Arbitrage and arbitrage-free pricing

Sapienza Università di Roma

Outline

Managerial motivation

Guided construction of the model

Duality is pricing: the risk-neutral probabilities

Worked example by hand

Case study

Results

Sensitivity analysis: the interval of consistent prices

Exercises

Learning objectives (1/2)

At the end of this chapter you will be able to:

- formulate the search for an arbitrage as an LP;
- read the risk-neutral probabilities in the duals of the constraints;

Learning objectives (2/2)

At the end of this chapter you will be able to:

- compute the interval of consistent prices of a security;
- price a new security (an option) in a complete or incomplete market.

Prerequisites

- From this module, Chapter 2 «Background: linear programming and sensitivity analysis»: duality, complementary slackness and strong duality
- From this module, Chapter 3 «The solver: building, solving, interpreting»: free variables in `gurobipy` and the reading of `Pi`

Managerial motivation

Introduction: Managerial motivation

- In this section we explain what an **arbitrage** is
- We see why the absence of arbitrage is the working assumption of pricing finance
- We anticipate the surprise of the chapter: all this is **linear programming**

What is an arbitrage

- An **arbitrage** is a trading strategy that creates money out of nothing:
  - it collects cash today with no risk of a loss tomorrow, or
  - it costs nothing today and can only make money tomorrow
- In a well-functioning market such opportunities cannot last: anyone would exploit them on an unlimited scale, and prices would adjust immediately

The absence-of-arbitrage assumption

"Absence of arbitrage" is the working assumption of the whole of pricing finance:

- of the trading desks that look for inconsistencies among quoted instruments
- of the risk managers who validate the prices in a price list
- of whoever has to price a new security consistently with the existing ones

The didactic surprise

- All this is **linear programming**: detecting an arbitrage means solving an LP
- Duality (shadow prices, complementary slackness, strong duality) becomes the **theory of pricing**
- It is the cleanest application of the primal-dual pair in the whole set of lecture notes

Summary

- An arbitrage creates money out of nothing, with no risk
- In a well-functioning market it does not last: absence of arbitrage is the basic assumption of pricing
- Detecting an arbitrage is an LP; its duality is the theory of prices

Guided construction of the model

Introduction: Guided construction of the model

- We describe the two-instant market and the states of the world
- We introduce data and variables and we write the **LP model for the search of an arbitrage**
- We learn to read the outcome of the LP: **type A** arbitrage, **type B** arbitrage, or consistent prices

The problem in words (1/2)

- The market has two instants:
  - **today** ($t = 0$): we buy or sell securities at the current prices
  - **tomorrow** ($t = 1$): one out of m *states of the world* occurs and each security pays a known amount that depends on the state
- *We decide* the quantities of each security in the portfolio: positive if bought, negative if sold short

The problem in words (2/2)

- *The objective* is to minimize the **cost of the portfolio today**
- *The constraints* impose that tomorrow, in *every* state of the world, the portfolio does not lose
- If under these rules one manages to spend less than zero (that is, to collect cash), that is an **arbitrage**

Data (input of the model) (1/2)

Symbol	Type	Meaning
n	$\in \mathbb{Z}_{\geq 1}$	number of risky securities; the securities are numbered $0, 1, \ldots, n$ and security 0 is risk-free
m	$\in \mathbb{Z}_{\geq 1}$	number of states of the world at $t = 1$
r	$\in \mathbb{Q}_{\geq 0}$	risk-free rate; $R = 1 + r$ is the gross return of security 0, the same in every state

Data (input of the model) (2/2)

Symbol	Type	Meaning
s_i^0	$\in \mathbb{Q}_{>0}$	price of security i today ($s_0^0 = 1$: security 0 costs 1 and pays R)
s_{ij}^1	$\in \mathbb{Q}_{\geq 0}$	payoff of security i in state j ($s_{0j}^1 = R$ for every state j)

Decision variables

We introduce one free variable for each of the $n+1$ securities:

$$x_i = \text{position in security } i, \qquad \forall i \in \{0, 1, \ldots, n\}$$

- $x_i > 0$: long position (bought)
- $x_i < 0$: short position (sold short)

