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Linear Programming

Volume 5 — Modelling

Techniques for writing a linear programming model from the description of a problem (production, blending, transportation, multi-period planning, linearizations), with worked examples. Complete models are in Exercises 5.

Fabio Furini

fabiofurini.github.io

Department of Computer, Control and Management Engineering — Sapienza University of Rome

Online version and downloadable material:
fabiofurini.github.io/linear-programming/
github.com/fabiofurini/linear-programming

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Linear Programming modeling techniques

16.1 From a problem description to an LP model

- A **model** of an optimization problem is a mathematical description of its feasible solutions and of the quantity to be optimized. Writing a model requires three steps, always in this order.

1. **Decision variables.** Identify the decisions to be taken and associate one variable with each of them. For every variable, state its unit of measure and its domain.
2. **Objective function.** Write, as a linear function of the decision variables, the quantity to be maximized or minimized.
3. **Constraints.** Write, as linear equalities or inequalities, every requirement that a feasible solution has to satisfy.

- The model is written with **symbolic input data**, so that it describes the problem for every instance, and not only for one of them. We use the following conventions:
 1. the sizes of the instance are integer numbers, for example $r \in \mathbb{Z}_{\geq 1}$ and $s \in \mathbb{Z}_{\geq 1}$;
 2. the numerical data are rational numbers, for example $a_i \in \mathbb{Q}_{>0}$ and $w_{ij} \in \mathbb{Q}_{\geq 0}$;
 3. the index sets are written explicitly, for example $\forall i \in \{1, 2, \dots, r\}$.
- A model is a **Linear Programming** model if its objective function and all its constraints are linear, and if all its variables are continuous. In this case the model can be solved with the simplex algorithm.

The three most frequent mistakes when writing an LP model are:

1. forgetting the **non-negativity constraints** of the variables;
2. writing constraints which are not dimensionally consistent, that is, which sum quantities with different units of measure;
3. using a variable in the objective function without defining it among the decision variables.

16.2 Production planning

- A company produces $r \in \mathbb{Z}_{\geq 1}$ different products using $s \in \mathbb{Z}_{\geq 1}$ limited resources. For each product $j \in \{1, 2, \dots, r\}$, the value $c_j \in \mathbb{Q}_{>0}$ is the profit of one unit of product; for each resource $i \in \{1, 2, \dots, s\}$, the value $b_i \in \mathbb{Q}_{>0}$ is the available quantity, and $a_{ij} \in \mathbb{Q}_{\geq 0}$ is the quantity of resource i consumed by one unit of product j . The company aims to determine a maximum profit production plan.
- We introduce the following r non-negative variables:

$$x_j = \text{units of product } j \text{ produced}, \quad \forall j \in \{1, 2, \dots, r\}.$$

$$\max \sum_{j=1}^r c_j x_j \quad (16.1a)$$

$$\text{subject to } \sum_{j=1}^r a_{ij} x_j \leq b_i, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.1b)$$

$$x_j \geq 0, \quad \forall j \in \{1, 2, \dots, r\}. \quad (16.1c)$$

- The objective function (16.1a) is the total profit. The constraints (16.1b) impose that the consumption of every resource does not exceed its availability. This is an LP problem in **canonical maximization form**.

Example 16.1: production planning

Consider an instance with $r = 2$ products and $s = 3$ resources:

	$j = 1$	$j = 2$	
c_j	3	5	b_1
a_{1j}	1	0	b_2
a_{2j}	0	2	b_3
a_{3j}	3	2	

The LP model for this instance reads as follows:

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ \text{subject to} \quad & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

An optimal solution is $\tilde{x}_1 = 2$ and $\tilde{x}_2 = 6$, with optimal value $z(LP) = 36$.

16.3 Blending

- A quantity $d \in \mathbb{Q}_{>0}$ of a final product has to be obtained by mixing $r \in \mathbb{Z}_{\geq 1}$ raw materials. For each raw material $j \in \{1, 2, \dots, r\}$, the value $c_j \in \mathbb{Q}_{>0}$ is the cost of one unit. The final product has to contain at least a quantity $q_k \in \mathbb{Q}_{\geq 0}$ of each of $s \in \mathbb{Z}_{\geq 1}$ components, and $p_{kj} \in \mathbb{Q}_{\geq 0}$ is the quantity of component k contained in one unit of raw material j . The company aims to determine a minimum cost blend.
- We introduce the following r non-negative variables:

$$x_j = \text{units of raw material } j \text{ used in the blend}, \quad \forall j \in \{1, 2, \dots, r\}.$$

$$\min \sum_{j=1}^r c_j x_j \quad (16.2a)$$

$$\text{subject to } \sum_{j=1}^r x_j = d, \quad (16.2b)$$

$$\sum_{j=1}^r p_{kj} x_j \geq q_k, \quad \forall k \in \{1, 2, \dots, s\}, \quad (16.2c)$$

$$x_j \geq 0, \quad \forall j \in \{1, 2, \dots, r\}. \quad (16.2d)$$

- The constraint (16.2b) imposes that the total quantity produced is d , and the constraints (16.2c) impose the minimum quantity of each component.
- A requirement of the form “the raw material j has to be at least a fraction α of the blend” is not written as $\frac{x_j}{\sum_{\ell} x_{\ell}} \geq \alpha$, which is not a linear constraint. It is written instead by multiplying both sides by the denominator, which is positive:

$$x_j \geq \alpha \sum_{\ell=1}^r x_{\ell} \quad \Longleftrightarrow \quad (1 - \alpha) x_j - \alpha \sum_{\ell \neq j} x_{\ell} \geq 0.$$

Example 16.2: blending

A company has to produce $d = 100$ kilograms of animal feed by mixing $r = 3$ raw materials, whose costs per kilogram are $c_1 = 0.80$, $c_2 = 0.50$ and $c_3 = 0.30$ euros. The feed has to contain at least $q_1 = 8$ kilograms of proteins, and one kilogram of each raw material contains, respectively, $p_{11} = 0.12$, $p_{12} = 0.08$ and $p_{13} = 0.03$ kilograms of proteins. Moreover, the first raw material has to be at least the 20% of the blend.

The LP model for this instance reads as follows:

$$\begin{aligned}
 \min \quad & 0.80 x_1 + 0.50 x_2 + 0.30 x_3 \\
 \text{subject to} \quad & x_1 + x_2 + x_3 = 100 \\
 & 0.12 x_1 + 0.08 x_2 + 0.03 x_3 \geq 8 \\
 & 0.80 x_1 - 0.20 x_2 - 0.20 x_3 \geq 0 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

where the third constraint is the proportion requirement $x_1 \geq 0.20 (x_1 + x_2 + x_3)$, rewritten as a linear constraint.

An optimal solution is $\tilde{x}_1 = 20$, $\tilde{x}_2 = 64$ and $\tilde{x}_3 = 16$, with optimal value

$$z(LP) = 0.80 \cdot 20 + 0.50 \cdot 64 + 0.30 \cdot 16 = 52.80 \text{ euros.}$$

16.4 Transportation

- A company has $s \in \mathbb{Z}_{\geq 1}$ warehouses and has to serve $r \in \mathbb{Z}_{\geq 1}$ clients. For each warehouse $i \in \{1, 2, \dots, s\}$, the value $a_i \in \mathbb{Q}_{>0}$ is the available quantity of product; for each client $j \in \{1, 2, \dots, r\}$, the value $d_j \in \mathbb{Q}_{>0}$ is the demand, and $c_{ij} \in \mathbb{Q}_{\geq 0}$ is the cost of transporting one unit of product from warehouse i to client j . The company aims to determine a minimum cost transportation plan.
- We introduce the following $s \cdot r$ non-negative variables:

$$\begin{aligned}
 x_{ij} = & \text{units transported from warehouse } i \text{ to client } j, \\
 & \forall i \in \{1, 2, \dots, s\}, \forall j \in \{1, 2, \dots, r\}.
 \end{aligned}$$

$$\min \sum_{i=1}^s \sum_{j=1}^r c_{ij} x_{ij} \quad (16.3a)$$

$$\text{subject to } \sum_{j=1}^r x_{ij} \leq a_i, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.3b)$$

$$\sum_{i=1}^s x_{ij} \geq d_j, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.3c)$$

$$x_{ij} \geq 0, \quad \forall i \in \{1, 2, \dots, s\}, \forall j \in \{1, 2, \dots, r\}. \quad (16.3d)$$

- The constraints (16.3b) impose that no warehouse ships more than its availability, and the constraints (16.3c) impose that every client receives at least its demand. The model is feasible if and only if $\sum_i a_i \geq \sum_j d_j$.

