Background: linear programming and sensitivity analysis

Outline

Linear programming theory
- Duality
- The optimality conditions: complementary slackness
- Sensitivity analysis and shadow prices

The sensitivity analysis protocol

Learning objectives (1/2)

At the end of this chapter you will be able to:

- write the dual of an LP in general form with the conversion rules;
- check optimality with complementary slackness;
- deduce the sign of a shadow price from the direction of the constraint and the sense of the objective;

Learning objectives (2/2)

At the end of this chapter you will be able to:

- read a shadow price and a reduced cost together with their validity ranges;
- apply the six steps of the sensitivity protocol.

Prerequisites

- From the module *Linear Programming*, Chapter 9 «Duality theory»: the dual of a Linear Programming problem, weak and strong duality, complementary slackness
- From the module *Linear Programming*, Chapter 14 «Sensitivity analysis»: shadow prices, reduced costs and their validity ranges

What this chapter is about

- We collect the **theoretical tools** used throughout the course
- The *theory of linear programming*: duality, optimality conditions, sensitivity analysis
- The complete theory is in the module **Linear Programming**
- It closes with the **sensitivity protocol** used in every laboratory session
- It also serves as a quick reference

Linear programming theory

Introduction: Linear programming theory

- Every maximization LP has a **dual** minimization problem
- The dual variables are the **shadow prices**: the value of one extra unit of resource
- **Complementary slackness** tells us when a primal-dual pair is optimal
- **Sensitivity analysis** tells us how the optimum changes when the data change: the question a decision maker cares about

Sets and data

- $M = \{1, 2, \ldots, m\}$: constraint indices; $N = \{1, 2, \ldots, n\}$: variable indices
- **Data**: costs $c_j \in \mathbb{R}$ $(j \in N)$, coefficients $a_{ij} \in \mathbb{R}$, right-hand sides $b_i \in \mathbb{R}$ $(i \in M)$
- $M_{\leq}$, $M_{=}$, $M_{\geq}$ partition M according to the direction of the constraint
- $N_{\geq 0}$, $N_{\gtrless 0}$, $N_{\leq 0}$ partition N according to the sign of the variable (non-negative, free, non-positive)
- **Variables**: the x_j of the primal and, in the dual, one π_i for each of the m constraints of the primal

The primal (P)

The primal (P), general form

$$
\begin{aligned}
\text{(P)} \qquad \max \quad & \sum_{j \in N} c_j \, x_j \\
\text{subject to} \quad & \sum_{j \in N} a_{ij} \, x_j \leq b_i, \quad \forall i \in M_{\leq}, \\
& \sum_{j \in N} a_{ij} \, x_j = b_i, \quad \forall i \in M_{=}, \\
& \sum_{j \in N} a_{ij} \, x_j \geq b_i, \quad \forall i \in M_{\geq}, \\
& x_j \geq 0, \quad \forall j \in N_{\geq 0}, \\
& x_j \gtreqless 0, \quad \forall j \in N_{\gtreqless 0}, \\
& x_j \leq 0, \quad \forall j \in N_{\leq 0}.
\end{aligned}
$$

The primal (P) (2/2)

- Three constraint directions ($\leq$, $=$, $\geq$) and three variable signs (non-negative, free, non-positive)

The dual (D)

The dual (D), general form

$$
\begin{aligned}
\text{(D)} \qquad \min \quad & \sum_{i \in M} b_i \, \pi_i \\
\text{subject to} \quad & \sum_{i \in M} a_{ij} \, \pi_i \geq c_j, \quad \forall j \in N_{\geq 0}, \\
& \sum_{i \in M} a_{ij} \, \pi_i = c_j, \quad \forall j \in N_{\gtreqless 0}, \\
& \sum_{i \in M} a_{ij} \, \pi_i \leq c_j, \quad \forall j \in N_{\leq 0}, \\
& \pi_i \geq 0, \quad \forall i \in M_{\leq}, \\
& \pi_i \gtreqless 0, \quad \forall i \in M_{=}, \\
& \pi_i \leq 0, \quad \forall i \in M_{\geq}.
\end{aligned}
$$

