Multi-period production and inventory

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Outline

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Learning objectives (1/2)

At the end of this chapter you will be able to:

- formulate a multi-period LP with inventory balance constraints;
- read and check the shadow prices of capacity and their validity ranges;

Learning objectives (2/2)

At the end of this chapter you will be able to:

- build the capacity value curve;
- explain how inventory moves capacity from the light months to the peak months.

Prerequisites

- From this module, Chapter 2 «Background: linear programming and sensitivity analysis»: shadow prices, their validity ranges and the sensitivity protocol
- From this module, Chapter 3 «The solver: building, solving, interpreting»: building an LP in `gurobipy` and reading `Pi` and `SARHS`

Managerial motivation

Introduction: Managerial motivation

- A firm decides how much to **produce today** and how much to **keep in inventory** for future demand
- Producing in advance costs **holding**
- Producing at the last moment risks crashing into the **capacity** limit precisely in the peak months
- Problem class: **LP**

The trade-off to balance

- The model balances four quantities:
  - margins
  - capacity
  - inventory
  - stability of the plan
- It is the ideal starting point of the laboratory: **every one of its numbers has an immediate economic interpretation**

Where this problem arises

- Every time a firm plans production over **several periods** with shared resources:
  - master production schedules (MPS)
  - aggregate planning
  - supply contracts with monthly profiles
- The core of the difficulty is the **intertemporal link**: today's decision (to produce in advance or not) changes tomorrow's constraints

Summary

- Decide how much to produce and how much to keep in inventory in every month
- The conflict is between the **holding cost** and the **capacity limit** in the peak months
- The intertemporal link prevents us from solving the months one at a time

Guided construction of the model

Introduction: Guided construction of the model

- We describe the problem in words, then we fix **data** and **variables**
- We write the minimum-cost production and inventory **LP model**
- We explain one by one the objective function and the families of constraints
- We will understand why the **inventory balance** is the heart of the model

The problem in words

- **We decide** how many units of each product to produce in each month and how many to keep in inventory
- **The objective** is to minimize the sum of the production and holding costs over the whole horizon
- **The constraints** are two:
  - the inventory accounting must add up in every month (what comes in equals what goes out, and all demand must be served)
  - the machine hours used in a month cannot exceed those available

Data (input of the model) (1/2)

Symbol	Type	Meaning
n	$\in \mathbb{Z}_{\geq 1}$	number of products; the products are indexed by $i \in \{1, 2, \ldots, n\}$
m	$\in \mathbb{Z}_{\geq 1}$	number of months of the horizon; the months are indexed by $t \in \{1, 2, \ldots, m\}$
d_{it}	$\in \mathbb{Q}_{\geq 0}$	demand for product i in month t (units)

Data (input of the model) (2/2)

Symbol	Type	Meaning
c_{it}	$\in \mathbb{Q}_{>0}$	production cost of one unit of i in month t (€)
h_{it}	$\in \mathbb{Q}_{>0}$	holding cost of one unit of i for month t (€)
a_i	$\in \mathbb{Q}_{>0}$	machine hours needed to produce one unit of product i
b_t	$\in \mathbb{Q}_{\geq 0}$	machine hours available in month t
s_{i0}	$\in \mathbb{Q}_{\geq 0}$	initial inventory of product i (units)

Decision variables

We introduce the following $2\,n\,m$ non-negative variables:

$$\begin{cases} x_{it} = \text{units of product } i \text{ produced in month } t \\ s_{it} = \text{units of product } i \text{ in stock at the end of month } t \end{cases}$$

$$\forall i \in \{1, 2, \ldots, n\},\ \forall t \in \{1, 2, \ldots, m\}.$$

Using these variables, we write an LP model for the problem.