Example 16.3: transportation

Consider an instance with $s = 2$ warehouses, of availabilities $a_1 = 50$ and $a_2 = 60$, and $r = 2$ clients, of demands $d_1 = 30$ and $d_2 = 80$. The transportation costs are:

c_{ij}	$j = 1$	$j = 2$
$i = 1$	4	6
$i = 2$	5	3

An optimal solution is

$$\tilde{x}_{11} = 30, \quad \tilde{x}_{12} = 20, \quad \tilde{x}_{21} = 0, \quad \tilde{x}_{22} = 60,$$

with optimal value

$$z(LP) = 4 \cdot 30 + 6 \cdot 20 + 5 \cdot 0 + 3 \cdot 60 = 420.$$

16.5 Multi-period planning

- A company has to satisfy the demand of a product over a time horizon of $n \in \mathbb{Z}_{\geq 1}$ periods. For each period $t \in \{1, 2, \dots, n\}$, the value $d_t \in \mathbb{Q}_{\geq 0}$ is the demand, $p_t \in \mathbb{Q}_{>0}$ is the maximum production, $c_t \in \mathbb{Q}_{>0}$ is the unit production cost and $h_t \in \mathbb{Q}_{>0}$ is the unit cost of storing one unit from period t to period $t + 1$.
- We introduce the following $2n$ non-negative variables:

$$\begin{cases} x_t = & \text{units produced in period } t \\ y_t = & \text{units stored at the end of period } t \end{cases} \quad \forall t \in \{1, 2, \dots, n\}.$$

$$\min \sum_{t=1}^n c_t x_t + \sum_{t=1}^n h_t y_t \quad (16.4a)$$

$$\text{subject to } y_{t-1} + x_t - y_t = d_t, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.4b)$$

$$x_t \leq p_t, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.4c)$$

$$x_t \geq 0, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.4d)$$

$$y_t \geq 0, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.4e)$$

with $y_0 = 0$.

- The constraints (16.4b) are the **inventory balance constraints**: in every period, what was in stock at the end of the previous period plus what is produced equals the demand plus what is left in stock. They are the key of every multi-period model, and they link consecutive periods.

16.6 Linearization techniques

- Some requirements are not linear as they are written, but they can be rewritten as linear constraints, possibly with the help of additional variables. We present the three most frequent cases.

16.6.1 Min-max and max-min objective functions

An objective function of the form

$$\min \max_{i \in \{1, 2, \dots, m\}} \left\{ \sum_{j=1}^n a_{ij} x_j \right\}$$

is linearized by introducing one additional free variable w and writing

$$\min w \quad \text{subject to} \quad w \geq \sum_{j=1}^n a_{ij} x_j, \quad \forall i \in \{1, 2, \dots, m\}.$$

- The m constraints impose that w is not smaller than each of the m linear functions, that is, that w is not smaller than their maximum. Since w is minimized, at an optimal solution it is exactly equal to that maximum.
- Symmetrically, a **max-min** objective function is linearized by introducing one additional free variable w , maximizing it and imposing $w \leq \sum_j a_{ij} x_j$ for every i .
- The linearization works only in these two directions: a maximization of a maximum, or a minimization of a minimum, cannot be linearized in this way.

16.6.2 Absolute values

A term $\left| \sum_{j=1}^n a_{ij} x_j \right|$ appearing with a positive coefficient in a minimized objective function is linearized by introducing one additional non-negative variable w_i and writing

$$w_i \geq \sum_{j=1}^n a_{ij} x_j \quad \text{and} \quad w_i \geq - \sum_{j=1}^n a_{ij} x_j.$$

- The two constraints impose $w_i \geq \left| \sum_j a_{ij} x_j \right|$, and the minimization makes the inequality tight at an optimal solution.
- Alternatively, under the same condition (positive coefficient in a minimized objective function, or negative coefficient in a maximized one), the same term is linearized by writing $\sum_j a_{ij} x_j = w_i^+ - w_i^-$ with $w_i^+, w_i^- \geq 0$, and by replacing the absolute value with $w_i^+ + w_i^-$. The optimization makes at least one of w_i^+ and w_i^- equal to 0 at an optimal solution, and then $w_i^+ + w_i^- = \left| \sum_j a_{ij} x_j \right|$; without that condition, both variables could be positive and the model would be wrong.

16.6.3 Ratios and percentages

A constraint of the form

$$\frac{\sum_{j=1}^n a_j x_j}{\sum_{j=1}^n e_j x_j} \leq \gamma, \quad \text{with} \quad \sum_{j=1}^n e_j x_j > 0,$$

is linearized by multiplying both sides by the denominator, which is positive:

$$\sum_{j=1}^n (a_j - \gamma e_j) x_j \leq 0.$$

- The same technique applies to the percentage requirements of the blending models. It is correct only when the denominator is known to be positive for every feasible solution: otherwise multiplying by it would change the direction of the inequality.

16.7 What cannot be modeled with Linear Programming

The following requirements cannot be written as linear constraints on continuous variables, and they need **integer** variables:

1. **indivisible quantities**, for example the number of machines to buy or of vehicles to use;
2. **yes/no decisions**, for example whether to open a warehouse or not;
3. **fixed costs**, that is, a cost which is paid only if the associated activity is performed;
4. **either-or constraints**, that is, requirements of which at least one out of two has to hold;
5. **minimum positive quantities**, for example “*either produce nothing, or produce at least q units*”.

These requirements are the subject of **Integer Linear Programming**.

- A frequent mistake is to write a fixed cost as a non-linear term of the objective function, or to impose a minimum positive quantity with the constraint $x_j \geq q$: the latter forbids the solution $x_j = 0$, which instead has to remain feasible.
- Whenever the requirements of a problem are of the kind listed above, the LP model obtained by ignoring them is still useful: it is a **relaxation** of the correct model, and then its optimal value is an upper bound on the optimal value of the correct model, for a maximization problem, and a lower bound for a minimization problem.

16.8 Examples of models

16.8.1 Symbolic models

Example 16.4 — Production with regular time and overtime

A company produces $r \in \mathbb{Z}_{\geq 1}$ different products. For each product $i \in \{1, 2, \dots, r\}$, the value $w_i \in \mathbb{Q}_{>0}$ is the maximum production quantity in liters of the product per day. Each product can be further treated and sold at a higher price. The maximum quantity per day of all products that can be treated is $d \in \mathbb{Q}_{>0}$ liters during regular time and $e \in \mathbb{Q}_{>0}$ liters during overtime. For each product $i \in \{1, 2, \dots, r\}$, the value $a_i \in \mathbb{Q}_{>0}$ is the profit in euros of one liter of product not treated; the values $b_i \in \mathbb{Q}_{>0}$ and $c_i \in \mathbb{Q}_{>0}$ are the profits in euros of one liter of product treated during regular time and overtime, respectively. The company aims to determine a maximum profit daily production plan subject to the constraints described above.

Questions

1. Write a *Linear Programming* (LP) model for the problem. Identify the decisions that must be taken and the corresponding decision variables. Identify and comment the objective function of the model and the constraints. What algorithm can be used to find an optimal solution of the model?
2. Consider now an instance of the problem in which there are $r = 3$ products. These are the other input data:

	$i = 1$	$i = 2$	$i = 3$		
w_i	35	30	25	d	e
a_i	3	2	2	30	20
b_i	6	4	4		
c_i	5	3	3		

Write the LP model for this specific instance.

3. Find a feasible solution of the LP model of the instance and accordingly a lower bound on the optimal solution value of the LP model for the instance.
4. Write the dual model of the LP model and the dual model for the instance. Find a feasible solution of the dual model of the instance and accordingly an upper bound on the optimal solution value of the LP model for the instance.

Solution

1. We introduce the following 3 r non-negative variables:

$$\begin{cases} x_i = & \text{liters of product } i \text{ not treated per day} \\ y_i = & \text{liters of product } i \text{ treated in regular time per day} \\ z_i = & \text{liters of product } i \text{ treated in overtime per day} \end{cases} \quad \forall i \in \{1, 2, \dots, r\}.$$

Using these variables, an LP model for the problem reads as follows:

$$\max \sum_{i=1}^r a_i x_i + \sum_{i=1}^r b_i y_i + \sum_{i=1}^r c_i z_i \quad (16.5a)$$

$$\text{subject to } x_i + y_i + z_i \leq w_i, \quad \forall i \in \{1, 2, \dots, r\}, \quad (16.5b)$$

$$\sum_{i=1}^r y_i \leq d, \quad (16.5c)$$

$$\sum_{i=1}^r z_i \leq e, \quad (16.5d)$$

$$x_i \geq 0, \quad \forall i \in \{1, 2, \dots, r\}, \quad (16.5e)$$

$$y_i \geq 0, \quad \forall i \in \{1, 2, \dots, r\}, \quad (16.5f)$$

$$z_i \geq 0, \quad \forall i \in \{1, 2, \dots, r\}. \quad (16.5g)$$

Description of the objective function and constraints of the LP model:

- the linear objective function (16.5a) maximizes the total profit of the production plan;
- the linear constraints (16.5b) ensure that, for each product, the total production does not exceed the available capacity (r linear constraints);
- the linear constraint (16.5c) ensures that the total quantity of products treated during regular time does not exceed the available capacity (one linear constraint);
- the linear constraint (16.5d) ensures that the total quantity of products treated during overtime does not exceed the available capacity (one linear constraint);
- the linear constraints (16.5e), (16.5f), and (16.5g) define the variables of the LP model.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. The LP model (16.5) for the instance reads as follows:

$$\begin{aligned}
 \max \quad & 3x_1 + 2x_2 + 2x_3 + 6y_1 + 4y_2 + 4y_3 + 5z_1 + 3z_2 + 3z_3 \\
 \text{subject to} \quad & x_1 + y_1 + z_1 \leq 35 \\
 & x_2 + y_2 + z_2 \leq 30 \\
 & x_3 + y_3 + z_3 \leq 25 \\
 & y_1 + y_2 + y_3 \leq 30 \\
 & z_1 + z_2 + z_3 \leq 20 \\
 & x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \geq 0
 \end{aligned}$$

