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UNIVERSITÀ DI ROMA

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# Mathematical Analysis 1

Exercises, volume 2 — Numbers and sets

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The worked exercises of the second part of the course, one sheet per chapter, with the solution right after each statement: maxima, minima, suprema and infima, equations and inequalities, summations, complex numbers. The sheets are by Fabio Furini and Gianluca Priori; many exercises are taken from M. Bramanti, *Esercitazioni di Analisi matematica 1*, Esculapio.

**Fabio Furini**

[fabiofurini.github.io](https://fabiofurini.github.io)

Department of Computer, Control and Management Engineering — Sapienza University of Rome

Online version and downloadable material:  
[fabiofurini.github.io/mathematical-analysis-1/](https://fabiofurini.github.io/mathematical-analysis-1/)  
[github.com/fabiofurini/mathematical-analysis-1](https://github.com/fabiofurini/mathematical-analysis-1)

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## Maxima, minima, suprema and infima

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### Exercise 1.1

Determine whether the set

$$A = [0, \sqrt{2}] \cap \mathbb{Q}$$

has a maximum, a minimum, a supremum, an infimum and, if so, determine these elements.

### Solution

The minimum of  $A$  is 0.  $A$  has no maximum; its supremum is  $\sqrt{2}$ .

### Exercise 1.2

Determine whether the set

$$A = [\sqrt{2}, \sqrt{3}] \cap \mathbb{Q}$$

has a maximum, a minimum, a supremum, an infimum and, if so, determine these elements.

### Solution

$A$  has neither a minimum nor a maximum. The infimum is  $\sqrt{2}$ , the supremum is  $\sqrt{3}$ .

### Exercise 1.3

Determine whether the set

$$A = [0, \sqrt{2}] \cap (\mathbb{R} - \mathbb{Q})$$

has a maximum, a minimum, a supremum, an infimum and, if so, determine these elements.

**Solution**

$A$  has no minimum; the infimum is 0. The maximum of  $A$  is  $\sqrt{2}$ .

**Exercise 1.4**

In each of the following cases, say whether the set

$$A = (-\infty, \sqrt{2}] \cap \mathbb{Q}$$

has a maximum, a minimum, a supremum, an infimum and, if so, determine these elements.

**Solution**

$A$  is not bounded below, hence it has no infimum in  $\mathbb{R}$ .  $A$  has no maximum; the supremum is  $\sqrt{2}$ .

**Exercise 1.5**

Say whether the following set  $A \subset \mathbb{R}$  has a maximum, a minimum, a supremum, an infimum, and whether it is bounded:

$$A = \left\{ \frac{2n+1}{n} : n \in \mathbb{N} \setminus \{0\} \right\}$$

**Solution**

From  $(2n+1)/n = 2 + (1/n)$  we have that the maximum of  $A$  is attained for  $n = 1$  and is equal to 3.  $A$  has no minimum; the infimum is 2. In particular,  $A$  is bounded.

**Exercise 1.6**

Say whether the following set  $A \subset \mathbb{R}$  has a maximum, a minimum, a supremum, an infimum, and whether it is bounded:

$$A = \left\{ \frac{n-1}{n} : n \in \mathbb{N} \setminus \{0\} \right\}$$

**Solution**

From  $(n-1)/n = 1 - (1/n)$  we have that the minimum of  $A$  is attained for  $n = 1$  and is equal to 0.  $A$  has no maximum; the supremum is 1. In particular,  $A$  is bounded.

**Exercise 1.7**

Say whether the following set  $A \subset \mathbb{R}$  has a maximum, a minimum, a supremum, an infimum, and whether it is bounded:

$$A = \left\{ \frac{3n^2 + 1}{n^2} : n \in \mathbb{N} \setminus \{0\} \right\}$$

**Solution**

From  $(3n^2 + 1)/n^2 = 3 + (1/n^2)$  we have that the maximum of  $A$  is attained for  $n = 1$  and is equal to 4.  $A$  has no minimum; the infimum is 3. In particular,  $A$  is bounded.

**Exercise 1.8**

Say whether the following set  $A \subset \mathbb{R}$  has a maximum, a minimum, a supremum, an infimum, and whether it is bounded:

$$A = \left\{ \frac{1}{n^2 + 1} : n \in \mathbb{N} \setminus \{0\} \right\}$$

**Solution**

The maximum value of  $A$  is attained for  $n = 1$  and is equal to  $1/2$ .  $A$  has no minimum; the infimum is 0. In particular,  $A$  is bounded.

**Exercise 1.9**

Let

$$A = \left\{ \left| \frac{3x}{x+1} \right| : -\frac{1}{2} < x \leq 2 \right\}.$$

Choose the correct statements among the following:

- (a)  $\sup A \notin A$                       (b)  $\inf A = \min A = 2$   
 (c)  $\max A = 3$                       (d) none of the other answers is correct

**Solution**

$A$  is the set of values of  $|f(x)|$  on the interval  $(-1/2, 2]$ , with  $f(x) = \frac{3x}{x+1}$ . From

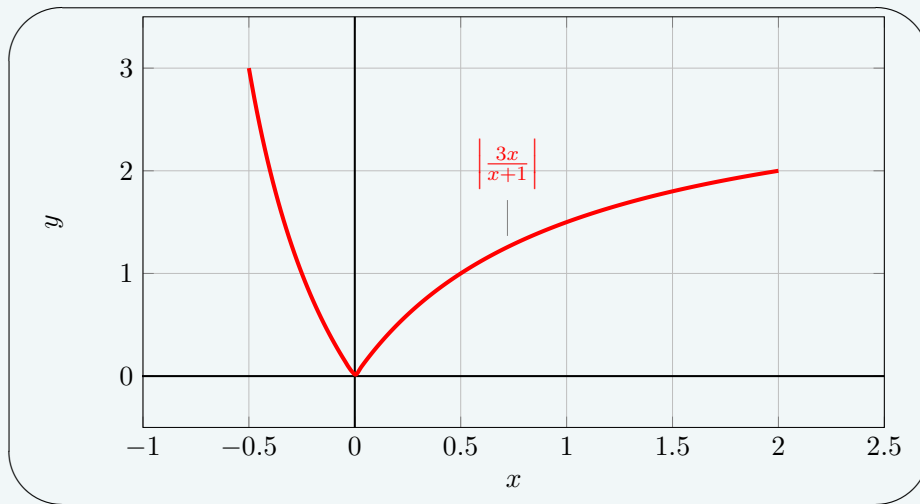
$$\frac{3x}{x+1} = 3 - \frac{3}{x+1}$$

we have that the function  $f$  is strictly increasing on the given interval  $(-1/2, 2]$ , negative

on  $(-1/2, 0)$ , zero for  $x = 0$ , and positive on  $(0, 2]$ . It follows that

$$|f(x)| = \left| \frac{3x}{x+1} \right|$$

is strictly decreasing on  $(-1/2, 0]$ , where it takes all the values in  $[0, 3]$ , and strictly increasing on  $[0, 2]$ , where it takes all the values in  $[0, 2]$ . The minimum value is therefore attained for  $x = 0$  and is equal to 0. The supremum is 3; there is no maximum value. The correct answer is (a).



