a \in \mathbb{R}, \quad b_n \rightarrow 0^+ \text{ or } b_n \rightarrow 0^- \quad \text{and} \quad c_n \rightarrow \infty,$$

we have:

$$a_n c_n \rightarrow \ell_a \infty = \infty \quad (\ell_a \neq 0), \quad \frac{a_n}{b_n} \rightarrow \frac{\ell_a}{0} = \infty \quad (\ell_a \neq 0), \quad \frac{a_n}{c_n} \rightarrow \frac{\ell_a}{\infty} = 0.$$

The hypothesis that $\{b_n\}$ tends to zero from above or from below (and hence that it eventually has constant sign, with $b_n \neq 0$) is necessary: with $a_n = 1$ and $b_n = \frac{(-1)^n}{n}$ we have $\frac{a_n}{b_n} = (-1)^n n$, which is irregular.

The four main indeterminate forms are:

$$[+\infty - \infty], \quad [0 \cdot \infty], \quad \left[\frac{0}{0}\right] \quad \text{and} \quad \left[\frac{\infty}{\infty}\right].$$

There are also three other indeterminate forms derived from $[0 \cdot \infty]$:

$$[1^\infty], \quad [0^0] \quad \text{and} \quad [\infty^0].$$

We have:

$$\lim_{n \rightarrow +\infty} n^\alpha = \begin{cases} +\infty & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha < 0 \end{cases} \quad \lim_{n \rightarrow +\infty} a^n = \begin{cases} +\infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } |a| < 1 \\ \text{does not exist} & \text{if } a \leq -1 \end{cases}$$

Exercise 3.1

Compute the following limit, which is an (elementary) indeterminate form:

$$\lim_{n \rightarrow +\infty} \frac{n^2 + 2n}{n + 1} = \left[\frac{\infty}{\infty}\right]$$

Solution

$$\lim_{n \rightarrow +\infty} \frac{n^2 + 2n}{n + 1} = \lim_{n \rightarrow +\infty} \frac{\overbrace{n^2}^{\rightarrow 1} \left(1 + \overbrace{\frac{2}{n}}^{\rightarrow 0} \right)}{\underbrace{n}_{\rightarrow 1} \left(1 + \underbrace{\frac{1}{n}}_{\rightarrow 0} \right)} = \lim_{n \rightarrow +\infty} \frac{n^2}{n} = \lim_{n \rightarrow +\infty} n = +\infty.$$

Exercise 3.2

Compute the following limit, which is an (elementary) indeterminate form:

$$\lim_{n \rightarrow +\infty} \frac{n^4 + 5}{n^5 + 7n - 1} = \left[\frac{\infty}{\infty}\right]$$

Solution

$$\lim_{n \rightarrow +\infty} \frac{n^4 + 5}{n^5 + 7n - 1} = \lim_{n \rightarrow +\infty} \frac{n^4 \left(1 + \frac{5}{n^4}\right)}{n^5 \left(1 + \frac{7}{n^4} - \frac{1}{n^5}\right)} = \lim_{n \rightarrow +\infty} \frac{n^4}{n^5} = \lim_{n \rightarrow +\infty} \frac{1}{n} = 0.$$

Exercise 3.3

Compute the following limit, which is an (elementary) indeterminate form:

$$\lim_{n \rightarrow +\infty} \frac{1 - n^2}{(n + 2)^2} = \left[\frac{\infty}{\infty} \right]$$

Solution

$$\lim_{n \rightarrow +\infty} \frac{1 - n^2}{(n + 2)^2} = \lim_{n \rightarrow +\infty} \frac{-n^2 \left(1 - \frac{1}{n^2}\right)}{n^2 \left(1 + \frac{4}{n} + \frac{4}{n^2}\right)} = \lim_{n \rightarrow +\infty} \frac{-n^2}{n^2} = \lim_{n \rightarrow +\infty} -1 = -1.$$

Exercise 3.4

Compute the following limit, which is an (elementary) indeterminate form:

$$\lim_{n \rightarrow +\infty} \sqrt{n^2 + 1} - \sqrt{n} = [+ \infty - \infty]$$

Solution

$$\lim_{n \rightarrow +\infty} \sqrt{n^2 + 1} - \sqrt{n} = \lim_{n \rightarrow +\infty} n \left(\underbrace{\sqrt{1 + \frac{1}{n^2}}}_{\substack{\rightarrow 0 \\ \rightarrow 1}} - \underbrace{\sqrt{\frac{1}{n}}}_{\rightarrow 0} \right) = \lim_{n \rightarrow +\infty} n = +\infty.$$

Solution

Alternative method: multiplying and dividing by $\sqrt{n^2 + 1} + \sqrt{n}$ we obtain

$$\frac{(\sqrt{n^2 + 1} - \sqrt{n})(\sqrt{n^2 + 1} + \sqrt{n})}{\sqrt{n^2 + 1} + \sqrt{n}} = \frac{(\sqrt{n^2 + 1})^2 - (\sqrt{n})^2}{\sqrt{n^2 + 1} + \sqrt{n}} = \frac{n^2 + 1 - n}{\sqrt{n^2 + 1} + \sqrt{n}},$$

recalling that $(a - b)(a + b) = a^2 - b^2$. Moreover, we have

$$\frac{n^2 + 1 - n}{\sqrt{n^2 + 1} + \sqrt{n}} = \frac{n^2 \left(1 + \frac{1}{n^2} - \frac{1}{n}\right)}{\sqrt{n^2 \left(1 + \frac{1}{n^2}\right)} + \sqrt{n}} = \frac{n^2 \left(1 + \frac{1}{n^2} - \frac{1}{n}\right)}{n \sqrt{\left(1 + \frac{1}{n^2}\right)} + n^{\frac{1}{2}}} = \frac{n^2 \left(1 + \frac{1}{n^2} - \frac{1}{n}\right)}{n \left(\sqrt{\left(1 + \frac{1}{n^2}\right)} + \frac{1}{n^{1/2}}\right)}.$$

Hence

$$\lim_{n \rightarrow +\infty} \sqrt{n^2 + 1} - \sqrt{n} = \lim_{n \rightarrow +\infty} n \frac{\overbrace{\left(1 + \frac{1}{n^2} - \frac{1}{n}\right)}^{\rightarrow 1}}{\underbrace{\left(\sqrt{\left(1 + \frac{1}{n^2}\right)} + \frac{1}{n^{1/2}}\right)}_{\rightarrow 1}} = \lim_{n \rightarrow +\infty} n = +\infty.$$

Exercise 3.5

Compute the following limit, which is an (elementary) indeterminate form:

$$\lim_{n \rightarrow +\infty} e^n - 2^n = [+ \infty - \infty]$$

Solution

$$\lim_{n \rightarrow +\infty} e^n - 2^n = \lim_{n \rightarrow +\infty} e^n \underbrace{\left(1 - \underbrace{\left(\frac{2}{e}\right)^n}_{\rightarrow 0}\right)}_{\rightarrow 1} = \lim_{n \rightarrow +\infty} e^n = +\infty$$

since

$$e^n \rightarrow +\infty \quad \text{and} \quad \left(\frac{2}{e}\right)^n \rightarrow 0.$$

Exercise 3.6

Compute the following limit, which is an (elementary) indeterminate form:

$$\lim_{n \rightarrow +\infty} 3^n + 4^n - 5^n = [+ \infty + \infty - \infty] = [+ \infty - \infty]$$

Solution

$$\lim_{n \rightarrow +\infty} 3^n + 4^n - 5^n = \lim_{n \rightarrow +\infty} -5^n \underbrace{\left(1 - \underbrace{\left(\frac{3}{5} \right)^n}_{\rightarrow 0} - \underbrace{\left(\frac{4}{5} \right)^n}_{\rightarrow 0} \right)}_{\rightarrow 1} = \lim_{n \rightarrow +\infty} -5^n = -\infty$$

since

$$-5^n \rightarrow -\infty, \quad \left(\frac{3}{5} \right)^n \rightarrow 0 \quad \text{and} \quad \left(\frac{4}{5} \right)^n \rightarrow 0.$$

Exercise 3.7

Compute the following simple limit

$$\lim_{n \rightarrow +\infty} n^{\sqrt{2}}$$

Solution

For every $\alpha > 0$, even irrational, $\lim_{n \rightarrow +\infty} n^\alpha = +\infty$.

Exercise 3.8

Compute the following simple limit

$$\lim_{n \rightarrow +\infty} n^{-e}$$

Solution

$$\lim_{n \rightarrow +\infty} n^{-e} = \lim_{n \rightarrow +\infty} \frac{1}{n^e} = 0.$$

Exercise 3.9

Compute the following simple limit

$$\lim_{n \rightarrow +\infty} \left(\frac{3n+5}{n^2+120} \right)^{2n}$$

Solution

We have

$$\frac{3n+5}{n^2+120} = \frac{3n \left(1 + \frac{5}{3n} \right)}{n^2 \left(1 + \frac{120}{n^2} \right)} = \frac{3}{n} \frac{\left(1 + \frac{5}{3n} \right)}{\left(1 + \frac{120}{n^2} \right)} \rightarrow 0 \quad \text{and} \quad 2n \rightarrow +\infty.$$

Hence

$$\lim_{n \rightarrow +\infty} \left(\frac{3n+5}{n^2+120} \right)^{2n} = 0^{+\infty} = 0.$$

Exercise 3.10

Compute the following simple limit

$$\lim_{n \rightarrow +\infty} \left(\frac{10n^2 - 5n}{n^3 - 8} \right)^{-3n+8}$$

Solution

We have

$$\frac{10n^2 - 5n}{n^3 - 8} = \frac{10n^2 \left(1 - \frac{5}{10n}\right)}{n^3 \left(1 - \frac{8}{n^3}\right)} = \frac{10}{n} \frac{\left(1 - \frac{1}{2n}\right)}{\left(1 - \frac{8}{n^3}\right)} \rightarrow 0 \quad \text{and} \quad -3n+8 \rightarrow -\infty.$$

Hence

$$\lim_{n \rightarrow +\infty} \left(\frac{10n^2 - 5n}{n^3 - 8} \right)^{-3n+8} = 0^{-\infty} = \frac{1}{0^{+\infty}} = \frac{1}{0} = +\infty.$$

3.2 Computing limits using the hierarchy of infinities

For every $a > 1$ and $\alpha > 0$ we have:

$$\lim_{n \rightarrow +\infty} \frac{\log_a n}{n^\alpha} = 0, \quad \lim_{n \rightarrow +\infty} \frac{n^\alpha}{a^n} = 0,$$

$$\lim_{n \rightarrow +\infty} \frac{a^n}{n!} = 0, \quad \lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0.$$