3. For example, by setting:

$$\bar{x}_1 = 5, \bar{x}_2 = 30, \bar{x}_3 = 25, \bar{y}_1 = 30, \bar{y}_2 = 0, \bar{y}_3 = 0, \bar{z}_1 = 0, \bar{z}_2 = 0, \bar{z}_3 = 0$$

we have a feasible solution. The objective function value of this feasible solution is:

$$\sum_{i=1}^r a_i \bar{x}_i + \sum_{i=1}^r b_i \bar{y}_i + \sum_{i=1}^r c_i \bar{z}_i = 3 \cdot 5 + 2 \cdot 30 + 2 \cdot 25 + 6 \cdot 30 + 4 \cdot 0 + 4 \cdot 0 + 5 \cdot 0 + 3 \cdot 0 + 3 \cdot 0 = 305$$

accordingly 305 is a lower bound for the optimal solution value of the LP model (16.5). So, for this instance, we have:

$$305 \leq z(LP)$$

4. By using

- a non-negative dual variable λ_i for each constraint (16.5b) ($i \in \{1, 2, \dots, r\}$)
- a non-negative dual variable π for the constraint (16.5c)
- a non-negative dual variable ϱ for the constraint (16.5d)

the dual of the LP model (16.5) reads as follows:

$$\min \sum_{i=1}^r w_i \lambda_i + d \pi + e \varrho \quad (16.6a)$$

$$\text{subject to } \lambda_i \geq a_i, \quad \forall i \in \{1, 2, \dots, r\}, \quad (16.6b)$$

$$\lambda_i + \pi \geq b_i, \quad \forall i \in \{1, 2, \dots, r\}, \quad (16.6c)$$

$$\lambda_i + \varrho \geq c_i, \quad \forall i \in \{1, 2, \dots, r\}, \quad (16.6d)$$

$$\lambda_i \geq 0, \quad \forall i \in \{1, 2, \dots, r\}, \quad (16.6e)$$

$$\pi \geq 0, \quad (16.6f)$$

$$\varrho \geq 0. \quad (16.6g)$$

The dual model (16.6) for the specific instance reads as follows:

$$\begin{array}{llllll}
 \min & 35 \lambda_1 + 30 \lambda_2 + 25 \lambda_3 + 30 \pi + 20 \varrho & & & & \\
 \text{subject to} & \lambda_1 & & & & \geq 3 \\
 & & \lambda_2 & & & \geq 2 \\
 & & & \lambda_3 & & \geq 2 \\
 & \lambda_1 & & + & \pi & \geq 6 \\
 & & \lambda_2 & & + & \pi & \geq 4 \\
 & & & \lambda_3 & + & \pi & \geq 4 \\
 & \lambda_1 & & & + & \varrho & \geq 5 \\
 & & \lambda_2 & & + & \varrho & \geq 3 \\
 & & & \lambda_3 & + & \varrho & \geq 3 \\
 & \lambda_1, & \lambda_2, & \lambda_3, & \pi, & \varrho & \geq 0
 \end{array}$$

For example, by setting:

$$\bar{\pi} = 0, \quad \bar{\varrho} = 0,$$

and by setting:

$$\bar{\lambda}_1 = \max\{3, 6, 5\} = 6, \quad \bar{\lambda}_2 = \max\{2, 4, 3\} = 4, \quad \bar{\lambda}_3 = \max\{2, 4, 3\} = 4$$

we have a feasible dual solution.

The objective function value of this feasible dual solution is:

$$\sum_{i=1}^r w_i \bar{\lambda}_i + d \bar{\pi} + e \bar{\varrho} = 35 \cdot 6 + 30 \cdot 4 + 25 \cdot 4 + 30 \cdot 0 + 20 \cdot 0 = 430$$

Accordingly 430 is an upper bound on the optimal solution value of the model. So, for this instance, we have:

$$305 \leq z(LP) \leq 430$$

Example 16.5 — Transportation from warehouses to clients

A company has $s \in \mathbb{Z}_{\geq 1}$ warehouses and transports its product to $r \in \mathbb{Z}_{\geq 1}$ clients. For each warehouse $i \in \{1, 2, \dots, s\}$, the value $d_i \in \mathbb{Q}_{>0}$ represents the maximum capacity in liters of product that can be transported from that warehouse to the clients per week. For each client $j \in \{1, 2, \dots, r\}$, the value $e_j \in \mathbb{Q}_{>0}$ denotes the demand in liters of product of that client per week. For each warehouse $i \in \{1, 2, \dots, s\}$ and each client $j \in \{1, 2, \dots, r\}$, the value $f_{ij} \in \mathbb{Q}_{>0}$ represents the cost in euros of transporting one liter of product from warehouse i to client j . The company aims to determine a weekly minimum-cost transportation plan for the product from the warehouses to the clients subject to the constraints described above.

Questions

1. Write a *Linear Programming* (LP) model for the problem. Identify the decisions that must be taken and the corresponding decision variables. Identify and comment the objective function of the model and the constraints. What algorithm can be used to find an optimal solution of the model?
2. Consider now an instance of the problem in which there are $s = 2$ warehouses and $r = 3$ clients. These are the other input data:

f_{ij}	$j = 1$	$j = 2$	$j = 3$
$i = 1$	3	2	1
$i = 2$	2	4	1

	$i = 1$	$i = 2$
d_i	120	110

	$j = 1$	$j = 2$	$j = 3$
e_j	50	60	100

Write the LP model for this instance.

3. Find a feasible solution of the LP model of the instance and accordingly an upper bound on the optimal solution value of the LP model for the instance.
4. Write the dual model of the LP model and the dual model for the instance. Find a feasible solution of the dual model of the instance and accordingly a lower bound on the optimal solution value of the LP model for the instance.

Solution

1. We introduce the following $r \times s$ non-negative variables:

$$x_{ij} = \text{liters of product transported from warehouse } i \text{ to client } j, \\ \forall i \in \{1, 2, \dots, s\}, \forall j \in \{1, 2, \dots, r\}.$$

Using these variables, an LP model for the problem reads as follows:

$$\min \sum_{i=1}^s \sum_{j=1}^r f_{ij} x_{ij} \quad (16.7a)$$

$$\text{subject to } \sum_{j=1}^r x_{ij} \leq d_i, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.7b)$$

$$\sum_{i=1}^s x_{ij} = e_j, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.7c)$$

$$x_{ij} \geq 0, \quad \forall i \in \{1, 2, \dots, s\}, \forall j \in \{1, 2, \dots, r\}. \quad (16.7d)$$

Description of the objective function and constraints of the LP model:

- the linear objective function (16.7a) minimizes the total transportation cost;

- the linear constraints (16.7b) ensure that the total quantity of product transported from each warehouse does not exceed its available capacity (s linear constraints);
- the linear constraints (16.7c) ensure that each client receives exactly the amount of product it has requested (r linear constraints);
- the linear constraints (16.7d) define the variables of the LP model.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. The LP model (16.7) for the specific instance reads as follows:

$$\begin{aligned}
 \min \quad & 3x_{11} + 2x_{12} + x_{13} + 2x_{21} + 4x_{22} + x_{23} \\
 \text{subject to} \quad & x_{11} + x_{12} + x_{13} \leq 120 \\
 & x_{21} + x_{22} + x_{23} \leq 110 \\
 & x_{11} + x_{21} = 50 \\
 & x_{12} + x_{22} = 60 \\
 & x_{13} + x_{23} = 100 \\
 & x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0
 \end{aligned}$$

3. For example, by setting:

$$\bar{x}_{11} = 50, \bar{x}_{12} = 60, \bar{x}_{13} = 10, \bar{x}_{21} = 0, \bar{x}_{22} = 0, \bar{x}_{23} = 90$$

we have a feasible solution. The objective function value of this feasible solution is:

$$\sum_{i=1}^s \sum_{j=1}^r f_{ij} \bar{x}_{ij} = 3 \cdot 50 + 2 \cdot 60 + 1 \cdot 10 + 2 \cdot 0 + 4 \cdot 0 + 1 \cdot 90 = 370$$

accordingly 370 is an upper bound on the optimal solution value of the LP model (16.7). So, for this instance, we have:

$$z(LP) \leq 370$$

4. By using

- a non-positive dual variable μ_i for each constraint (16.7b) ($i \in \{1, 2, \dots, s\}$)
- a free dual variable π_j for each constraint (16.7c) ($j \in \{1, 2, \dots, r\}$)

the dual model of the LP model (16.7) reads as follows:

$$\max \sum_{i=1}^s d_i \mu_i + \sum_{j=1}^r e_j \pi_j \quad (16.8a)$$

$$\text{subject to } \mu_i + \pi_j \leq f_{ij}, \quad \forall i \in \{1, 2, \dots, s\}, \forall j \in \{1, 2, \dots, r\}, \quad (16.8b)$$

$$\mu_i \leq 0, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.8c)$$

$$\pi_j \geq 0, \quad \forall j \in \{1, 2, \dots, r\}. \quad (16.8d)$$

The dual model (16.8) for the specific instance reads as follows:

$$\begin{aligned} \max \quad & 120 \mu_1 + 110 \mu_2 + 50 \pi_1 + 60 \pi_2 + 100 \pi_3 \\ \text{subject to} \quad & \mu_1 + \pi_1 \leq 3 \\ & \mu_1 + \pi_2 \leq 2 \\ & \mu_1 + \pi_3 \leq 1 \\ & \mu_2 + \pi_1 \leq 2 \\ & \mu_2 + \pi_2 \leq 4 \\ & \mu_2 + \pi_3 \leq 1 \\ & \mu_1, \mu_2 \leq 0 \\ & \pi_1, \pi_2, \pi_3 \geq 0 \end{aligned}$$

