



SAPIENZA
UNIVERSITÀ DI ROMA

Mathematical Analysis 1

Exercises, volume 1 — Logic and proof methods

The worked exercises of the first part of the course, with the solution right after each statement: proofs by mathematical induction. The sheets are by Fabio Furini and Gianluca Priori; many exercises are taken from M. Bramanti, *Esercitazioni di Analisi matematica 1*, Esculapio.

Fabio Furini

fabiofurini.github.io

Department of Computer, Control and Management Engineering — Sapienza University of Rome

Online version and downloadable material:
fabiofurini.github.io/mathematical-analysis-1/
github.com/fabiofurini/mathematical-analysis-1

Academic year 2026–2027

Contents

1	Principle of mathematical induction	2
---	-------------------------------------	---



Principle of mathematical induction

Exercise 1.1

Prove by induction that for every integer $n \geq 1$ the following holds:

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

(sum of the squares of the first n natural numbers)

Solution

By induction on n .

- **Base case**

Let $n = 1$. Then the statement becomes:

$$\sum_{k=1}^1 k^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6} \quad \text{i.e.} \quad 1 = 1$$

which is clearly true.

- **Inductive step**

Assume that it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have:

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Hence we can write:

$$\begin{aligned}
 \sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^n k^2 + (n+1)^2 \\
 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
 &= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} = \frac{(n+1)(2n^2 + n + 6n + 6)}{6} \\
 &= \frac{(n+1)(2n^2 + 4n + 6)}{6} = \frac{(n+1)((2n)(n+2) + 3(n+2))}{6} \\
 &= \frac{(n+1)(n+2)(2n+3)}{6} = \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}
 \end{aligned}$$

which is exactly the desired statement for $n+1$.

Exercise 1.2

Prove by induction that for every integer $n \geq 1$ the following holds:

$$\sum_{k=0}^{n-1} (2k+1) = n^2 \quad \text{or} \quad \sum_{k=1}^n (2k-1) = n^2$$

(sum of the first n odd numbers)

Solution

By induction on n .

- **Base case**

Let $n = 1$. Then the statement becomes:

$$\sum_{k=0}^0 (2k+1) = 1^2 \quad \text{i.e.} \quad 1 = 1$$

which is clearly true.

- **Inductive step**

Assume that it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have:

$$\sum_{k=0}^{n-1} (2k+1) = n^2$$

Hence we can write:

$$\begin{aligned}\sum_{k=0}^{n-1+1} (2k+1) &= \sum_{k=0}^{n-1} (2k+1) + 2n+1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2\end{aligned}$$

which is exactly the desired statement for $n+1$.

Exercise 1.3

Prove by induction that for every integer $n \geq 1$ the following holds:

$$\sum_{k=1}^n 2k = n(n+1)$$

(sum of the first n even numbers)

Solution

By induction on n .

- **Base case**

Let $n = 1$. Then the statement becomes:

$$\sum_{k=1}^1 2k = 1 \cdot 2 \quad \text{i.e.} \quad 2 = 2$$

which is clearly true.

- **Inductive step**

Assume that it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have:

$$\sum_{k=1}^n 2k = n(n+1)$$

Hence we can write:

$$\begin{aligned}
 \sum_{k=1}^{n+1} 2k &= \sum_{k=1}^n 2k + 2(n+1) \\
 &= 2 \left(\sum_{k=1}^n k + (n+1) \right) \\
 &= 2 \left(\frac{n(n+1)}{2} + (n+1) \right) \\
 &= 2 \left(\frac{n(n+1) + 2(n+1)}{2} \right) \\
 &= (n+1)(n+2) = (n+1)(n+1+1)
 \end{aligned}$$

which is exactly the desired statement for $n+1$.

Exercise 1.4

Prove by induction that for every integer $n \geq 1$ the following holds:

$$\sum_{k=1}^n \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

Solution

By induction on n .

- **Base case**

Let $n = 1$. Then the statement becomes:

$$\sum_{k=1}^1 \frac{k}{2^k} = 2 - \frac{1+2}{2^1} \quad \text{i.e.} \quad \frac{1}{2} = \frac{1}{2}$$

which is clearly true.