Hence the following infinities are listed in increasing order:

$$\log n, \quad n^\alpha, \quad a^n, \quad n!, \quad n^n.$$

Exercise 3.11

Compute the limit of the following sequence, if it exists

$$a_n = \frac{2^{1/n} + n^2 + 3^{-n}}{\log^6 n + 2 + n}$$

Solution

Consider the numerator; we have:

$$2^{1/n} \rightarrow 1, \quad n^2 \rightarrow +\infty, \quad 3^{-n} \rightarrow 0$$

The power of n with the highest degree is n^2 , and moreover we have

$$\frac{3^{-n}}{n^2} = \frac{1}{3^n n^2} \quad \text{hence} \quad \frac{3^{-n}}{n^2} \rightarrow 0 \quad \text{and} \quad \frac{2^{1/n}}{n^2} \rightarrow 0.$$

Consider the denominator; we have:

$$\log^6 n \rightarrow +\infty, \quad 2 \rightarrow 2, \quad n \rightarrow +\infty$$

By the hierarchy of infinities theorem, we have

$$\frac{\log^6 n}{n} = \left(\underbrace{\frac{\log n}{n^{1/6}}}_{\rightarrow 0} \right)^6 \rightarrow 0.$$

We therefore factor out n^2 in the numerator and n in the denominator and obtain:

$$\begin{aligned} \frac{n^2 \left(1 + \frac{2^{1/n}}{n^2} + \frac{3^{-n}}{n^2} \right)}{n \left(1 + \frac{\log^6 n}{n} + \frac{2}{n} \right)} &= n \frac{\left(1 + \frac{\overbrace{2^{1/n}}^{\rightarrow 0}}{n^2} + \frac{\overbrace{3^{-n}}^{\rightarrow 0}}{n^2} \right)}{\left(1 + \underbrace{\frac{\log^6 n}{n}}_{\rightarrow 0} + \underbrace{\frac{2}{n}}_{\rightarrow 0} \right)} \\ \lim_{n \rightarrow +\infty} \frac{2^{1/n} + n^2 + 3^{-n}}{\log^6 n + 2 + n} &= \lim_{n \rightarrow +\infty} n \frac{\overbrace{\left(1 + \frac{2^{1/n}}{n^2} + \frac{3^{-n}}{n^2} \right)}^{\rightarrow 1}}{\left(1 + \frac{\log^6 n}{n} + \frac{2}{n} \right)} = +\infty \end{aligned}$$

Exercise 3.12

Compute the limit of the following sequence, if it exists

$$a_n = \frac{3n^3 + 3^n + \log n}{\log^6 n + 2^{2n} + n^5}$$

Solution

Consider the numerator; we have:

$$3n^3 \rightarrow +\infty, \quad 3^n \rightarrow +\infty, \quad \log n \rightarrow +\infty$$

that is, a sum of sequences tending to infinity. We also have:

$$\frac{3n^3}{3^n} \rightarrow 0 \quad \text{and} \quad \frac{\log n}{3^n} \rightarrow 0,$$

by the hierarchy of infinities theorem. Hence the principal part is 3^n . Consider the denominator; we have:

$$\log^6 n \rightarrow +\infty, \quad 2^{2n} = 4^n \rightarrow +\infty, \quad n^5 \rightarrow +\infty$$

that is, a sum of sequences tending to infinity. We also have:

$$\frac{\log^6 n}{4^n} \rightarrow 0 \quad \text{and} \quad \frac{n^5}{4^n} \rightarrow 0,$$

by the hierarchy of infinities theorem. Hence the principal part is 4^n .

We therefore factor out 3^n in the numerator and 4^n in the denominator and obtain:

$$\frac{3^n \left(1 + \frac{3n^3}{3^n} + \frac{\log n}{3^n} \right)}{4^n \left(1 + \frac{\log^6 n}{4^n} + \frac{n^5}{4^n} \right)} = \left(\frac{3}{4} \right)^n \frac{\left(1 + \overbrace{\frac{3n^3}{3^n}}^{\rightarrow 0} + \overbrace{\frac{\log n}{3^n}}^{\rightarrow 0} \right)}{\left(1 + \underbrace{\frac{\log^6 n}{4^n}}_{\rightarrow 0} + \underbrace{\frac{n^5}{4^n}}_{\rightarrow 0} \right)}$$

hence

$$\lim_{n \rightarrow +\infty} \frac{3n^3 + 3^n + \log n}{\log^6 n + 2^{2n} + n^5} = \lim_{n \rightarrow +\infty} \left(\frac{3}{4} \right)^n \underbrace{\frac{\left(1 + \frac{3n^3}{3^n} + \frac{\log n}{3^n} \right)}{\left(1 + \frac{\log^6 n}{4^n} + \frac{n^5}{4^n} \right)}}_{\rightarrow 1} = 0$$

Exercise 3.13

Compute the following simple limit

$$\lim_{n \rightarrow +\infty} \sqrt[n]{n^2}$$

Solution

We have

$$\sqrt[n]{n} = n^{\frac{1}{n}} = e^{\log n^{1/n}} = e^{\frac{\log n}{n}}$$

and, studying the sequence in the exponent,

$$a_n = \frac{\log n}{n} \quad \text{we have} \quad a_n \rightarrow 0$$

thanks to the hierarchy of infinities theorem. Hence we have:

$$\lim_{n \rightarrow +\infty} \sqrt[n]{n} = \lim_{n \rightarrow +\infty} e^{\frac{\log n}{n}} = 1.$$

Consequently

$$\lim_{n \rightarrow +\infty} \sqrt[n]{n^2} = \lim_{n \rightarrow +\infty} (\sqrt[n]{n})^2 = 1^2 = 1.$$

3.3 Computing limits using asymptotic estimates

We have $a_n \sim b_n$ if $\frac{a_n}{b_n} \rightarrow 1$ or if $a_n = b_n c_n$ with $c_n \rightarrow 1$.

Moreover, we have $a_n + b_n \sim a_n$ if $\frac{b_n}{a_n} \rightarrow 0$

Exercise 3.14

Compute the following limit

$$\lim_{n \rightarrow +\infty} \frac{2^n + n^2 + 1}{5^n + 2^n + n}$$

Solution

We have

$$2^n + n^2 + 1 \sim 2^n \text{ since } \frac{n^2}{2^n} \rightarrow 0 \text{ and } \frac{1}{2^n} \rightarrow 0$$

and

$$5^n + 2^n + n \sim 5^n \text{ since } \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n \rightarrow 0 \text{ and } \frac{n}{5^n} \rightarrow 0$$

hence

$$\lim_{n \rightarrow +\infty} \frac{2^n + n^2 + 1}{5^n + 2^n + n} = \lim_{n \rightarrow +\infty} \frac{2^n}{5^n} = \lim_{n \rightarrow +\infty} \left(\frac{2}{5}\right)^n = 0$$

Exercise 3.15

Compute the following limit

$$\lim_{n \rightarrow +\infty} \frac{n! - 5^n}{7^n}$$

Solution

We have

$$n! - 5^n \sim n! \text{ since } \frac{5^n}{n!} \rightarrow 0$$

hence

$$\lim_{n \rightarrow +\infty} \frac{n! - 5^n}{7^n} = \lim_{n \rightarrow +\infty} \frac{n!}{7^n} = +\infty$$

Exercise 3.16

Compute the following limit

$$\lim_{n \rightarrow +\infty} \frac{2^{3n-1} - n^2}{(2n)! - 5^n}$$

Solution

We have

$$2^{3n-1} - n^2 \sim 2^{3n-1} \quad \text{since} \quad -\frac{n^2}{2^{3n-1}} \rightarrow 0$$

and

$$(2n)! - 5^n \sim (2n)! \quad \text{since} \quad -\frac{5^n}{(2n)!} \rightarrow 0$$

hence

$$\lim_{n \rightarrow +\infty} \frac{2^{3n-1} - n^2}{(2n)! - 5^n} = \lim_{n \rightarrow +\infty} \frac{2^{3n-1}}{(2n)!} = 0$$

Exercise 3.17

Compute the limit of the following sequence, if it exists

$$\lim_{n \rightarrow +\infty} \frac{3^n}{n! n^{1/n}}$$

Solution

We have

$$n! n^{1/n} \sim n! \quad \text{since} \quad n^{1/n} = \sqrt[n]{n} \rightarrow 1$$

because

$$n^{1/n} = e^{\log n^{1/n}} = e^{\frac{\log n}{n}} \quad \text{and} \quad \frac{\log n}{n} \rightarrow 0.$$

Hence

$$\lim_{n \rightarrow +\infty} \frac{3^n}{n! n^{1/n}} = \lim_{n \rightarrow +\infty} \frac{3^n}{n!} = 0.$$

Exercise 3.18

Compute the limit of the following sequence, if it exists

$$\lim_{n \rightarrow +\infty} \frac{\log(1 + e^n)}{\sqrt{1 + n^2}}$$

Solution

Observing that

$$\log(1 + e^n) \sim \log(e^n) = n \log e = n$$

we have

$$\lim_{n \rightarrow +\infty} \frac{\log(1 + e^n)}{\sqrt{1 + n^2}} = \lim_{n \rightarrow +\infty} \frac{n}{\sqrt{1 + n^2}} = \lim_{n \rightarrow +\infty} \sqrt{\frac{n^2}{1 + n^2}} = 1$$

because

$$\frac{n^2}{1 + n^2} \rightarrow 1.$$

Exercise 3.19

Compute the limit of the following sequence, if it exists

$$\lim_{n \rightarrow +\infty} \frac{n + 1}{\log_2(3 + n^n)}$$

Solution

Observing that

$$\log_2(3 + n^n) \sim \log_2(n^n) = n \log_2 n$$

we have

$$\lim_{n \rightarrow +\infty} \frac{n + 1}{\log_2(3 + n^n)} = \lim_{n \rightarrow +\infty} \frac{n + 1}{n \log_2 n} = 0$$

because

$$\frac{n + 1}{n} \rightarrow 1 \quad \text{and} \quad \frac{1}{\log_2 n} \rightarrow 0.$$

3.4 Computing limits with the sequence converging to Euler's number e

For indeterminate forms $1^{\pm\infty}$, one can use:

$$\left(1 + \frac{1}{a_n}\right)^{a_n} \rightarrow e \quad \text{as } a_n \rightarrow \pm\infty.$$