For example, by setting:

$$\bar{\mu}_1 = 0, \quad \bar{\mu}_2 = 0$$

and by setting:

$$\bar{\pi}_1 = \min\{3, 2\} = 2, \quad \bar{\pi}_2 = \min\{2, 4\} = 2, \quad \bar{\pi}_3 = \min\{1, 1\} = 1$$

we have a feasible dual solution.

The objective function value of this feasible dual solution is:

$$\sum_{i=1}^s d_i \bar{\mu}_i + \sum_{j=1}^r e_j \bar{\pi}_j = 120 \cdot 0 + 110 \cdot 0 + 50 \cdot 2 + 60 \cdot 2 + 100 \cdot 1 = 320$$

Accordingly 320 is a lower bound on the optimal solution value of the model. So, for this instance, we have:

$$320 \leq z(LP) \leq 370$$

Example 16.6 — Diet planning

A school canteen needs to determine the monthly diet for its students. It is considering $s \in \mathbb{Z}_{\geq 1}$ possible types of food (e.g., milk, rice, bread, potatoes, etc.), each containing $r \in \mathbb{Z}_{\geq 1}$ nutrients (e.g., iron, sodium, calcium, etc.). For each food $i \in \{1, 2, \dots, s\}$, the value $w_i \in \mathbb{Q}_{>0}$ represents the cost in euros per kilogram of that food. For each food $i \in \{1, 2, \dots, s\}$ and each nutrient $j \in \{1, 2, \dots, r\}$, the value $g_{ij} \in \mathbb{Q}_{\geq 0}$ denotes the grams of nutrient j provided by one kilogram of food i . For each nutrient $j \in \{1, 2, \dots, r\}$, the diet must contain at least $a_j \in \mathbb{Q}_{\geq 0}$ grams and at most $b_j \in \mathbb{Q}_{>0}$ grams of that nutrient per month. Additionally, for each food $i \in \{1, 2, \dots, s\}$, the diet must include at least $c_i \in \mathbb{Q}_{\geq 0}$ kilograms and at most $d_i \in \mathbb{Q}_{>0}$ kilograms of that food per month. The school canteen aims to determine a minimum-cost monthly diet subject to the constraints described above.

Questions

1. Write a *Linear Programming* (LP) model for the problem. Identify the decisions that must be taken and the corresponding decision variables. Identify and comment the objective function of the model and the constraints. What algorithm can be used to find an optimal solution of the model?
2. Consider now an instance of the problem with $s = 4$ types of food and $r = 3$ nutrients. These are the other input data:

	$j = 1$	$j = 2$	$j = 3$		$i = 1$	$i = 2$	$i = 3$	$i = 4$
a_j	10	20	10	w_i	5	8	2	2
b_j	40	60	50	c_i	1	1	0	1
				d_i	10	15	20	10

g_{ij}	$j = 1$	$j = 2$	$j = 3$
$i = 1$	2	4	2
$i = 2$	1	1	1
$i = 3$	4	6	4
$i = 4$	4	6	2

Write the LP model for this instance.

3. Find a feasible solution of the LP model of the instance and accordingly an upper bound on the optimal solution value of the LP model for the instance.
4. Write the dual model of the LP model and the dual model for the instance. Find a feasible solution of the dual model of the instance and accordingly a lower bound on the optimal solution value of the LP model for the instance.

Solution

1. We introduce the following s non-negative variables:

$$x_i = \text{kilograms of food } i \text{ in the diet per month, } \forall i \in \{1, 2, \dots, s\}.$$

Using these variables, an LP model for the problem reads as follows:

$$\min \sum_{i=1}^s w_i x_i \quad (16.9a)$$

$$\text{subject to } \sum_{i=1}^s g_{ij} x_i \geq a_j, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.9b)$$

$$\sum_{i=1}^s g_{ij} x_i \leq b_j, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.9c)$$

$$x_i \geq c_i, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.9d)$$

$$x_i \leq d_i, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.9e)$$

$$x_i \geq 0, \quad \forall i \in \{1, 2, \dots, s\}. \quad (16.9f)$$

Description of the objective function and constraints of the LP model:

- the linear objective function (16.9a) minimizes the total cost of the diet;
- the linear constraints (16.9b) ensure that the total amount of each nutrient provided by the selected foods is at least equal to its minimum requirement (r linear constraints);
- the linear constraints (16.9c) ensure that the total amount of each nutrient provided by the selected foods does not exceed its maximum requirement (r linear constraints);
- the linear constraints (16.9d) ensure that each type of food included in the diet meets its minimum quantity requirement (s linear constraints);
- the linear constraints (16.9e) ensure that each type of food included in the diet does not exceed its maximum quantity requirement (s linear constraints);
- the linear constraints (16.9f) define the variables of the LP model.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. The LP model (16.9) for the specific instance reads as follows:

$$\begin{array}{ll}
 \min & 5x_1 + 8x_2 + 2x_3 + 2x_4 \\
 \text{subject to} & 2x_1 + x_2 + 4x_3 + 4x_4 \geq 10 \\
 & 4x_1 + x_2 + 6x_3 + 6x_4 \geq 20 \\
 & 2x_1 + x_2 + 4x_3 + 2x_4 \geq 10 \\
 & 2x_1 + x_2 + 4x_3 + 4x_4 \leq 40 \\
 & 4x_1 + x_2 + 6x_3 + 6x_4 \leq 60 \\
 & 2x_1 + x_2 + 4x_3 + 2x_4 \leq 50 \\
 & x_1 \geq 1 \\
 & x_2 \geq 1 \\
 & x_3 \geq 0 \\
 & x_4 \geq 1 \\
 & x_1 \leq 10 \\
 & x_2 \leq 15 \\
 & x_3 \leq 20 \\
 & x_4 \leq 10 \\
 & x_1, x_2, x_3, x_4 \geq 0
 \end{array}$$

3. For example, by setting:

$$\bar{x}_1 = 5, \quad \bar{x}_2 = 1, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 1$$

we have a feasible solution. The objective function value of this feasible solution is:

$$\sum_{i=1}^s w_i \bar{x}_i = 5 \cdot 5 + 8 \cdot 1 + 2 \cdot 0 + 2 \cdot 1 = 35$$

accordingly 35 is an upper bound on the optimal solution value of the LP model (16.9). So, for this instance, we have:

$$z(LP) \leq 35$$

4. By using

- a non-negative variable α_j for each constraint (16.9b) ($j \in \{1, 2, \dots, r\}$)
- a non-positive variable β_j for each constraint (16.9c) ($j \in \{1, 2, \dots, r\}$)
- a non-negative variable γ_i for each constraint (16.9d) ($i \in \{1, 2, \dots, s\}$)
- a non-positive variable δ_i for each constraint (16.9e) ($i \in \{1, 2, \dots, s\}$)

the dual of the LP model (16.9) reads as follows:

$$\begin{aligned} \max \quad & \sum_{j=1}^r a_j \alpha_j + \sum_{j=1}^r b_j \beta_j \\ & + \sum_{i=1}^s c_i \gamma_i + \sum_{i=1}^s d_i \delta_i \end{aligned} \quad (16.10a)$$

$$\text{subject to} \quad \sum_{j=1}^r g_{ij} \alpha_j + \sum_{j=1}^r g_{ij} \beta_j + \gamma_i + \delta_i \leq w_i, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.10b)$$

$$\alpha_j \geq 0, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.10c)$$

$$\beta_j \leq 0, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.10d)$$

$$\gamma_i \geq 0, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.10e)$$

$$\delta_i \leq 0, \quad \forall i \in \{1, 2, \dots, s\}. \quad (16.10f)$$

The dual model (16.10) for the instance reads as follows:

$$\begin{aligned} \max \quad & 10 \alpha_1 + 20 \alpha_2 + 10 \alpha_3 + 40 \beta_1 + 60 \beta_2 + 50 \beta_3 \\ & + \gamma_1 + \gamma_2 + \gamma_4 + 10 \delta_1 + 15 \delta_2 + 20 \delta_3 + 10 \delta_4 \\ \text{subject to} \quad & 2 \alpha_1 + 4 \alpha_2 + 2 \alpha_3 + 2 \beta_1 + 4 \beta_2 + 2 \beta_3 \\ & + \gamma_1 + \delta_1 \leq 5 \\ & \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 \\ & + \gamma_2 + \delta_2 \leq 8 \\ & 4 \alpha_1 + 6 \alpha_2 + 4 \alpha_3 + 4 \beta_1 + 6 \beta_2 + 4 \beta_3 \\ & + \gamma_3 + \delta_3 \leq 2 \\ & 4 \alpha_1 + 6 \alpha_2 + 2 \alpha_3 + 4 \beta_1 + 6 \beta_2 + 2 \beta_3 \\ & + \gamma_4 + \delta_4 \leq 2 \\ & \alpha_1, \alpha_2, \alpha_3, \\ & \gamma_1, \gamma_2, \gamma_3, \gamma_4 \geq 0 \\ & \beta_1, \beta_2, \beta_3, \\ & \delta_1, \delta_2, \delta_3, \delta_4 \leq 0 \end{aligned}$$