- **Inductive step**

Assume that it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have:

$$\sum_{k=1}^n \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

Hence we can write:

$$\begin{aligned}
 \sum_{k=1}^{n+1} \frac{k}{2^k} &= \sum_{k=1}^n \frac{k}{2^k} + \frac{n+1}{2^{n+1}} \\
 &= 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}} \\
 &= 2 - \frac{2(n+2)}{2^{n+1}} + \frac{n+1}{2^{n+1}} \\
 &= 2 - \left(\frac{2(n+2) - n - 1}{2^{n+1}} \right) \\
 &= 2 - \left(\frac{n+3}{2^{n+1}} \right) \\
 &= 2 - \frac{(n+1)+2}{2^{n+1}}
 \end{aligned}$$

which is exactly the desired statement for $n+1$.

Exercise 1.5

Given a set X with n elements, prove by induction that the power set $\mathcal{P}(X)$ has 2^n elements, that is,

$$|\mathcal{P}(X)| = 2^n \quad \text{for } n \geq 0$$

Solution

By induction on n .

- **Base case**

Let $n = 0$. A set with 0 elements is the empty set $(\emptyset)$. The power set of the empty set contains only the empty set as an element. Then the statement becomes:

$$|\mathcal{P}(\emptyset)| = 2^0 \quad \text{i.e.} \quad 1 = 1$$

which is clearly true.

- **Inductive step**

Assume the statement is true for a set of n elements, and let us prove it for a set of $(n+1)$ elements. By the inductive hypothesis, the power set of a set of n elements has 2^n elements.

A set Y of $(n+1)$ elements can be written as:

$$Y = \{a_1, a_2, \dots, a_n, a_{n+1}\}$$

hence:

$$\mathcal{P}(Y) = \{S : S \subseteq Y\} = \{S : S \subseteq \{a_1, a_2, \dots, a_n, a_{n+1}\}\}$$

The subsets of Y either contain the element a_{n+1} or do not contain it. We therefore define the set Z_a formed by the first n elements of Y :

$$Z_a = \{a_1, a_2, \dots, a_n\} \quad \text{hence} \quad |\mathcal{P}(Z_a)| = 2^n$$

We now define the following set Z_b :

$$Z_b = \{S \cup \{a_{n+1}\} : S \subseteq \underbrace{\{a_1, a_2, \dots, a_n\}}_{=Z_a}\} = \{S \cup \{a_{n+1}\} : S \subseteq Z_a\} \quad \text{and} \quad |Z_b| = 2^n$$

since Z_b has one element for each element of $\mathcal{P}(Z_a)$.

In other words, $\mathcal{P}(Z_a)$ contains all the subsets of Y in which the element a_{n+1} is not present, while Z_b contains all the subsets of Y in which the element a_{n+1} is present.

We also note that the sets $\mathcal{P}(Z_a)$ and Z_b are disjoint, that is: $\mathcal{P}(Z_a) \cap Z_b = \emptyset$. Hence $|\mathcal{P}(Z_a) \cup Z_b| = |\mathcal{P}(Z_a)| + |Z_b|$. Finally, we have:

$$|\mathcal{P}(Y)| = |\mathcal{P}(Z_a) \cup Z_b| = 2^n + 2^n = 2^{n+1}$$

which is exactly the desired statement for $n + 1$.

Exercise 1.6

Prove by induction that for every integer $n \geq 1$ the following holds:

$$\sum_{k=1}^n k^3 = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n+1)^2}{4}$$

(sum of the cubes of the first n natural numbers). Then compute the sum of the cubes of the first 10, 100 and 1000 natural numbers (without zero) and prove that the sum of the cubes of the first n natural numbers equals the square of the sum of the first n natural numbers:

$$\sum_{k=1}^n k^3 = \left(\sum_{k=1}^n k \right)^2$$

Solution

By induction on n .

- **Base case**

Let $n = 1$. Then the statement becomes:

$$\sum_{k=1}^1 k^3 = \frac{1 + 2 + 1}{4} \quad \text{i.e.} \quad 1 = 1$$

which is clearly true.