LP model for the search of an arbitrage

LP model for the search of an arbitrage

$$\min \quad \sum_{i=0}^{n} s_i^0 \, x_i \tag{9.1a}$$

$$\text{subject to} \quad \sum_{i=0}^{n} s_{ij}^1 \, x_i \geq 0, \quad \forall j \in \{1, 2, \ldots, m\}, \tag{9.1b}$$

$$x_i \gtreqless 0, \quad \forall i \in \{0, 1, \ldots, n\}. \tag{9.1c}$$

Description of the model

- **Objective function:** it minimizes the cost of the portfolio today; a negative value is an immediate cash inflow
- **State constraints:** tomorrow the portfolio does not lose in any of the m states of the world (m linear constraints)
- **Free variables:** one can buy and sell short ($n + 1$ variables)

Reading the outcome: type A, type B, normalization (1/2)

- The null portfolio is always feasible, hence the optimum is always ≤ 0. Three possible outcomes
- **Optimum** < 0 (*type A arbitrage*): one collects cash today with no risk tomorrow
- The constraints are homogeneous: the strategy can be scaled at will and the model is *unbounded*
- For a finite certificate one adds the *normalization* $\sum_{i=0}^{n} s_i^0 x_i \geq -1$ ("collect at most 1 today"): the optimum becomes -1 and the solution is a concrete strategy

Reading the outcome: type A, type B, normalization (2/2)

- **Optimum $= 0$ with some state constraint slack in every optimal solution** (*type B arbitrage*): today it costs nothing, tomorrow one never loses and in at least one state one gains
- **Optimum $= 0$ and all constraints active at the optimum**: no arbitrage, the prices are *consistent*

Summary

- Free variables x_i (positions), objective: cost today, constraints: payoff ≥ 0 in every state
- Optimum < 0: type A arbitrage (unbounded LP, normalized with cost ≥ -1)
- Optimum $= 0$ with a state constraint always slack: type B arbitrage
- Optimum $= 0$ with all constraints active: consistent prices

Duality is pricing

Introduction: Duality is pricing

- We write the **dual** of the arbitrage search model
- The dual variables become **risk-neutral probabilities**: today's price is a discounted expected value
- We reach the **fundamental theorem of asset pricing** and we prove it with duality

From the primal to the dual

We apply the duality rules (minimization primal: table read from right to left):

- each state constraint, of type $\geq$, generates a dual variable $p_j \geq 0$
- each free variable x_i generates a dual equality constraint
- the right-hand sides (all zero) become the objective of the dual

The dual: pricing measures

The dual: pricing measures

$$\max \quad \sum_{j=1}^{m} 0 \cdot p_j \tag{9.2a}$$

$$\text{subject to} \quad \sum_{j=1}^{m} s_{ij}^1 \, p_j = s_i^0, \quad \forall i \in \{0, 1, \ldots, n\}, \tag{9.2b}$$

$$p_j \geq 0, \quad \forall j \in \{1, 2, \ldots, m\}. \tag{9.2c}$$

Description of the dual

- **Objective** identically zero: of the dual only *feasibility* matters (every feasible solution is optimal)
- **Pricing constraints:** today's price of every security is the sum of its payoffs weighted with the p_j; the duals *price* the securities ($n + 1$ linear constraints)
- **Last constraints:** they define the variables (m non-negative variables)

The risk-neutral probabilities

- The dual constraint of security 0 gives $\sum_{j=1}^{m} R\, p_j = 1$, that is, $\sum_{j=1}^{m} p_j = 1/R$
- The numbers $q_j = R\, p_j$ are non-negative and sum to 1: they are *probabilities*, called **risk-neutral**
- With them the pricing constraints read:

$$s_i^0 = \frac{1}{R} \sum_{j=1}^{m} q_j\, s_{ij}^1, \qquad \forall i \in \{0, 1, \ldots, n\}$$

- *Today's price is the discounted expected value of tomorrow's payoffs*, under the probabilities q

Fundamental theorem of asset pricing

Fundamental theorem of asset pricing

There is no arbitrage (neither of type A nor of type B) *if and only if* there exists a risk-neutral probability measure with $q_j > 0$ for every $j \in \{1, 2, \ldots, m\}$.