LP model

Minimum-cost production and inventory LP model

$$\min \quad \sum_{i=1}^{n}\sum_{t=1}^{m}\left(c_{it}\,x_{it} + h_{it}\,s_{it}\right) \tag{4.1a}$$

$$\text{subject to} \quad s_{i,t-1} + x_{it} = d_{it} + s_{it}, \quad \forall i \in \{1,2,\ldots,n\},\ \forall t \in \{1,2,\ldots,m\}, \tag{4.1b}$$

$$\sum_{i=1}^{n} a_i\,x_{it} \le b_t, \qquad \forall t \in \{1,2,\ldots,m\}, \tag{4.1c}$$

$$x_{it} \ge 0, \qquad \forall i \in \{1,2,\ldots,n\},\ \forall t \in \{1,2,\ldots,m\}, \tag{4.1d}$$

$$s_{it} \ge 0, \qquad \forall i \in \{1,2,\ldots,n\},\ \forall t \in \{1,2,\ldots,m\}. \tag{4.1e}$$

Objective function and inventory balances

- The linear **objective function** minimizes the total cost of the plan: the sum of the production costs $c_{it}\, x_{it}$ and of the holding costs $h_{it}\, s_{it}$ over all products and months
- The **inventory balances**: for every product and every month, what comes in (previous inventory plus production) is equal to what goes out (demand served plus subsequent inventory)
  - for $t = 1$ the initial inventory s_{i0} is used
  - they are $n\, m$ linear constraints

Capacity and domain of the variables

- The **capacity constraints**

$$\sum_{i=1}^{n} a_i \, x_{it} \leq b_t$$

  ensure that, in every month, the machine hours used by all the products do not exceed those available

- They are m linear constraints
- The constraints $x_{it} \geq 0$ and $s_{it} \geq 0$ define the variables of the model

Why the balance is the heart of the model

- It is the constraint that **links the months to one another**
- Without it the problem would break up into m **independent problems**
- And inventory would make no sense: nothing would connect what is left in the warehouse at the end of a month with the next one

What happens if... we remove the balance?

- The model **produces zero** (minimum cost!) because nothing forces it to serve demand
- If instead total demand exceeds the cumulated capacity, the model becomes **infeasible**; in that case either:
  - lost demand is allowed with a variable $u_{it} \geq 0$ and a penalty π_{it} in the objective
  - or capacity has to be revised

Summary

- Variables: x_{it} (production) and s_{it} (end-of-month inventory)
- Objective: **production** costs plus **holding** costs
- Constraints: **inventory balance** for every product and month, **capacity** in machine hours for every month
- The balance is what makes the problem genuinely **multi-period**

Worked example by hand

Introduction: Worked example by hand

- A minimal example: **one product** and **two months**
- We solve it entirely by hand, with no solver
- We also compute the **shadow price** of capacity by perturbation, and we will find the same mechanism again in the case study

Example: One product, two months (1/3)

Data:

- demand $d_1 = 60$, $d_2 = 100$
- capacity $b_1 = b_2 = 80$ units/month (with $a = 1$)
- production cost $c = 10$ €, holding $h = 1$ €/unit/month
- zero initial inventory

Step 1 — is it feasible? Total demand $160 \leq$ total capacity 160: yes, but only by producing at full capacity in *both* months.

Example: One product, two months (2/3)

Step 2 — balances.

$$x_1 = 60 + s_1, \qquad s_1 + x_2 = 100$$

- With $x_2 \leq 80$ we need $s_1 \geq 20$, hence $x_1 \geq 80$
- But $x_1 \leq 80$: therefore $x_1 = 80$, $s_1 = 20$, $x_2 = 80$

Step 3 — cost.

$$10 \cdot (80 + 80) + 1 \cdot 20 = 1620 \text{ €}$$

The solution is **forced**: both capacity constraints are active.

Example: One product, two months (3/3)

Step 4 — shadow price by hand.

- With one more hour in month 2 ($b_2 = 81$): $x_2 = 81$, $s_1 = 19$, $x_1 = 79$
- The cost decreases by 1 € (one unit no longer sits in inventory for a month): the dual of the capacity of month 2 is -1 €/unit
- With one more hour in month 1, instead, nothing changes (month 1 already produces what is needed): dual 0

This mechanism — **the capacity of a month is worth as much as the holding it saves** — reappears identically in the case study.

