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Mathematical Analysis 1

Exercises, volume 3 — Functions

The worked exercises on functions, one sheet per chapter, with the solution right after each statement: domains of functions and inverse functions. The sheets are by Fabio Furini and Gianluca Priori; many exercises are taken from M. Bramanti, *Esercitazioni di Analisi matematica 1*, Esculapio.

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Online version and downloadable material:
fabiofurini.github.io/mathematical-analysis-1/
github.com/fabiofurini/mathematical-analysis-1

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Domains of functions

Exercise 1.1

Determine the domain of the function:

$$f(x) = \frac{\log_2(3+x)}{\sqrt[3]{x+2}}$$

it is determined by studying the numerator and the denominator of the function

- The numerator:

The function $\log_2 t$ is defined for $t > 0$, hence the function $\log_2(3+x)$ is defined for $3+x > 0$, i.e., $x > -3$.

- The denominator:

The function $\sqrt[3]{t}$ is always defined; however, since it is in the denominator, it must not vanish. Hence we impose $\sqrt[3]{x+2} \neq 0$, i.e., $x \neq -2$.

Hence the domain is

$$(-3, -2) \cup (-2, +\infty)$$

Exercise 1.2

Determine the domain of the function:

$$f(x) = \sqrt{2^x - 8} \cdot \log|x-4|$$

Exercise 1.3

Determine the domain of the function:

$$f(x) = \tan(\log x)$$

Exercise 1.4

Determine the domain of the function:

$$f(x) = \log(\tan x)$$

Inverse functions

A function $f : D \rightarrow \mathbb{R}$ is **invertible** on the domain D if one of the following equivalent conditions holds:

$$\forall x_1, x_2 \in D, \quad x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

$$\forall x_1, x_2 \in D, \quad f(x_1) = f(x_2) \implies x_1 = x_2$$

$$\forall y \in f(D), \quad \exists! x \in D : y = f(x)$$

that is, if f is **injective**. In the following exercises, the second condition will mostly be used to verify the injectivity of a function f .

Given an invertible function $f : D \rightarrow \mathbb{R}$, the function that assigns to each output $y \in f(D)$ the unique input $x \in D$ such that $f(x) = y$ is called the **inverse function** of f and is denoted by the symbol f^{-1} .

Recall that the invertibility condition is equivalent to requiring that the graph of f be intersected in at most one point by every line parallel to the x -axis.

If a function $f : D \rightarrow \mathbb{R}$ is strictly increasing (decreasing) on D , then it is invertible on D . Moreover, its inverse function is strictly increasing (decreasing).

In conclusion, to establish whether or not a function $f : D \rightarrow \mathbb{R}$ is invertible on D , it suffices to verify any one of the following three conditions:

- f is **injective**;
- f is bijective, since every bijective function is both injective and surjective;
- f is strictly increasing (or decreasing) on D , since this is a *sufficient* condition for injectivity.

The relations between an invertible function f and its inverse function f^{-1} :

$$f : \begin{cases} y = f(x) \\ x \in D \end{cases} \quad f^{-1} : \begin{cases} x = f^{-1}(y) \\ y \in f(D) \end{cases}$$

show that if the point (x_0, y_0) is on the graph of f , then the point (y_0, x_0) is on the graph of f^{-1} . Since the points (x_0, y_0) and (y_0, x_0) are **symmetric with respect to the bisector** of equation $y = x$, the graph of f^{-1} is obtained from that of f by reflection across the bisector.

Exercise 2.1

Prove that the function

$$f(x) = \log(2 + 3x)$$

is invertible on its domain and determine the analytic expression of the inverse f^{-1} . Specify the domain of f^{-1} .

