



SAPIENZA
UNIVERSITÀ DI ROMA

Linear Programming

Volume 4 — Matrix methods, sensitivity and geometry

The revised primal and dual simplex, the primal-dual method, sensitivity analysis, the geometry of linear programming (faces, vertices and basic solutions). Examples worked out step by step and exercises with complete solutions at the end of each chapter.

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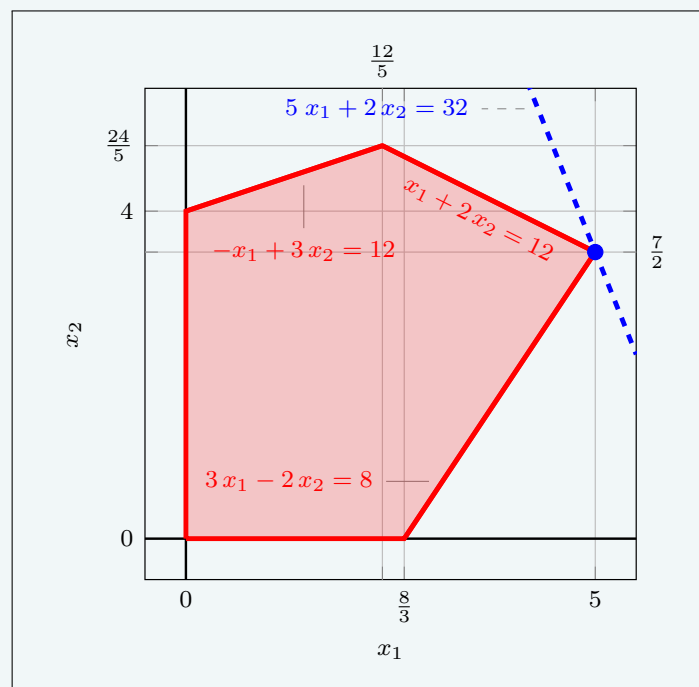
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Revised simplex algorithm

Example 11.1: an LP problem in canonical form and its standard form

We consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{aligned}
 \max \quad & 5x_1 + 2x_2 \\
 \text{subject to} \quad & -x_1 + 3x_2 \leq 12 \\
 & x_1 + 2x_2 \leq 12 \\
 & 3x_1 - 2x_2 \leq 8 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$



Its equivalent standard form with $n + m = 5$ variables and $m = 3$ constraints is:

$$\begin{array}{rcllcl}
 \max & 5 & x_1 & + & 2 & x_2 & & & & \\
 \text{subject to} & -x_1 & + & 3 & x_2 & + & x_3 & & & = & 12 \\
 & & x_1 & + & 2 & x_2 & & + & x_4 & = & 12 \\
 & 3 & x_1 & - & 2 & x_2 & & & + & x_5 & = & 8 \\
 & & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
 \end{array}$$

For this LP problem in standard form, we have:

$$\mathbf{c} = \begin{pmatrix} 5 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix}$$

Since $\begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, with $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$ we have the following initial basic feasible solution:

$$\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{\mathbf{b}} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix}$$

11.1 Pivoting operations with two systems of linear equations

- Given a basis $\mathbf{A}_{\mathcal{B}}$, all the information necessary for the pivoting operations can be determined by solving **two systems of linear equations**.

1. First system of linear equations

- The reduced costs and the basic feasible solution value can be determined by computing directly $\mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$ since:

$$\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} \quad \text{and} \quad \bar{z} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$$

By setting $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, the row vector $\boldsymbol{\pi} \in \mathbb{R}^{1 \times m}$ of m columns can be found by solving the following system of linear equations:

$$\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}} \implies \boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \quad \text{where} \quad \boldsymbol{\pi} = [\pi_j] \quad \text{with} \quad j \in \mathcal{B}$$

In the examples the components of $\boldsymbol{\pi}$ are labelled with the indices $j \in \mathcal{B}$ of the basic variables, but each component of $\boldsymbol{\pi}$ corresponds to a constraint, i.e., to a row of $\mathbf{A}_{\mathcal{B}}$: the i -th component of $\boldsymbol{\pi}$ is associated with the i -th constraint.