Summary

- With tight capacity the solution can be **forced**: the balances determine it on their own
- The **shadow price** of a capacity is checked by adding one hour and recomputing the cost
- A **non-binding** capacity has zero shadow price
- The value of one more hour is the **holding it saves**

Case study

Introduction: Case study

- We apply the model to a realistic instance: **three products** over **six months**
- Demand is **seasonal and increasing**, with a peak in months 4–5, and the resource is single and shared
- We then see how the model is written in **Python with Gurobi**

The data of the products

Three products ($n = 3$) over six months ($m = 6$):

Product	c_i (€/u)	h_i (€/u/month)	a_i (hours/u)	s_{i0}
1	12.00	0.80	0.9	30
2	18.00	1.20	1.4	20
3	25.00	1.60	2.1	10

Product 3 is the most expensive one **and** the heaviest in machine hours: it is the one that will use the warehouse.

Capacity and monthly demand

month	1	2	3	4	5	6
b_t (hours)	420	420	460	460	460	420
$d_{1,t}$	110	130	150	190	210	160
$d_{2,t}$	70	80	110	140	150	100
$d_{3,t}$	50	55	60	80	95	70

- Data in the CSV files `produzione_domanda` and `produzione_capacita`
- Capacity grows only a little ($420 \rightarrow 460$ hours) while demand grows a lot

The model in Gurobi (1/2)

Variables and inventory balances:

```
m = gp.Model("produzione_scorte")
x = m.addVars(prodotti, mesi, name="x")  # produced
s = m.addVars(prodotti, mesi, name="s")  # in stock

# balance: incoming inventory + production
#                = demand + outgoing inventory
m.addConstrs(
    ((s0[i] if t == 1 else s[i, t-1]) + x[i, t]
     == domanda[i, t] + s[i, t]
     for i in prodotti for t in mesi), name="bilancio")
```

The model in Gurobi (2/2)

Shared capacity and objective function:

```
# shared capacity (keep the constraints for the duals)
v_cap = m.addConstrs(
    (gp.quicksum(a[i] * x[i, t] for i in prodotti)
     <= capacita[t] for t in mesi), name="capacita")

m.setObjective(
    gp.quicksum(c[i]*x[i, t] + h[i]*s[i, t]
                for i in prodotti for t in mesi),
    GRB.MINIMIZE)
m.optimize()
```

Summary

- Instance: 3 products, 6 months, a single shared resource
- Seasonal demand with a peak in months 4–5, almost constant capacity
- In Gurobi: `addVars` for x and s, `addConstrs` for balances and capacity
- The capacity constraints are **kept in a variable** (`v_cap`) so that their duals can be read

Results

Introduction: Results

- We read the **optimal production plan** and the **inventory** month by month
- We look at the plots of the plan and of the inventory
- We will discover the strategy that the solver found all by itself: the **pre-build**

The optimal production plan

Optimal total cost: **32,889.33 €**

units/month	1	2	3	4	5	6
prod. 1	80.0	130.0	150.0	190.0	210.0	160.0
prod. 2	50.0	80.0	110.0	140.0	150.0	100.0
prod. 3	89.5	91.0	81.4	44.3	29.0	64.8