Solution

The domain of the given function is

$$\left(-\frac{2}{3}, +\infty\right),$$

the interval where the argument of the logarithm is strictly positive. To prove that f is invertible, it suffices to verify that it is injective, i.e.,

$$\forall x_1, x_2 \in D \quad f(x_1) = f(x_2) \implies x_1 = x_2$$

Imposing $f(x_1) = f(x_2)$ is equivalent to imposing that

$$\log(2 + 3x_1) = \log(2 + 3x_2)$$

$$2 + 3x_1 = 2 + 3x_2$$

$$3x_1 = 3x_2 \iff x_1 = x_2$$

The function f is injective and, consequently, invertible. We look for the analytic expression of the inverse function f^{-1} by solving, with respect to the variable x , the equation

$$\log(2 + 3x) = y.$$

which has the unique solution

$$x = \frac{1}{3}(e^y - 2)$$

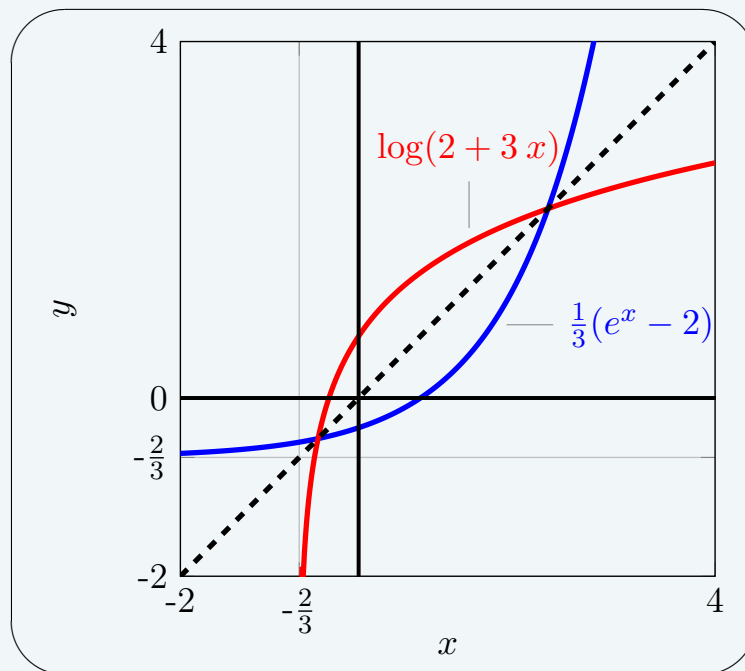
Solution

More formally, the inverse function of f is

$$f^{-1}(y) = \frac{1}{3}(e^y - 2)$$

whose domain is the image of the original function f , that is, $\mathbb{R}$. Hence $f : (-\frac{2}{3}, +\infty) \rightarrow (-\infty, +\infty)$, while $f^{-1} : (-\infty, +\infty) \rightarrow (-\frac{2}{3}, +\infty)$.

The graphs $y = \log(2 + 3x)$ and $y = \frac{1}{3}(e^x - 2)$ are symmetric with respect to the bisector $y = x$:

**Exercise 2.2**

Prove that the function

$$f(x) = \sqrt{1 - 2x}$$

is invertible on its domain and determine the analytic expression of the inverse f^{-1} . Specify the domain of f^{-1} .

Solution

The domain of the given function is

$$\left(-\infty, \frac{1}{2}\right],$$

the interval where the radicand is non-negative.

Solution

To prove that f is invertible, it suffices to verify that it is injective, i.e.,

$$\forall x_1, x_2 \in D \quad f(x_1) = f(x_2) \implies x_1 = x_2$$

Imposing $f(x_1) = f(x_2)$ is equivalent to imposing that

$$\begin{aligned} \sqrt{1-2x_1} &= \sqrt{1-2x_2} \\ 1-2x_1 &= 1-2x_2 \\ -2x_1 &= -2x_2 \iff x_1 = x_2 \end{aligned}$$

The function f is injective and, consequently, invertible. We look for the analytic expression of the inverse function f^{-1} by solving, with respect to the variable x , the equation

$$\sqrt{1-2x} = y.$$

which, for $y \geq 0$ (a square root is never negative in $\mathbb{R}$), has the unique solution