2. Second system of linear equations

- Given a pivot column x_p with $p \in \mathcal{N}$, the pivot row and the update of the current basic feasible solution can be done by directly computing $\mathbf{A}_B^{-1} \mathbf{A}_p$ since:

$$\mathbf{x}_B = \bar{\mathbf{b}} - \mathbf{A}_B^{-1} \mathbf{A}_p x_p$$

By setting $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_p$, the column vector $\mathbf{d} \in \mathbb{R}^{m \times 1}$ of m rows (associated with the basic variables) can be found by solving the following system of linear equations:

$$\mathbf{A}_B \mathbf{d} = \mathbf{A}_p \implies \mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_p \quad \text{where} \quad \mathbf{d} = [d_i] \quad \text{with} \quad i \in \mathcal{B}$$

Example 11.2: Systems of linear equations

For the LP problem of [Example 11.1](#) with $\mathcal{B} = \{1, 3, 4\}$, $\mathcal{N} = \{2, 5\}$ and x_2 as the pivot column, we have:

$$\mathbf{A}_B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \quad \mathbf{c}_B = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

We solve $\boldsymbol{\pi} \mathbf{A}_B = \mathbf{c}'_B$ to compute $\boldsymbol{\pi} = \mathbf{c}'_B \mathbf{A}_B^{-1}$, first system of linear equations:

$$\begin{cases} -\pi_1 + \pi_3 + 3\pi_4 = 5 \\ \pi_1 = 0 \\ \pi_3 = 0 \end{cases} \implies \boldsymbol{\pi} = \begin{pmatrix} 0 & 0 & \frac{5}{3} \end{pmatrix}$$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} -d_1 + d_3 = 3 \\ d_1 + d_4 = 2 \\ 3d_1 = -2 \end{cases} \implies \mathbf{d} = \begin{pmatrix} -\frac{2}{3} \\ \frac{7}{3} \\ \frac{8}{3} \end{pmatrix}$$

The revised simplex algorithm never computes the inverse matrix $\mathbf{A}_B^{-1}$: at every iteration it solves two systems of linear equations, one for the row vector $\boldsymbol{\pi}$ and one for the column vector $\mathbf{d}$. Solving a system of m linear equations is computationally cheaper than inverting a matrix of size m , and the matrix $\mathbf{A}_B$ of the two systems is the same.

11.2 Pseudo-code of the revised simplex algorithm

Algorithm

Step 0: Initialization

Find a basic feasible solution $\mathbf{x}_B = \bar{\mathbf{b}}$ and $\mathbf{x}_N = \mathbf{0}$ of the LP problem in standard maximization form. Go to Step 1.

Step 1: Optimality check

- Solve $\boldsymbol{\pi} \mathbf{A}_B = \mathbf{c}'_B$ to compute $\boldsymbol{\pi} = \mathbf{c}'_B \mathbf{A}_B^{-1}$
- Compute the reduced costs and the basic feasible solution value:

$$\bar{\mathbf{c}}' = \mathbf{c}'_N - \underbrace{\mathbf{c}'_B \mathbf{A}_B^{-1}}_{=\boldsymbol{\pi}} \mathbf{A}_N \quad \quad \bar{z} = \underbrace{\mathbf{c}'_B \mathbf{A}_B^{-1}}_{=\boldsymbol{\pi}} \mathbf{b}$$

If $\bar{\mathbf{c}}' \leq \mathbf{0}$ then the basic feasible solution $\mathbf{x}_B = \bar{\mathbf{b}}$ and $\mathbf{x}_N = \mathbf{0}$ is an optimal solution and $\bar{z}$ is the optimal value **else** go to Step 2.