Why it is true: strong duality and complementary slackness (1/2)

- If there is no type A arbitrage, the optimum of the primal is 0
- By *strong duality* the dual is feasible: a pricing measure $p \geq 0$ exists
- If there is no type B arbitrage either, all the state constraints are active at the optimum
- By *complementary slackness* there exists a dual solution with $p_j > 0$ for every state: multiplying by R one obtains the $q_j > 0$

Why it is true: strong duality and complementary slackness (2/2)

Conversely, if there exists $q > 0$, every strategy with non-negative payoffs has cost

$$\sum_i s_i^0 x_i = \frac{1}{R} \sum_j q_j \left(\sum_i s_{ij}^1 x_i \right) \geq 0$$

hence no arbitrage exists.

Summary

- The dual has a zero objective: only its feasibility matters
- The dual variables p_j, multiplied by R, are risk-neutral probabilities q_j
- Today's price = discounted expected value of the payoffs under q
- No arbitrage $\iff$ there exists q with $q_j > 0$ for every state

Worked example by hand

Introduction: Worked example by hand

- We build by hand an arbitrage in a market with **three states**
- We check that it carries no risk tomorrow
- We see why, without the normalization, the model is **unbounded**

Example: three states, an obvious arbitrage (1/3)

- Market with $m = 3$ states, $r = 4\%$ ($R = 1.04$)
- Two risky securities with payoffs

$$s^1_{1\cdot} = (10, 15, 13), \qquad s^1_{2\cdot} = (30, 15, 25)$$

- Today's prices: $s^0_1 = 6$ and $s^0_2 = 20$

Example: three states, an obvious arbitrage (2/3)

Step 1: a candidate strategy.

- We buy one unit of each risky security ($x_1 = x_2 = 1$, outlay 26)
- We sell short the risk-free security for 27 ($x_0 = -27$)
- Cost today:

$$1 \cdot (-27) + 6 \cdot 1 + 20 \cdot 1 = -1 :$$

we collect 1

Example: three states, an obvious arbitrage (3/3)

Step 2: no risk tomorrow. In the three states the portfolio pays

$$-27 \cdot 1.04 + 10 + 30 = 11.92$$
$$-27 \cdot 1.04 + 15 + 15 = 1.92$$
$$-27 \cdot 1.04 + 13 + 25 = 9.92$$

all ≥ 0: it is a type A arbitrage.

Step 3: why the model is unbounded. By doubling the strategy one collects 2 today with payoffs still ≥ 0: without the normalization the cost can go down as far as one wishes.

Summary

- By buying the risky securities and selling short the risk-free one we collect 1 today
- Tomorrow the portfolio pays 11.92, 1.92 and 9.92: no risk, a type A arbitrage
- The strategy scales: without the normalization the LP is unbounded

Case study

Introduction: Case study

- We analyse with the script the same market as in the example
- Two price configurations: one **with arbitrage** and a **consistent** one
- We see how the model is written in Gurobi, with the optional normalization

The case

- Same market as in the example: $R = 1.04$, payoffs in the file `dati/arbitraggio_payoff.csv`
- Two price configurations:
  - with arbitrage: $s_1^0 = 6$, $s_2^0 = 20$
  - consistent: $s_1^0 = 13$, $s_2^0 = 18.6923$

The model in Gurobi

```
x = m.addVars(n1, lb=-GRB.INFINITY, name="x")  # free positions
v_stato = m.addConstrs(
    (gp.quicksum(s1[i, j] * x[i] for i in range(n1)) >= 0
     for j in range(n_stati)), name="stato")
if normalizza:  # without it: with arbitrage the model is
    unbounded
    m.addConstr(gp.quicksum(s0[i] * x[i] for i in range(n1))
                >= -1)
m.setObjective(gp.quicksum(s0[i] * x[i] for i in range(n1)),
               GRB.MINIMIZE)
```

Summary

- Free positions: `lb=-GRB.INFINITY`
- One constraint per state of the world: payoff of the portfolio ≥ 0
- The normalization (cost ≥ -1) avoids unboundedness in the presence of arbitrage

Results

Introduction: Results

- We read the output of the script in the two price configurations
- With consistent prices, the **duals** of the state constraints give the pricing measure

Prices with arbitrage: $(6, 20)$

- Without normalization: **UNBOUNDED** (type A arbitrage)
- With normalization: optimum -1
  - we collect 1 today
  - payoff $(0;\ 0;\ 0.20) \geq 0$ tomorrow