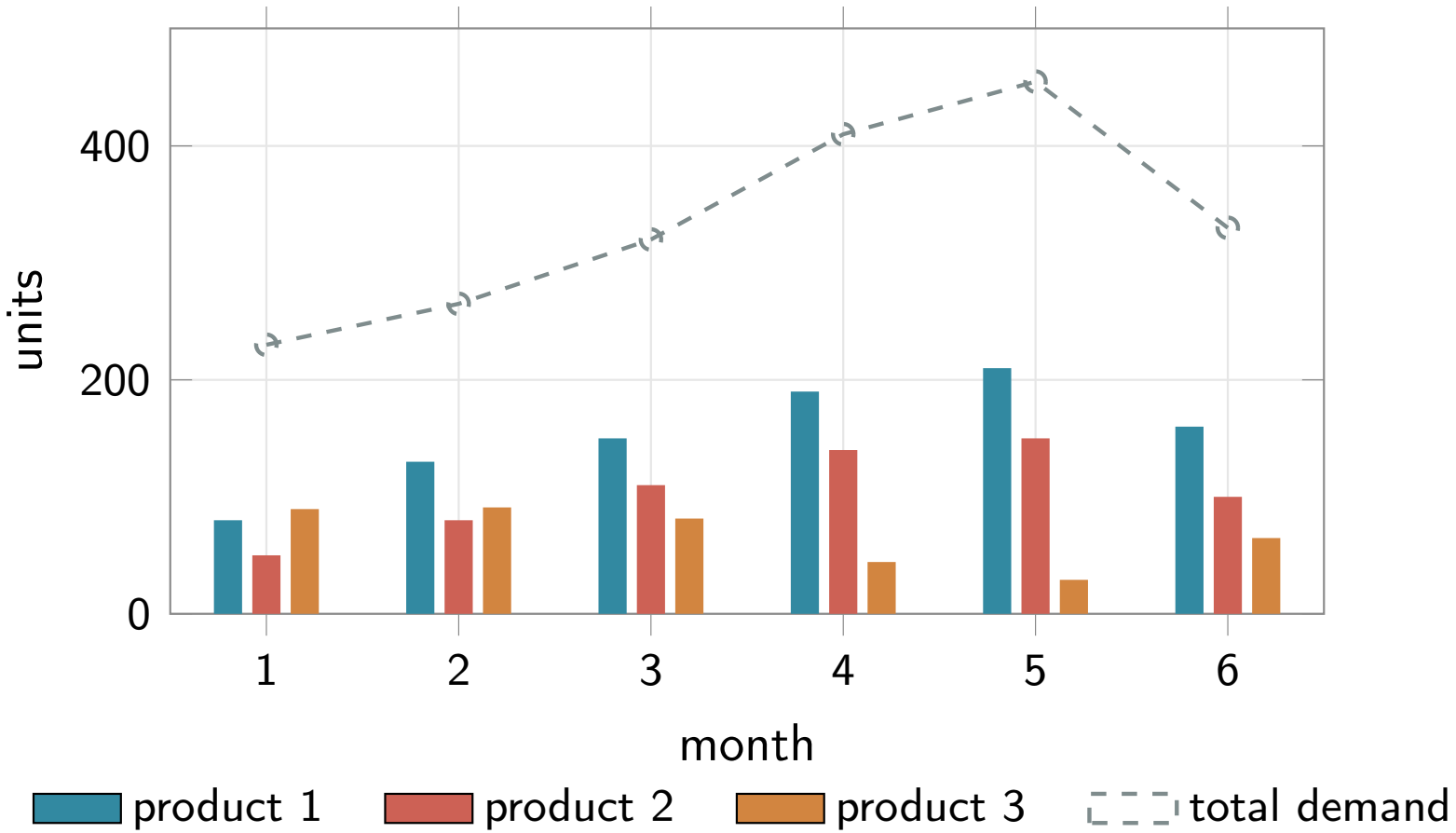
Products 1 and 2 follow demand; product 3 does not.

End-of-month inventory

units	1	2	3	4	5	6
prod. 1	0.0	0.0	0.0	0.0	0.0	0.0
prod. 2	0.0	0.0	0.0	0.0	0.0	0.0
prod. 3	49.5	85.5	106.9	71.2	5.2	0.0

- Only product 3 uses the warehouse
- The inventory goes to **zero** in the final month