Step 2: Pivoting operations

The **pivot column** is a non-basic variable x_p with $p \in \mathcal{N}$ such that:

$$c_p > \underbrace{\mathbf{c}'_B \mathbf{A}_B^{-1}}_{=\boldsymbol{\pi}} \mathbf{A}_p$$

- Solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_p$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_p$
- If $\mathbf{d} \leq \mathbf{0}$, then the LP problem is **unbounded**. STOP.
- Find the largest value t (called t_m) such that:

$$\bar{\mathbf{b}} - t \underbrace{\mathbf{A}_B^{-1} \mathbf{A}_p}_{=\mathbf{d}} \geq \mathbf{0} \quad \implies \quad t_m \leftarrow \min \left\{ \frac{\bar{b}_i}{d_i} : i \in \mathcal{B}, d_i > 0 \right\}$$

The **pivot row** is a basic variable x_r with $r \in \mathcal{B}$ such that:

$$\bar{b}_r - t_m \underbrace{\bar{a}_{rp}}_{=d_r} = 0$$

- Update the basis with the new partition of the indices of the variables:

$$\mathcal{B} \leftarrow (\mathcal{B} \setminus \{r\}) \cup \{p\} \quad \text{and} \quad \mathcal{N} \leftarrow (\mathcal{N} \setminus \{p\}) \cup \{r\}$$

- Update the basic feasible solution (new vector $\bar{\mathbf{b}}$):

$$x_i = \bar{b}_i - t_m \underbrace{\bar{a}_{ip}}_{=d_i}, \quad \forall i \in \mathcal{B} \setminus \{p\}, \quad x_p = t_m \quad \text{and} \quad x_i = 0, \quad \forall i \in \mathcal{N}$$

Go to Step 1.

Example 11.3: revised simplex algorithm – first iteration
Initialization

In the LP problem of [Example 11.1](#), $\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a basic feasible solution

with $\bar{\mathbf{b}} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix}$.

Optimality check

With $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$, we solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$.

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{c}'_{\mathcal{B}} = (0 \quad 0 \quad 0)$$

$$\text{then } \begin{cases} \pi_3 & & = & 0 \\ & \pi_4 & = & 0 \\ & & \pi_5 & = & 0 \end{cases} \Rightarrow \boldsymbol{\pi} = (0 \quad 0 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \quad 0 \quad 0) \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = 0$. We compute

the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (5 \quad 2) - (0 \quad 0 \quad 0) \begin{pmatrix} -1 & 3 \\ 1 & 2 \\ 3 & -2 \end{pmatrix} = \underbrace{(5 \quad 2)}_{c_1 \quad c_2} - \underbrace{(0 \quad 0)}_{\pi \mathbf{A}_1 \quad \pi \mathbf{A}_2} = \underbrace{(5 \quad 2)}_{\bar{c}_1 \quad \bar{c}_2}$$

Pivoting operation

Since $c_1 > \pi \mathbf{A}_1$, x_1 is the pivot column. We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_1$.

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{A}_1 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$$\text{then } \begin{cases} d_3 & & = & -1 \\ & d_4 & = & 1 \\ & & d_5 & = & 3 \end{cases} \Rightarrow \mathbf{d} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{12 - t \cdot (-1)}_{x_3 \geq 0} \geq 0 \\ \underbrace{12 - t \cdot 1}_{x_4 \geq 0} \geq 0 \\ \underbrace{8 - t \cdot 3}_{x_5 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \geq -12 \\ t \leq 12 \\ t \leq \frac{8}{3} \end{cases} \quad \text{then } t_m = \frac{8}{3}$$

The value of the pivot column becomes t_m , i.e., $x_1 = \frac{8}{3}$. The value of x_5 becomes zero and x_5 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_3 = 12 - \frac{8}{3} \cdot (-1) = \frac{44}{3}, \quad x_4 = 12 - \frac{8}{3} \cdot 1 = \frac{28}{3} \quad \text{and accordingly} \quad \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{44}{3} \\ \frac{28}{3} \end{pmatrix}$$

The new partition of the indices of the variables is $\mathcal{B} = \{1, 3, 4\}$ and $\mathcal{N} = \{2, 5\}$.

Example 11.4: revised simplex algorithm – second iteration

Optimality check

With $\mathcal{B} = \{1, 3, 4\}$ and $\mathcal{N} = \{2, 5\}$, we solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$.