Exercise 3.20

Compute the following limit

$$\lim_{n \rightarrow +\infty} \left(\frac{n-1}{n-3}\right)^n$$

Solution

We have

$$\left(\frac{n-1}{n-3}\right)^n = \left(\frac{n-3-1+3}{n-3}\right)^n = \left(1 + \frac{2}{n-3}\right)^n = \left(1 + \frac{1}{\frac{n-3}{2}}\right)^n$$

hence

$$\lim_{n \rightarrow +\infty} \left(\frac{n-1}{n-3}\right)^n = \lim_{n \rightarrow +\infty} \left[\left(1 + \frac{1}{\frac{n-3}{2}}\right)^{\frac{n-3}{2}} \right]^{\frac{2n}{n-3}} = e^2,$$

since

$$\left(1 + \frac{1}{\frac{n-3}{2}}\right)^{\frac{n-3}{2}} \rightarrow e, \quad 2 \left(\frac{n}{n-3}\right) = 2 \left(\frac{n-3+3}{n-3}\right) = 2 \left(1 + \underbrace{\frac{3}{n-3}}_{\rightarrow 0}\right) \rightarrow 2$$

and

$$\frac{n-3}{2} \rightarrow +\infty.$$

Exercise 3.21

Compute the following limit

$$\lim_{n \rightarrow +\infty} \left(\frac{n^2+1}{n^2}\right)^n$$

Solution

$$\lim_{n \rightarrow +\infty} \left(\frac{n^2+1}{n^2}\right)^n = \lim_{n \rightarrow +\infty} \left[\left(1 + \frac{1}{n^2}\right)^{n^2} \right]^{\frac{1}{n}} = e^0 = 1$$

since

$$\left(1 + \frac{1}{n^2}\right)^{n^2} \rightarrow e, \quad \frac{1}{n} \rightarrow 0$$

and

$$n^2 \rightarrow +\infty.$$

Exercise 3.22

Compute the following limit

$$\lim_{n \rightarrow +\infty} \left(\frac{2n-5}{2n}\right)^{-n}$$

Solution

$$\lim_{n \rightarrow +\infty} \left(\frac{2n-5}{2n} \right)^{-n} = \lim_{n \rightarrow +\infty} \left[\left(1 + \frac{1}{-\frac{2n}{5}} \right)^{-\frac{2n}{5}} \right]^{\frac{5}{2}} = \sqrt{e^5}.$$

since

$$\left(1 + \frac{1}{-\frac{2n}{5}} \right)^{-\frac{2n}{5}} \rightarrow e \quad \text{and} \quad -\frac{2n}{5} \rightarrow -\infty.$$

Exercise 3.23

Compute the following limit

$$\lim_{n \rightarrow +\infty} \frac{n^{n-1}}{(n-1)^n}$$

Solution

We have

$$\frac{n^{n-1}}{(n-1)^n} = \frac{1}{n} \left(\frac{n}{n-1} \right)^n = \frac{1}{n} \left(\frac{n-1}{n} \right)^{-n} = \frac{1}{n} \left(1 + \frac{1}{-n} \right)^{-n}$$

hence

$$\lim_{n \rightarrow +\infty} \frac{n^{n-1}}{(n-1)^n} = \lim_{n \rightarrow +\infty} \frac{1}{n} \left(1 + \frac{1}{-n} \right)^{-n} = 0$$

since

$$\left(1 + \frac{1}{-n} \right)^{-n} \rightarrow e \quad \text{and} \quad \frac{1}{n} \rightarrow 0.$$

Exercise 3.24

Compute the limit of the following sequence, if it exists

$$\lim_{n \rightarrow +\infty} \left(\frac{n^2 + 2}{n^2 + n + 1} \right)^{2n}$$

Solution

Since

$$\left(\frac{n^2 + 2}{n^2 + n + 1} \right) \rightarrow 1 \quad \text{and} \quad 2n \rightarrow +\infty$$

we have an indeterminate form of type $[1^{+\infty}]$.

Since

$$\left(\frac{n^2+2}{n^2+n+1}\right)^{2n} = \left(\frac{n^2+n+1-n+1}{n^2+n+1}\right)^{2n} = \left(1 + \frac{-n+1}{n^2+n+1}\right)^{2n},$$

we can therefore write

$$\left(1 + \frac{-n+1}{n^2+n+1}\right)^{2n} = \left(1 + \frac{1}{\frac{n^2+n+1}{-n+1}}\right)^{2n} = \left[\left(1 + \frac{1}{\frac{n^2+n+1}{-n+1}}\right)^{\frac{n^2+n+1}{-n+1}}\right]^{\left(\frac{-n+1}{n^2+n+1}\right)2n}.$$

Hence

$$\lim_{n \rightarrow +\infty} \left(\frac{n^2+2}{n^2+n+1}\right)^{2n} = \lim_{n \rightarrow +\infty} \left[\left(1 + \frac{1}{\frac{n^2+n+1}{-n+1}}\right)^{\frac{n^2+n+1}{-n+1}}\right]^{\left(\frac{-n+1}{n^2+n+1}\right)2n} = e^{-2},$$

since

$$\left(1 + \frac{1}{\frac{n^2+n+1}{-n+1}}\right)^{\frac{n^2+n+1}{-n+1}} \rightarrow e, \quad \left(\frac{-n+1}{n^2+n+1}\right)2n = \left(\frac{\overbrace{-2n^2+2n}^{\sim -2n^2}}{\underbrace{n^2+n+1}_{\sim n^2}}\right) \rightarrow -2$$

and

$$\frac{n^2+n+1}{-n+1} = \frac{n^2 \overbrace{\left(1 + \frac{1}{n} + \frac{1}{n^2}\right)}^{\rightarrow 1}}{\underbrace{-n \left(1 - \frac{1}{n}\right)}_{\rightarrow 1}} \rightarrow -\infty.$$

3.5 Computing limits with the comparison theorem

- If $c_n \rightarrow 0$ and $|b_n| \leq c_n$ eventually, then $b_n \rightarrow 0$.
- If $c_n \rightarrow 0$ and b_n is bounded (even if not convergent), then $c_n b_n \rightarrow 0$. The product of an infinitesimal sequence and a bounded one is infinitesimal.
- A sum of infinitesimals and bounded sequences is asymptotically equivalent to the infinity of highest order.

Exercise 3.25

Compute the following limit

$$\lim_{n \rightarrow +\infty} \frac{n + \sin n}{\cos n + \log n}$$

Solution

We have

$$n + \sin n \sim n \quad \text{since} \quad \frac{\sin n}{n} \rightarrow 0$$

the product of an infinitesimal sequence $\{\frac{1}{n}\}$ and a bounded one $\{\sin n\}$. Moreover,

$$\cos n + \log n \sim \log n \quad \text{since} \quad \frac{\cos n}{\log n} \rightarrow 0$$

the product of an infinitesimal sequence $\{\frac{1}{\log n}\}$ and a bounded one $\{\cos n\}$. Hence

$$\lim_{n \rightarrow +\infty} \frac{n + \sin n}{\cos n + \log n} = \lim_{n \rightarrow +\infty} \frac{n}{\log n} = +\infty$$

Exercise 3.26

Compute the limit of the following sequence, if it exists

$$\lim_{n \rightarrow +\infty} \frac{n \sin n + \sin n^2}{n^2 + 1}$$

Solution

We have

$$\frac{n \sin n + \sin n^2}{n^2 + 1} = \frac{n \sin n}{n^2 + 1} + \frac{\sin n^2}{n^2 + 1}$$

and

$$\frac{n}{n^2 + 1} \sin n \rightarrow 0 \quad \text{since} \quad \frac{n}{n^2 + 1} \rightarrow 0 \quad \text{and} \quad \sin n \text{ is bounded.}$$

Moreover

$$\frac{1}{n^2 + 1} \sin n^2 \rightarrow 0 \quad \text{since} \quad \frac{1}{n^2 + 1} \rightarrow 0 \quad \text{and} \quad \sin n^2 \text{ is bounded.}$$

Hence

$$\lim_{n \rightarrow +\infty} \frac{n \sin n + \sin n^2}{n^2 + 1} = \lim_{n \rightarrow +\infty} \underbrace{\frac{n \sin n}{n^2 + 1}}_{\rightarrow 0} + \underbrace{\frac{\sin n^2}{n^2 + 1}}_{\rightarrow 0} = 0$$

Solution

Alternative method.

Note that the sequences $\{\sin n\}$ and $\{\sin n^2\}$ are irregular but bounded. Hence we can write the following upper bound

$$\left| \frac{n \sin n + \sin n^2}{n^2 + 1} \right| \leq \frac{n + 1}{n^2 + 1}$$

and we have

$$\lim_{n \rightarrow +\infty} \frac{n+1}{n^2+1} = \lim_{n \rightarrow +\infty} \frac{n}{n^2} = \lim_{n \rightarrow +\infty} \frac{1}{n} = 0.$$

Hence, by the comparison theorem, we have:

$$\lim_{n \rightarrow +\infty} \frac{n \sin n + \sin n^2}{n^2+1} = 0$$

3.6 Computing limits with the ratio test

Given a positive sequence ($a_n > 0$ for every n):

if there exists $\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = \ell$ and $\ell < 1$, then $a_n \rightarrow 0$,

if there exists $\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = \ell$ and $\ell > 1$ (or $\ell = +\infty$), then $a_n \rightarrow +\infty$.

Exercise 3.27

Compute the limit of the following sequence, if it exists

$$\lim_{n \rightarrow +\infty} \frac{3^{n-1}}{n(n+1)!}$$

Solution

Since this is a sequence with positive terms whose general term is the ratio of simpler sequences of exponential and factorial type, it is natural to apply the ratio test.