### Exercise 1.10

For  $I = [1/3, +\infty)$  consider the function

$$f : I \rightarrow \mathbb{R}, f(x) = \exp \left( \left| \frac{2x-1}{x} \right| \right).$$

Determine, among the following intervals, the set  $J = f(I)$  of the values taken by  $f$ .

- |                    |                   |                      |
|--------------------|-------------------|----------------------|
| (a) $[e, e^2)$     | (b) $(e^{-2}, e]$ | (c) $[1, e^2)$       |
| (d) $(e, +\infty)$ | (e) $(0, e^2)$    | (f) another interval |

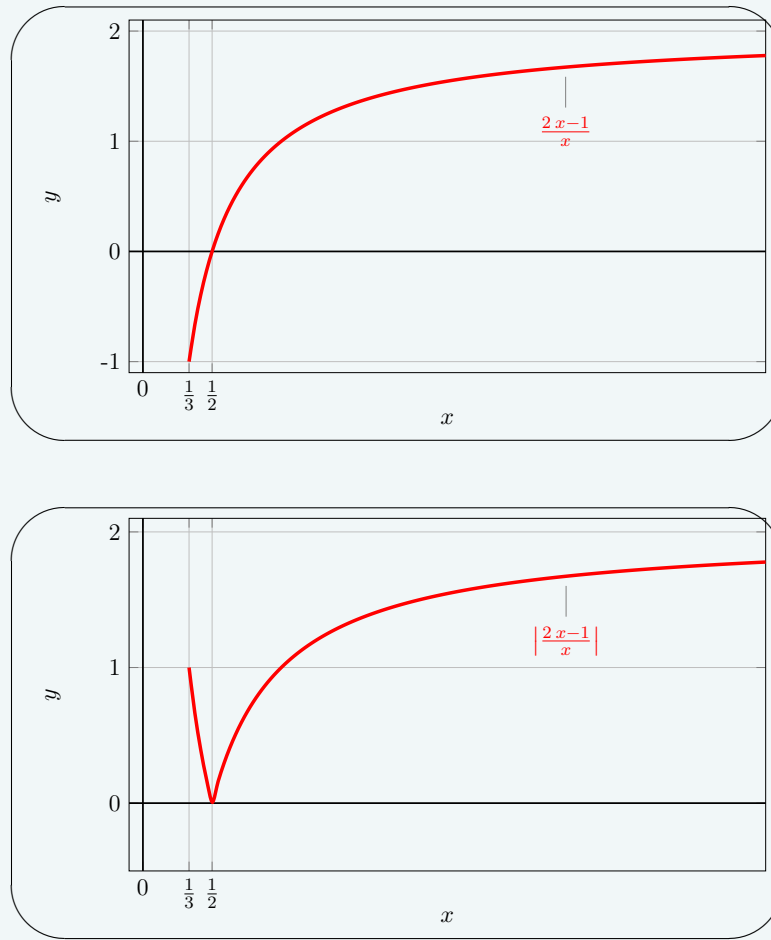
### Solution

The function  $f$  has the same monotonicity behavior as

$$|g(x)| = \left| \frac{2x-1}{x} \right|$$

with

$$g(x) = \frac{2x-1}{x} = 2 - \frac{1}{x}.$$

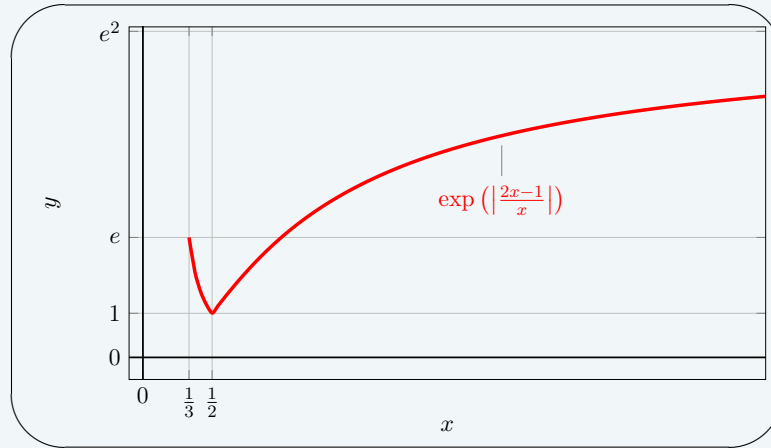


### Solution

The function  $g$  is strictly increasing on the interval  $I = [1/3, +\infty)$ , negative on  $[1/3, 1/2)$ , zero for  $x = 1/2$ , and positive on  $(1/2, +\infty)$ . It follows that  $|g(x)|$  is strictly decreasing on  $[1/3, 1/2]$ , where it takes all the values in  $[0, 1]$ , and strictly increasing on  $[1/2, +\infty)$ , where it takes all the values in  $[0, 2)$ . The set of values  $|g|(I)$  is  $[0, 2)$ , hence for  $f(x) = e^{|g(x)|}$  we have

$$f(I) = [1, e^2).$$

The correct answer is (c).



More precisely,  $f$  is strictly decreasing on  $[1/3, 1/2]$ , where it takes all the values in  $[1, e]$ , and strictly increasing on  $[1/2, +\infty)$ , where it takes all the values in  $[1, e^2]$ .

### Exercise 1.11

Let  $A = A_+ \cup A_-$  with

$$A_+ = \left\{ x + \frac{2}{x} : x > 0 \right\}, \quad A_- = \left\{ x + \frac{2}{x} : x < 0 \right\}.$$

Choose the correct statements among the following:

- (a)  $\sup A = +\infty$       (b)  $\inf A_+ = 2\sqrt{2}$       (c)  $A_+$  has no minimum  
 (d)  $A_-$  has no maximum      (e) none of the other answers is correct

### Solution

Clearly,  $A_+$  is not bounded above, hence statement (a) is correct (and statement (e) is false). For  $x \neq 0$ , we consider the equation