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \quad \mathbf{c}'_{\mathcal{B}} = (5 \quad 0 \quad 0)$$

$$\text{then } \begin{cases} -\pi_1 + \pi_3 + 3\pi_4 = 5 \\ \pi_1 = 0 \\ \pi_3 = 0 \end{cases} \Rightarrow \boldsymbol{\pi} = (0 \quad 0 \quad \frac{5}{3})$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \quad 0 \quad \frac{5}{3}) \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = \frac{40}{3}$. We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (2 \quad 0) - (0 \quad 0 \quad \frac{5}{3}) \begin{pmatrix} 3 & 0 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} = \underbrace{(2 \quad 0)}_{c_2 \quad c_5} - \underbrace{\left(-\frac{10}{3} \quad \frac{5}{3}\right)}_{\pi \mathbf{A}_2 \quad \pi \mathbf{A}_5} = \underbrace{\left(\frac{16}{3} \quad -\frac{5}{3}\right)}_{\bar{c}_2 \quad \bar{c}_5}$$

Pivoting operation

Since $c_2 > \pi \mathbf{A}_2$, x_2 is the pivot column. We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_2$.

$$\mathbf{A}_B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

$$\text{then } \begin{cases} -d_1 + d_3 = 3 \\ d_1 + d_4 = 2 \\ 3d_1 = -2 \end{cases} \Rightarrow \mathbf{d} = \begin{pmatrix} -\frac{2}{3} \\ \frac{7}{3} \\ \frac{8}{3} \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{\frac{8}{3} - t \cdot \left(-\frac{2}{3}\right)}_{x_1 \geq 0} \geq 0 \\ \underbrace{\frac{44}{3} - t \cdot \frac{7}{3}}_{x_3 \geq 0} \geq 0 \\ \underbrace{\frac{28}{3} - t \cdot \frac{8}{3}}_{x_4 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \geq -4 \\ t \leq \frac{44}{7} \\ t \leq \frac{7}{2} \end{cases} \quad \text{then } t_m = \frac{7}{2}$$

The value of the pivot column becomes t_m , i.e., $x_2 = \frac{7}{2}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = \frac{8}{3} - \frac{7}{2} \cdot \left(-\frac{2}{3}\right) = 5, \quad x_3 = \frac{44}{3} - \frac{7}{2} \cdot \frac{7}{3} = \frac{13}{2} \quad \text{and accordingly } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ \frac{7}{2} \\ \frac{13}{2} \end{pmatrix}$$

The new partition of the indices of the variables is $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$.

Example 11.5: revised simplex algorithm – third iteration

Optimality check

With $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$, we solve $\boldsymbol{\pi} \mathbf{A}_B = \mathbf{c}'_B$ to compute $\boldsymbol{\pi} = \mathbf{c}'_B \mathbf{A}_B^{-1}$.

$$\mathbf{A}_B = \begin{pmatrix} -1 & 3 & 1 \\ 1 & 2 & 0 \\ 3 & -2 & 0 \end{pmatrix} \quad \mathbf{c}'_B = (5 \quad 2 \quad 0)$$

$$\text{then } \begin{cases} -\pi_1 + \pi_2 + 3\pi_3 = 5 \\ 3\pi_1 + 2\pi_2 - 2\pi_3 = 2 \\ \pi_1 = 0 \end{cases} \Rightarrow \boldsymbol{\pi} = (0 \quad 2 \quad 1)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \ 2 \ 1) \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = 32$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (0 \ 0) - (0 \ 2 \ 1) \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{(0 \ 0)}_{c_4 \ c_5} - \underbrace{(2 \ 1)}_{\boldsymbol{\pi} \mathbf{A}_4 \ \boldsymbol{\pi} \mathbf{A}_5} = \underbrace{(-2 \ -1)}_{\bar{c}_4 \ \bar{c}_5}$$

All reduced costs are non-positive, then the basic feasible solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ \frac{7}{2} \\ \frac{13}{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = 32$ is the optimal value.

Revised dual simplex algorithm

12.1 Pivoting operations with two systems of linear equations

- We consider an LP problem in standard maximization form with $n + m$ non-negative variables and m equality linear constraints:

$$\max \quad \mathbf{c}' \mathbf{x} \quad (12.1a)$$

$$\text{subject to} \quad \mathbf{A} \mathbf{x} = \mathbf{b}, \quad (12.1b)$$

$$\mathbf{x} \geq \mathbf{0}. \quad (12.1c)$$

- The dual simplex algorithm starts from a **dual feasible** basis, i.e., a basis whose dictionary satisfies $\bar{\mathbf{c}} \leq \mathbf{0}$, and it performs dual pivoting operations until the basic solution becomes feasible.
- Exactly as the revised simplex algorithm does for the simplex algorithm, the **revised dual simplex algorithm** never computes the inverse matrix $\mathbf{A}_B^{-1}$ and it never updates the whole dictionary: at every iteration it determines all the information needed by a dual pivoting operation by solving **two systems of linear equations**, both of them with the same matrix $\mathbf{A}_B$.