We therefore compute:

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{\frac{3^n}{(n+1) \cdot (n+2)!}}{\frac{3^{n-1}}{n(n+1)!}} = \frac{3^n}{(n+1) \cdot \underbrace{(n+2)!}_{=(n+1)! \cdot (n+2)}} \cdot \frac{n(n+1)!}{3^{n-1}} = \\ &= \frac{3n}{(n+1)(n+2)} = \frac{3n}{n^2 + 2n + 2} = \frac{3n}{n^2 \left(1 + \frac{2}{n} + \frac{2}{n^2}\right)} = \frac{3}{n \left(1 + \frac{2}{n} + \frac{2}{n^2}\right)} \end{aligned}$$

We have:

$$\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow +\infty} \frac{3}{n \left(1 + \frac{2}{n} + \frac{2}{n^2}\right)} = 0$$

Hence, by the ratio test:

$$\lim_{n \rightarrow +\infty} \frac{3^{n-1}}{n(n+1)!} = 0$$

Exercise 3.28

Compute the limit of the following sequence, if it exists

$$\lim_{n \rightarrow +\infty} \frac{n^3 2^n}{n!}$$

Solution

We have a positive sequence; we compute:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3 2^{(n+1)}}{(n+1)!} \frac{n!}{n^3 2^n} = \underbrace{\left(\frac{n+1}{n}\right)^3}_{\rightarrow 1} \frac{2}{n+1} \sim \frac{2}{n+1}$$

We therefore have:

$$\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow +\infty} \frac{2}{n+1} = 0$$

Then, by the ratio test:

$$\lim_{n \rightarrow +\infty} \frac{n^3 2^n}{n!} = 0$$

Exercise 3.29

Compute the limit of the following sequence, if it exists

$$\lim_{n \rightarrow +\infty} \frac{n^n}{(n+1)!}$$

Solution

We have a positive sequence; we compute:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1} (n+1)!}{(n+2)!} \frac{n!}{n^n} = \underbrace{\frac{n+1}{n+2}}_{\rightarrow 1} \left(\frac{n+1}{n}\right)^n \sim \left(1 + \frac{1}{n}\right)^n$$

We therefore have:

$$\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n = e > 1$$

Then, by the ratio test:

$$\lim_{n \rightarrow +\infty} \frac{n^n}{(n+1)!} = +\infty$$

Exercise 3.30

Compute the limit of the following sequence, if it exists

$$\lim_{n \rightarrow +\infty} \frac{n^{\frac{1}{n}} 2^n}{(n+1)!}$$

Solution

We have:

$$\frac{\overbrace{n^{\frac{1}{n}}}^{\rightarrow 1} 2^n}{(n+1)!} \sim \frac{2^n}{(n+1)!} \equiv b_n$$

We now have the positive sequence $\{b_n\}$; we compute:

$$\frac{b_{n+1}}{b_n} = \frac{2^{n+1}}{(n+2)!} \frac{(n+1)!}{2^n} = \frac{2}{n+2}$$

We therefore have:

$$\lim_{n \rightarrow +\infty} \frac{b_{n+1}}{b_n} = \lim_{n \rightarrow +\infty} \frac{2}{n+2} = 0$$

Then, by the ratio test:

$$\lim_{n \rightarrow +\infty} \frac{n^{\frac{1}{n}} 2^n}{(n+1)!} = 0$$

3.7 Discussion depending on a parameter

Exercise 3.31

Compute, as the parameter $\alpha \in \mathbb{R}$ varies, the limit of the sequence

$$\lim_{n \rightarrow +\infty} \frac{5^{3\alpha n}}{2^{3n+1}}$$

Solution

We have

$$\frac{5^{3\alpha n}}{2^{3n+1}} = \frac{1}{2} \left(\frac{5^\alpha}{2} \right)^{3n}$$

hence

$$\lim_{n \rightarrow +\infty} \frac{5^{3\alpha n}}{2^{3n+1}} = \lim_{n \rightarrow +\infty} \frac{1}{2} \left(\frac{5^\alpha}{2} \right)^{3n} = \frac{1}{2} \lim_{n \rightarrow +\infty} \left(\frac{5^\alpha}{2} \right)^{3n} = \ell$$

At this point, we can study three cases:

- *Case 1*

$$\frac{5^\alpha}{2} > 1 \iff \alpha > \log_5 2$$

then $\ell = +\infty$;

- *Case 2*

$$0 < \frac{5^\alpha}{2} < 1 \iff \alpha < \log_5 2$$

then $\ell = 0$;

- *Case 3*

$$\frac{5^\alpha}{2} = 1 \iff \alpha = \log_5 2$$

then $\ell = \frac{1}{2}$.

In conclusion:

$$\lim_{n \rightarrow +\infty} \frac{5^{3\alpha n}}{2^{3n+1}} = \begin{cases} +\infty & \text{if } \alpha > \log_5 2 \\ \frac{1}{2} & \text{if } \alpha = \log_5 2 \\ 0 & \text{if } \alpha < \log_5 2 \end{cases}$$

Exercise 3.32

Compute, as the parameter $\alpha \in \mathbb{R}$ varies, the limit of the sequence

$$\lim_{n \rightarrow +\infty} (e^{(2-\alpha)n} + 1) \log \left(1 + \frac{1}{e^n} \right)$$

Solution

We have

$$\begin{aligned} a_n &= (e^{(2-\alpha)n} + 1) \log \left(1 + \frac{1}{e^n} \right) = (e^{(2-\alpha)n} + 1) \frac{e^n}{e^n} \log \left(1 + \frac{1}{e^n} \right) \\ &= (e^{(2-\alpha)n} + 1) \frac{1}{e^n} \log \left(1 + \frac{1}{e^n} \right)^{e^n} \\ &\sim (e^{(2-\alpha)n} + 1) \frac{1}{e^n} \end{aligned}$$

since

$$\left(1 + \frac{1}{e^n}\right)^{e^n} \rightarrow e \quad \text{and} \quad \log \left(1 + \frac{1}{e^n}\right)^{e^n} \rightarrow 1$$

Hence

$$\lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow +\infty} (e^{(2-\alpha)n} + 1) \frac{1}{e^n}$$

On the other hand

$$(e^{(2-\alpha)n} + 1) \frac{1}{e^n} = e^{(1-\alpha)n} + e^{-n} \rightarrow \begin{cases} +\infty & \text{if } 1 - \alpha > 0 \\ 1 & \text{if } 1 - \alpha = 0 \\ 0 & \text{if } 1 - \alpha < 0 \end{cases}$$

Summarizing, we obtain

$$\lim_{n \rightarrow +\infty} (e^{(2-\alpha)n} + 1) \log \left(1 + \frac{1}{e^n}\right) = \begin{cases} +\infty & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1 \end{cases}$$

3.8 Multiple choice exercises

Exercise 3.33

Let

$$\lim_{n \rightarrow +\infty} \left(\frac{\sqrt{3} + n}{e - 2n} \right)^n = \ell$$

- (a) $\ell = 0$
- (b) $\ell = e^{(\sqrt{3} - \frac{e}{2})}$
- (c) $\ell = +\infty$
- (d) None of the other answers is correct.

Solution

Setting

$$a_n = \frac{\sqrt{3} + n}{e - 2n}$$

we have $a_n < 0$ for $n \geq 2$ and

$$|a_n|^n = \left(\frac{\sqrt{3} + n}{2n - e} \right)^n \rightarrow 0$$

since $|a_n| \rightarrow 1/2$. From $|a_n|^n \rightarrow 0$ it follows that $(a_n)^n \rightarrow 0$. The correct answer is (a).

Exercise 3.34

Let

$$\lim_{n \rightarrow +\infty} (-1)^n n^\alpha \left(\frac{1+3n}{2-n} \right)^n = \ell$$

- (a) ℓ does not exist if $\alpha \geq 0$
- (b) $\ell = +\infty$ for every $\alpha \in \mathbb{R}$
- (c) $\ell = 0$ if $\alpha < 0$
- (d) None of the other answers is correct.

Solution

We set

$$a_n = (-1)^n \left(\frac{1+3n}{2-n} \right)^n = \left(\frac{1+3n}{n-2} \right)^n.$$

We have $a_n \rightarrow +\infty$ since $(1+3n)/(n-2) \rightarrow 3$. Let us compare a_n with 3^n :

$$\frac{a_n}{3^n} = \left(\frac{1+3n}{3n-6} \right)^n = \left[\left(1 + \frac{1}{\frac{3n-6}{7}} \right)^{\frac{3n-6}{7}} \right]^{\frac{7n}{3n-6}} \rightarrow e^{7/3}$$

from which

$$a_n \sim 3^n e^{7/3}.$$

It follows that

$$\lim_{n \rightarrow +\infty} (-1)^n n^\alpha \left(\frac{1+3n}{2-n} \right)^n = \lim_{n \rightarrow +\infty} e^{7/3} n^\alpha 3^n = +\infty$$

even when $\alpha < 0$, because 3^n is an infinity of higher order than any power of n . The correct answer is (b).

Exercise 3.35

Let

$$\lim_{n \rightarrow +\infty} \left(\frac{an+2}{en+a} \right)^n = \ell$$

- (a) $\ell = e^{(\frac{2}{e}-1)}$ if $a = e$
- (b) $\ell = 1$ if $a \leq e$
- (c) $\ell = 0$ if $a > 0$
- (d) None of the other answers is correct.

Solution

Setting

$$a_n = \frac{an + 2}{en + a}$$

we have $a_n \rightarrow a/e$, hence

$$(a_n)^n \rightarrow \begin{cases} +\infty, & a > e \\ 0, & 0 < a < e \end{cases}$$

therefore answers (b) and (c) are not correct. In the case $a = e$ we have an indeterminate form 1^∞ :

$$\left(\frac{en + 2}{en + e}\right)^n = \left[\left(1 + \frac{1}{\frac{en+e}{2-e}}\right)^{\frac{en+e}{2-e}} \right]^{\frac{(2-e)n}{en+e}} \rightarrow e^{-1+\frac{2}{e}}.$$

The correct answer is (a).