• **Passo induttivo**

Suppose it is true for n , and let us prove it for $(n + 1)$. By the inductive hypothesis, we have:

$$\sum_{k=1}^n k^3 = \frac{n^4 + 2n^3 + n^2}{4}.$$

Hence we can write:

$$\begin{aligned} \sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^n k^3 + (n+1)^3 = \frac{n^4 + 2n^3 + n^2}{4} + (n+1)^3 \\ &= \frac{n^4 + 2n^3 + n^2 + 4(n+1)^3}{4} = \frac{n^4 + 2n^3 + n^2 + 4n^3 + 12n^2 + 12n + 4}{4} \\ &= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4} \\ &= \frac{(n^4 + 4n^3 + 6n^2 + 4n + 1) + 2(n^3 + 3n^2 + 3n + 1) + (n^2 + 2n + 1)}{4} \\ &= \frac{(n+1)^4 + 2(n+1)^3 + (n+1)^2}{4} \end{aligned}$$

which is exactly the desired statement for $n + 1$.

The sum of the cubes of the first 10, 100 and 1000 natural numbers (without zero) is:

$$\begin{aligned} \sum_{k=1}^{10} k^3 &= \frac{10^4 + 2 \cdot 10^3 + 10^2}{4} = 3025, & \sum_{k=1}^{100} k^3 &= \frac{100^4 + 2 \cdot 100^3 + 100^2}{4} = 25,502,500 \\ \sum_{k=1}^{1000} k^3 &= \frac{1000^4 + 2 \cdot 1000^3 + 1000^2}{4} = 250,500,250,000 \end{aligned}$$

Finally, the sum of the cubes of the first n natural numbers equals the square of the sum of the first n natural numbers, since:

$$\sum_{k=1}^n k^3 = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n^2 + 2n + 1)}{4} = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^n k\right)^2$$

Exercise 1.7

Prove by induction that for every integer $n \geq 1$ the following holds:

$$\sum_{k=1}^n \frac{1}{k(k+1)} = \frac{n}{n+1} = 1 - \frac{1}{n+1}$$

Then compute the value of the sum for $n = 10$, $n = 100$ and $n = 1000$.

Solution

By induction on n .

- **Base case**

Let $n = 1$. Then the statement becomes:

$$\sum_{k=1}^1 \frac{1}{k(k+1)} = \frac{1}{1+1} \quad \text{i.e.} \quad \frac{1}{2} = \frac{1}{2}$$

which is clearly true.

- **Passo induttivo**

Suppose it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have:

$$\sum_{k=1}^n \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

Hence we can write:

$$\begin{aligned} \sum_{k=1}^{n+1} \frac{1}{k(k+1)} &= \sum_{k=1}^n \frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{(n+1)+1} \end{aligned}$$

which is exactly the desired statement for $n+1$.

The value of the sum for $n = 10$, $n = 100$ and $n = 1000$ is:

$$\sum_{k=1}^{10} \frac{1}{k(k+1)} = \frac{10}{11}, \quad \sum_{k=1}^{100} \frac{1}{k(k+1)} = \frac{100}{101}, \quad \sum_{k=1}^{1000} \frac{1}{k(k+1)} = \frac{1000}{1001}$$

Exercise 1.8

Prove by induction that for every integer $n \geq 1$ the following holds:

$$\sum_{k=1}^n k(k+1) = \frac{2n+3}{3} \frac{n^2+n^3}{3} = \frac{n(n+1)(n+2)}{3}$$

Then compute the value of the sum for $n = 10$, $n = 100$ and $n = 1000$.

Solution

By induction on n .

- **Base case**

Let $n = 1$. Then the statement becomes:

$$\sum_{k=1}^1 k(k+1) = \frac{2+3+1}{3} \quad \text{i.e.} \quad 2 = 2$$

which is clearly true.

- **Passo induttivo**

Suppose it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have:

$$\sum_{k=1}^n k(k+1) = \frac{2n + 3n^2 + n^3}{3}.$$

Hence we can write:

$$\begin{aligned} & \sum_{k=1}^{n+1} k(k+1) \\ &= \sum_{k=1}^n k(k+1) + (n+1)((n+1)+1) = \frac{2n + 3n^2 + n^3}{3} + (n+1)(n+2) \\ &= \frac{2n + 3n^2 + n^3 + 3(n+1)(n+2)}{3} = \frac{n^3 + 6n^2 + 11n + 6}{3} \\ &= \frac{2(n+1) + 3(n^2 + 2n + 1) + (n^3 + 3n^2 + 3n + 1)}{3} \\ &= \frac{2(n+1) + 3(n+1)^2 + (n+1)^3}{3} \end{aligned}$$

which is exactly the desired statement for $n+1$.