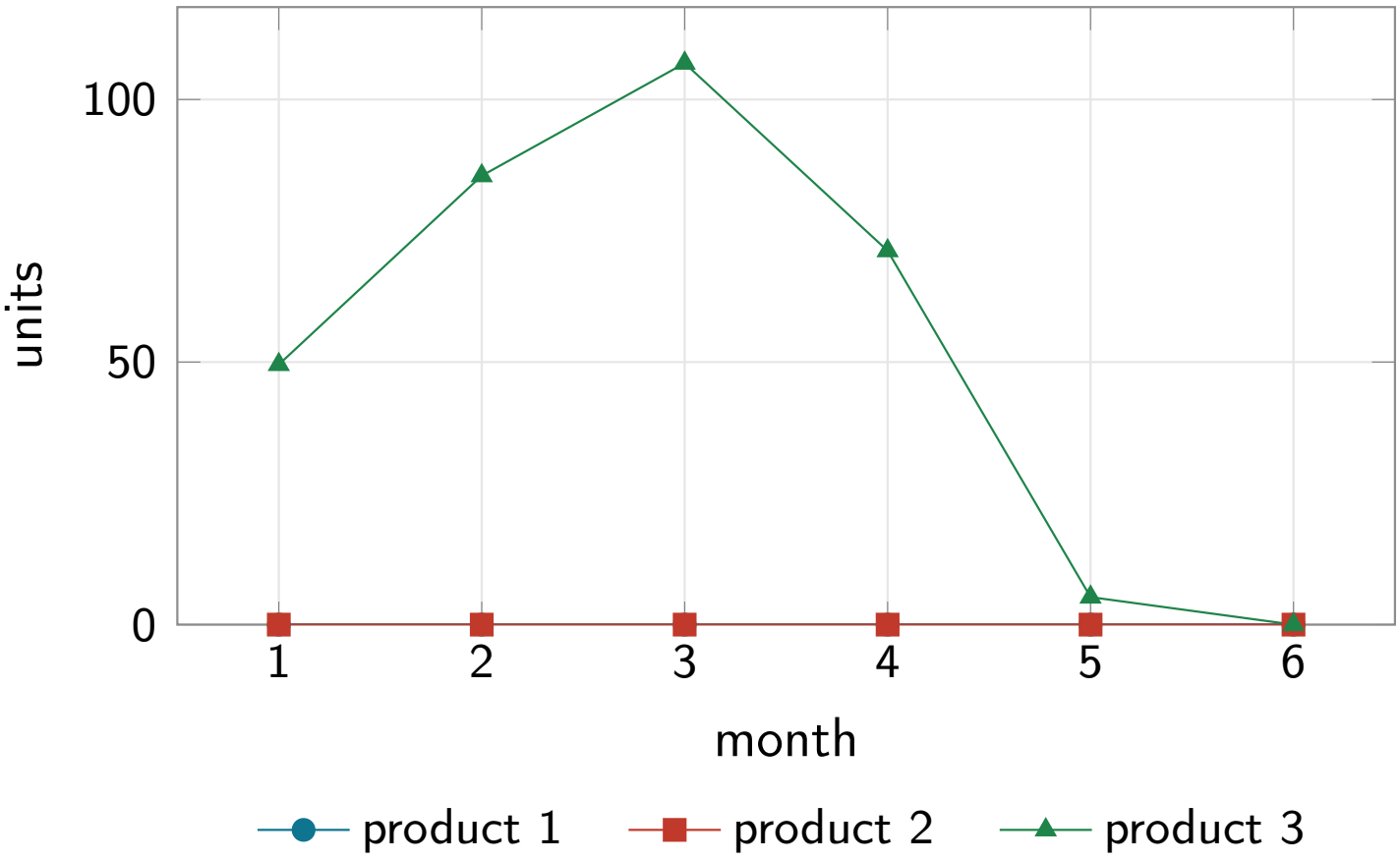
Optimal production plan and total demand



How to read the plot of the plan

- The **bars** are the units produced of the three products in each month
- The **dashed line** is the total demand of the month
- In the peak months (4–5) total production stays **below** demand
- The difference is covered by the **inventory of product 3** accumulated in the first months

End-of-month inventory



Only product 3 uses the warehouse (**pre-build** strategy).

The pre-build strategy (1/2)

- The solver discovered the *pre-build* strategy **all by itself**
- Products 1 and 2 (high holding costs relative to the machine hours they absorb) are produced **to demand**
- Product 3 — the most expensive one in machine hours — is produced in advance in months 1–3 and stockpiled (up to 107 units in inventory)
- In this way **capacity is freed up** in the peak months

The pre-build strategy (2/2)

- The inventory goes back to **zero** in the final month
- A positive terminal inventory would be wasted money, and the model knows it
- Nobody "taught" the pre-build to the solver: it is a consequence of the balances and of the capacity constraints

Summary

- Optimal total cost: **32,889.33 €**
- Products 1 and 2 produced to demand; product 3 in **pre-build**
- Inventory is a device to **move capacity** from the light months to the peak months
- Zero terminal inventory: emptying the warehouse at the end is always optimal

Sensitivity analysis

Introduction: Sensitivity analysis

- We read the **shadow prices** of capacity month by month, with their validity ranges
- We will derive an **exact** economic interpretation of them
- We finally build the **capacity value curve**

Shadow prices of capacity

month	use	Π (€/hour)	validity of b_t
1	330.0 / 420	0.000	$[330.0, +\infty)$
2	420.0 / 420	-0.762	$[330.0, 524.0]$
3	460.0 / 460	-1.524	$[370.0, 564.0]$
4	460.0 / 460	-2.286	$[370.0, 564.0]$
5	460.0 / 460	-3.048	$[399.0, 564.0]$
6	420.0 / 420	-3.810	$[330.0, 431.0]$

Check: $+1$ hour in month 6 takes the cost from 32889.33 to 32885.52 (delta -3.810).