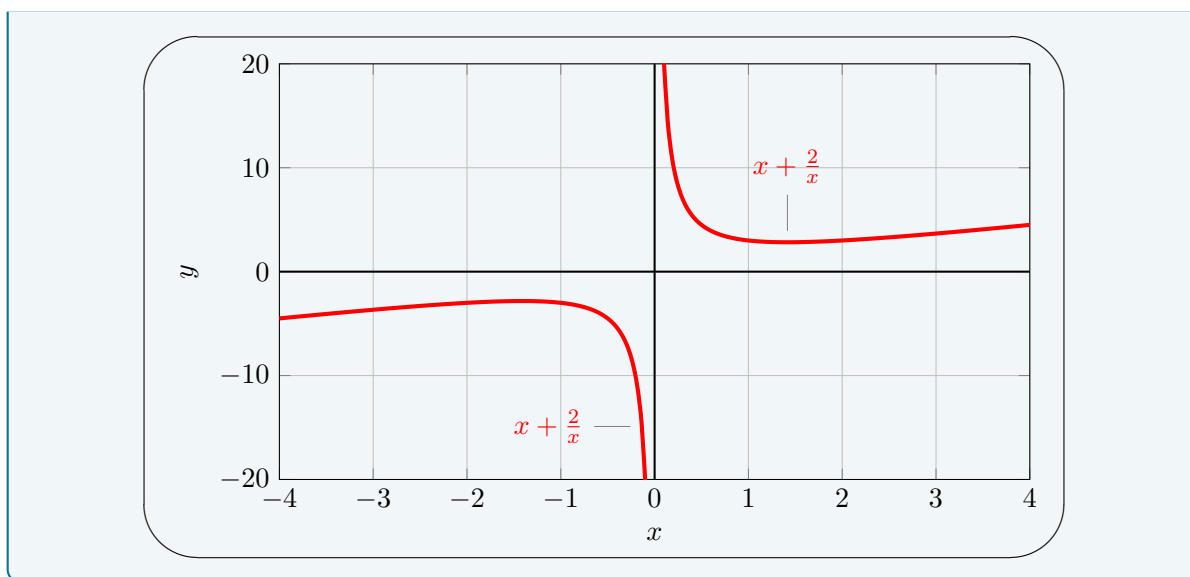
$$x + \frac{2}{x} = y,$$

$$x^2 - yx + 2 = 0.$$

There are solutions  $x \in \mathbb{R}$  if and only if  $y \in (-\infty, -2\sqrt{2}] \cup [2\sqrt{2}, +\infty)$ . Moreover, for every  $y \in (-\infty, -2\sqrt{2}]$  the solutions  $x$  are negative, while for every  $y \in [2\sqrt{2}, +\infty)$  the solutions  $x$  are positive. This means

$$A_+ = [2\sqrt{2}, +\infty), \quad A_- = (-\infty, -2\sqrt{2}].$$

Therefore (b) is true, (c) is false, (d) is false.



# Equations and inequalities in one real variable

## Exercise 2.1

Solve the following inequality in the real variable  $x$ :

$$\frac{x^2 - 2x}{x^2 - 4x + 3} > 0$$

## Solution

The numerator  $x^2 - 2x$  is positive for  $x < 0$  or for  $x > 2$ , zero for  $x = 0$ ,  $x = 2$ , and negative for  $0 < x < 2$ . The denominator  $x^2 - 4x + 3$  is positive for  $x < 1$  or for  $x > 3$ , and negative for  $1 < x < 3$ . The quotient is defined and has the required positive sign for

$$x \in (-\infty, 0) \cup (1, 2) \cup (3, +\infty).$$

## Exercise 2.2

Solve the following inequality in the real variable  $x$ :

$$\log(x-1)^2 - \log(x-2) > 0$$

## Solution

The function  $f(x) = \log(x-1)^2 - \log(x-2)$  is defined for  $x > 2$ , since for such  $x$ , and only for such  $x$ , we have  $x-2 > 0$ ,  $(x-1)^2 > 0$ . In its domain, we have, as required,  $f(x) > 0$  if and only if

$$\log(x-1)^2 > \log(x-2), \quad x > 2$$

hence if and only if

$$\begin{cases} (x-1)^2 > x-2 \\ x > 2 \end{cases}$$

We obtain

$$\begin{cases} x^2 - 3x + 3 > 0 \\ x > 2 \end{cases}$$

which is equivalent to

$$x > 2$$

since  $x^2 - 3x + 3 > 0$  is satisfied for every  $x$ . On its whole domain  $(2, +\infty)$  the function  $f(x)$  takes positive values.

### Exercise 2.3

Solve the following inequality in the real variable  $x$ :

$$e^x + e^{-x} < \frac{10}{3}$$

### Solution

For every  $x \in \mathbb{R}$

$$e^x + e^{-x} < \frac{10}{3}$$

is equivalent to

$$3e^{2x} - 10e^x + 3 < 0$$

hence to

$$\frac{1}{3} < e^x < 3.$$

It follows that the inequality is solved by

$$-\log 3 < x < \log 3.$$

### Exercise 2.4

Solve the following inequality in the real variable  $x$ :

$$8^{x+1} \geq 2^{x^2}$$

### Solution

For every  $x \in \mathbb{R}$

$$8^{x+1} \geq 2^{x^2}$$

is equivalent to

$$2^{3x+3} \geq 2^{x^2}$$

hence to

$$3x + 3 \geq x^2$$

since  $2^x$  is a strictly increasing function on  $\mathbb{R}$ . It follows that the inequality is solved by

$$\frac{3 - \sqrt{21}}{2} \leq x \leq \frac{3 + \sqrt{21}}{2}.$$

### Exercise 2.5

Solve the following inequality in the real variable  $x$ :

$$\log(1 - \sin x) \geq 0$$

### Solution

The function  $f(x) = \log(1 - \sin x)$  is defined for  $\sin x < 1$ , hence for

$$x \neq \frac{\pi}{2} + 2k\pi, \quad k \in \mathbf{Z}.$$

For such  $x$  we have  $f(x) \geq 0$  if and only if

$$1 - \sin x \geq 1$$

hence for

$$\sin x \leq 0.$$

The given inequality is solved for

$$\pi + 2k\pi \leq x \leq 2\pi + 2k\pi, \quad k \in \mathbf{Z}$$

i.e., for

$$x \in \bigcup_{k \in \mathbf{Z}} [\pi + 2k\pi, 2\pi + 2k\pi].$$

### Exercise 2.6

Solve the following inequality in the real variable  $x$ :

$$|x|\sqrt{1 - 2x^2} > 2x^2 - 1$$

### Solution

$\sqrt{1 - 2x^2}$  is defined for  $1 - 2x^2 \geq 0$ . When  $1 - 2x^2 = 0$  the inequality is not satisfied, since it reduces to  $0 > 0$ . When  $1 - 2x^2 > 0$  the inequality is satisfied, since the left-hand side  $|x|\sqrt{1 - 2x^2}$  is the product of two positive numbers while the right-hand side  $2x^2 - 1 = -(1 - 2x^2)$  is negative. Therefore the inequality is solved for

$$-\sqrt{\frac{1}{2}} < x < \sqrt{\frac{1}{2}}.$$

**Exercise 2.7**

Solve the following equation in the real variable  $x$ :