The value of the sum for $n = 10$, $n = 100$ and $n = 1000$ is:

$$\sum_{k=1}^{10} k(k+1) = \frac{10 \cdot 11 \cdot 12}{3} = 440, \quad \sum_{k=1}^{100} k(k+1) = \frac{100 \cdot 101 \cdot 102}{3} = 343,400$$

$$\sum_{k=1}^{1000} k(k+1) = \frac{1000 \cdot 1001 \cdot 1002}{3} = 334,334,000$$

Exercise 1.9

Prove by induction that for every integer $n \geq 1$ the following holds:

$$n! \geq 2^{n-1}$$

Solution

By induction on n .

- **Base case**

Let $n = 1$. Then the statement becomes:

$$1! \geq 2^{1-1} \quad \text{i.e.} \quad 1 \geq 1$$

which is clearly true.

- **Passo induttivo**

Suppose it is true for n , and let us prove it for $(n + 1)$. By the inductive hypothesis, we have $n! \geq 2^{n-1}$. Moreover, for every $n \geq 1$:

$$(n + 1) 2^{n-1} \geq 2^n \iff n + 1 \geq \frac{2^n}{2^{n-1}} = 2 \iff n \geq 1$$

Hence we can write:

$$(n + 1)! = (n + 1) n! \geq (n + 1) 2^{n-1} \geq 2^n = 2^{(n+1)-1}$$

which is exactly the desired statement for $n + 1$.

Exercise 1.10

Prove by induction that for every integer $n \geq 1$ the following holds:

$$n^n \geq 2^{n-1} n!$$

Solution

By induction on n .

- **Base case**

Let $n = 1$. Then the statement becomes:

$$1^1 \geq 2^{1-1} \cdot 1! \quad \text{i.e.} \quad 1 \geq 1$$

which is clearly true.

- **Passo induttivo**

Suppose it is true for n , and let us prove it for $(n + 1)$. By the inductive hypothesis, we have $n^n \geq 2^{n-1} n!$. Moreover, for every $n \geq 1$:

$$(n + 1)^n \geq 2 n^n \iff \left(\frac{n + 1}{n}\right)^n \geq 2 \iff \left(1 + \frac{1}{n}\right)^n \geq 2$$

and the last inequality holds by Bernoulli's inequality, $(1 + x)^m \geq 1 + m x$ for every integer $m \geq 0$ and $x \geq -1$: with $x = \frac{1}{n}$ and $m = n$ we get

$$\left(1 + \frac{1}{n}\right)^n \geq 1 + n \frac{1}{n} = 2$$

Hence we can write:

$$\begin{aligned} (n + 1)^{n+1} &= (n + 1) (n + 1)^n \geq (n + 1) 2 n^n \geq (n + 1) 2 \cdot 2^{n-1} n! \\ &= 2^n (n + 1)! = 2^{(n+1)-1} (n + 1)! \end{aligned}$$

which is exactly the desired statement for $n + 1$.

Exercise 1.11

Prove by induction that for every integer $n \geq 1$ the following holds:

$$n! \leq n^n$$

Solution

By induction on n .

- **Base case**

Let $n = 1$. Then the statement becomes:

$$1! \leq 1^1 \quad \text{i.e.} \quad 1 \leq 1$$

which is clearly true.

- **Passo induttivo**

Suppose it is true for n , and let us prove it for $(n + 1)$. By the inductive hypothesis, we have $n! \leq n^n$. Moreover, for every $n \geq 1$:

$$(n + 1)^n \geq n^n \iff \left(\frac{n + 1}{n}\right)^n \geq 1 \iff \left(1 + \frac{1}{n}\right)^n \geq 1$$

and the last inequality is clearly true, since $1 + \frac{1}{n} > 1$. Hence we can write:

$$(n + 1)! = (n + 1) n! \leq (n + 1) n^n \leq (n + 1) (n + 1)^n = (n + 1)^{n+1}$$

which is exactly the desired statement for $n + 1$.