How to read the shadow prices

- Month 1 is **not binding** (330 hours out of 420): its shadow price is zero
- All the other months are binding and have a **negative** shadow price: one more hour *reduces* the cost
- The value grows month after month: the most precious hour is the one of **month 6**
- The validity range says between which values of b_t the shadow price stays the same

Where the number 0.762 comes from (1/2)

- The duals grow by exactly 0.762 € per month: this is no accident

$$0.762 = \frac{h_3}{a_3} = \frac{1.6}{2.1}$$

- One more hour in month t allows us to produce $1/a_3$ units of product 3 *one month later*
- One month of holding is thus saved: $h_3 \cdot (1/a_3)$

Where the number 0.762 comes from (2/2)

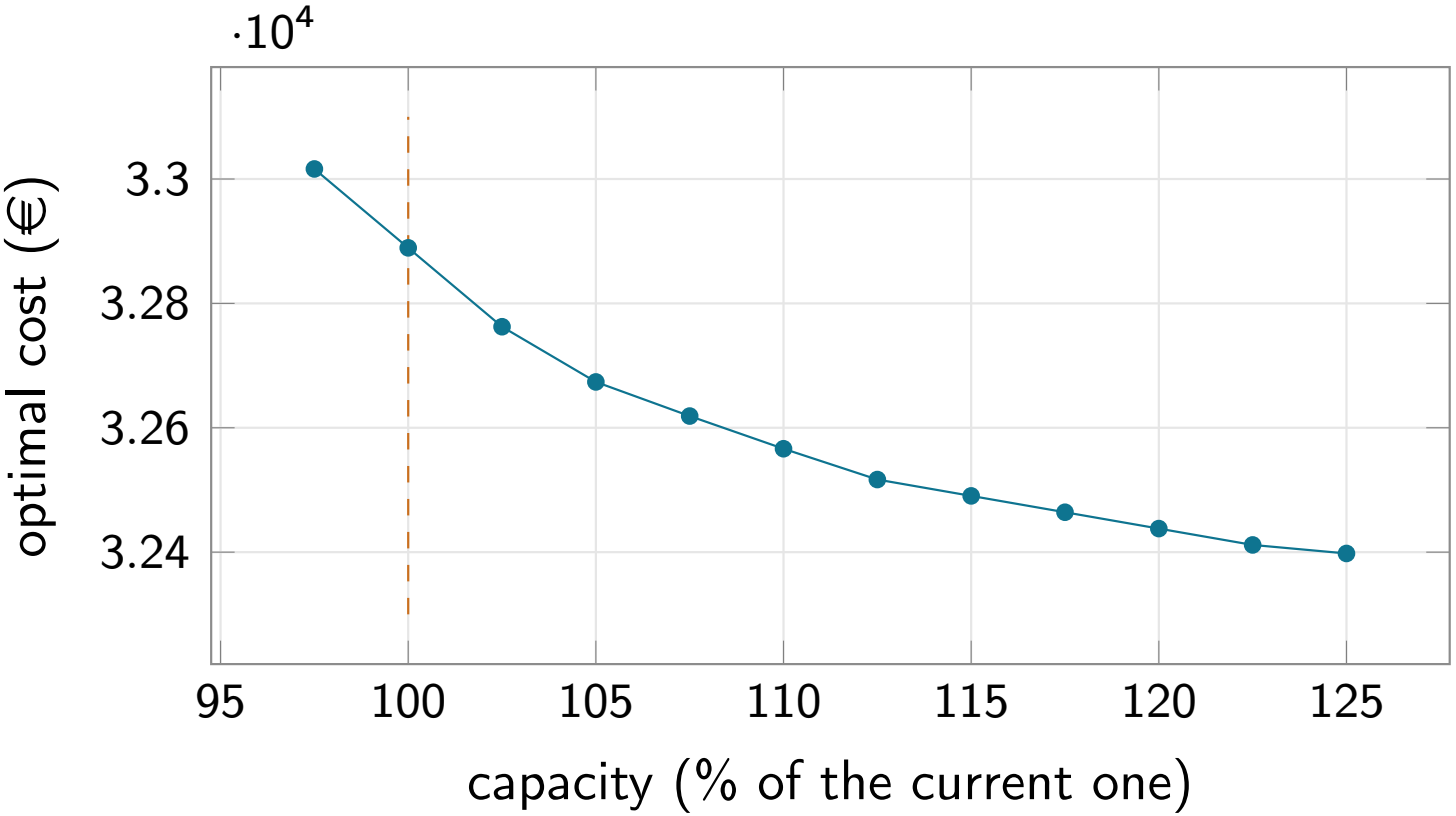
- The dual of month 6 is worth

$$5 \times 0.762 = 3.81 \text{ €/hour}$$

  because one hour there avoids **five months** of holding for a unit produced in month 1

- When the solver and an elementary economic argument give the same number, the model is probably **right**

Capacity value curve



The dashed orange line marks the current capacity.

How to read the capacity value curve

- The curve is **convex and decreasing**: the first extra hours are worth a lot, then less and less
- Below **97%** of the current capacity the model becomes **infeasible**: demand can no longer be served in full
- This is managerial information as precious as the optimum itself

The quadratic variant

- A **smooth** plan penalizes the changes of production from one month to the next
- It needs a quadratic term in the objective: the model is **no longer an LP**
- It is the quadratic variant of this model, in the module *NLP Modelling*: "Production with quadratic smoothing"

The managerial recommendation

Typical recommendation from a lab report:

- It is worth buying **overtime hours** in months 5–6 up to 3–3.8 €/hour below the internal price
- Below **97%** of the current capacity full service cannot be guaranteed

Summary

- The shadow prices of capacity grow by $h_3/a_3 = 0.762$ € per month