For example, by setting:

$$\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_3 = \bar{\beta}_1 = \bar{\beta}_2 = \bar{\beta}_3 = \bar{\delta}_1 = \bar{\delta}_2 = \bar{\delta}_3 = \bar{\delta}_4 = 0$$

and by setting:

$$\bar{\gamma}_1 = 5, \quad \bar{\gamma}_2 = 8, \quad \bar{\gamma}_3 = 2, \quad \bar{\gamma}_4 = 2$$

we have a feasible dual solution.

The objective function value of this feasible dual solution is:

$$\sum_{j=1}^r a_j \bar{\alpha}_j + \sum_{j=1}^r b_j \bar{\beta}_j + \sum_{i=1}^s c_i \bar{\gamma}_i + \sum_{i=1}^s d_i \bar{\delta}_i =$$

$$= 10 \cdot 0 + 20 \cdot 0 + 10 \cdot 0 + 40 \cdot 0 + 60 \cdot 0 + 50 \cdot 0 + 1 \cdot 5 + 1 \cdot 8 + 0 \cdot 2 + 1 \cdot 2 + 10 \cdot 0 + 15 \cdot 0 + 20 \cdot 0 + 10 \cdot 0 = 15$$

Accordingly 15 is a lower bound on the optimal solution value of the model. So, for this instance, we have:

$$15 \leq z(LP) \leq 35$$

Example 16.7 — Diet planning for several classes

A school canteen needs to determine $n \in \mathbb{Z}_{\geq 1}$ monthly diets for its classes of students. It is considering $s \in \mathbb{Z}_{\geq 1}$ possible types of food (e.g., milk, rice, bread, etc.), each containing $r \in \mathbb{Z}_{\geq 1}$ nutrients (e.g., iron, sodium, etc.). For each food $i \in \{1, 2, \dots, s\}$, the value $w_i \in \mathbb{Q}_{>0}$ represents the cost in euros per kilogram of that food. For each food $i \in \{1, 2, \dots, s\}$ and each nutrient $j \in \{1, 2, \dots, r\}$, the value $g_{ij} \in \mathbb{Q}_{\geq 0}$ denotes the grams of nutrient j provided by one kilogram of food i . For each diet $k \in \{1, 2, \dots, n\}$ and each nutrient $j \in \{1, 2, \dots, r\}$, the diet must contain at least $a_{kj} \in \mathbb{Q}_{\geq 0}$ grams and at most $b_{kj} \in \mathbb{Q}_{>0}$ grams of that nutrient per month. Additionally, for each food $i \in \{1, 2, \dots, s\}$, the diets must include at least $c_i \in \mathbb{Q}_{\geq 0}$ kilograms and at most $d_i \in \mathbb{Q}_{>0}$ kilograms of that food per month. The school canteen aims to determine a minimum-cost monthly diets for all classes of students subject to the constraints described above.

Questions

1. Write a *Linear Programming* (LP) model for the problem. Identify the decisions that must be taken and the corresponding decision variables. Identify and comment the objective function of the model and the constraints. What algorithm can be used to find an optimal solution of the model?
2. Consider now an instance of the problem with $n = 2$ diets, $s = 3$ types of food and $r = 2$ nutrients. These are the other input data:

a_{kj}	$j = 1$	$j = 2$	b_{kj}	$j = 1$	$j = 2$
$k = 1$	20	10	$k = 1$	40	20
$k = 2$	10	20	$k = 2$	20	40

	$i = 1$	$i = 2$	$i = 3$	g_{ij}	$j = 1$	$j = 2$
w_i	5	8	6	$i = 1$	6	2
c_i	1	1	0	$i = 2$	2	1
d_i	5	4	10	$i = 3$	2	6

Write the LP model for this instance.

3. Find a feasible solution of the LP model of the instance and accordingly an upper bound on the optimal solution value of the LP model for the instance.
4. Write the dual model of the LP model and the dual model for the instance. Find a feasible solution of the dual model of the instance and accordingly a lower bound on the optimal solution value of the LP model for the instance.

Solution

1. We introduce the following $s \cdot n$ non-negative variables:

x_{ik} = kilograms of food i in the diet k per month,

$$\forall i \in \{1, 2, \dots, s\}, \forall k \in \{1, 2, \dots, n\}.$$

Using these variables, an LP model for the problem reads as follows:

$$\min \sum_{i=1}^s \sum_{k=1}^n w_i x_{ik} \quad (16.11a)$$

$$\text{subject to } \sum_{i=1}^s g_{ij} x_{ik} \geq a_{kj}, \quad \forall k \in \{1, 2, \dots, n\}, \forall j \in \{1, 2, \dots, r\}, \quad (16.11b)$$

$$\sum_{i=1}^s g_{ij} x_{ik} \leq b_{kj}, \quad \forall k \in \{1, 2, \dots, n\}, \forall j \in \{1, 2, \dots, r\}, \quad (16.11c)$$

$$\sum_{k=1}^n x_{ik} \geq c_i, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.11d)$$

$$\sum_{k=1}^n x_{ik} \leq d_i, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.11e)$$

$$x_{ik} \geq 0, \quad \forall i \in \{1, 2, \dots, s\}, \forall k \in \{1, 2, \dots, n\}. \quad (16.11f)$$

Description of the objective function and constraints of the LP model:

- the linear objective function (16.11a) minimizes the total cost of the diets for all classes of students.
- the linear constraints (16.11b) ensure that, for every diet and every nutrient, the total amount of the nutrient provided by the foods included in that diet satisfies the minimum requirement ($n \cdot r$ linear constraints);
- the linear constraints (16.11c) ensure that, for every diet and every nutrient, the total amount of the nutrient provided by the foods included in that diet does not exceed the maximum requirement ($n \cdot r$ linear constraints);
- the linear constraints (16.11d) ensure that, for each food, the total quantity distributed across all diets satisfies the minimum requirement (s linear constraints);
- the linear constraints (16.11e) ensure that, for each food, the total quantity distributed across all diets does not exceed the maximum requirement (s linear constraints);

- the linear constraints (16.11f) define the variables of the LP model.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. The LP model (16.11) for the specific instance reads as follows:

$$\begin{aligned}
 \min \quad & 5x_{11} + 5x_{12} + 8x_{21} + 8x_{22} + 6x_{31} + 6x_{32} \\
 \text{subject to} \quad & 6x_{11} + 2x_{21} + 2x_{31} \geq 20 \\
 & 2x_{11} + x_{21} + 6x_{31} \geq 10 \\
 & 6x_{12} + 2x_{22} + 2x_{32} \geq 10 \\
 & 2x_{12} + x_{22} + 6x_{32} \geq 20 \\
 & 6x_{11} + 2x_{21} + 2x_{31} \leq 40 \\
 & 2x_{11} + x_{21} + 6x_{31} \leq 20 \\
 & 6x_{12} + 2x_{22} + 2x_{32} \leq 20 \\
 & 2x_{12} + x_{22} + 6x_{32} \leq 40 \\
 & x_{11} + x_{12} \geq 1 \\
 & x_{21} + x_{22} \geq 1 \\
 & x_{31} + x_{32} \geq 0 \\
 & x_{11} + x_{12} \leq 5 \\
 & x_{21} + x_{22} \leq 4 \\
 & x_{31} + x_{32} \leq 10 \\
 & x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32} \geq 0
 \end{aligned}$$

3. For example, by setting:

$$\bar{x}_{11} = 4, \bar{x}_{12} = 1, \bar{x}_{21} = 2, \bar{x}_{22} = 1, \bar{x}_{31} = 0, \bar{x}_{32} = 3$$

we have a feasible solution. The objective function value of this feasible solution is:

$$\sum_{i=1}^s w_i \sum_{k=1}^n \bar{x}_{ik} = 5 \cdot 4 + 5 \cdot 1 + 8 \cdot 2 + 8 \cdot 1 + 6 \cdot 0 + 6 \cdot 3 = 67$$

accordingly 67 is an upper bound on the optimal solution value of the LP model (16.11). So, for this instance, we have:

$$z(LP) \leq 67$$

4. By using

- a non-negative variable α_{kj} for each constraint (16.11b) ($k \in \{1, 2, \dots, n\}, j \in \{1, 2, \dots, r\}$)