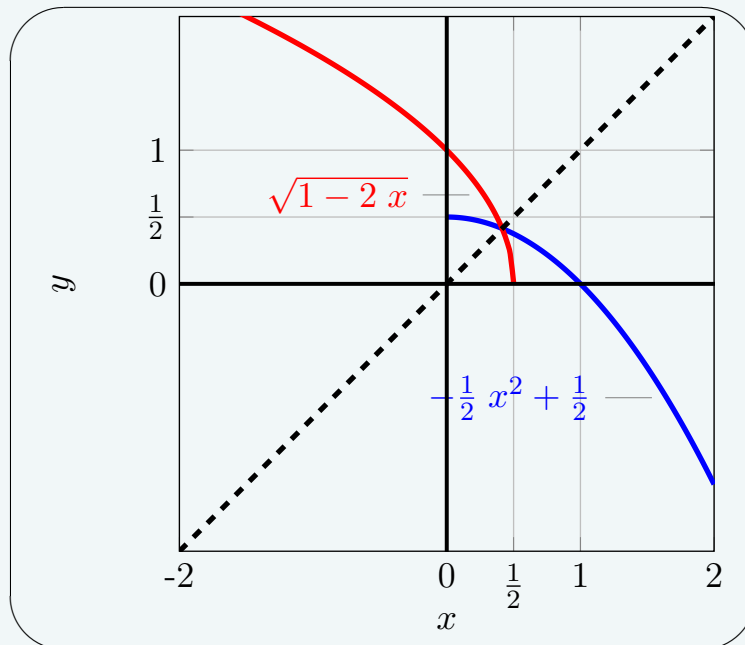
$$x = -\frac{1}{2}y^2 + \frac{1}{2}$$

More formally, the inverse function of f is

$$f^{-1}(y) = -\frac{1}{2}y^2 + \frac{1}{2}$$

whose domain is the image of the original function f , that is, $[0, +\infty)$. Hence $f : (-\infty, \frac{1}{2}] \rightarrow [0, +\infty)$, $f^{-1} : [0, +\infty) \rightarrow (-\infty, \frac{1}{2}]$.

The graphs $y = \sqrt{1-2x}$ and $y = -\frac{1}{2}x^2 + \frac{1}{2}$ are symmetric with respect to the bisector $y = x$:



Exercise 2.3

Prove that the function

$$f(x) = e^{2x-3}$$

is invertible on its domain and determine the analytic expression of the inverse f^{-1} . Specify the domain of f^{-1} .

Solution

The domain of the given function is

$$(-\infty, +\infty),$$

since the exponential function is defined for every $x \in \mathbb{R}$. To prove that f is invertible, it suffices to verify that it is injective, i.e.,

$$\forall x_1, x_2 \in D \quad f(x_1) = f(x_2) \implies x_1 = x_2$$

Imposing $f(x_1) = f(x_2)$ is equivalent to imposing that

$$\begin{aligned} e^{2x_1-3} &= e^{2x_2-3} \\ 2x_1 - 3 &= 2x_2 - 3 \\ 2x_1 &= 2x_2 \iff x_1 = x_2 \end{aligned}$$

The function f is injective and, consequently, invertible. We look for the analytic expression of the inverse function f^{-1} by solving, with respect to the variable x , the equation

$$e^{2x-3} = y.$$

It makes sense to solve this equation only for $y > 0$, since it would have no solutions when $y \leq 0$ (the exponential never vanishes and is never negative). The equation has the unique solution

$$x = \frac{3}{2} + \frac{1}{2} \log y$$

More formally, the inverse function of f is

$$f^{-1}(y) = \frac{3}{2} + \frac{1}{2} \log y$$

whose domain is the image of the original function f , that is, $(0, +\infty)$. Hence $f : (-\infty, +\infty) \rightarrow (0, +\infty)$, $f^{-1} : (0, +\infty) \rightarrow (-\infty, +\infty)$.

Exercise 2.4

Prove that the function

$$f(x) = \arctan(2x - 1)$$

is invertible on its domain and determine the analytic expression of the inverse f^{-1} . Specify the domain of f^{-1} .

Solution

The arctangent function, which we recall is the inverse function of the restriction of $y = \tan(x)$ to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, has the following properties:

- its *domain* is the set $\mathbb{R}$;
- its image is the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$;
- it is a **strictly increasing** monotonic function.