- The non-binding month has zero shadow price; month 6 is the most precious one (3.81 €/hour)
- The capacity value curve is **convex and decreasing**, and it is interrupted at 97%

Exercises

Introduction: Exercises

- Five exercises to be carried out with the script of the case study
- They check the shadow prices, extend the model (lost demand, profit, emissions) and test a safety stock policy

Exercise: check

- Verify by perturbation the duals of months 3 and 4 (add one hour and reoptimize)
- Why is the dual of month 1 zero?

Exercise: lost demand

- Add the variable u_{it} (unserved demand) with penalty $\pi = 40$ €/unit
- Reduce capacity to 90%
- How much demand is lost, of which product, in which months? Interpret the choice of the model

Exercise: maximum profit

- Turn the model into a profit maximization with revenues $p = (20, 30, 42)$ €
- Make service optional with $y_{it} \leq d_{it}$
- Which demand is it worth *not* serving, and why?

Exercise: emissions

- Every unit of product 3 emits 2.5 kg of CO_2, products 1 and 2 one kg
- Add to the objective the term

$$\tau \sum_{i=1}^{n} \sum_{t=1}^{m} e_i \, x_{it}$$

- Draw the cost-emissions frontier for $\tau \in [0, 10]$ €/kg

Exercise: safety stock

- Impose the 10% safety stock:

$$s_{it} \geq 0.1\, d_{i,t+1}$$

- How much does this policy cost?
- Which capacity constraint becomes the most critical?

Summary

- Check of the duals by perturbation
- Extensions of the model: **lost demand**, **maximum profit**, **emissions**
- Warehouse policies: **safety stock** and its cost

Final summary (1/2)

Model:

- Multi-period LP with production variables x_{it} and inventory variables s_{it}
- Inventory balance for every product and month, capacity in machine hours
- The balance is what links the months to one another

Optimal solution:

- Cost 32,889.33 €; pre-build of product 3 in months 1–3
- Zero terminal inventory

Final summary (2/2)

Economic reading:

- Shadow prices of capacity = holding saved, h_3/a_3 per month
- Capacity value curve convex and decreasing, infeasible below 97%