1. First system of linear equations

- The dual simplex algorithm needs the values of the basic variables, in order to determine the pivot row. They are the entries of the column vector $\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b}$, which is found by solving the following system of linear equations:

$$\mathbf{A}_B \bar{\mathbf{b}} = \mathbf{b} \implies \bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b} \quad \text{where} \quad \bar{\mathbf{b}} = (\bar{b}_i)_{i \in \mathcal{B}}.$$

2. Second system of linear equations

- Once the pivot row x_r with $r \in \mathcal{B}$ is determined, the ratio test needs the row of the matrix $\bar{\mathbf{A}}$ associated with x_r , that is, the values $\bar{a}_{rj}$ for $j \in \mathcal{N}$. This row is obtained from the row vector $\mathbf{u} \in \mathbb{R}^{1 \times m}$, which is found by solving the following system of linear equations:

$$\mathbf{u} \mathbf{A}_{\mathcal{B}} = \mathbf{e}'_r \implies \mathbf{u} = \mathbf{e}'_r \mathbf{A}_{\mathcal{B}}^{-1},$$

where $\mathbf{e}_r$ is the column vector of m entries, all equal to 0 except the one associated with the basic variable x_r , which is equal to 1. The required row is then

$$\bar{a}_{rj} = \mathbf{u} \mathbf{A}_j, \quad \forall j \in \mathcal{N}.$$

The two systems of linear equations have the same matrix $\mathbf{A}_{\mathcal{B}}$: the first one has it on the left of the unknown vector, the second one on its right. The vector $\mathbf{u}$ selects the row of $\bar{\mathbf{A}}$ associated with the pivot row, without computing the whole matrix $\bar{\mathbf{A}}$.

12.2 Updating the reduced costs

- The ratio test needs the reduced costs $\bar{c}_j$ with $j \in \mathcal{N}$. They are not recomputed at every iteration: they are **updated**, exactly as the revised simplex algorithm updates the values of the basic variables, and the update uses only the row $\bar{a}_{rj}$ already determined by the second system.
- At the first iteration the basis is the one of the m slack variables, and then $\mathbf{A}_{\mathcal{B}} = \mathbf{I}$ and $\mathbf{c}_{\mathcal{B}} = \mathbf{0}$, which gives

$$\bar{c}_j = c_j, \quad \forall j \in \mathcal{N}.$$

After a dual pivoting operation with pivot row x_r and pivot column x_p , the reduced costs are updated as follows:

$$\bar{c}_j \longleftarrow \bar{c}_j - \frac{\bar{c}_p}{\bar{a}_{rp}} \bar{a}_{rj}, \quad \forall j \in \mathcal{N} \setminus \{p\}, \quad \text{and} \quad \bar{c}_r \longleftarrow -\frac{\bar{c}_p}{\bar{a}_{rp}},$$

where $\bar{c}_r$ is the reduced cost of the variable x_r , which leaves the basis and becomes non-basic.

- The quantity $\frac{\bar{c}_p}{\bar{a}_{rp}}$ is exactly the minimum of the ratio test, which has already been computed to determine the pivot column. The update is then a single scalar multiplication of the row $\bar{a}_{rj}$, and it costs n operations.
- Since $\frac{\bar{c}_p}{\bar{a}_{rp}} \leq \frac{\bar{c}_j}{\bar{a}_{rj}}$ for every $j \in \mathcal{N}$ with $\bar{a}_{rj} < 0$, the updated reduced costs remain non-positive: the basis stays **dual feasible**, as required.