Since it is strictly increasing, the function f is invertible (note that this is only a *sufficient* condition). We look for the analytic expression of the inverse function f^{-1} by solving, with respect to the variable x , the equation

$$\arctan(2x - 1) = y,$$

where $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The equation becomes

$$\begin{aligned} 2x - 1 &= \tan(y) \\ x &= \frac{1}{2} + \frac{\tan(y)}{2} \end{aligned}$$

More formally, the inverse function of f is

$$f^{-1}(y) = \frac{1}{2} + \frac{\tan(y)}{2}$$

whose domain is the image of the original function f , that is, $(-\frac{\pi}{2}, \frac{\pi}{2})$. Hence $f : \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$, while $f^{-1} : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$

Exercise 2.5

Prove that the function

$$f(x) = \frac{x - 1}{x + 2}$$

is invertible on its domain and determine the analytic expression of the inverse f^{-1} . Specify the domain of f^{-1} .

Solution

The domain of the given function is

$$(-\infty, -2) \cup (-2, +\infty),$$

since it is defined $\forall x \in \mathbb{R}, x \neq -2$. To prove that f is invertible, it suffices to verify that it is injective, i.e.,

$$\forall x_1, x_2 \in D \quad f(x_1) = f(x_2) \implies x_1 = x_2$$

To simplify the subsequent computations, it is convenient to rewrite the function f as:

$$f(x) = \frac{x - 1}{x + 2} = \frac{x + 2 - 3}{x + 2} = 1 - \frac{3}{x + 2}$$

Imposing $f(x_1) = f(x_2)$ is equivalent to imposing that

$$\begin{aligned} 1 - \frac{3}{x_1 + 2} &= 1 - \frac{3}{x_2 + 2} \\ \frac{3}{x_1 + 2} &= \frac{3}{x_2 + 2} \\ x_1 + 2 &= x_2 + 2 \iff x_1 = x_2 \end{aligned}$$

The function f is injective and, consequently, invertible. We look for the analytic expression of the inverse function f^{-1} by solving, with respect to the variable x , the equation

$$\frac{x - 1}{x + 2} = y.$$

Rewriting the function again as before

$$1 - \frac{3}{x + 2} = y$$

we obtain

$$x = \frac{3}{1 - y} - 2 = \frac{2y + 1}{1 - y}$$

More formally, the inverse function of f is

$$f^{-1}(y) = \frac{2y + 1}{1 - y}$$

whose domain is the image of the original function f , that is, $(-\infty, 1) \cup (1, +\infty)$. Hence $f : (-\infty, -2) \cup (-2, +\infty) \rightarrow (-\infty, 1) \cup (1, +\infty)$, $f^{-1} : (-\infty, 1) \cup (1, +\infty) \rightarrow (-\infty, -2) \cup (-2, +\infty)$.

Exercise 2.6

Prove that the function

$$f(x) = e^{1+x^2}$$

is invertible on a restriction of its domain and determine the analytic expression of the inverse f^{-1} . Specify the domain of f^{-1} .

Solution

The domain of the given function is

$$(-\infty, +\infty),$$

since the exponential function is defined for every $x \in \mathbb{R}$. It is clear that f is not invertible on all of $\mathbb{R}$, since it is an even function, i.e., symmetric with respect to the y -axis:

$$f(-x) = e^{1+(-x)^2} = e^{1+x^2} = f(x)$$

This implies that any horizontal line intersects the graph of the function in two points. The function is not injective, hence not invertible on $\mathbb{R}$. However, we can restrict the domain

by considering only the positive x -half-axis including the origin, i.e., the set

$$[0, +\infty)$$

Let us verify that, on this restricted domain, the function f is injective. To this end, we impose $f(x_1) = f(x_2)$, i.e.,

$$\begin{aligned} e^{1+x_1^2} &= e^{1+x_2^2} \\ 1 + x_1^2 &= 1 + x_2^2 \\ x_1^2 &= x_2^2 \iff x_1 = x_2 \end{aligned}$$

where the last equivalence holds since $x \in [0, +\infty)$, i.e., $x \geq 0$, so the solution $x_1 = -x_2$ is excluded. We have thus proved that the function f is injective on the restricted domain and, consequently, invertible on $[0, +\infty)$.

We look for the analytic expression of the inverse function f^{-1} by solving, with respect to the variable x , the equation

$$e^{1+x^2} = y.$$

The equation is solved by

$$x = \pm \sqrt{\log(y) - 1}$$

where, however, it is now clear that the negative solution must be excluded, since x varies in the restriction $[0, +\infty)$.