$$\sin^2 x = 2 \cos^2 x - \frac{1}{2}$$

**Solution**

The equation

$$\sin^2 x = 2 \cos^2 x - \frac{1}{2}$$

is equivalent to

$$2 \sin^2 x = 4(1 - \sin^2 x) - 1$$

$$6 \sin^2 x = 3$$

$$\sin x = \pm \sqrt{\frac{1}{2}}.$$

The solutions are given by

$$x = \frac{\pi}{4} + \frac{k\pi}{2}, \quad k \in \mathbf{Z}.$$

**Exercise 2.8**

Solve the following inequality in the real variable  $x$ :

$$e^{\sin^2 x - \sin x} \leq 1$$

**Solution**

For every  $x \in \mathbb{R}$ ,  $e^{\sin^2 x - \sin x} \leq 1$  is equivalent to

$$\sin^2 x - \sin x \leq 0$$

hence to

$$0 \leq \sin x \leq 1.$$

The solutions are given by

$$2k\pi \leq x \leq \pi + 2k\pi, \quad k \in \mathbf{Z}$$

i.e., by

$$x \in \bigcup_{k \in \mathbf{Z}} [2k\pi, \pi + 2k\pi].$$

**Exercise 2.9**

Solve the following inequality in the real variable  $x$ :

$$2|x^2 - x| > |x|$$

**Solution**

The inequality  $2|x^2 - x| > |x|$  is not satisfied for  $x = 0$ . For  $x \neq 0$ , dividing by the positive term  $|x|$ , it is equivalent to

$$2|x - 1| > 1$$

hence to

$$|x - 1| > 1/2.$$

The solutions are given by

$$(x - 1 < -1/2, x \neq 0) \vee (x - 1 > 1/2),$$

$$(x < 1/2, x \neq 0) \vee x > 3/2$$

hence by

$$x \in (-\infty, 0) \cup (0, 1/2) \cup (3/2, +\infty).$$

**Exercise 2.10**

Solve the following inequality in the real variable  $x$ :

$$(x + 1)^{x^2 - 1} > 1$$

**Solution**

The function  $f(x) = (x + 1)^{x^2 - 1}$  is defined for  $x > -1$ . For such  $x$ ,  $f(x) > 1$  is equivalent to

$$e^{(x^2 - 1)\log(x + 1)} > 1$$

hence to

$$(x^2 - 1)\log(x + 1) > 0, \quad x > -1.$$

Dividing by the positive term  $x + 1$  we obtain

$$(x - 1)\log(x + 1) > 0, \quad x > -1.$$

The factor  $x - 1$  is positive for  $x > 1$ , zero for  $x = 1$ , and negative for  $-1 < x < 1$ . The factor  $\log(x + 1)$  is positive for  $x > 0$ , zero for  $x = 0$ , and negative for  $-1 < x < 0$ . By the rule of signs, the solutions are given by

$$x \in (-1, 0) \cup (1, +\infty).$$

**Exercise 2.11**

Solve the following equation in the real variable  $x$ :

$$3^{|x^2-4|} = 0$$

**Solution**

$3^{|x^2-4|} = 0$  has no solutions, since  $3^y > 0$  for every  $y \in \mathbb{R}$ .

**Exercise 2.12**

Solve the following inequality in the real variable  $x$ :

$$\frac{\log_a(4x-3)}{\log_a(2x-1)} > 1 \quad 0 < a < 1$$

**Solution**

Because of the existence conditions of the logarithm and of the denominator, we must impose the system

$$\begin{cases} 4x-3 > 0 \\ 2x-1 > 0 \\ \log_a(2x-1) \neq 0 \end{cases}$$

which gives  $\frac{3}{4} < x < 1, x > 1$ . Now moving everything to the left-hand side, finding the least common denominator and using the properties of logarithms, we obtain

$$\frac{\log_a\left(\frac{4x-3}{2x-1}\right)}{\log_a(2x-1)} > 0$$

Studying the sign of the numerator and of the denominator, and keeping in mind that  $a < 1$ , for the numerator we obtain

$$\log_a\left(\frac{4x-3}{2x-1}\right) > 0 \iff \frac{4x-3}{2x-1} < 1 \iff \frac{1}{2} < x < 1$$

while for the denominator we obtain

$$\log_a(2x-1) > 0 \iff x < 1$$

Combining the signs, the inequality is satisfied for  $\frac{1}{2} < x < 1, x > 1$ . Intersecting this set of solutions with the existence conditions studied above, we obtain that the inequality is satisfied for every  $x$  belonging to the set  $(\frac{3}{4}; 1) \cup (1, +\infty)$ .

**Exercise 2.13**

Solve the following inequality in the real variable  $x$ :

$$\sqrt{x^2 - 3x + 5} \leq x + 3$$

**Solution**

This irrational inequality has the form  $\sqrt{A(x)} \leq B(x)$ , with  $A(x)$  and  $B(x)$  polynomials in  $x$ . First of all, we must impose the existence condition of the root, i.e.,  $x^2 - 3x + 5 \geq 0$ . Moreover, since a square root is always positive or zero in  $\mathbb{R}$ , we impose  $x + 3 \geq 0$ . We therefore solve the system:

$$\begin{cases} x^2 - 3x + 5 \geq 0 \\ x + 3 \geq 0 \\ x^2 - 3x + 5 \leq (x + 3)^2 \end{cases}$$

where, in the last inequality, we were allowed to square both sides, since they are positive. The system reduces to:

$$\begin{cases} \forall x \in \mathbb{R} \\ x \geq -3 \\ x \geq -\frac{4}{9} \end{cases}$$

whose set of solutions is the interval  $[-\frac{4}{9}, +\infty)$ .

**Exercise 2.14**

Solve the following rational inequality in the real variable  $x$ :

$$\frac{x^3 - 3x^2 + 2x - 6}{x^2 - 2x - 3} > \frac{2}{x + 1}$$

**Solution**

First of all, we must write the inequality in canonical form (0 on the right-hand side). The existence conditions (denominators different from zero) will be automatically included in the sign study. We obtain

$$\frac{x^3 - 3x^2 + 2x - 6}{x^2 - 2x - 3} - \frac{2}{x + 1} > 0$$

We factor the numerator and the denominator of the first fraction, obtaining:

$$\frac{(x^2 + 2)(x - 3)}{(x - 3)(x + 1)} - \frac{2}{x + 1} > 0$$

Since we can cancel  $x - 3$  and obtain an equivalent fraction, provided that  $x \neq 3$  by the existence condition, we have

$$\frac{x^2 + 2}{x + 1} - \frac{2}{x + 1} > 0$$

that is,

$$\frac{x^2}{x+1} > 0$$

The numerator is positive  $\forall x \in \mathbb{R}, x \neq 0$ , and the denominator is positive for  $x > -1$ . Combining the signs, and taking into account the condition  $x \neq 3$ , we obtain the solution set  $(-1, 0) \cup (0, 3) \cup (3, +\infty)$ .