The dual (D) (2/2)

- The sign of π_i depends on the direction of constraint i of the primal

The rules of duality (1/2)

Every constraint of the primal generates a dual variable, every variable of the primal a dual constraint:

Primal (max)		Dual (min)
constraint $\leq b_i$	$(i \in M_{\leq})$	variable $\pi_i \geq 0$
constraint $= b_i$	$(i \in M_{=})$	variable $\pi_i \gtreqless 0$
constraint $\geq b_i$	$(i \in M_{\geq})$	variable $\pi_i \leq 0$
variable $x_j \geq 0$	$(j \in N_{\geq 0})$	constraint $\geq c_j$
variable $x_j \gtreqless 0$	$(j \in N_{\gtreqless 0})$	constraint $= c_j$
variable $x_j \leq 0$	$(j \in N_{\leq 0})$	constraint $\leq c_j$

The rules of duality (2/2)

Two quick readings:

- the *equalities* of the primal give *free* dual variables (and vice versa)
- in a maximization a resource constraint $\leq$ has $\pi_i \geq 0$ (more resource, more value)

For a *minimization* primal the table is read from right to left

The duality theorems

- The primal-dual pair is linked by the **two duality theorems**
- In the first, the *bar* denotes any feasible solution
- In the second, the *tilde* denotes an optimal solution

Weak duality

Weak duality theorem

For every feasible solution $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ of the primal (P) and every feasible solution $(\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_m)$ of the dual (D) we have

$$\sum_{j \in N} c_j \, \bar{x}_j \;\leq\; \sum_{i \in M} b_i \, \bar{\pi}_i .$$

Strong duality

Strong duality theorem

The primal (P) has an optimal solution $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ *if and only if* the dual (D) has an optimal solution $(\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_m)$, and in that case the two optimal values coincide:

$$\sum_{j \in N} c_j \, \tilde{x}_j \;=\; \sum_{i \in M} b_i \, \tilde{\pi}_i .$$

What the two theorems say

- **Weak duality**: every feasible solution of the dual is an *upper bound* for the value of the maximization primal (and every feasible solution of the primal a lower bound for the dual)
- If two feasible solutions have the same value, they are both optimal
- **Strong duality**: at the optimum this always happens, the gap closes
- Existence does not mean uniqueness: there may be more than one optimal solution, but the optimal values coincide

Example: the dual of the 2×2 LP (1/3)

Let us apply the rules to the 2×2 LP, which is exactly in the canonical maximization form (constraints $\leq$, non-negative variables):

$$\begin{array}{rrcrcr} \max & 30\,x_1 & + & 50\,x_2 & & \\ \text{subject to} & x_1 & + & 3\,x_2 & \leq & 90 \\ & 2\,x_1 & + & x_2 & \leq & 80 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

Example: the dual of the 2×2 LP (2/3)

- **Step 1 — every constraint generates a dual variable.**
  Two constraints $\leq$ in a maximization $\Rightarrow \pi_1, \pi_2 \geq 0$
- **Step 2 — every variable generates a dual constraint.**
  The coefficients are the *columns* of the primal, the right-hand sides are the costs:
  - from $x_1 \geq 0$ we get $\pi_1 + 2\pi_2 \geq 30$
  - from $x_2 \geq 0$ we get $3\pi_1 + \pi_2 \geq 50$
- **Step 3 — the objective swaps the roles**: the right-hand sides $(90, 80)$ become the costs of the dual, which is a *minimization* problem

Example: the dual of the 2×2 LP (3/3)

The dual is:

$$
\begin{array}{rrcl}
\min & 90\,\pi_1 + 80\,\pi_2 & & \\
\text{subject to} & \pi_1 + 2\,\pi_2 & \geq & 30 \\
& 3\,\pi_1 + \pi_2 & \geq & 50 \\
& \pi_1, \quad \pi_2 & \geq & 0
\end{array}
$$