- a non-positive variable β_{kj} for each constraint (16.11c) ($k \in \{1, 2, \dots, n\}, j \in \{1, 2, \dots, r\}$)
- a non-negative variable γ_i for each constraint (16.11d) ($i \in \{1, 2, \dots, s\}$)
- a non-positive variable δ_i for each constraint (16.11e) ($i \in \{1, 2, \dots, s\}$)

the dual of the LP model (16.11) reads as follows:

$$\begin{aligned}
 \max \quad & \sum_{k=1}^n \sum_{j=1}^r a_{kj} \alpha_{kj} \\
 & + \sum_{k=1}^n \sum_{j=1}^r b_{kj} \beta_{kj} \\
 & + \sum_{i=1}^s c_i \gamma_i + \sum_{i=1}^s d_i \delta_i
 \end{aligned} \tag{16.12a}$$

$$\begin{aligned}
 \text{subject to} \quad & \sum_{j=1}^r g_{ij} \alpha_{kj} \\
 & + \sum_{j=1}^r g_{ij} \beta_{kj} + \gamma_i + \delta_i \leq w_i, \quad \forall i \in \{1, 2, \dots, s\}, \forall k \in \{1, 2, \dots, n\},
 \end{aligned} \tag{16.12b}$$

$$\alpha_{kj} \geq 0, \quad \forall k \in \{1, 2, \dots, n\}, \forall j \in \{1, 2, \dots, r\}, \tag{16.12c}$$

$$\beta_{kj} \leq 0, \quad \forall k \in \{1, 2, \dots, n\}, \forall j \in \{1, 2, \dots, r\}, \tag{16.12d}$$

$$\gamma_i \geq 0, \quad \forall i \in \{1, 2, \dots, s\}, \tag{16.12e}$$

$$\delta_i \leq 0, \quad \forall i \in \{1, 2, \dots, s\}. \tag{16.12f}$$

The dual model (16.12) for the instance reads as follows:

$$\begin{aligned}
 \max \quad & 20\alpha_{11} + 10\alpha_{12} + 10\alpha_{21} + 20\alpha_{22} + 40\beta_{11} + 20\beta_{12} + 20\beta_{21} + 40\beta_{22} \\
 & + \gamma_1 + \gamma_2 + 5\delta_1 + 4\delta_2 + 10\delta_3 \\
 \text{subject to} \quad & 6\alpha_{11} + 2\alpha_{12} + 6\beta_{11} + 2\beta_{12} \leq 5 \\
 & + \gamma_1 + \delta_1 \\
 & 6\alpha_{21} + 2\alpha_{22} + 6\beta_{21} + 2\beta_{22} \leq 5 \\
 & + \gamma_1 + \delta_1 \\
 & 2\alpha_{11} + \alpha_{12} + 2\beta_{11} + \beta_{12} \leq 8 \\
 & + \gamma_2 + \delta_2 \\
 & 2\alpha_{21} + \alpha_{22} + 2\beta_{21} + \beta_{22} \leq 8 \\
 & + \gamma_2 + \delta_2 \\
 & 2\alpha_{11} + 6\alpha_{12} + 2\beta_{11} + 6\beta_{12} \leq 6 \\
 & + \gamma_3 + \delta_3 \\
 & 2\alpha_{21} + 6\alpha_{22} + 2\beta_{21} + 6\beta_{22} \leq 6 \\
 & + \gamma_3 + \delta_3 \\
 & \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, \gamma_1, \gamma_2, \gamma_3 \geq 0 \\
 & \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \delta_1, \delta_2, \delta_3 \leq 0
 \end{aligned}$$

For example, by setting:

$$\bar{\alpha}_{11} = \bar{\alpha}_{12} = \bar{\alpha}_{21} = \bar{\alpha}_{22} = \bar{\beta}_{11} = \bar{\beta}_{12} = \bar{\beta}_{21} = \bar{\beta}_{22} = \bar{\delta}_1 = \bar{\delta}_2 = \bar{\delta}_3 = 0$$

and by setting:

$$\bar{\gamma}_1 = \min\{5, 5\} = 5, \quad \bar{\gamma}_2 = \min\{8, 8\} = 8, \quad \bar{\gamma}_3 = \min\{6, 6\} = 6$$

we have a feasible dual solution.

The objective function value of this feasible dual solution is:

$$\begin{aligned}
 & \sum_{k=1}^n \sum_{j=1}^r a_{kj} \bar{\alpha}_{kj} + \sum_{k=1}^n \sum_{j=1}^r b_{kj} \bar{\beta}_{kj} + \sum_{i=1}^s c_i \bar{\gamma}_i + \sum_{i=1}^s d_i \bar{\delta}_i = \\
 & = 20 \cdot 0 + 10 \cdot 0 + 10 \cdot 0 + 20 \cdot 0 + 40 \cdot 0 + 20 \cdot 0 + 20 \cdot 0 + 40 \cdot 0 + 1 \cdot 5 + 1 \cdot 8 + 0 \cdot 6 + 5 \cdot 0 + 4 \cdot 0 + 10 \cdot 0 = 13
 \end{aligned}$$

Accordingly 13 is a lower bound on the optimal solution value of the model. So, for this instance, we have:

$$13 \leq z(LP) \leq 67$$

Example 16.8 — Multi-period production with purchases and storage

A company needs to satisfy the forecast demand for a chemical product over a time horizon of $n \in \mathbb{Z}_{\geq 1}$ days. Each day $t \in \{1, 2, \dots, n\}$, the quantity $d_t \in \mathbb{Q}_{\geq 0}$ is the forecast demand in liters of the product. On each day $t \in \{1, 2, \dots, n\}$, the company can produce up to $m_t \in \mathbb{Q}_{\geq 0}$ liters at a cost of $p_t \in \mathbb{Q}_{> 0}$ euros per liter, buy from the market at a cost of $u_t \in \mathbb{Q}_{> 0}$ euros per liter, and store product at the end of the day in a warehouse at a holding cost of $h_t \in \mathbb{Q}_{> 0}$ euros per liter. The initial stock in the warehouse is $\ell_0 \in \mathbb{Q}_{\geq 0}$ liters while a safety stock of $\ell_n \in \mathbb{Q}_{\geq 0}$ liters is required at the end of the time horizon. Storage and buying capacities are assumed to be unlimited. The company wants to determine a minimum-cost plan to satisfy the demand subject to the constraints described above.

Questions

1. Write a linear programming (LP) model for the problem. Identify the decisions that must be taken and the corresponding decision variables. Identify and comment the objective function of the model and the constraints. What algorithm can be used to find an optimal solution of the model?
2. Consider now an instance of the problem with 4 days ($n = 4$) and the following input data:

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
d_t	45	60	75	75
p_t	10	10	10	10
u_t	15	15	15	15
h_t	2	2	2	2
m_t	40	50	150	40

Moreover $\ell_0 = 10$ and $\ell_n = 5$. Write the LP model for this instance.

3. Find a feasible solution of the LP model of the instance and accordingly an upper bound on the optimal solution value of the LP model for the instance.
4. Write the dual model of the LP model and the dual model for the instance. Find a feasible solution of the dual model of the instance and accordingly a lower bound on the optimal solution value of the LP model for the instance.

Solution

1. We introduce the following $2n + n - 1$ non-negative variables:

$$\begin{cases} x_t = & \text{liters produced on day } t \\ b_t = & \text{liters bought from the market on day } t \end{cases} \quad \forall t \in \{1, 2, \dots, n\}$$

$$s_t = \text{liters stored in the warehouse at the end of day } t, \quad \forall t \in \{1, 2, \dots, n-1\}.$$

Using these variables, an LP model for the problem reads as follows:

$$\underbrace{h_n \ell_n}_{\text{constant term}} + \min \sum_{t=1}^n p_t x_t + \sum_{t=1}^n u_t b_t + \sum_{t=1}^{n-1} h_t s_t \quad (16.13a)$$

$$\text{subject to } x_1 + b_1 - s_1 = d_1 - \ell_0, \quad (16.13b)$$

$$x_t + b_t + s_{t-1} - s_t = d_t, \quad \forall t \in \{2, 3, \dots, n-1\}, \quad (16.13c)$$

$$x_n + b_n + s_{n-1} = d_n + \ell_n, \quad (16.13d)$$

$$x_t \leq m_t, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.13e)$$

$$x_t \geq 0, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.13f)$$

$$b_t \geq 0, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.13g)$$

$$s_t \geq 0, \quad \forall t \in \{1, 2, \dots, n-1\}. \quad (16.13h)$$

Description of the objective function and constraints of the LP model:

- the linear objective function (16.13a) minimizes the total cost of the production, market purchases, and inventory holding over the planning horizon. The constant term $h_n \ell_n$ accounts for the holding cost of the final safety stock, and it is written outside the minimization since it does not depend on the decision variables.
- the linear constraint (16.13b) ensures that the demand on the first day is satisfied using production, purchases, and the initial stock level, while accounting for the stock stored at the end of the first day (one linear constraint);
- the linear constraints (16.13c) ensure that, on intermediate days, the demand is satisfied using daily production, purchases, and inventory carried over from the previous day ($n - 2$ linear constraints);
- the linear constraint (16.13d) ensures that the final day's demand is satisfied, and the required safety stock is left in storage at the end of the planning horizon (one linear constraint);
- the linear constraint (16.13e) ensures that daily production does not exceed the available production capacity (n linear constraints);
- the linear constraints (16.13f), (16.13g), and (16.13h) define the variables of the LP model.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. The LP model (16.13) for the specific instance (without the constant term) reads as follows:

$$\begin{aligned}
 \min \quad & 10x_1 + 10x_2 + 10x_3 + 10x_4 + 15b_1 + 15b_2 + 15b_3 + 15b_4 \\
 & + 2s_1 + 2s_2 + 2s_3 \\
 \text{subject to} \quad & x_1 + b_1 = 35 \\
 & -s_1 + x_2 + b_2 = 60 \\
 & +s_1 - s_2 + x_3 + b_3 = 75 \\
 & +s_2 - s_3 + x_4 + b_4 = 80 \\
 & +s_3 + x_1 \leq 40 \\
 & +x_2 \leq 50 \\
 & +x_3 \leq 150 \\
 & +x_4 \leq 40 \\
 & x_1, x_2, x_3, x_4, b_1, b_2, b_3, b_4, \\
 & s_1, s_2, s_3 \geq 0
 \end{aligned}$$

3. For example, by setting:

$$\begin{aligned}
 \bar{x}_1 &= 35, \quad \bar{x}_2 = 50, \quad \bar{x}_3 = 75, \quad \bar{x}_4 = 40, \\
 \bar{b}_1 &= 0, \quad \bar{b}_2 = 10, \quad \bar{b}_3 = 0, \quad \bar{b}_4 = 40, \\
 \bar{s}_1 &= 0, \quad \bar{s}_2 = 0, \quad \bar{s}_3 = 0
 \end{aligned}$$

we have a feasible solution. The objective function value of this feasible solution is:

$$\begin{aligned}
 \underbrace{h_n \ell_n}_{\text{constant term}} + \sum_{t=1}^n p_t \bar{x}_t + \sum_{t=1}^n u_t \bar{b}_t + \sum_{t=1}^{n-1} h_t \bar{s}_t &= 2 \cdot 5 + \\
 &+ 10 \cdot 35 + 10 \cdot 50 + 10 \cdot 75 + 10 \cdot 40 + \\
 &+ 15 \cdot 0 + 15 \cdot 10 + 15 \cdot 0 + 15 \cdot 40 + \\
 &+ 2 \cdot 0 + 2 \cdot 0 + 2 \cdot 0 \\
 &= 2760
 \end{aligned}$$

accordingly 2760 is an upper bound on the optimal solution value of the LP model (16.13). So, for this instance, we have:

$$z(LP) \leq 2760$$

4. By using

- a free dual variable μ_1 for the constraint (16.13b)

- a free dual variable μ_t for each constraint (16.13c) ($t \in \{2, 3, \dots, n-1\}$)
- a free dual variable μ_n for the constraint (16.13d)
- a non-positive dual variable π_t for each constraint (16.13e) ($t \in \{1, 2, \dots, n\}$)

the dual of the LP model (16.13) (without the constant term) reads as follows:

$$\begin{aligned} \max \quad & (d_1 - \ell_0) \mu_1 + \sum_{t=2}^{n-1} d_t \mu_t \\ & + (d_n + \ell_n) \mu_n + \sum_{t=1}^n m_t \pi_t \end{aligned} \quad (16.14a)$$

$$\text{subject to} \quad \mu_t + \pi_t \leq p_t, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.14b)$$

$$\mu_t \leq u_t, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.14c)$$

$$-\mu_t + \mu_{t+1} \leq h_t, \quad \forall t \in \{1, 2, \dots, n-1\}, \quad (16.14d)$$

$$\mu_t \geq 0, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.14e)$$

$$\pi_t \leq 0, \quad \forall t \in \{1, 2, \dots, n\}. \quad (16.14f)$$

The dual model (16.14) for the instance (without the constant term) reads as follows:

$$\begin{aligned} \max \quad & 35 \mu_1 + 60 \mu_2 + 75 \mu_3 + 80 \mu_4 + 40 \pi_1 + 50 \pi_2 + 150 \pi_3 + 40 \pi_4 \\ \text{subject to} \quad & \mu_1 + \pi_1 \leq 10 \\ & \mu_2 + \pi_2 \leq 10 \\ & \mu_3 + \pi_3 \leq 10 \\ & \mu_4 + \pi_4 \leq 10 \\ & \mu_1 \leq 15 \\ & \mu_2 \leq 15 \\ & \mu_3 \leq 15 \\ & \mu_4 \leq 15 \\ & -\mu_1 + \mu_2 \leq 2 \\ & -\mu_2 + \mu_3 \leq 2 \\ & -\mu_3 + \mu_4 \leq 2 \\ & \mu_1, \mu_2, \mu_3, \mu_4 \geq 0 \\ & \pi_1, \pi_2, \pi_3, \pi_4 \leq 0 \end{aligned}$$

For example, by setting:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0$$

and by setting:

$$\begin{aligned}\bar{\mu}_1 &= \min\{10, 15\} = 10, & \bar{\mu}_2 &= \min\{10, 15\} = 10, \\ \bar{\mu}_3 &= \min\{10, 15\} = 10, & \bar{\mu}_4 &= \min\{10, 15\} = 10\end{aligned}$$

we have a feasible dual solution.

The objective function value of this feasible dual solution is:

$$\begin{aligned}& \underbrace{h_n \ell_n}_{\text{constant term}} + (d_1 - \ell_0) \bar{\mu}_1 + \sum_{t=2}^{n-1} d_t \bar{\mu}_t + (d_n + \ell_n) \bar{\mu}_n + \\& + \sum_{t=1}^n m_t \bar{\pi}_t = 2 \cdot 5 + \\& + 35 \cdot 10 + 60 \cdot 10 + 75 \cdot 10 + 80 \cdot 10 + \\& + 40 \cdot 0 + 50 \cdot 0 + 150 \cdot 0 + 40 \cdot 0 \\& = 2510\end{aligned}$$

Accordingly, 2510 is a lower bound on the optimal solution value of the primal LP model. So, for this instance, we have:

$$2510 \leq z(LP) \leq 2760$$

16.8.2 Numerical models

Example 16.9 — Choice of desserts

A person wishes to determine which desserts to eat and in what quantities the next day. The available desserts are fruit cake and ice cream. According to nutritional labels, each portion of fruit cake weighs 50 grams, provides 120 calories, and contains 5 grams of fat. Each portion of ice cream weighs 75 grams, provides 160 calories, and contains 10 grams of fat. The person aims to consume at least 120 grams of dessert, without exceeding 450 calories and 25 grams of fat. Additionally, a personal taste score is assigned to each gram of dessert on a scale from 0 (least preferred) to 10 (most preferred): 8 for fruit cake and 9 for ice cream. The goal of the person is to determine the quantities of desserts to consume that satisfy all nutritional constraints while maximizing the total taste score.

Questions

1. Write a Linear Programming optimization model for the problem. Identify the decisions that must be made and define the corresponding decision variables. Describe the objective function of the model and its constraints. What algorithm can be used to find an optimal solution to the model?

Solution

Warning

We assume that the person can consume any fractional portions of each dessert.

1. We introduce the following two non-negative decision variables:

$$\begin{cases} x_1 = \text{number of portions of fruit cake} \\ x_2 = \text{number of portions of ice cream} \end{cases}$$

Using these variables, a Linear Programming (LP) model for the problem reads as follows:

$$\begin{aligned} \max \quad & \overbrace{(8 \cdot 50)}^{=400} x_1 + \overbrace{(9 \cdot 75)}^{=675} x_2 \\ \text{subject to} \quad & 120 x_1 + 160 x_2 \leq 450 \\ & 5 x_1 + 10 x_2 \leq 25 \\ & 50 x_1 + 75 x_2 \geq 120 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Description of the objective function and constraints of the LP model:

- the linear objective function maximizes the total taste score;
- the first linear constraint ensures that the total caloric intake from the selected portions does not exceed the specified limit;
- the second linear constraint restricts the total fat intake to the specified maximum;
- the third linear constraint guarantees that the person consumes at least the specified number of grams of dessert per day;
- the final constraint defines the variables of the model.

Example 16.10 — Production and storage of fruit cakes

A bakery must plan fruit cake production over the next four days to meet daily demands of 10, 20, 30, and 20 kilograms. It starts with 5 kilograms in stock. Daily production is limited to 30 kilograms, and storage capacity at the end of each day is 10 kilograms.

Production costs per kilogram are 10, 20, 30, and 40 euros, respectively. Holding one kilogram in inventory at the end of a day costs 10% of its production cost. The bakery aims to minimize total production and inventory holding costs while satisfying all demands and respecting capacity constraints.