## Summations

### Exercise 3.1

Prove, using the properties of summations, that for every integer  $n \geq 1$  we have:

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

(sum of the squares of the first  $n$  natural numbers, excluding zero).

### Solution

We start by writing the sum  $\sum_{k=0}^n (k+1)^3$  in two different ways:

$$\begin{aligned} 1) \quad \sum_{k=0}^n (k+1)^3 &= \sum_{k=0}^n (k^3 + 3k^2 + 3k + 1) \\ &= \left( \sum_{k=1}^n k^3 + 3 \sum_{k=1}^n k^2 + 3 \sum_{k=1}^n k + \sum_{k=1}^n 1 \right) + 1 \\ &= \sum_{k=1}^n k^3 + 3 \sum_{k=1}^n k^2 + \frac{3n(n+1)}{2} + n + 1 \end{aligned}$$

$$2) \quad \sum_{k=0}^n (k+1)^3 = \sum_{k=1}^{n+1} k^3 = \sum_{k=1}^n k^3 + (n+1)^3$$

Equating the two expressions and canceling the sum with  $k^3$ , we obtain:

$$3 \sum_{k=1}^n k^2 + \frac{3n(n+1)}{2} + n + 1 = (n+1)^3$$

Isolating the sum with  $k^2$ , we obtain:

$$\begin{aligned}
3 \sum_{k=1}^n k^2 &= (n+1)^3 - \frac{3n(n+1)}{2} - n - 1 \\
&= n^3 + 3n^2 + 3n + 1 - \frac{3n^2 + 3n}{2} - n - 1 \\
&= \frac{2n^3 + 3n^2 + n}{2}
\end{aligned}$$

Hence:

$$\sum_{k=1}^n k^2 = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

### Exercise 3.2

Prove, using the properties of summations, that for every integer  $n \geq 1$  we have:

$$\sum_{k=1}^n k^3 = \frac{n^2 (n+1)^2}{4}$$

(sum of the cubes of the first  $n$  natural numbers, excluding zero). That is, we have:

$$\sum_{k=1}^n k^3 = \frac{n^2 (n+1)^2}{4} = \left( \frac{n(n+1)}{2} \right)^2 = \left( \sum_{k=1}^n k \right)^2$$

and hence the sum of the cubes is equal to the square of the sum of the natural numbers.

## Solution

We start by writing the following sum in two different ways:  $\sum_{k=0}^n (k+1)^4$ .

$$\begin{aligned}
 1) \quad \sum_{k=0}^n (k+1)^4 &= \sum_{k=0}^n (k^4 + 4k^3 + 6k^2 + 4k + 1) \\
 &= \left( \sum_{k=1}^n k^4 + 4 \sum_{k=1}^n k^3 + 6 \sum_{k=1}^n k^2 + 4 \sum_{k=1}^n k + \sum_{k=1}^n 1 \right) + 1 \\
 &= \sum_{k=1}^n k^4 + 4 \sum_{k=1}^n k^3 + n(n+1)(2n+1) + 2n(n+1) + n + 1 \\
 &= \sum_{k=1}^n k^4 + 4 \sum_{k=1}^n k^3 + 2n^3 + 5n^2 + 4n + 1 \\
 2) \quad \sum_{k=0}^n (k+1)^4 &= \sum_{k=1}^{n+1} k^4 = \sum_{k=1}^n k^4 + (n+1)^4
 \end{aligned}$$

Equating the two expressions and canceling the sum with  $k^4$ , we obtain:

$$4 \sum_{k=1}^n k^3 + 2n^3 + 5n^2 + 4n + 1 = (n+1)^4$$

Isolating the sum with  $k^3$ , we obtain:

$$4 \sum_{k=1}^n k^3 = n^4 + 4n^3 + 6n^2 + 4n + 1 - 2n^3 - 5n^2 - 4n - 1 = n^4 + 2n^3 + n^2$$

Hence:

$$\sum_{k=1}^n k^3 = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2 (n+1)^2}{4}$$

## Complex numbers

### Exercise 4.1

Write the following complex numbers  $z$  in algebraic form, identifying their real part and imaginary part. Then, for each of them, find the modulus, the conjugate and the reciprocal.

$$z = \frac{2+5i}{1+i} \qquad z = \frac{2+i}{1-3i}$$

### Solution

$$z = \frac{2+5i}{1+i} \cdot \frac{1-i}{1-i} = \frac{7+3i}{2} = \frac{7}{2} + i\frac{3}{2}$$

from which

$$\Re z = \frac{7}{2}, \Im z = \frac{3}{2}, |z| = \sqrt{\frac{49}{4} + \frac{9}{4}} = \frac{\sqrt{58}}{2}, \bar{z} = \frac{7}{2} - i\frac{3}{2}.$$

Finally

$$z^{-1} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{7-3i}{2} \cdot \frac{4}{58} = \frac{7}{29} - i\frac{3}{29}.$$

### Solution

$$z = \frac{2+i}{1-3i} \cdot \frac{1+3i}{1+3i} = \frac{-1+7i}{10} = -\frac{1}{10} + i\frac{7}{10}$$

from which

$$\Re z = -\frac{1}{10}, \Im z = \frac{7}{10}, |z| = \sqrt{\frac{1}{100} + \frac{49}{100}} = \frac{\sqrt{2}}{2}, \bar{z} = -\frac{1}{10} - i\frac{7}{10}.$$

Finally

$$z^{-1} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = -\frac{1+7i}{10} \cdot 2 = -\frac{1}{5} - i\frac{7}{5}.$$

**Exercise 4.2**

Determine all the complex numbers  $z$  that satisfy:

$$|z| \leq |z - 2i|$$

**Solution**

It is convenient to use the form  $z = x + iy$ ,  $x, y \in \mathbb{R}$ .

$$|x + iy| \leq |x + i(y - 2)|$$

$$|x + iy|^2 \leq |x + i(y - 2)|^2$$

$$x^2 + y^2 \leq x^2 + (y - 2)^2$$

$$0 \leq -4y + 4$$

$$y \leq 1.$$

Representing the solutions in the Cartesian plane, we obtain a closed half-plane.

**Exercise 4.3**

Determine all the complex numbers  $z$  that satisfy:

$$|z - (3 + i)| \leq 2$$

**Solution**

This inequality is satisfied by those complex numbers  $z = x + iy$  which, in the Gauss plane (complex plane), are represented by the points  $P(x, y)$  whose distance from the point  $(3, 1)$  is less than or equal to 2.