Example: all the cases in a single LP (1/3)

A *minimization* LP with all the cases of the table: three variables with the three signs and three constraints with the three directions (the table is read from right to left):

$$
\begin{array}{rrrrrrr}
\min & 5\,x_1 & + 8\,x_2 & - 9\,x_3 & & & \\
\text{subject to} & x_1 & + x_2 & & \geq & 30 \\
& x_1 & + x_2 & - x_3 & = & 100 \\
& x_1 & - 2\,x_2 & & \leq & -20 \\
& x_1 & & & \geq & 0 \\
& & x_2 & & \lesseqgtr & 0, \\
& & & x_3 & \leq & 0
\end{array}
$$

Example: all the cases in a single LP (2/3)

- **Step 1 — the dual variables**, one per constraint, with the sign given by the direction:
  - $\pi_1 \geq 0$ (constraint $\geq$ in a minimization)
  - $\pi_2 \gtreqless 0$ (equality)
  - $\pi_3 \leq 0$ (constraint $\leq$ in a minimization)
- **Step 2 — the dual constraints**, one per variable, with the direction given by the sign:
  - from $x_1 \geq 0$ we get $\pi_1 + \pi_2 + \pi_3 \leq 5$
  - from $x_2 \gtreqless 0$ we get $\pi_1 + \pi_2 - 2\pi_3 = 8$
  - from $x_3 \leq 0$ we get $-\pi_2 \geq -9$
- **Step 3 — the objective**: the dual is a maximization, $\max\ 30\pi_1 + 100\pi_2 - 20\pi_3$

Example: all the cases in a single LP (3/3)

The dual is:

$$
\begin{array}{rrrrrrrrrr}
\max & 30\,\pi_1 & + & 100\,\pi_2 & - & 20\,\pi_3 & & \\
\text{subject to} & \pi_1 & + & \pi_2 & + & \pi_3 & \leq & 5 \\
& \pi_1 & + & \pi_2 & - & 2\,\pi_3 & = & 8 \\
& & - & \pi_2 & & & \geq & -9 \\
& \pi_1 & & & & & \geq & 0 \\
& & & \pi_2 & & & \lessgtr & 0, \\
& & & & & \pi_3 & \leq & 0
\end{array}
$$

Sapienza Università di Roma

Slacks and reduced costs

To every primal constraint we associate its *slack* $\bar{s}_i$, and to every variable its *reduced cost* $\bar{c}_j$, that is the slack of the corresponding *dual* constraint:

$$\bar{s}_i = \sum_{j \in N} a_{ij}\, \bar{x}_j - b_i, \quad \forall i \in M, \qquad \bar{c}_j = c_j - \sum_{i \in M} a_{ij}\, \bar{\pi}_i, \quad \forall j \in N$$

- The slack measures how far constraint i is from being binding
- The reduced cost measures the coefficient c_j net of the value of the resources consumed by x_j, evaluated at the dual variables

Optimality conditions

LP optimality conditions (complementary slackness)

A pair of feasible solutions, $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ for the primal (P) and $(\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_m)$ for the dual (D), is optimal for both problems *if and only if*

$$\bar{\pi}_i \cdot \bar{s}_i = 0, \quad \forall i \in M,$$

$$\bar{x}_j \cdot \bar{c}_j = 0, \quad \forall j \in N.$$

Complementary slackness in words

- **Slack constraint $\Rightarrow$ dual at zero**

$$\bar{s}_i \neq 0 \Rightarrow \bar{\pi}_i = 0$$

- **Variable different from zero $\Rightarrow$ null reduced cost**

$$\bar{x}_j \neq 0 \Rightarrow \bar{c}_j = 0$$

- At the optimum these are *implications*, not equivalences: because of degeneracy an active constraint may have a null dual, and a variable at zero a null reduced cost
- The verification examples further on check exactly these conditions