Additional exercises

Exercise 4.1

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \frac{2^{-n} + 3n^3}{\log^6 n + 2 - n^5}$$

Solution**Exercise 4.2**

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \frac{\log n + 3n^3 \log n}{2^{1/n} + n^5}$$

Solution**Exercise 4.3**

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \frac{n^2 + \log^3 n}{n \log n + 1}$$

Solution

Exercise 4.4

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \frac{n^{2+\frac{1}{n}}}{1+n^{3-\frac{2}{n}}}$$

Solution**Exercise 4.5**

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \left(\frac{n}{n+1} \right)^{2n}$$

Solution**Exercise 4.6**

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \left(\frac{n+2}{n+1} \right)^{n^2}$$

Solution**Exercise 4.7**

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{3}{n} \right)^{-n}$$

Solution

Exercise 4.8

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \left(1 - \frac{2}{n}\right)^n$$

Solution**Exercise 4.9**

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \frac{n + \log(n^2) - 2^n}{(\log n)^3 + n^2}$$

Solution**Exercise 4.10**

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \frac{8 + 2n - n^{4/3}}{n^{-1/3} + 3n - n^{5/3}}$$

Solution**Exercise 4.11**

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} n \left(\sqrt{n^2 + 3} - n \right)$$

Solution

Exercise 4.12

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \frac{n^2 + \log(5n^3) + (\log n)^2}{2(\log n)^3 + n \log n}$$

Solution**Exercise 4.13**

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \frac{8 + 2n^2 - n^{4/3}}{n^{-1/3} + 3n^2 - n^{5/3}}$$

Solution**Exercise 4.14**

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \left(\sqrt{n^4 + 1} - n \sqrt{n^2 + 1} \right)$$

Solution**Exercise 4.15**

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \frac{n^2 + 3^n + 2^{-n}}{2^{2n} + (\log n)^3}$$

Solution

Exercise 4.16

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} (\sin^2 n) \cdot \sin(n^2)$$

Solution**Exercise 4.17**

Compute, if it exists, the limit of the sequence, justifying the steps taken:

$$\lim_{n \rightarrow +\infty} \frac{\cos(\pi n)}{\pi n} + \log \left(\sin \frac{\pi}{n} \right)$$

Solution

Part II

Limits of functions and continuity



Limits of real functions of a real variable

We have the following fundamental limit:

$$\left(1 + \frac{\alpha}{\eta(x)}\right)^{\eta(x)} \rightarrow e^{\alpha} \quad \text{as } \eta(x) \rightarrow \pm\infty, \quad \alpha \in \mathbb{R}, \alpha \neq 0$$

We have the following 5 main fundamental limits:

$$\frac{\sin \varepsilon(x)}{\varepsilon(x)} \rightarrow 1 \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$\frac{1 - \cos \varepsilon(x)}{\varepsilon^2(x)} \rightarrow \frac{1}{2} \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$\frac{\log(1 + \varepsilon(x))}{\varepsilon(x)} \rightarrow 1 \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$\frac{e^{\varepsilon(x)} - 1}{\varepsilon(x)} \rightarrow 1 \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$\frac{(1 + \varepsilon(x))^\alpha - 1}{\varepsilon(x)} \rightarrow \alpha \quad \text{as } \varepsilon(x) \rightarrow 0, \quad \alpha \in \mathbb{R}, \alpha \neq 0$$

We also have the following fundamental limits of the hyperbolic functions:

$$\frac{\sinh \varepsilon(x)}{\varepsilon(x)} \rightarrow 1 \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$\frac{\cosh \varepsilon(x) - 1}{\varepsilon^2(x)} \rightarrow \frac{1}{2} \quad \text{as } \varepsilon(x) \rightarrow 0$$

By the hierarchy of infinities theorem, we have:

$$\varepsilon^\beta(x) [\log_a \varepsilon(x)]^\alpha \rightarrow 0 \quad \text{as } \varepsilon(x) \rightarrow 0^+, \quad \alpha, \beta > 0, \quad a > 1$$

From the 5 main fundamental limits, the following asymptotic equivalences follow:

$$\sin \varepsilon(x) \sim \varepsilon(x) \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$1 - \cos \varepsilon(x) \sim \frac{1}{2} \varepsilon^2(x) \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$e^{\varepsilon(x)} - 1 \sim \varepsilon(x) \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$\log(1 + \varepsilon(x)) \sim \varepsilon(x) \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$(1 + \varepsilon(x))^\alpha - 1 \sim \alpha \varepsilon(x) \quad \text{as } \varepsilon(x) \rightarrow 0$$

From the 5 main fundamental limits, the following asymptotic expansions follow:

$$\sin \varepsilon(x) = \varepsilon(x) + o(\varepsilon(x)) \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$\cos \varepsilon(x) = 1 - \frac{1}{2} \varepsilon^2(x) + o(\varepsilon^2(x)) \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$\log(1 + \varepsilon(x)) = \varepsilon(x) + o(\varepsilon(x)) \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$e^{\varepsilon(x)} = 1 + \varepsilon(x) + o(\varepsilon(x)) \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$(1 + \varepsilon(x))^\alpha = 1 + \alpha \varepsilon(x) + o(\varepsilon(x)) \quad \text{as } \varepsilon(x) \rightarrow 0$$

From the fundamental limits, the following asymptotic equivalences also follow:

$$\sinh \varepsilon(x) \sim \varepsilon(x) \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$\cosh \varepsilon(x) - 1 \sim \frac{1}{2} \varepsilon^2(x) \quad \text{as } \varepsilon(x) \rightarrow 0$$

For the hyperbolic functions we have the asymptotic expansions:

$$\sinh \varepsilon(x) = \varepsilon(x) + o(\varepsilon(x)) \quad \text{as } \varepsilon(x) \rightarrow 0$$

$$\cosh \varepsilon(x) = 1 + \frac{1}{2} \varepsilon^2(x) + o(\varepsilon^2(x)) \quad \text{as } \varepsilon(x) \rightarrow 0$$

Operations with “little o”:

$o(1)$ denotes a generic infinitesimal;

$$C \cdot o(x^n) = o(x^n);$$

$$x^m \cdot o(x^n) = o(x^{m+n});$$

$$o(x^m) \cdot o(x^n) = o(x^{m+n})$$

$$\frac{o(x^n)}{x^m} = o(x^{n-m}) \quad (n \geq m);$$

$$o(x^n) \pm o(x^m) = o(x^M) \quad \text{where } M = \min \{m, n\}.$$

Exercise 1.1

From $e^x = 1 + x + o(x)$, $x \rightarrow 0$, deduce that

$$\sinh(x) = x + o(x), \quad x \rightarrow 0.$$

Solution

For $x \rightarrow 0$, we have $e^{-x} = 1 - x + o(x)$, hence

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2} \left((1 + x + o(x)) - (1 - x + o(x)) \right) = x + o(x).$$

Exercise 1.2

Using $\cosh^2 x - 1 = \sinh^2 x$ and the previous exercise, prove that

$$\lim_{x \rightarrow 0} \frac{\cosh x - 1}{x^2} = \frac{1}{2}$$

and deduce

$$\cosh x = 1 + \frac{1}{2}x^2 + o(x^2), \quad x \rightarrow 0.$$

Solution

From the previous exercise we have $\sinh^2 x \sim x^2$ as $x \rightarrow 0$ and moreover $\cosh 0 = 1$, hence

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cosh x - 1}{x^2} \cdot \frac{\cosh x + 1}{\cosh x + 1} &= \lim_{x \rightarrow 0} \frac{\cosh^2 x - 1}{x^2} \cdot \underbrace{\lim_{x \rightarrow 0} \frac{1}{\cosh x + 1}}_{\rightarrow \frac{1}{2}} = \\ &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sinh^2 x}{x^2} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{x^2}{x^2} = \frac{1}{2}. \end{aligned}$$

Moreover

$$\left(\lim_{x \rightarrow 0} \frac{\cosh x - 1}{x^2} \right) - \frac{1}{2} = \lim_{x \rightarrow 0} \frac{\cosh x - 1 - \frac{1}{2}x^2}{x^2} = 0$$

which means

$$\begin{aligned} \cosh x - 1 - \frac{1}{2}x^2 &= o(x^2), \\ \cosh x &= 1 + \frac{1}{2}x^2 + o(x^2), \quad x \rightarrow 0. \end{aligned}$$

Exercise 1.3

Refine the equivalence

$$\log(x^2 + 1) \sim 2 \log x, \quad x \rightarrow +\infty$$

by proving that

$$\log(x^2 + 1) = 2 \log x + \frac{1}{x^2} + o\left(\frac{1}{x^2}\right), \quad x \rightarrow +\infty.$$

Solution

$$\log(x^2 + 1) = \log\left(x^2 \left(1 + \frac{1}{x^2}\right)\right) = \log x^2 + \log\left(1 + \frac{1}{x^2}\right).$$

Now

$$1/x^2 \rightarrow 0 \quad \text{as } x \rightarrow +\infty$$

and

$$\log(1 + y) = y + o(y) \quad \text{as } y \rightarrow 0,$$

hence

$$\log(x^2 + 1) = 2 \log x + \frac{1}{x^2} + o\left(\frac{1}{x^2}\right), \quad x \rightarrow +\infty.$$

Exercise 1.4

Compute the following limit of a function of a real variable:

$$\lim_{x \rightarrow +\infty} \log x - \sqrt{x}$$

Solution

$$\lim_{x \rightarrow +\infty} \log x - \sqrt{x} = \lim_{x \rightarrow +\infty} -\sqrt{x} = -\infty.$$

Exercise 1.5

Compute the following limit of a function of a real variable:

$$\lim_{x \rightarrow +\infty} 2^x - x^2$$

Solution

$$\lim_{x \rightarrow +\infty} 2^x - x^2 = \lim_{x \rightarrow +\infty} 2^x = +\infty.$$

Exercise 1.6

Compute the following limit of a function of a real variable:

$$\lim_{x \rightarrow +\infty} \frac{\log(x^2 + 1)}{2^x}$$

Solution

We use the equivalence $\log(x^2 + 1) \sim 2 \log x$ as $x \rightarrow +\infty$:

$$\lim_{x \rightarrow +\infty} \frac{\log(x^2 + 1)}{2^x} = \lim_{x \rightarrow +\infty} \frac{2 \log x}{2^x} = 0.$$

Exercise 1.7

Compute the following limit of a function of a real variable:

$$\lim_{x \rightarrow +\infty} \left(\frac{x+2}{x+1} \right)^x$$

Solution

We use the fundamental limit

$$\lim_{x \rightarrow \pm\infty} \left(1 + \frac{1}{x} \right)^x = e.$$

$$\lim_{x \rightarrow +\infty} \left(\frac{x+2}{x+1} \right)^x = \lim_{x \rightarrow +\infty} \left[\left(1 + \frac{1}{x+1} \right)^{x+1} \right]^{\frac{x}{x+1}} = e^1 = e.$$

Exercise 1.8

Compute the following limit of a function of a real variable:

$$\lim_{x \rightarrow 0^+} x^{\log x}$$

Solution

We have

$$x^{\log x} = e^{\log x^{\log x}} = e^{\log x \log x},$$

hence

$$\lim_{x \rightarrow 0^+} x^{\log x} = \lim_{x \rightarrow 0^+} e^{\overbrace{\log x}^{\rightarrow -\infty} \overbrace{\log x}^{\rightarrow -\infty}} = e^{+\infty} = +\infty.$$

Moreover, the base x tends to 0 and the exponent $\log x$ tends to $-\infty$, hence the limit is not an indeterminate form:

$$\lim_{x \rightarrow 0^+} x^{\log x} = 0^{-\infty} = +\infty$$

Exercise 1.9

Let

$$\lim_{x \rightarrow +\infty} x (\log(x+2) - \log x) = L.$$

- (a) $L = 2$
- (b) L does not exist
- (c) $L = 1$
- (d) None of the other answers is correct.