That is, all and only the points of the **closed disk** with center  $C(3, 1)$  and radius 2 (careful: it is not the circle, but the disk, i.e., the interior together with the boundary). The circle has equation:

$$\gamma: (x - 3)^2 + (y - 1)^2 = 4$$

while the closed disk has equation:

$$(x - 3)^2 + (y - 1)^2 \leq 4$$

Alternatively, the inequality can be handled through the substitution  $z = x + iy$ , obtaining

$$|x + iy - 3 - i| \leq 2$$

in which, factoring out the imaginary unit,

$$|x - 3 + i(y - 1)| \leq 2$$

Applying the definition of the modulus of a complex number, we have

$$\sqrt{(x-3)^2 + (y-1)^2} \leq 2$$

Since we can square both non-negative sides, we finally obtain:

$$(x-3)^2 + (y-1)^2 \leq 4$$

#### Exercise 4.4

Determine all the complex numbers  $z$  that satisfy:

$$(1-i)z - (1+i)\bar{z} = i$$

#### Solution

$$(1-i)(x+iy) - (1+i)(x-iy) = i$$

$$x+y-ix+iy-x-y-ix+iy = i$$

$$i(2y-2x) = i$$

$$2y-2x = 1.$$

Representing the solutions in the Cartesian plane, we obtain a line.

#### Exercise 4.5

Determine all the complex numbers  $z$  that satisfy:

$$|z + \bar{z}| + |z - \bar{z}| \leq 2$$

#### Solution

$$|z + \bar{z}| + |z - \bar{z}| \leq 2$$

$$2|\Re z| + 2|\Im z| \leq 2$$

$$|x| + |y| \leq 1.$$

Representing the solutions in the Cartesian plane, we obtain a closed square with vertices  $\pm 1, \pm i$ .

#### Exercise 4.6

Determine all the complex numbers  $z$  that satisfy:

$$\Im(z) - |z + \bar{z}|^2 < 1$$

#### Solution

$$\Im(z) - |z + \bar{z}|^2 < 1$$

$$\Im(z) - 4 |\Re z|^2 < 1$$

$$y < 4x^2 + 1.$$

Representing the solutions in the Cartesian plane, we obtain the open region below the parabola  $y = 4x^2 + 1$ .

#### Exercise 4.7

Determine all the complex numbers  $z$  that satisfy:

$$\begin{cases} z\bar{z} \leq 2 \\ z + \bar{z} \leq 2 \end{cases}$$

#### Solution

$$\begin{cases} |z|^2 \leq 2 \\ 2\Re z \leq 2 \end{cases} \quad \begin{cases} x^2 + y^2 \leq 2 \\ x \leq 1 \end{cases}$$

Representing the solutions in the Cartesian plane, we obtain the part of the plane common to the closed disk with center at the origin and radius  $\sqrt{2}$  and to the closed half-plane to the left of the vertical line  $x = 1$ .

#### Exercise 4.8

The system in  $\mathbb{C}$

$$\begin{cases} z^2 + \bar{z}^2 = 0 \\ |\Re(z)| + |\Im(z)| = 1 \end{cases}$$

has:

- (a) two solutions      (b) three solutions  
(c) four solutions      (d) none of the other answers is correct

**Solution**

Setting  $z = x + iy$ ,  $x, y \in \mathbb{R}$ , we have

$$\begin{cases} (x + iy)^2 + (x - iy)^2 = 0 \\ |x| + |y| = 1 \end{cases}$$

$$\begin{cases} x^2 - y^2 + 2ixy + x^2 - y^2 - 2ixy = 0 \\ |x| + |y| = 1 \end{cases}$$

$$\begin{cases} x^2 = y^2 \\ |x| + |y| = 1 \end{cases} \quad \begin{cases} |x| = |y| \\ |x| + |y| = 1 \end{cases} \quad \begin{cases} |x| = 1/2 \\ |y| = 1/2 \end{cases}$$

We obtain the four points  $(1 + i)/2$ ,  $(-1 + i)/2$ ,  $(1 - i)/2$ ,  $(-1 - i)/2$ . The correct answer is (c).

**Exercise 4.9**

The solutions  $z = x + iy$  of the complex equation

$$z^2 - z\bar{z} + iz = -3 + 2i$$

satisfy

$$\begin{aligned} \text{(a)} \quad \left(x + \frac{1}{6}\right)^2 + \left(y + \frac{1}{4}\right)^2 &= 1 & \text{(b)} \quad y &= \frac{2}{3}x + 1 \\ \text{(c)} \quad \left(x - \frac{1}{6}\right)^2 + \left(y - \frac{1}{4}\right)^2 &= 1 & \text{(d)} \quad y &= \frac{3}{2}x \end{aligned}$$

**Solution**

$$(x + iy)^2 - (x^2 + y^2) + ix - y = -3 + 2i$$

$$-2y^2 - y + i(x + 2xy) = -3 + 2i$$

$$\begin{cases} 2y^2 + y - 3 = 0 \\ x + 2xy = 2 \end{cases} \quad \begin{cases} y = 1 \vee y = -3/2 \\ x = 2/(1 + 2y) \end{cases}$$

We obtain the two points  $(2/3) + i$ ,  $-1 - i(3/2)$ , which lie on the line  $y = (3/2)x$ . The correct answer is (d).

**Exercise 4.10**

Find the modulus and the principal argument of

$$(a) z = -1 - i\sqrt{3}, \quad (b) z = -4i.$$

**Solution**

(a) We have  $|z| = \sqrt{1+3} = 2$ . The principal argument  $\text{Arg } z$  is the unique  $\vartheta \in [-\pi, \pi)$  such that

$$\cos \vartheta = \Re z / |z| = -1/2, \quad \sin \vartheta = \Im z / |z| = -\sqrt{3}/2$$

hence  $\text{Arg } z = -2\pi/3$ .

**Solution**

(b) We have  $|z| = 4$ . The principal argument  $\text{Arg } z$  is the unique  $\vartheta \in [-\pi, \pi)$  such that

$$\cos \vartheta = \Re z / |z| = 0, \quad \sin \vartheta = \Im z / |z| = -1$$

hence  $\text{Arg } z = -\pi/2$ .

**Exercise 4.11**

Let

$$w = \frac{z+1-i}{z+i}$$

Determine, locating them in the Gauss plane, the complex numbers  $z$  for which  $w$  is a purely imaginary number.