Two readings for the laboratory (1/3)

Shadow price $\tilde{\pi}_i$: the marginal value of the right-hand side b_i,

$$\tilde{\pi}_i = \frac{\partial \tilde{z}}{\partial b_i}$$

- It holds for b_i within the *validity range of the right-hand side* $[b_i^{\min}, b_i^{\max}]$
- It is the largest interval in which, varying only b_i, the shadow price remains exactly $\tilde{\pi}_i$ (beyond it, it changes and must be recomputed)
- Special case: inactive constraint $\Rightarrow \tilde{\pi}_i = 0$

Two readings for the laboratory (2/3)

Reduced cost $\bar{c}_j$ (the slack of the dual constraint):

- for a variable at zero it says by how much c_j must improve for it to become worth activating
- it holds for c_j within the *validity range of the coefficient* $[c_j^{\min}, c_j^{\max}]$: the largest interval in which, varying only c_j, the optimal solution does not change
- for a variable at zero the profitability threshold is precisely one of the two endpoints
- special case: variable different from zero $\Rightarrow \bar{c}_j = 0$

Two readings for the laboratory (3/3)

Beware of degeneracy:

- there may exist a variable *at zero with null reduced cost*, even though it is not used
- hence only the implication holds,

$$\bar{c}_j \neq 0 \;\Rightarrow\; \text{variable on one of its bounds}$$

not the converse

The signs of shadow prices (1/4)

The shadow price is always, with a single convention,

$$\tilde{\pi}_i = \frac{\partial \tilde{z}}{\partial b_i}$$

the derivative of the optimal value with respect to the right-hand side of constraint i: by how much $\tilde{z}$ changes if b_i increases by one (small) unit

The sign is always deduced by answering **two questions**

The signs of shadow prices (2/4)

- **Does increasing b_i enlarge or shrink the feasible region?**
  - with a constraint $\leq$ it enlarges it (the ceiling is raised)
  - with a constraint $\geq$ it shrinks it (the requirement grows)
  - with an equality it merely shifts it
- **How does a wider region change the optimum?**
  - it can never worsen it
  - in a minimization the optimal value decreases or stays the same
  - in a maximization it increases or stays the same

The signs of shadow prices (3/4)

Combining the two answers:

Direction of the constraint	minimization	maximization
$\leq$ (b_i grows $\Rightarrow$ wider region)	$\tilde{\pi}_i \leq 0$	$\tilde{\pi}_i \geq 0$
$\geq$ (b_i grows $\Rightarrow$ narrower region)	$\tilde{\pi}_i \geq 0$	$\tilde{\pi}_i \leq 0$
$=$	any sign (free dual)	

The signs of shadow prices (4/4)

Two typical readings in a *minimization*:

- a constraint $\leq$ (a capacity) has $\tilde{\pi}_i \leq 0$: relaxing it reduces the cost, and $|\tilde{\pi}_i|$ is the saving for each extra unit of capacity
- a constraint $\geq$ (a requirement to be met) has $\tilde{\pi}_i \geq 0$: raising the requirement costs, and $\tilde{\pi}_i$ is the marginal cost

In a *maximization* the readings are reversed

In every case an inactive constraint has $\tilde{\pi}_i = 0$ by complementarity

Example: duality on the 2×2 LP (1/4)

Let us take up again the 2×2 LP and its dual and verify the optimality conditions.

Step 1 — solving we obtain:

$$(\tilde{x}_1, \tilde{x}_2) = (30, 20), \qquad \tilde{z} = 1900,$$

$$(\tilde{\pi}_1, \tilde{\pi}_2) = (14, 8), \qquad (\bar{c}_1, \bar{c}_2) = (0, 0)$$

with validity ranges

$$[b_1^{\min}, b_1^{\max}] = [40, 240], \qquad [b_2^{\min}, b_2^{\max}] = [30, 180],$$
$$[c_1^{\min}, c_1^{\max}] = [50/3, 100], \quad [c_2^{\min}, c_2^{\max}] = [15, 90].$$

Both constraints are active.