Questions

1. Write a Linear Programming optimization model for the problem. Identify the decisions that must be made and define the corresponding decision variables. Describe the objective function of the model and its constraints. What algorithm can be used to find an optimal solution to the model?

Solution

Warning

We assume that the bakery can produce any fractional quantity of fruit cake.

1. We introduce the following seven non-negative decision variables:

$$\begin{cases} x_t = \text{kilograms of fruit cake produced on day } t, & \forall t \in \{1, 2, 3, 4\}, \\ s_t = \text{kilograms of fruit cake stored at the end of day } t, & \forall t \in \{1, 2, 3\}. \end{cases}$$

Using these variables, a Linear Programming (LP) model for the problem reads as follows:

$$\begin{array}{llllllll} \min & 10x_1 + 20x_2 + 30x_3 + 40x_4 + s_1 + 2s_2 + 3s_3 \\ \text{subject to} & x_1 & & & & -s_1 & & = 10 - 5 \\ & & x_2 & & & +s_1 - s_2 & & = 20 \\ & & & x_3 & & & +s_2 - s_3 & = 30 \\ & & & & x_4 & & +s_3 & = 20 \\ & x_1 & & & & & & \leq 30 \\ & & x_2 & & & & & \leq 30 \\ & & & x_3 & & & & \leq 30 \\ & & & & x_4 & & & \leq 30 \\ & & & & & s_1 & & \leq 10 \\ & & & & & & s_2 & \leq 10 \\ & & & & & & & s_3 \leq 10 \\ & x_1, & x_2, & x_3, & x_4, & s_1, & s_2, & s_3 \geq 0 \end{array}$$

Description of the objective function and constraints of the LP model:

- the linear objective function minimizes the total production and inventory holding costs over the 4-day planning horizon;
- the first four linear constraints ensure that, in each day, the quantity of fruit cake available—given by the initial stock and the amount produced during the day—matches the daily customer demand after accounting for what is stored at the end of the day;
- the next eight linear constraints enforce the daily production and storage capacity limits;
- the final constraints define the variables of the optimization model.

Example 16.11 — Blending of crude oil in a refinery

A refinery has 5000 barrels of crude oil of type 1 and 10000 barrels of crude oil of type 2. The refinery produces and sells two products: gasoline and heating oil. These products are obtained by blending the two types of crude oil in appropriate proportions. Each barrel of type 1 crude oil has a quality level of 10, while each barrel of type 2 has a quality level of 5. To meet market standards, gasoline must have an average quality level of at least 8, and heating oil must have an average quality level of at least 6. Gasoline is sold at 25 dollars per barrel, and heating oil at 20 dollars per barrel. The goal of the refinery is to determine a production plan that maximizes total profit while satisfying all constraints.

Questions

1. Write a Linear Programming optimization model for the problem. Identify the decisions that must be made and define the corresponding decision variables. Describe the objective function of the model and its constraints. What algorithm can be used to find an optimal solution to the model?

Solution

Warning

We assume that the refinery can blend any fractional quantity of crude oil to produce gasoline and heating oil.

1. We introduce the following four non-negative decision variables:

$$\begin{cases} x_1 = \text{barrels of crude oil type 1 used for gasoline,} \\ x_2 = \text{barrels of crude oil type 2 used for gasoline,} \\ y_1 = \text{barrels of crude oil type 1 used for heating oil,} \\ y_2 = \text{barrels of crude oil type 2 used for heating oil.} \end{cases}$$

Using these variables, a Linear Programming (LP) model for the problem reads as follows:

$$\begin{array}{llllll}
 \max & 25x_1 + 25x_2 + 20y_1 + 20y_2 & & & & \\
 \text{subject to} & x_1 & + & y_1 & \leq & 5000 \\
 & & & x_2 & + & y_2 \leq 10000 \\
 & 2x_1 - 3x_2 & & & \geq & 0 \\
 & & & 4y_1 - y_2 & \geq & 0 \\
 & x_1, & x_2, & y_1, & y_2 & \geq 0
 \end{array}$$

Description of the objective function and constraints of the LP model:

- the linear objective function maximizes the total revenue from the sale of gasoline and heating oil;
- the first two linear constraints impose the availability limits on the total usage of crude oil type 1 and type 2, respectively;
- the third linear constraint enforces the quality requirement for gasoline. It ensures that the average quality of the crude oil blend used to produce gasoline is at least the desired amount.

$$\frac{10x_1 + 5x_2}{x_1 + x_2} \geq 8 \iff 10x_1 + 5x_2 \geq 8(x_1 + x_2) \iff 2x_1 - 3x_2 \geq 0$$

- the fourth linear constraint enforces the quality requirement for heating oil. It ensures that the average quality of the crude oil blend used to produce heating oil is at least the desired amount.

$$\frac{10y_1 + 5y_2}{y_1 + y_2} \geq 6 \iff 10y_1 + 5y_2 \geq 6(y_1 + y_2) \iff 4y_1 - y_2 \geq 0$$

- the final constraint defines the variables of the model.

Example 16.12 — Production of three drugs

A pharmaceutical company produces three drugs: drug 1, drug 2, and drug 3, which are sold at unit prices of 8 euros, 70 euros, and 100 euros, respectively. The production of 1 unit of drug 1 requires 1 hour of labor. The production of 1 unit of drug 2 requires 2 hours of labor and consumes 2 units of drug 1. The production of 1 unit of drug 3 requires 3 hours of labor and consumes 1 unit of drug 2. A total of 4000 labor hours are available. The goal of the company is to determine a production plan that maximizes the total revenue.

Questions

1. Write a Linear Programming optimization model for the problem. Identify the decisions that must be made and define the corresponding decision variables. Describe the objective function of the model and its constraints. What algorithm can be used to find an optimal solution to the model?

Solution

Warning

We assume that the company can produce and sell any fractional quantities of each drug.

1. We introduce the following three non-negative decision variables:

$$\begin{cases} y_1 = \text{units of drug 1 sold,} \\ y_2 = \text{units of drug 2 sold,} \\ y_3 = \text{units of drug 3 sold.} \end{cases}$$

Using these variables, a Linear Programming (LP) model for the problem reads as follows:

$$\begin{array}{ll} \max & 8y_1 + 70y_2 + 100y_3 \\ \text{subject to} & y_1 + 4y_2 + 7y_3 \leq 4000 \\ & y_1, \quad y_2, \quad y_3 \geq 0 \end{array}$$

Description of the objective function and constraints:

- the linear objective function maximizes total revenue: drug 1 brings 8 euros/unit, drug 2 brings 70 euros/unit, and drug 3 brings 100 euros/unit.
- the linear constraint limits total labor usage. Each unit of drug 1 sold requires 1 hour of labor; each unit of drug 2 requires 4 hours (2 units of drug 1 + 1 unit of drug 2); and each unit of drug 3 requires 7 hours (2 units of drug 1 + 1 unit of drug 2 + 1 unit of drug 3).
- the final constraint defines the variables of the model.

Another LP model.

We introduce the following three non-negative decision variables:

$$\begin{cases} x_1 = \text{units of drug 1 produced,} \\ x_2 = \text{units of drug 2 produced,} \\ x_3 = \text{units of drug 3 produced.} \end{cases}$$

Using these variables, a Linear Programming (LP) model for the problem reads as follows:

$$\begin{aligned} \max \quad & 8x_1 + 54x_2 + 30x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 \leq 4000 \\ & -x_1 + 2x_2 \leq 0 \\ & \quad -x_2 + x_3 \leq 0 \\ & x_1, \quad x_2, \quad x_3 \geq 0 \end{aligned}$$

Description of the objective function and constraints of the LP model:

- the linear function maximizes the total revenue generated by the sale of the three drugs, after accounting for the internal consumption of intermediate drugs. The coefficients of each variable are computed by considering only the units that are actually sold on the market.

- the quantity of drug 1 sold is equal to the quantity produced, x_1 , minus the quantity used to produce drug 2, which requires 2 units of drug 1 for each unit of drug 2 produced. Therefore, the effective contribution to revenue from drug 1 is:

$$8(x_1 - 2x_2)$$

- the quantity of drug 2 sold is equal to the quantity produced, x_2 , minus the quantity used to produce drug 3, which requires 1 unit of drug 2 for each unit of drug 3 produced. Therefore, the effective contribution to revenue from drug 2 is:

$$70(x_2 - x_3)$$

- all units of drug 3 are sold directly, so their contribution to revenue is simply:

$$100x_3$$

Therefore, we have:

$$\max 8(x_1 - 2x_2) + 70(x_2 - x_3) + 100x_3 \iff \max 8x_1 + 54x_2 + 30x_3$$

- the first linear constraint limits total labor usage, ensuring that the total time required for all production activities does not exceed the available hours;

- the second linear constraint ensures that drug 1 production is enough to support the production of drug 2:

$$x_1 - 2x_2 \geq 0 \iff 2x_2 \leq x_1$$

- the third linear constraint ensures that drug 2 production is enough to support the production of drug 3:

$$x_2 - x_3 \geq 0 \iff x_3 \leq x_2$$

- the final constraint defines the variables of the model.