**Solution**

The complex number  $w$  is defined for  $z \neq -i$ . Moreover,

$$w = \frac{z+1-i}{z+i} \cdot \frac{z-i}{z-i} = \frac{z^2 + z - 1 + i(-2z-1)}{z^2 + 1}$$

That is, separating the real part of  $w$  from its imaginary part:

$$w = \frac{z^2 + z - 1}{z^2 + 1} + i \frac{-2z - 1}{z^2 + 1}$$

The complex number  $w$  is purely imaginary when  $\Re(w) = 0$ , i.e.,

$$\Re(w) = \frac{z^2 + z - 1}{z^2 + 1} = 0$$

This rational equation, under the condition  $z \neq \pm\sqrt{-1}$ , i.e.,  $z \neq \pm i$ , gives the real numbers

$$z_1 = -\frac{1}{2} + \frac{\sqrt{5}}{2}, \quad z_2 = -\frac{1}{2} - \frac{\sqrt{5}}{2}$$

which, in the Gauss plane, can be represented as two points on the real axis. In conclusion,  $w$  is purely imaginary for the two values of  $z$  found.

#### Exercise 4.12

Let  $z = \sqrt{3} - 3i$ . The number of solutions  $w$  with  $\Im(w) < 0$  of the equation

$$w^6 = |z|(z + \bar{z})$$

is:

- (a) two      (b) three  
(c) zero      (d) none of the other answers is correct

#### Solution

We have  $|z| = \sqrt{12}$ ,  $z + \bar{z} = 2\sqrt{3}$ , hence

$$w^6 = 12.$$

The complex sixth roots of a positive real number are six, pairwise opposite, and two of them are real. Of the remaining four, two have positive imaginary part and two have negative imaginary part. The correct answer is (a). We nevertheless want to determine their values. Setting  $w = r(\cos \vartheta + i \sin \vartheta)$  and taking into account that  $|12| = 12$ ,  $\arg 12 = 2k\pi$ ,  $k \in \mathbb{Z}$ , we obtain

$$\begin{cases} r^6 = 12 \\ 6\vartheta = 2k\pi \end{cases} \quad \begin{cases} r = 12^{1/6} \\ \vartheta = k\pi/3 \end{cases}$$

We therefore have the six roots

$$w_k = 12^{1/6}(\cos k\pi/3 + i \sin k\pi/3), \quad k = 0, 1, 2, 3, 4, 5,$$

i.e.,

$$w_0 = 12^{1/6}, \quad w_1 = 12^{1/6}(1/2 + i\sqrt{3}/2), \quad w_2 = 12^{1/6}(-1/2 + i\sqrt{3}/2)$$

$$w_3 = -12^{1/6}, \quad w_4 = 12^{1/6}(-1/2 - i\sqrt{3}/2), \quad w_5 = 12^{1/6}(1/2 - i\sqrt{3}/2).$$

The two with negative imaginary part are  $w_4$  and  $w_5$ .

#### Exercise 4.13

Solve the following system in the unknown  $z \in \mathbb{C}$ .

$$\begin{cases} z^6 = i \\ \Re(z)\Im(z) < 0 \end{cases}$$

**Solution**

We first solve the equation  $z^6 = i$ . Setting  $z = r(\cos \vartheta + i \sin \vartheta)$  and taking into account that  $|i| = 1$ ,  $\arg(i) = \pi/2 + 2k\pi$ ,  $k \in \mathbf{Z}$ , we obtain

$$\begin{cases} r^6 = 1 \\ 6\vartheta = \pi/2 + 2k\pi \end{cases} \quad \begin{cases} r = 1 \\ \vartheta = \pi/12 + k\pi/3. \end{cases}$$

We therefore have the six roots

$$z_k = \cos((\pi + 4k\pi)/12) + i \sin((\pi + 4k\pi)/12), \quad k = 0, 1, 2, 3, 4, 5.$$

Those satisfying the condition  $\Re(z)\Im(z) < 0$  (points in the second or fourth quadrant) are:

$$z_2 = -\sqrt{1/2} + i\sqrt{1/2}, \quad z_5 = \sqrt{1/2} - i\sqrt{1/2}.$$

**Exercise 4.14**

Solve the following equation in the complex variable  $z$ .

$$z^3 + 8 = 0$$

**Solution**

The equation to be solved is, clearly,

$$z^3 = -8$$

Setting  $z = r(\cos \vartheta + i \sin \vartheta)$  and taking into account that  $|-8| = 8$ ,  $\arg(-8) = \pi + 2k\pi$ ,  $k \in \mathbf{Z}$ , we obtain

$$\begin{cases} r^3 = 8 \\ 3\vartheta = \pi + 2k\pi \end{cases} \quad \begin{cases} r = 2 \\ \vartheta = (2k + 1)\pi/3. \end{cases}$$

We therefore have the three roots

$$z_k = 2 \cos((2k + 1)\pi/3) + 2i \sin((2k + 1)\pi/3), \quad k = 0, 1, 2,$$

i.e.,

$$z_0 = 1 + \sqrt{3}i, \quad z_1 = -2, \quad z_2 = 1 - \sqrt{3}i.$$

**Exercise 4.15**

Solve the following equation in the complex variable  $z$ .

$$z^3 = -27i$$

**Solution**

Setting  $z = r(\cos \vartheta + i \sin \vartheta)$  and taking into account that  $|-27i| = 27$ ,  $\arg(-27i) = -\pi/2 + 2k\pi$ ,  $k \in \mathbb{Z}$ , we obtain

$$\begin{cases} r^3 = 27 \\ 3\vartheta = -\pi/2 + 2k\pi \end{cases} \quad \begin{cases} r = 3 \\ \vartheta = (4k - 1)\pi/6. \end{cases}$$

We therefore have the three roots

$$z_k = 3 \cos((4k - 1)\pi/6) + 3i \sin((4k - 1)\pi/6), \quad k = 0, 1, 2,$$

i.e.,

$$z_0 = 3\sqrt{3}/2 - 3i/2, \quad z_1 = 3i, \quad z_2 = -3\sqrt{3}/2 - 3i/2.$$

**Exercise 4.16**

Solve the following equation in the complex variable  $z$ .

$$z^5 + (3 + i\sqrt{3})z = 0$$

**Solution**

We have

$$z(z^4 + 3 + i\sqrt{3}) = 0$$

hence one solution is  $z = 0$ . The other four are found by solving

$$z^4 = -3 - i\sqrt{3}.$$

Setting  $z = r(\cos \vartheta + i \sin \vartheta)$  and taking into account that

$$|-3 - i\sqrt{3}| = 2\sqrt{3}, \quad \arg(-3 - i\sqrt{3}) = -5\pi/6 + 2k\pi, \quad k \in \mathbb{Z}$$

we obtain

$$\begin{cases} r^4 = 12^{1/2} \\ 4\vartheta = -5\pi/6 + 2k\pi \end{cases} \quad \begin{cases} r = 12^{1/8} \\ \vartheta = -5\pi/24 + k\pi/2. \end{cases}$$

Besides the solution  $z = 0$ , we therefore have four more roots

$$z_k = 12^{1/8} \cos((12k - 5)\pi/24) + 12^{1/8}i \sin((12k - 5)\pi/24), \quad k = 0, 1, 2, 3.$$