Example: duality on the 2×2 LP (2/4)

Step 2 — verification of the shadow prices.

- One extra unit of b_1 is worth 14, one extra unit of b_2 is worth 8
- $\tilde{x}_1$ and $\tilde{x}_2$ are different from zero: by complementary slackness their dual constraints must be active ($\bar{c}_1 = \bar{c}_2 = 0$):

$$\tilde{\pi}_1 + 2\tilde{\pi}_2 = 14 + 16 = 30 = c_1, \qquad 3\tilde{\pi}_1 + \tilde{\pi}_2 = 42 + 8 = 50 = c_2 \quad \checkmark$$

Example: duality on the 2×2 LP (3/4)

Step 3 — verification of strong duality.

$$\sum_{i \in M} b_i \, \tilde{\pi}_i = 90 \cdot 14 + 80 \cdot 8 = 1260 + 640 = 1900 = \tilde{z} \quad \checkmark$$

Example: duality on the 2×2 LP (4/4)

Step 4 — verification of the reduced costs. With $\bar{c}_j = c_j - \sum_{i \in M} a_{ij}\, \tilde{\pi}_i$:

$$\bar{c}_1 = 30 - (1 \cdot 14 + 2 \cdot 8) = 0, \qquad \bar{c}_2 = 50 - (3 \cdot 14 + 1 \cdot 8) = 0$$

- Both null, as for every positive variable at the optimum: the two variables repay exactly the resources they consume
- Only a variable sitting on one of its bounds can have a reduced cost different from zero

Example: verification on all the cases (1/5)

Let us take up again the "all the cases" LP and its dual: the cases are the same ones that will come back, one at a time, in the application chapters.

Step 1 — solving we obtain:

$$(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (60,\ 40,\ 0), \qquad \tilde{z} = 620,$$

$$(\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3) = (0,\ 6,\ -1), \qquad (\bar{c}_1, \bar{c}_2, \bar{c}_3) = (0,\ 0,\ -3)$$

- ranges of the right-hand sides: $(-\infty, 100]$, $[30, +\infty)$, $[-200, +\infty)$
- ranges of the coefficients: $(-\infty, 8]$, $[5, 17]$, $(-\infty, -6]$
- the second and third constraints are active, with $\tilde{x}_3 = 0$

Example: verification on all the cases (2/5)

Step 2 — shadow prices with the rules of the table.

- $\tilde{\pi}_1 = 0$, consistent with complementarity: the first constraint is slack ($\tilde{x}_1 + \tilde{x}_2 = 100 > 30$)
- $\tilde{\pi}_2 = 6$ (free dual variable, here positive)
- $\tilde{\pi}_3 = -1 \leq 0$, as it must be for a constraint $\leq$ in a minimization

Example: verification on all the cases (3/5)

The dual constraints of x_1 and x_2 (different from zero) must be active:

$$\underbrace{0+6-1}_{x_1} = 5 = c_1, \qquad \underbrace{0+6-2\cdot(-1)}_{x_2} = 8 = c_2 \quad \checkmark$$

(x_2 is free: its dual constraint is an equality)

Verification by perturbation:

$$b_2 = 101 \;\Rightarrow\; \tilde{z} = 626\,(+6), \qquad b_3 = -19 \;\Rightarrow\; \tilde{z} = 619\,(-1) \quad \checkmark$$

Example: verification on all the cases (4/5)

Step 3 — verification of strong duality.

$$\sum_{i \in M} b_i \, \tilde{\pi}_i = 30 \cdot 0 + 100 \cdot 6 + (-20) \cdot (-1) = 620 = \tilde{z} \quad \checkmark$$