Consistent prices: (13; 18.69)

- Optimum 0: no arbitrage
- Duals of the state constraints: $p^* = (0.2846;\ 0.6769;\ 0)$
- Risk-neutral probabilities: $q = R\,p^* = (0.296;\ 0.704;\ 0)$, sum 1
- Pricing check: $s^0 = (1;\ 13;\ 18.6923)$ reproduced exactly

The duals do the pricing

- With consistent prices the LP returns optimum 0 and, in the duals of the state constraints, the pricing measure $q = (0.296; 0.704; 0)$
- Every security, and every *new* security, is priced as the discounted expected value of the payoffs under q
- $q_3 = 0$ signals an informational degeneracy: state 3 is not needed to price these securities (its payoff can be replicated with the other two states)
- This is why the "if and only if" of the theorem requires $q > 0$: with $q_3 = 0$ a latent type B arbitrage on state 3 remains

Summary

- Prices $(6, 20)$: unbounded LP; once normalized, optimum -1
- Consistent prices: optimum 0 and $q = (0.296; 0.704; 0)$ from the duals
- $q_3 = 0$: state not needed for the pricing, latent type B arbitrage

Sensitivity analysis: the interval of consistent prices

Introduction: the interval of consistent prices

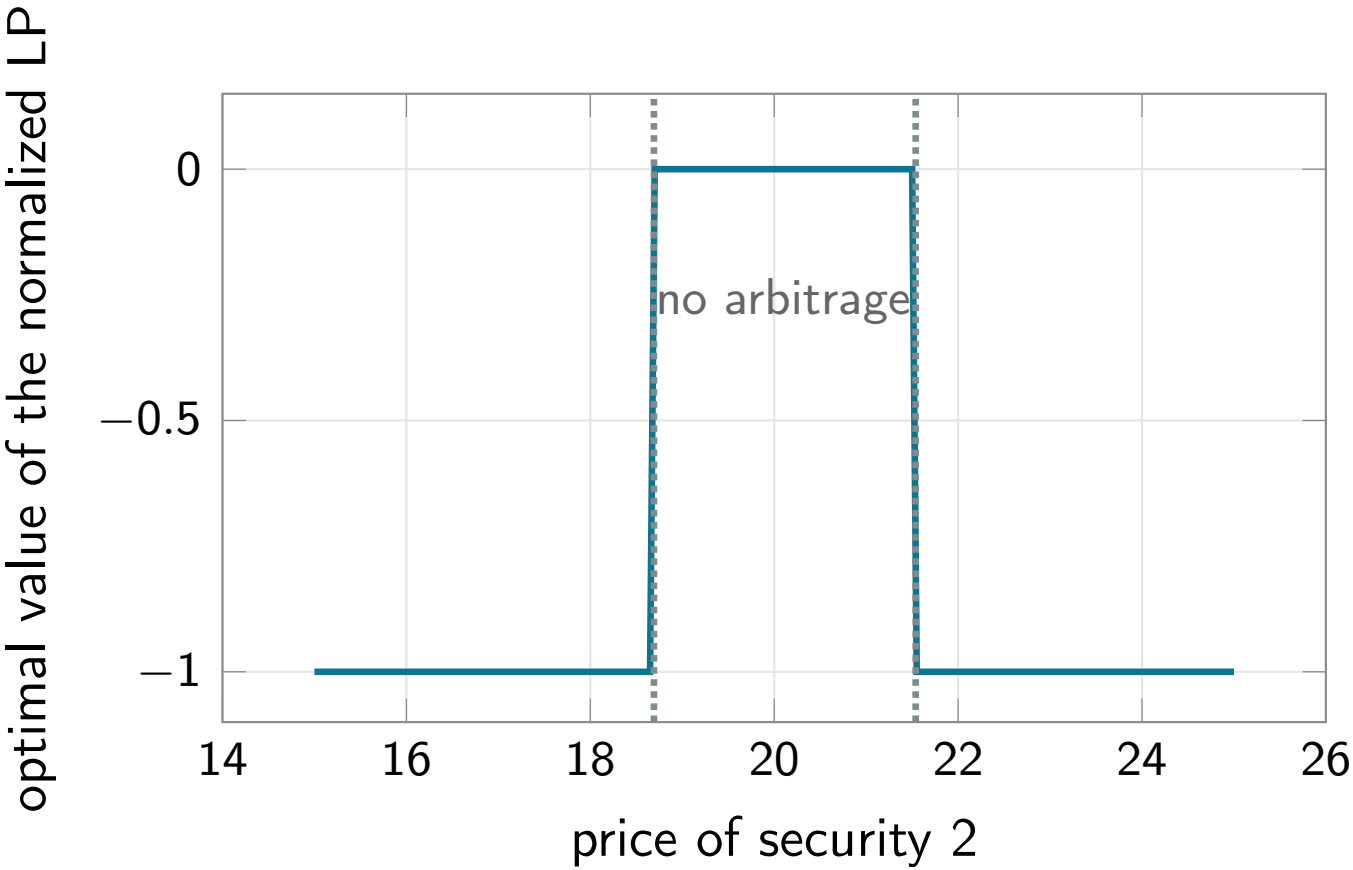
- If some securities are not quoted, which of their prices are **consistent**?
- Two LPs on the pricing measures give an **interval** of arbitrage-free prices
- With the same idea we price a new security (a **call**) in a complete and in an incomplete market