**Exercise 4.17**

Solve the following equation in the complex variable  $z$ .

$$\left(\frac{z+i}{1+i}\right)^3 = -8i$$

### Solution

Setting  $w = (z+i)/(1+i)$ , we solve  $w^3 = -8i$ . Writing  $w = r(\cos \vartheta + i \sin \vartheta)$  and taking into account that  $|-8i| = 8$ ,  $\arg(-8i) = -\pi/2 + 2k\pi$ ,  $k \in \mathbb{Z}$ , we obtain

$$\begin{cases} r^3 = 8 \\ 3\vartheta = -\pi/2 + 2k\pi \end{cases} \quad \begin{cases} r = 2 \\ \vartheta = (4k-1)\pi/6. \end{cases}$$

We therefore have the three roots

$$w_k = 2 \cos((4k-1)\pi/6) + 2i \sin((4k-1)\pi/6), \quad k = 0, 1, 2,$$

i.e.,

$$w_0 = \sqrt{3} - i, \quad w_1 = 2i, \quad w_2 = -\sqrt{3} - i.$$

Going back to the variable  $z$ , from  $z = (1+i)w - i$ , the three solutions of the given equation are

$$z_0 = \sqrt{3} + 1 + i\sqrt{3} - 2i, \quad z_1 = -2 + i, \quad z_2 = 1 - \sqrt{3} - i\sqrt{3} - 2i.$$

### Exercise 4.18

Solve the following equation in the complex variable  $z$ .

$$\left(\frac{z-i}{i}\right)^4 = -16$$

### Solution

From  $i^4 = 1$  we have  $(z-i)^4 = -16$ . Setting  $w = z-i$ , we solve  $w^4 = -16$ . Writing  $w = r(\cos \vartheta + i \sin \vartheta)$  and taking into account that  $|-16| = 16$ ,  $\arg(-16) = -\pi + 2k\pi$ ,  $k \in \mathbb{Z}$ , we obtain

$$\begin{cases} r^4 = 16 \\ 4\vartheta = -\pi + 2k\pi \end{cases} \quad \begin{cases} r = 2 \\ \vartheta = -\pi/4 + k\pi/2. \end{cases}$$

We therefore have the four roots

$$w_k = 2 \cos((2k-1)\pi/4) + 2i \sin((2k-1)\pi/4), \quad k = 0, 1, 2, 3,$$

i.e.,

$$w_0 = \sqrt{2} - i\sqrt{2}, \quad w_1 = \sqrt{2} + i\sqrt{2}, \quad w_2 = -\sqrt{2} + i\sqrt{2}, \quad w_3 = -\sqrt{2} - i\sqrt{2}.$$

Going back to the variable  $z$ , from  $z = w + i$ , the four solutions of the given equation are

$$z_0 = \sqrt{2} - i\sqrt{2} + i, \quad z_1 = \sqrt{2} + i\sqrt{2} + i, \quad z_2 = -\sqrt{2} + i\sqrt{2} + i, \quad z_3 = -\sqrt{2} - i\sqrt{2} + i.$$

**Exercise 4.19**

Solve the following equation in the complex variable  $z$ .

$$z^5 = -\bar{z}$$

**Solution**

The equation can be seen as

$$z^5 = (-1)\bar{z}$$

Setting  $z = r(\cos \vartheta + i \sin \vartheta)$  and taking into account that  $|\bar{z}| = r$ ,  $\arg(\bar{z}) = -\vartheta + 2k\pi$ ,  $k \in \mathbb{Z}$ , and recalling that  $-1 = e^{i\pi}$ , we obtain

$$\begin{cases} r^5 = r \\ 5\vartheta = \pi - \vartheta + 2k\pi \end{cases} \quad \begin{cases} r = 0, r = 1 \\ \vartheta = (2k+1)\pi/6. \end{cases}$$

At this point,

- for  $r = 0$ , we obtain  $z_0 = 0$ ;
- for  $r = 1$ , we obtain the roots  $z = \cos((2k+1)\pi/6) + i \sin((2k+1)\pi/6)$ ,  $k = 0, 1, 2, 3, 4, 5$ .

**Exercise 4.20**

The equation

$$z^4 - 2z^3 + 4z^2 - 2z + 3 = 0$$

has  $z_1 = i$  as a solution. Denoting by  $z_2, z_3, z_4$  the other solutions, we have:

- (a)  $\Im(z_1 z_2 z_3 z_4) = 0$       (b)  $\Re(z_1 + z_2 + z_3 + z_4) = 0$   
 (c)  $\Re((z_1 + z_2 + z_3 + z_4)^{-1}) = 1/2$       (d) none of the other answers is correct

**Solution**

The equation has real coefficients, hence the solutions come in conjugate pairs; in particular, another solution is  $z_2 = -i$ . It follows that the polynomial

$$p(z) = z^4 - 2z^3 + 4z^2 - 2z + 3$$

is divisible by  $(z - i)(z + i)$ , i.e., by  $z^2 + 1$ . Performing the division we obtain

$$p(z) = (z^2 + 1)(z^2 - 2z + 3)$$

hence among the solutions we also have the roots of

$$z^2 - 2z + 3 = 0$$

which are  $z_3 = 1 - i\sqrt{2}$ ,  $z_4 = 1 + i\sqrt{2}$ .

Summarizing, the solutions of the given equation are

$$z_1 = i, \quad z_2 = -i, \quad z_3 = 1 - i\sqrt{2}, \quad z_4 = 1 + i\sqrt{2}.$$

Since  $z_1 + z_2 + z_3 + z_4 = 2$  and  $z_1 z_2 z_3 z_4 = 3$ , the correct answers are (a) and (c).

#### Exercise 4.21

Let  $z_h, h = 1, \dots, 6$  be the solutions of

$$z^6 - (8 - i)z^3 - 8i = 0.$$

We have:

$$(a) \sum_{h=1}^6 \Re(z_h) = -1 \quad (b) \sum_{h=1}^6 \Im(z_h) = -1$$

$$(c) \sum_{h=1}^6 \Re(z_h) = 1 \quad (d) \sum_{h=1}^6 \Im(z_h) = 0$$

$$(e) \sum_{h=1}^6 \Re(z_h) = 2 \quad (f) \text{ none of the other answers is correct}$$

#### Solution

It is convenient to factor it as

$$(z^3 - 8)(z^3 + i) = 0$$

and then solve the two equations

$$z^3 = 8, \quad z^3 = -i$$

with the usual method illustrated several times in the solutions of the previous exercises. We obtain the values:

$$z_1 = 2, \quad z_2 = -1 + i\sqrt{3}, \quad z_3 = -1 - i\sqrt{3},$$

$$z_4 = \sqrt{3}/2 - i/2, \quad z_5 = i, \quad z_6 = -\sqrt{3}/2 - i/2.$$

The correct answer is (d).