Step 4 — reduced costs, one per sign. With $\bar{c}_j = c_j - \sum_{i \in M} a_{ij} \, \tilde{\pi}_i$:

$$\bar{c}_1 = 5 - (0 + 6 - 1) = 0, \qquad \bar{c}_2 = 8 - (0 + 6 + 2) = 0$$

consistent with $\tilde{x}_1, \tilde{x}_2 \neq 0$

Example: verification on all the cases (5/5)

x_3 equals zero, which for a variable ≤ 0 is its *upper bound*:

$$\bar{c}_3 = -9 - (-1) \cdot 6 = -3$$

- forcing $x_3 = -1$ would worsen the optimum by 3
- x_3 would become active only if its coefficient rose to at least -6 (precisely the endpoint $c_3^{\max}$ of the validity range)

Verification:

$$x_3 = -1 \;\Rightarrow\; \tilde{z} = 623\,(+3) \quad \checkmark$$

Sensitivity analysis

- Solving an LP gives an optimal solution
- **Sensitivity analysis** answers the next question, the one that really matters to a decision maker:

  how does the optimum change if the data change?
- In LPs a large part of the answer comes for free together with the solution, in the form of **four numbers** for each constraint and variable

The four numbers (1/2)

- The **shadow price** $\tilde{\pi}_i$ of a constraint: by how much the optimal value improves if the right-hand side of the constraint increases by one unit

  "how much I would pay for one extra unit of that resource"
- The **validity range of the right-hand side** $[b_i^{\min}, b_i^{\max}]$: where that shadow price remains exact (outside it, the optimization must be redone)

The four numbers (2/2)

- The **reduced cost** $\bar{c}_j$ of a variable at zero: by how much its objective coefficient must improve for it to become worth activating
- The **validity range of the objective coefficient** $[c_j^{\min}, c_j^{\max}]$: where the coefficient can vary without the optimal solution changing
- In the 2×2 LP: c_1 can vary in $[50/3 \approx 16.7,\ 100]$ and c_2 in $[15, 90]$ without moving the solution $(30, 20)$

Example: shadow prices in practice (1/2)

2×2 LP: optimum 1900 with duals $\tilde{\pi}_1 = 14$ and $\tilde{\pi}_2 = 8$.

Reading. One extra unit of b_1 makes the optimum grow by 14; if obtaining it cost 12, it would be worth it ($14 > 12$).

Verification by perturbation. Re-solving:

$$b_1 = 91 \;\Rightarrow\; 1914 = 1900{+}14, \qquad b_1 = 100 \;\Rightarrow\; 2040 = 1900{+}10{\cdot}14$$

the shadow price is still exact because 100 lies inside the validity range $[40, 240]$.

Example: shadow prices in practice (2/2)

Where it stops holding.

- At $b_1 = 240$ the first constraint stops being the scarce resource
- Beyond that, every additional unit is worth less than 14 and everything must be recomputed
- This is why every shadow price is quoted *together* with its validity range

Inactive constraints.

- If we had $b_1 = 300$ (constraint inactive at the optimum), its shadow price would be 0
- A resource left over is worth nothing at the margin

Example: is a third variable worth it? (1/3)

- In the 2×2 LP both variables are positive at the optimum and their reduced costs equal 0: this is always the case for variables different from zero
- To see a reduced cost at work we need a variable at zero
- Let us add to the model a variable $x_3 \geq 0$ with coefficient $c_3 = 20$ in the objective and consumptions $a_{13} = a_{23} = 1$ in the two constraints

Example: is a third variable worth it? (2/3)

Step 1 — valuation with the shadow prices. The resources absorbed by one unit of x_3 are worth, at the shadow prices,

$$1 \cdot 14 + 1 \cdot 8 = 22 > 20 = c_3$$

every unit of x_3 would take resources away from better uses.

Step 2 — the reduced cost. The difference is precisely the reduced cost:

$$\bar{c}_3 = 20 - 22 = -2$$

Solving confirms it: the optimal solution remains $(30, 20, 0)$ with $\tilde{z} = 1900$, x_3 is at zero with $\bar{c}_3 = -2$ and $c_3^{\max} = 22$.