Solution

More formally, the inverse function of f is

$$f^{-1}(y) = \sqrt{\log(y) - 1}$$

whose domain is the image of the original function f , that is, $[e, +\infty)$. Hence $f : [0, +\infty) \rightarrow [e, +\infty)$, $f^{-1} : [e, +\infty) \rightarrow [0, +\infty)$.

Exercise 2.7

Prove that the function

$$f(x) = \frac{4}{5^{3x}}$$

is invertible on its domain and determine the analytic expression of the inverse f^{-1} . Specify the domain of f^{-1} .

Solution

The domain of the given function is

$$(-\infty, +\infty),$$

since the denominator is always nonzero (the exponential never vanishes). To prove that f

is invertible, it suffices to verify that it is injective, i.e.,

$$\forall x_1, x_2 \in D \quad f(x_1) = f(x_2) \implies x_1 = x_2$$

Imposing $f(x_1) = f(x_2)$ is equivalent to imposing that

$$\begin{aligned} \frac{4}{5^{3x_1}} &= \frac{4}{5^{3x_2}} \\ 5^{3x_1} &= 5^{3x_2} \\ 3x_1 &= 3x_2 \iff x_1 = x_2 \end{aligned}$$

The function f is injective and, consequently, invertible. We look for the analytic expression of the inverse function f^{-1} by solving, with respect to the variable x , the equation

$$\frac{4}{5^{3x}} = y,$$

with $y > 0$, which has the unique solution

$$x = \frac{1}{3} \log_5 \left(\frac{4}{y} \right) = -\frac{1}{3} \log_5 \left(\frac{y}{4} \right)$$

Solution

More formally, the inverse function of f is

$$f^{-1}(y) = -\frac{1}{3} \log_5 \left(\frac{y}{4} \right)$$

whose domain is the image of the original function f , that is, $(0, +\infty)$. Hence $f : \mathbb{R} \rightarrow (0, +\infty)$, while $f^{-1} : (0, +\infty) \rightarrow \mathbb{R}$.

Exercise 2.8

Prove that the function

$$f(x) = -1 + \sqrt[5]{1+x}$$

is invertible on its domain and determine the analytic expression of the inverse f^{-1} . Specify the domain of f^{-1} .

Solution

The domain of the given function is

$$(-\infty, +\infty),$$

since roots of odd index are always defined. To prove that f is invertible, it suffices to verify that it is injective, i.e.,

$$\forall x_1, x_2 \in D \quad f(x_1) = f(x_2) \implies x_1 = x_2$$

Imposing $f(x_1) = f(x_2)$ is equivalent to imposing that

$$\begin{aligned} -1 + \sqrt[5]{1+x_1} &= -1 + \sqrt[5]{1+x_2} \\ \sqrt[5]{1+x_1} &= \sqrt[5]{1+x_2} \\ 1+x_1 &= 1+x_2 \iff x_1 = x_2 \end{aligned}$$

The function f is injective (it is also strictly increasing) and, consequently, invertible. We look for the analytic expression of the inverse function f^{-1} by solving, with respect to the variable x , the equation

$$-1 + \sqrt[5]{1+x} = y,$$

which has the unique solution

$$x = -1 + (y+1)^5$$

Solution

More formally, the inverse function of f is

$$f^{-1}(y) = -1 + (y+1)^5$$

whose domain is the image of the original function f , that is, $\mathbb{R}$. Hence $f : \mathbb{R} \rightarrow \mathbb{R}$, and also $f^{-1} : \mathbb{R} \rightarrow \mathbb{R}$.

Exercise 2.9

Prove that the function

$$f(x) = \frac{1-3x}{x+1}$$

is invertible on its domain and determine the analytic expression of the inverse f^{-1} . Specify the domain of f^{-1} .

(Hint: perform the division of $1-3x$ by $x+1$).

Exercise 2.10

Prove that the function

$$f(x) = 3^{x^2}$$

is invertible on a restriction of its domain and determine the analytic expression of the inverse f^{-1} on the restriction. Specify the domain of f^{-1} .

Exercise 2.11

Prove that the function

$$f(x) = \sqrt{\log\left(\frac{x-1}{x}\right)}$$

is invertible on its domain and determine the analytic expression of the inverse f^{-1} . Specify

the domain of f^{-1} .