Solution

$$\lim_{x \rightarrow +\infty} x (\log(x+2) - \log x) = \lim_{x \rightarrow +\infty} x \log \left(1 + \frac{2}{x} \right) = \lim_{x \rightarrow +\infty} x \cdot \frac{2}{x} = 2.$$

The correct answer is (a).

Exercise 1.10

Compute the following limit of a quotient using equivalent power functions for each term:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin x}$$

Solution

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin x} = \lim_{x \rightarrow 0} \frac{x}{x} = 1.$$

Exercise 1.11

Compute the following limit of a quotient using equivalent power functions for each term:

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin^2 x}$$

Solution

We have

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2} \text{ hence } 1 - \cos x \sim \frac{1}{2} x^2 \text{ as } x \rightarrow 0.$$

Moreover, we have

$$\sin x \sim x \text{ as } x \rightarrow 0 \text{ hence } (\sin x)^2 \sim x^2 \text{ as } x \rightarrow 0.$$

We then have:

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{\frac{1}{2} x^2}{x^2} = \frac{1}{2}$$

Exercise 1.12

Compute the following limit of a quotient using equivalent power functions for each term:

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 4x}$$

Solution

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 4x} = \lim_{x \rightarrow 0} \frac{3x}{4x} = \frac{3}{4}$$

Exercise 1.13

Compute the following limit of a quotient using equivalent power functions for each term:

$$\lim_{x \rightarrow 0} \frac{(1 - e^{2x})^2}{1 - \cos 5x}$$

Solution

$$\lim_{x \rightarrow 0} \frac{(1 - e^{2x})^2}{1 - \cos 5x} = \lim_{x \rightarrow 0} \frac{(-2x)^2}{\frac{1}{2}(5x)^2} = \frac{8}{25}$$

Exercise 1.14

Compute the following limit of a quotient using equivalent power functions for each term:

$$\lim_{x \rightarrow 0} \frac{\sin x^3}{(1 - e^x)^3}$$

Solution

$$\lim_{x \rightarrow 0} \frac{\sin x^3}{(1 - e^x)^3} = \lim_{x \rightarrow 0} \frac{x^3}{(-x)^3} = -1.$$

Exercise 1.15

Compute

$$\lim_{x \rightarrow 0} x^2 e^{\frac{\sqrt{\pi}}{x} \sin(x \log 7)}$$

Solution

We analyze the exponent using the equivalence $\sin(x \log 7) \sim x \log 7$ as $x \rightarrow 0$:

$$\lim_{x \rightarrow 0} \frac{\sqrt{\pi}}{x} \sin(x \log 7) = \lim_{x \rightarrow 0} \frac{\sqrt{\pi}}{x} \cdot x \log 7 = \sqrt{\pi} \log 7.$$

The given limit equals

$$\lim_{x \rightarrow 0} x^2 e^{\frac{\sqrt{\pi}}{x} \sin(x \log 7)} = 0 \cdot e^{\sqrt{\pi} \log 7} = 0 \cdot 7^{\sqrt{\pi}} = 0.$$

Exercise 1.16

Compute

$$\lim_{x \rightarrow 0^+} \left(\frac{1}{\sin x} + \log x \right)$$

Solution

Using

$$\lim_{x \rightarrow 0^+} \sin x \log x = \lim_{x \rightarrow 0^+} x \log x = 0,$$

we have

$$\lim_{x \rightarrow 0^+} \left(\frac{1}{\sin x} + \log x \right) = \lim_{x \rightarrow 0^+} \frac{1 + \sin x \log x}{\sin x} = +\infty$$

since the numerator $1 + \sin x \log x$ tends to 1 while the denominator $\sin x$ is a positive function tending to 0.

Equivalently, again from

$$\lim_{x \rightarrow 0^+} \frac{\log x}{\frac{1}{\sin x}} = \lim_{x \rightarrow 0^+} \sin x \log x = \lim_{x \rightarrow 0^+} x \log x = 0,$$

we have that $1/\sin x$ is an infinity of higher order than $\log x$ as $x \rightarrow 0^+$, hence

$$\lim_{x \rightarrow 0^+} \left(\frac{1}{\sin x} + \log x \right) = \lim_{x \rightarrow 0^+} \frac{1}{\sin x} = +\infty.$$

We also prove that

$$\lim_{x \rightarrow 0^+} x \log x = 0$$

We make the change of variable $x = \frac{1}{y}$, hence

$$\lim_{x \rightarrow 0^+} x \log x = \lim_{y \rightarrow +\infty} \frac{1}{y} \log \frac{1}{y} = \lim_{y \rightarrow +\infty} \frac{-\log y}{y} = 0$$

Exercise 1.17

Compute

$$\lim_{x \rightarrow 0} \frac{(1 - \cos x)^3 - \sin x^6}{x^6}$$

Solution

From

$$1 - \cos x \sim \frac{1}{2}x^2, \quad \sin x \sim x, \quad x \rightarrow 0,$$

we have

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{(1 - \cos x)^3 - \sin x^6}{x^6} &= \lim_{x \rightarrow 0} \frac{(1 - \cos x)^3}{x^6} - \lim_{x \rightarrow 0} \frac{\sin x^6}{x^6} = \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{8}x^6}{x^6} - \lim_{x \rightarrow 0} \frac{x^6}{x^6} = \frac{1}{8} - 1 = -\frac{7}{8}.\end{aligned}$$

Equivalently, using the calculus of Landau symbols, from

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^2), \quad \sin x = x + o(x), \quad x \rightarrow 0,$$

we have

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{(1 - \cos x)^3 - \sin x^6}{x^6} &= \lim_{x \rightarrow 0} \frac{\frac{1}{8}x^6 - x^6 + o(x^6)}{x^6} = \\ \lim_{x \rightarrow 0} \frac{-\frac{7}{8}x^6 + o(x^6)}{x^6} &= \lim_{x \rightarrow 0} \frac{-\frac{7}{8}x^6}{x^6} = -\frac{7}{8}.\end{aligned}$$

Exercise 1.18

Compute

$$\lim_{x \rightarrow 0} \frac{8(1 - \cos x)^3 - \sin x^6}{x^6}$$

Solution

From

$$1 - \cos x \sim \frac{1}{2}x^2, \quad \sin x \sim x, \quad x \rightarrow 0,$$

we have

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{8(1 - \cos x)^3 - \sin x^6}{x^6} &= \lim_{x \rightarrow 0} \frac{8(1 - \cos x)^3}{x^6} - \lim_{x \rightarrow 0} \frac{\sin x^6}{x^6} = \\ &= \lim_{x \rightarrow 0} \frac{x^6}{x^6} - \lim_{x \rightarrow 0} \frac{x^6}{x^6} = 1 - 1 = 0.\end{aligned}$$

Equivalently, using the calculus of Landau symbols, from

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^2), \quad \sin x = x + o(x), \quad x \rightarrow 0,$$

we have

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{8(1 - \cos x)^3 - \sin x^6}{x^6} &= \lim_{x \rightarrow 0} \frac{x^6 - x^6 + o(x^6)}{x^6} = \\ \lim_{x \rightarrow 0} \frac{o(x^6)}{x^6} &= 0\end{aligned}$$

since the numerator $o(x^6)$, by the very definition of the Landau symbol, is an infinitesimal of higher order than the denominator x^6 .

We have the asymptotic expansions as $x \rightarrow 0$:

$$\sin x = x + o(x), \quad \cos x = 1 - \frac{1}{2}x^2 + o(x^2), \quad e^x = 1 + x + o(x),$$

$$\sinh x = x + o(x), \quad \cosh x = 1 + \frac{1}{2}x^2 + o(x^2).$$

Exercise 1.19

Compute the following limits using the comparison of infinitesimals

$$\lim_{x \rightarrow 0} \frac{(\sin x)^2 + x}{x^3 - \sin x}$$

Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{(\sin x)^2 + x}{x^3 - \sin x} &= \lim_{x \rightarrow 0} \frac{(x + o(x))^2 + x}{x^3 - x + o(x)} = \\ &= \lim_{x \rightarrow 0} \frac{x^2 + o(x^2) + x}{x^3 - x + o(x)} = \lim_{x \rightarrow 0} \frac{x + o(x)}{-x + o(x)} = \lim_{x \rightarrow 0} \frac{x}{-x} = -1. \end{aligned}$$

Exercise 1.20

Compute the following limits using the comparison of infinitesimals

$$\lim_{x \rightarrow 0} \frac{|1 - \cos x + \sin x|}{(e^x - 1)^2}$$

Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{|1 - \cos x + \sin x|}{(e^x - 1)^2} &= \lim_{x \rightarrow 0} \frac{|\frac{1}{2}x^2 + o(x^2) + x + o(x)|}{(x + o(x))^2} = \\ \lim_{x \rightarrow 0} \frac{|x + o(x)|}{x^2 + o(x^2)} &= \lim_{x \rightarrow 0} \frac{|x|}{x^2} = \lim_{x \rightarrow 0} \frac{1}{|x|} = +\infty. \end{aligned}$$

Exercise 1.21

Compute the following limits using the comparison of infinitesimals

$$\lim_{x \rightarrow 0} \frac{\sin 2x - \sin^2 x}{e^x - 1}$$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 2x - \sin^2 x}{e^x - 1} &= \lim_{x \rightarrow 0} \frac{2x + o(x) - (x + o(x))^2}{x + o(x)} = \\ &= \lim_{x \rightarrow 0} \frac{2x + o(x) - x^2 + o(x^2)}{x + o(x)} = \lim_{x \rightarrow 0} \frac{2x + o(x)}{x + o(x)} = \lim_{x \rightarrow 0} \frac{2x}{x} = 2.\end{aligned}$$

Exercise 1.22

Compute the following limits using the comparison of infinitesimals

$$\lim_{x \rightarrow 0} \frac{\sin^3 x - \sin 4x}{\cosh x - 1 + x}$$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin^3 x - \sin 4x}{\cosh x - 1 + x} &= \lim_{x \rightarrow 0} \frac{(x + o(x))^3 - 4x + o(x)}{\frac{1}{2}x^2 + o(x^2) + x} = \\ &= \lim_{x \rightarrow 0} \frac{x^3 + o(x^3) - 4x + o(x)}{x + o(x)} = \lim_{x \rightarrow 0} \frac{-4x + o(x)}{x + o(x)} = \lim_{x \rightarrow 0} \frac{-4x}{x} = -4.\end{aligned}$$