Exercise 1.12

Prove by induction that for every integer $n \geq 0$ the natural number $10^n - 1$ is divisible by 9, that is:

$$\forall n \in \mathbb{N}, \exists m \in \mathbb{N} : 10^n - 1 = 9m$$

Solution

By induction on n .

- **Base case**

Let $n = 0$. Then:

$$10^0 - 1 = 0 = 9 \cdot 0$$

and, since 0 is a natural number, the statement is true.

- **Passo induttivo**

Suppose it is true for n , and let us prove it for $(n + 1)$. By the inductive hypothesis, there exists $m \in \mathbb{N}$ such that $10^n - 1 = 9m$. Hence we can write:

$$\begin{aligned} 10^{n+1} - 1 &= 10 \cdot 10^n - 1 = 10 \cdot 10^n - 10 + 9 = 10(10^n - 1) + 9 \\ &= 10(9m) + 9 = 9(10m + 1) \end{aligned}$$

and, since $10m + 1$ is a natural number, the statement is true for $n + 1$.

Exercise 1.13

Prove by induction that for every integer $n \geq 1$ the natural number $n(n + 1)$ is divisible by 2, that is:

$$\forall n \in \mathbb{N}, n \geq 1, \exists m \in \mathbb{N} : n(n + 1) = 2m$$

Solution

By induction on n .

- **Base case**

Let $n = 1$. Then:

$$1(1 + 1) = 2 = 2 \cdot 1$$

and, since 1 is a natural number, the statement is true.

- **Passo induttivo**

Suppose it is true for n , and let us prove it for $(n + 1)$. By the inductive hypothesis,

there exists $m \in \mathbb{N}$ such that $n(n+1) = 2m$. Hence we can write:

$$\begin{aligned}(n+1)((n+1)+1) &= (n+1)(n+2) = n^2 + n + 2n + 2 = n(n+1) + 2n + 2 \\ &= 2m + 2n + 2 = 2(m+n+1)\end{aligned}$$

and, since $m+n+1$ is a natural number, the statement is true for $n+1$.

Exercise 1.14

Prove by induction that for every integer $n \geq 2$ the natural number $n^3 - n$ is divisible by 6, that is:

$$\forall n \in \mathbb{N}, n \geq 2, \exists m \in \mathbb{N} : n^3 - n = 6m$$

(hint: use the previous exercise).

Solution

By induction on n .

- **Base case**

Let $n = 2$. Then:

$$2^3 - 2 = 6 = 6 \cdot 1$$

and, since 1 is a natural number, the statement is true.

- **Passo induttivo**

Suppose it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, there exists $m \in \mathbb{N}$ such that $n^3 - n = 6m$. Moreover, by the previous exercise, $n(n+1)$ is even: there exists $h \in \mathbb{N}$ such that $n(n+1) = 2h$. Hence we can write:

$$\begin{aligned}(n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 = (n^3 - n) + (3n^2 + 3n) \\ &= 6m + 3n(n+1) = 6m + 3 \cdot 2h = 6(m+h)\end{aligned}$$

and, since $m+h$ is a natural number, the statement is true for $n+1$.

Exercise 1.15

Prove by induction that for every integer $n \geq 0$ and every $x \in \mathbb{R}$, $x \geq -\frac{2}{3}$, the following holds:

$$(1+x)e^n \geq \frac{1}{n+3}$$

Solution

By induction on n .

- **Base case**

Let $n = 0$. Then the statement becomes:

$$(1+x)e^0 \geq \frac{1}{0+3} \quad \text{i.e.} \quad x \geq -\frac{2}{3}$$

which is true by hypothesis.

- **Passo induttivo**

Suppose it is true for n , and let us prove it for $(n+1)$. By the inductive hypothesis, we have $(1+x)e^n \geq \frac{1}{n+3}$. Hence, since $e > 1$, we can write:

$$(1+x)e^{n+1} = e((1+x)e^n) \geq \frac{e}{n+3} \geq \frac{1}{n+3} \geq \frac{1}{(n+1)+3}$$

which is exactly the desired statement for $n+1$.