Example: is a third variable worth it? (3/3)

- The coefficient c_3 would have to rise to at least 22 for it to be worth activating x_3
- **Step 3 — counter-check.** With $c_3 = 23 > 22$ the solution really does change, and not by a little: the new optimal solution is $(0, 5, 75)$ with value 1975
- The reduced cost tells us *when* a variable at zero enters the solution, not *how* the solution changes: that has to be recomputed

Summary: Linear programming theory (1/2)

Duality:

- Every constraint generates a dual variable, every variable a dual constraint
- Weak duality: the dual gives a bound; strong duality: at the optimum the values coincide

Complementary slackness:

- $\bar{\pi}_i \cdot \bar{s}_i = 0$ and $\bar{x}_j \cdot \bar{c}_j = 0$
- Slack constraint $\Rightarrow$ null dual; non-null variable $\Rightarrow$ null reduced cost

Summary: Linear programming theory (2/2)

Sensitivity:

- Shadow price $\tilde{\pi}_i = \partial \tilde{z} / \partial b_i$, valid within its range $[b_i^{\min}, b_i^{\max}]$
- The sign is deduced from: does the region widen or narrow? Min or max?
- Reduced cost: by how much c_j must improve for a variable at zero to be worth it
- A number is always quoted *together* with its validity range

The sensitivity analysis protocol

Introduction: The sensitivity protocol

- Sensitivity analysis is **part of the model**, not an appendix
- In every laboratory session we follow the same **six-step protocol**
- We close with an exercise on duality

The six-step protocol (1/2)

1. Base scenario	solve, verify feasibility, identify the active constraints
2. One-at-a-time	vary one key parameter over a grid; plot value/decisions
3. Shadow prices and reduced costs	compare the marginal prediction of the dual with a re-optimization after a small perturbation; read the reduced cost $\bar{c}_j$ of every variable at zero: the threshold beyond which it is worth activating it

The six-step protocol (2/2)

4. Scenarios	build at least three consistent scenarios (pessimistic, central, optimistic)
5. Trade-off	vary a weight or a threshold to trace a frontier (cost-service, risk-return, cost-emissions)
6. Stability	flag abrupt changes in the optimal mix and fragile estimated parameters

Exercise: duality

- In the 2×2 LP of the example it is possible to increase b_1 by 10 units by paying 12 per unit. Is it worth it?
- Up to which unit price is it worth it?
- Up to which value of b_1 does the shadow price of 14 hold?

Hint: recompute the optimal vertex when the first constraint moves to $90 + \Delta$ and find the Δ beyond which the solution changes structure.

Solution

- With the first constraint at $90 + \Delta$ the optimal vertex solves $x_1 + 3x_2 = 90 + \Delta$, $2x_1 + x_2 = 80$:

$$\tilde{x}_1 = \frac{150 - \Delta}{5}, \qquad \tilde{x}_2 = \frac{100 + 2\Delta}{5}$$

- $\tilde{x}_1 \geq 0$ as long as $\Delta \leq 150$, that is up to $b_1 = 240$: up to there the shadow price 14 is exact
- 10 units are worth $10 \cdot 14 = 140$ and cost $10 \cdot 12 = 120$: it is worth it (net gain 20)
- It is worth it as long as the unit price is below 14

Summary: The sensitivity protocol

- Sensitivity is part of the model
- Six steps: base scenario, one-at-a-time, shadow prices and reduced costs, scenarios, trade-off, stability
- Every marginal prediction must be compared with a re-optimization

Final summary (1/2)

Linear programming:

- Primal and dual, linked by weak and strong duality
- Complementary slackness as optimality conditions
- Shadow prices, reduced costs and validity ranges

Final summary (2/2)

In the laboratory:

- Shadow prices are verified by perturbation
- Every shadow price is quoted with its validity range
- Every laboratory session follows the six-step sensitivity protocol