Exercise 1.23

Compute the following limits using the comparison of infinitesimals

$$\lim_{x \rightarrow 0} \frac{\sin x - \sin x^2}{\sinh(2x) - \sinh(2x)^2}$$

Solution

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin x - \sin x^2}{\sinh(2x) - \sinh(2x)^2} &= \lim_{x \rightarrow 0} \frac{x + o(x) - x^2 + o(x^2)}{2x + o(x) - (2x + o(x))^2} = \\ &= \lim_{x \rightarrow 0} \frac{x + o(x)}{2x + o(x) - 4x^2 + o(x^2)} = \lim_{x \rightarrow 0} \frac{x + o(x)}{2x + o(x)} = \lim_{x \rightarrow 0} \frac{x}{2x} = \frac{1}{2}.\end{aligned}$$

Exercise 1.24

Compute

$$\lim_{x \rightarrow 1^+} \frac{\log(1 + \sqrt{x-1})}{\sqrt{x^2-1}}$$

Solution

We have

$$\lim_{x \rightarrow 1^+} \frac{\log(1 + \sqrt{x-1})}{\sqrt{x^2-1}} = \lim_{x \rightarrow 1^+} \frac{\log(1 + \sqrt{x-1})}{\sqrt{x-1}\sqrt{x+1}} = \frac{1}{\sqrt{2}} \lim_{x \rightarrow 1^+} \frac{\log(1 + \sqrt{x-1})}{\sqrt{x-1}}.$$

Setting $y = \sqrt{x-1}$, the limit becomes

$$\frac{1}{\sqrt{2}} \lim_{y \rightarrow 0^+} \frac{\log(1+y)}{y} = \frac{1}{\sqrt{2}}$$

by the fundamental limit

$$\lim_{y \rightarrow 0} \frac{\log(1+y)}{y} = 1.$$

Exercise 1.25

Compute

$$\lim_{x \rightarrow +\infty} \left(\frac{x^2 + 3x}{x^2 - 5x} \right)^{(x+\log x)}$$

Solution

We have

$$\begin{aligned} \lim_{x \rightarrow +\infty} \left(\frac{x^2 + 3x}{x^2 - 5x} \right)^{(x+\log x)} &= \lim_{x \rightarrow +\infty} \left(\frac{x+3}{x-5} \right)^{(x+\log x)} = \lim_{x \rightarrow +\infty} \left(1 + \frac{8}{x-5} \right)^{(x+\log x)} \\ &= \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{\frac{x-5}{8}} \right)^{(x+\log x)} \end{aligned}$$

Using the fundamental limit

$$\lim_{x \rightarrow \pm\infty} \left(1 + \frac{1}{x} \right)^x = e$$

we obtain

$$\lim_{x \rightarrow +\infty} \left(\frac{x^2 + 3x}{x^2 - 5x} \right)^{(x+\log x)} = \lim_{x \rightarrow +\infty} \left[\left(1 + \frac{1}{\frac{x-5}{8}} \right)^{\frac{x-5}{8}} \right]^{\frac{8(x+\log x)}{x-5}} = e^8$$

since

$$\frac{8(x+\log x)}{x-5} \sim \frac{8x}{x-5} \xrightarrow{x \rightarrow +\infty} 8.$$

Exercise 1.26

Compute

$$\lim_{x \rightarrow 0} \frac{x(\pi^x - e^x)}{\cos(x) - 1}$$

Solution

We have

$$\lim_{x \rightarrow 0} \frac{x(\pi^x - e^x)}{\cos(x) - 1} = \lim_{x \rightarrow 0} \frac{x\pi^x(1 - (\frac{e}{\pi})^x)}{\cos(x) - 1} = \lim_{x \rightarrow 0} \pi^x \frac{x(1 - (\frac{e}{\pi})^x)}{\cos(x) - 1}$$

Since

$$1 - \left(\frac{e}{\pi}\right)^x \sim -x \log\left(\frac{e}{\pi}\right)$$

and also

$$\cos(x) - 1 \sim -\frac{1}{2}x^2$$

we obtain

$$\lim_{x \rightarrow 0} \frac{x(\pi^x - e^x)}{\cos(x) - 1} = \lim_{x \rightarrow 0} \pi^x \frac{-x^2 \log\left(\frac{e}{\pi}\right)}{-\frac{1}{2}x^2} = 2 \log\left(\frac{e}{\pi}\right).$$

Exercise 1.27

Compute

$$\lim_{x \rightarrow 0} \frac{\sin(\ln(1-x))}{1-2^x}$$

Solution

The limit is in fact immediate if we observe that, as $x \rightarrow 0$,

$$\ln(1-x) \sim -x$$

and also

$$1 - 2^x \sim -x \log 2$$

thus obtaining

$$\lim_{x \rightarrow 0} \frac{\sin(\ln(1-x))}{1-2^x} = \lim_{x \rightarrow 0} \frac{\sin(-x)}{-x \log 2} = \frac{1}{\log 2}$$

since

$$\sin(-x) \sim -x$$

Exercise 1.28

Compute

$$\lim_{x \rightarrow +\infty} \frac{x^4 \sin^2(\pi - 2 \arctan(x))}{x^2 + 3}$$

Solution

Recalling that

$$\arctan(x) \xrightarrow{x \rightarrow +\infty} \frac{\pi}{2}$$

and hence that

$$\pi - 2 \arctan(x) \xrightarrow{x \rightarrow +\infty} 0$$

we can immediately use the asymptotic estimate

$$\sin^2(\pi - 2 \arctan(x)) \sim (\pi - 2 \arctan(x))^2$$

as well as

$$\frac{x^4}{x^2 + 3} \sim x^2$$

and say that

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{x^4 \sin^2(\pi - 2 \arctan(x))}{x^2 + 3} &= \lim_{x \rightarrow +\infty} \frac{x^4 (\pi - 2 \arctan(x))^2}{x^2 + 3} \\ &= \lim_{x \rightarrow +\infty} x^2 (\pi - 2 \arctan(x))^2 \end{aligned}$$

which is still an indeterminate form of the type $[0 \cdot +\infty]$. However, with the change of variable $y = \pi - 2 \arctan(x)$ and observing that

$$x = \tan\left(\frac{\pi}{2} - \frac{y}{2}\right) = \cot\left(\frac{y}{2}\right)$$

we obtain

$$\begin{aligned} \lim_{x \rightarrow +\infty} x^2 (\pi - 2 \arctan(x))^2 &= \lim_{y \rightarrow 0} y^2 \cot^2\left(\frac{y}{2}\right) \\ &= \lim_{y \rightarrow 0} \frac{y^2}{\sin^2\left(\frac{y}{2}\right)} \cos^2\left(\frac{y}{2}\right) = 4. \end{aligned}$$

Exercise 1.29

Compute

$$\lim_{x \rightarrow 1} \frac{\log(x^x) - \log x}{1 - \cos(x - 1)}$$

Solution

Recalling that

$$\log(x^x) - \log x = \log(e^{x \log x}) - \log x = x \log x - \log x = (x - 1) \log x$$

we immediately have

$$\lim_{x \rightarrow 1} \frac{\log(x^x) - \log x}{1 - \cos(x - 1)} = \lim_{x \rightarrow 1} \frac{(x - 1) \log x}{1 - \cos(x - 1)}$$

Setting, for greater clarity in the use of the asymptotic estimates, $x - 1 = t$, we obtain by substitution

$$\lim_{x \rightarrow 1} \frac{(x - 1) \log x}{1 - \cos(x - 1)} = \lim_{t \rightarrow 0} \frac{t \log(1 + t)}{1 - \cos t} = 2$$

since

$$\frac{t \log(1 + t)}{1 - \cos t} \sim \frac{t^2}{\frac{1}{2}t^2}$$

Exercise 1.30

Compute

$$\lim_{x \rightarrow +\infty} x \log \left(\frac{3x + x^2}{1 + x + x^2} \right)$$

Solution

We immediately observe that the limit is in the indeterminate form of the type $[+\infty \cdot 0]$, since

$$\frac{3x + x^2}{1 + x + x^2} \xrightarrow{x \rightarrow +\infty} 1$$

In order to resolve the indeterminate form, we can add and subtract 1 inside the argument of the logarithm:

$$\log \left(\frac{3x + x^2}{1 + x + x^2} \right) = \log \left(1 + \frac{3x + x^2}{1 + x + x^2} - 1 \right)$$

At this point, since

$$\frac{3x + x^2}{1 + x + x^2} - 1 \xrightarrow{x \rightarrow +\infty} 0$$

we can use the well-known asymptotic estimate of the logarithm, valid for some $\varepsilon(x) \rightarrow 0$, obtaining

$$\log \left(1 + \frac{3x + x^2}{1 + x + x^2} - 1 \right) \sim \frac{3x + x^2}{1 + x + x^2} - 1 = \frac{2x - 1}{1 + x + x^2}$$

The limit reduces to

$$\lim_{x \rightarrow +\infty} x \left(\frac{2x - 1}{1 + x + x^2} \right) = 2.$$

Exercise 1.31

Compute

$$\lim_{x \rightarrow +\infty} \left(e^{\sqrt{x^2+x}} - e^{\sqrt{x^2-1}} \right)$$

Solution

Factoring out $e^{\sqrt{x^2-1}}$ we obtain

$$\lim_{x \rightarrow +\infty} e^{\sqrt{x^2-1}} \left(e^{\sqrt{x^2+x} - \sqrt{x^2-1}} - 1 \right).$$

We examine the indeterminate form $\sqrt{x^2+x} - \sqrt{x^2-1}$ as $x \rightarrow +\infty$:

$$\begin{aligned} \lim_{x \rightarrow +\infty} \sqrt{x^2+x} - \sqrt{x^2-1} &= \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+x} + \sqrt{x^2-1}}{\sqrt{x^2+x} + \sqrt{x^2-1}} = \lim_{x \rightarrow +\infty} \frac{x^2+x-x^2+1}{\sqrt{x^2+x} + \sqrt{x^2-1}} \\ &= \lim_{x \rightarrow +\infty} \frac{x+1}{x \left(\sqrt{1+\frac{1}{x}} + \sqrt{1-\frac{1}{x^2}} \right)} = \frac{1}{2}. \end{aligned}$$

Hence the given limit equals

$$\lim_{x \rightarrow +\infty} e^{\sqrt{x^2-1}} \left(e^{\sqrt{x^2+x}-\sqrt{x^2-1}} - 1 \right) = (\sqrt{e} - 1) \lim_{x \rightarrow +\infty} e^{\sqrt{x^2-1}} = +\infty.$$