The interval of consistent prices

- Only security 0 (price 1) and security 1 (price 13) are quoted
- Which prices of security 2 do *not* create arbitrage?
- Two LPs: minimizing and maximizing $\sum_j s^1_{2j} p_j$ over the $p \geq 0$ consistent with the quoted securities
- Result:

$$s^0_2 \in [18.69,\ 21.54]$$

The arbitrage gain as the price varies



How to read the figure

- Optimal value of the normalized arbitrage LP as the price of security 2 varies (securities 0 and 1 quoted at 1 and 13)
- Inside the interval [18.69, 21.54] (dashed lines) the value is 0: **consistent** prices
- Outside it, the value drops to -1: an **arbitrage** exists, in one direction or the other

Pricing a new security: complete vs incomplete

- The same pair of LPs prices any new security
- A *call* on security 1 with strike 12 pays $\max(s^1_{1j} - 12, 0) = (0, 3, 1)$ in the three states
- **Complete market** (securities 0, 1, 2 quoted): the pricing measure is unique and so is the price: 2.0308
- **Incomplete market** (only securities 0 and 1): the consistent measures are infinitely many and the price is an *interval*, $[1.4615, 2.0308]$

Why an interval

- In the incomplete market, below the minimum or above the maximum of the interval anyone could build an arbitrage by combining the call and the quoted securities
- Inside the interval every price is consistent with the quoted securities

Summary

- Two LPs (min and max over the pricing measures) give the interval of consistent prices: [18.69, 21.54] for security 2
- Outside the interval the normalized LP is worth -1: arbitrage
- Complete market: a unique price (2.0308 for the call); incomplete: an interval ([1.4615, 2.0308])

Exercises

Introduction: Exercises

- Three exercises to consolidate the concepts of the chapter
- Scale of the arbitrage, endpoints of the interval of consistent prices, pricing of a call

Exercise: scale of the arbitrage

- With prices $(6, 20)$, find a strategy that collects 5 today with no risk tomorrow
- Hint: the solutions of the homogeneous LP can be scaled

Exercise: endpoint of the interval

- Set the price of security 2 at the upper endpoint 21.5385 and solve the arbitrage LP
- Check that the optimum is 0 and compute the associated pricing measure
- What happens just above, at 21.60?

Exercise: pricing a call

- In the complete market (prices 1, 13, 18.6923), price a call on security 2 with strike 20

$$\text{payoff} = \max(s_{2j}^1 - 20, 0)$$

- Check the price with the risk-neutral probabilities of the case study

Final summary (1/2)

Arbitrage search model:

- LP with free positions: minimum cost today, payoff ≥ 0 in every state
- Optimum < 0: type A (normalize); optimum 0 with a constraint always slack: type B

Duality and pricing:

- The duals p_j, multiplied by R, are risk-neutral probabilities
- Price = discounted expected value of the payoffs under q

Final summary (2/2)

Fundamental theorem:

- No arbitrage $\iff$ there exists a risk-neutral q with $q_j > 0$

Consistent prices:

- Two LPs on the pricing measures give the interval of arbitrage-free prices
- Complete market: a unique price; incomplete: an interval