Equivalently, from $1/x \rightarrow 0$ as $x \rightarrow +\infty$ and from

$$\sqrt{1+y} = 1 + (1/2)y + o(y)$$

as $y \rightarrow 0$, we have

$$\sqrt{x^2+x} = x\sqrt{1+\frac{1}{x}} = x \left(1 + \frac{1}{2x} + o\left(\frac{1}{x}\right) \right) = x + \frac{1}{2} + o(1), \quad x \rightarrow +\infty,$$

where $o(1)$ denotes a generic infinitesimal, and

$$\sqrt{x^2-1} = x\sqrt{1-\frac{1}{x^2}} = x \left(1 - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right) \right) = x - \frac{1}{2x} + o\left(\frac{1}{x}\right), \quad x \rightarrow +\infty.$$

Hence the given limit equals

$$\begin{aligned} \lim_{x \rightarrow +\infty} \left(e^{\sqrt{x^2+x}} - e^{\sqrt{x^2-1}} \right) &= \lim_{x \rightarrow +\infty} \left(e^{x+1/2+o(1)} - e^{x-1/(2x)+o(1/x)} \right) = \\ &= \lim_{x \rightarrow +\infty} e^x \left(e^{1/2+o(1)} - e^{-1/(2x)+o(1/x)} \right) = (\sqrt{e} - 1) \lim_{x \rightarrow +\infty} e^x = +\infty. \end{aligned}$$

We have that $1/x \rightarrow 0$ as $x \rightarrow +\infty$ and we have the asymptotic expansions as $y \rightarrow 0$

$$\sin y = y + o(y), \quad \cos y = 1 - \frac{1}{2}y^2 + o(y^2), \quad e^y = 1 + y + o(y), \quad \log(1+y) = y + o(y).$$

Exercise 1.32

Compute the following limit

$$\lim_{x \rightarrow +\infty} \frac{1 + x^4 \sin(1/x^4)}{x^2(1 - \cos(1/x^2))}$$

Solution

From the asymptotic expansion we have:

$$\cos\left(\frac{1}{x^2}\right) = 1 - \frac{1}{2} \left(\frac{1}{x^2}\right)^2 + o\left(\left(\frac{1}{x^2}\right)^2\right) = 1 - \frac{1}{2} \frac{1}{x^4} + o\left(\frac{1}{x^4}\right) \quad \text{as } x \rightarrow +\infty.$$

Expanding the ratio asymptotically, we have:

- for the numerator

$$1 + x^4 \sin\left(\frac{1}{x^4}\right) = 1 + x^4 \left(\frac{1}{x^4} + o\left(\frac{1}{x^4}\right)\right) = 1 + 1 + o(1) \quad \text{as } x \rightarrow +\infty;$$

- for the denominator

$$\begin{aligned} x^2 \left[1 - \cos\left(\frac{1}{x^2}\right)\right] &= x^2 \left[1 - \left(1 - \frac{1}{2} \frac{1}{x^4} + o\left(\frac{1}{x^4}\right)\right)\right] = \\ &= x^2 \left[\frac{1}{2} \frac{1}{x^4} - o\left(\frac{1}{x^4}\right)\right] = \frac{1}{2} \frac{1}{x^2} - o\left(\frac{1}{x^2}\right) = \frac{1}{2} \frac{1}{x^2} + o\left(\frac{1}{x^2}\right) \quad \text{as } x \rightarrow +\infty. \end{aligned}$$

Hence we obtain:

$$\lim_{x \rightarrow +\infty} \frac{1 + x^4 \sin\left(\frac{1}{x^4}\right)}{x^2 \left[1 - \cos\left(\frac{1}{x^2}\right)\right]} = \lim_{x \rightarrow +\infty} \frac{2 + o(1)}{\frac{1}{2} \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)} = +\infty$$

where, as always, $o(1)$ denotes a generic infinitesimal.

Exercise 1.33

Compute the following limit

$$\lim_{x \rightarrow +\infty} \frac{x^2 e^{-1/x^4} - x^2}{\log(x^2 + 1) - 2 \log x}$$

Solution

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{x^2(e^{-1/x^4} - 1)}{\log(x^2 + 1) - \log x^2} &= \lim_{x \rightarrow +\infty} \frac{x^2\left(-\frac{1}{x^4} + o\left(\frac{1}{x^4}\right)\right)}{\log\left(1 + \frac{1}{x^2}\right)} = \\ &= \lim_{x \rightarrow +\infty} \frac{-\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)}{\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)} = \lim_{x \rightarrow +\infty} \frac{-\frac{1}{x^2}}{\frac{1}{x^2}} = -1. \end{aligned}$$

Exercise 1.34

Compute

$$\lim_{x \rightarrow 0} \frac{e^{-x^4} - x^4 - 1}{x^2 \cos x - x^2}$$

Solution

We use the expansion

$$e^{-x^4} = 1 - x^4 + o(x^4) \quad \text{as } x \rightarrow 0$$

and the equivalence

$$\cos x - 1 \sim -x^2/2 \text{ as } x \rightarrow 0.$$

Hence we have

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^{-x^4} - x^4 - 1}{x^2(\cos x - 1)} &= \lim_{x \rightarrow 0} \frac{1 - x^4 + o(x^4) - x^4 - 1}{x^2 \left(-\frac{x^2}{2} \right)} = \\ &= \lim_{x \rightarrow 0} \frac{-2x^4 + o(x^4)}{-\frac{x^4}{2}} = \lim_{x \rightarrow 0} \frac{-2x^4}{-\frac{x^4}{2}} = 4. \end{aligned}$$

Exercise 1.35

Compute

$$\lim_{x \rightarrow \pi/4} (2 \sin^2 x)^{\frac{1}{\cos 2x}}$$

Solution

Using $\cos 2x = 1 - 2 \sin^2 x$ and setting $y = 1 - 2 \sin^2 x$, we observe that $y \rightarrow 0$ as $x \rightarrow \pi/4$ and the given limit equals

$$\lim_{y \rightarrow 0} (1 - y)^{\frac{1}{y}} = \lim_{y \rightarrow 0} e^{\frac{\log(1-y)}{y}} = e^{-1} = \frac{1}{e}$$

also by virtue of the fundamental limit

$$\lim_{y \rightarrow 0} \frac{\log(1 - y)}{y} = -1.$$

Exercise 1.36

Compute

$$\lim_{x \rightarrow 0^+} \frac{(1 + x^5)^{\frac{1}{x^2 \sin 2x}} - 1}{2 \log(1 + x^3)}$$

Solution

We write the numerator in the form

$$e^{\frac{\log(1+x^5)}{x^2 \sin 2x}} - 1$$

and we analyze the exponent. As $x \rightarrow 0$ we have

$$\frac{\log(1 + x^5)}{x^2 \sin 2x} \sim \frac{x^5}{x^2 \cdot 2x} = \frac{x^2}{2}.$$

From this and from $e^y - 1 \sim y$ as $y \rightarrow 0$, it follows that

$$e^{\frac{\log(1+x^5)}{x^2 \sin 2x}} - 1 \sim \frac{\log(1 + x^5)}{x^2 \sin 2x} \sim \frac{x^2}{2}.$$

The given limit equals

$$\lim_{x \rightarrow 0^+} \frac{(1+x^5)^{\frac{1}{x^2 \sin 2x}} - 1}{2 \log(1+x^3)} = \lim_{x \rightarrow 0^+} \frac{\frac{x^2}{2}}{2x^3} = \frac{1}{4} \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty.$$

Exercise 1.37

Compute

$$\lim_{x \rightarrow 0^+} \frac{1 - x^x}{x^2}$$

Solution

Writing the numerator in the form

$$1 - e^{x \log x},$$

the exponent $x \log x$ tends to zero as $x \rightarrow 0^+$. Since $1 - e^y \sim -y$ as $y \rightarrow 0$, it follows that

$$1 - e^{x \log x} \sim -x \log x, \quad x \rightarrow 0^+.$$

The given limit equals

$$\lim_{x \rightarrow 0^+} \frac{1 - x^x}{x^2} = \lim_{x \rightarrow 0^+} \frac{-x \log x}{x^2} = \lim_{x \rightarrow 0^+} \frac{-\log x}{x} = +\infty.$$

Exercise 1.38

Compute

$$\lim_{x \rightarrow 0} \frac{\sin(e^{x^3} - 1)}{\sqrt{1+x^3} - 1}$$

Solution

We have

$$\sin(e^{x^3} - 1) \sim e^{x^3} - 1 \sim x^3, \quad x \rightarrow 0^+$$

and

$$\sqrt{1+x^3} - 1 \sim \frac{1}{2}x^3, \quad x \rightarrow 0^+.$$

The given limit equals

$$\lim_{x \rightarrow 0} \frac{\sin(e^{x^3} - 1)}{\sqrt{1+x^3} - 1} = \lim_{x \rightarrow 0} \frac{x^3}{\frac{1}{2}x^3} = 2.$$

Exercise 1.39

Compute

$$\lim_{x \rightarrow 0^+} \frac{x^x - e^x + x^\alpha}{\sin \sqrt{x}} = L$$

as the parameter $\alpha > 0$ varies.

Solution

We have

$$\lim_{x \rightarrow 0^+} \frac{x \log x}{x^\alpha} = \begin{cases} 0, & \alpha < 1 \\ -\infty, & \alpha = 1 \end{cases}$$

hence, as $x \rightarrow 0^+$, the infinitesimal $x \log x$ is of lower order than x ($x = o(x \log x)$) but of higher order than x^α if $\alpha < 1$ ($x \log x = o(x^\alpha)$, $\alpha < 1$).

Hence, from the expansion

$$e^y = 1 + y + o(y), \quad y \rightarrow 0,$$

it follows

$$\begin{aligned} x^x - e^x + x^\alpha &= e^{x \log x} - e^x + x^\alpha = \\ &= 1 + x \log x + o(x \log x) - 1 - x + o(x) + x^\alpha = \\ &= \begin{cases} x \log x + o(x \log x), & \alpha \geq 1 \\ x^\alpha + o(x^\alpha), & 0 < \alpha < 1. \end{cases} \end{aligned}$$

The given limit equals

$$\lim_{x \rightarrow 0^+} \frac{x^x - e^x + x^\alpha}{\sin \sqrt{x}} = \begin{cases} \lim_{x \rightarrow 0^+} \frac{x \log x}{\sqrt{x}} = \lim_{x \rightarrow 0^+} \sqrt{x} \log x = 0, & \alpha \geq 1 \\ \lim_{x \rightarrow 0^+} \frac{x^\alpha}{\sqrt{x}} = 0, & 1/2 < \alpha < 1 \\ \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{x}} = 1, & \alpha = 1/2 \\ \lim_{x \rightarrow 0^+} \frac{x^\alpha}{\sqrt{x}} = +\infty, & 0 < \alpha < 1/2. \end{cases}$$