The two revised algorithms are perfectly symmetric: each of them solves **two** systems of linear equations per iteration, one with $\mathbf{A}_{\mathcal{B}}$ on the right of the unknown vector and one with $\mathbf{A}_{\mathcal{B}}$ on its left, and **updates** the other vector instead of recomputing it.

	revised simplex	revised dual simplex
first system	$\pi \mathbf{A}_B = \mathbf{c}'_B$	$\mathbf{A}_B \bar{\mathbf{b}} = \mathbf{b}$
it gives	the reduced costs $\bar{\mathbf{c}}$	the basic values $\bar{\mathbf{b}}$
used to choose	the pivot <u>column</u>	the pivot <u>row</u>
second system	$\mathbf{A}_B \mathbf{d} = \mathbf{A}_p$	$\mathbf{u} \mathbf{A}_B = \mathbf{e}'_r$
it gives	the <u>column</u> $\bar{\mathbf{A}}_p$	the <u>row</u> $\bar{a}_r$.
used for	the ratio test	the ratio test
updated, not recomputed	$\bar{\mathbf{b}}$	$\bar{\mathbf{c}}$

12.3 Pseudo-code of the revised dual simplex algorithm

Algorithm

Input

An LP problem (12.1) obtained from an LP problem in canonical maximization form with $\mathbf{c} \leq \mathbf{0}$ by adding the m slack variables, so that the basis of the slack variables is **dual feasible**.

Output

An optimal solution of the LP problem (12.1), if one exists; otherwise, a certificate that the LP problem is infeasible.

Step 0: Initialization

Consider the basis of the m slack variables, for which $\mathbf{A}_B = \mathbf{I}$ and $\bar{c}_j = c_j \leq 0$ for every $j \in \mathcal{N}$, i.e., the basis is dual feasible. Go to Step 1.

Step 1: Feasibility check

a. Solve $\mathbf{A}_B \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b}$.

If $\bar{\mathbf{b}} \geq \mathbf{0}$, then the basic feasible solution $\mathbf{x}_B = \bar{\mathbf{b}}$ and $\mathbf{x}_N = \mathbf{0}$ is an optimal solution, of optimal value $\bar{z} = \mathbf{c}'_B \bar{\mathbf{b}}$. **STOP**. **Else** go to Step 2.

Step 2: Dual pivoting operations

The **pivot row** is the basic variable x_r with the smallest index $r \in \mathcal{B}$ among those with $\bar{b}_r < 0$.

a. Solve $\mathbf{u} \mathbf{A}_B = \mathbf{e}'_r$ and compute $\bar{a}_{rj} = \mathbf{u} \mathbf{A}_j$ for every $j \in \mathcal{N}$.

b. If $\bar{a}_{rj} \geq 0$ for every $j \in \mathcal{N}$, then the LP problem (12.1) is **infeasible**. **STOP**.

c. Determine the **pivot column**, i.e., a non-basic variable x_p with $p \in \mathcal{N}$ such that

$$\frac{\bar{c}_p}{\bar{a}_{rp}} = \min \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} : j \in \mathcal{N}, \bar{a}_{rj} < 0 \right\};$$

in case of ties, the non-basic variable with the smallest index is chosen.

- d. Update the reduced costs:

$$\bar{c}_j \leftarrow \bar{c}_j - \frac{\bar{c}_p}{\bar{a}_{rp}} \bar{a}_{rj}, \quad \forall j \in \mathcal{N} \setminus \{p\}, \quad \text{and} \quad \bar{c}_r \leftarrow -\frac{\bar{c}_p}{\bar{a}_{rp}}.$$

- e. Update the basis with the new partition of the indices of the variables:

$$\mathcal{B} \leftarrow (\mathcal{B} \setminus \{r\}) \cup \{p\} \quad \text{and} \quad \mathcal{N} \leftarrow (\mathcal{N} \setminus \{p\}) \cup \{r\}.$$

Go to Step 1.

The revised dual simplex algorithm does not update the dictionary: at every iteration it updates only the partition $\mathcal{B}$ and $\mathcal{N}$ and the reduced costs $\bar{c}$, and it recomputes $\bar{b}$ and the row $\bar{a}_r$ by solving the two systems of linear equations.

12.4 Execution of the revised dual simplex algorithm

- We consider the following LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 3$ less-than-or-equal-to linear constraints, which is solved in the examples of this section:

$$\begin{aligned} \max \quad & -x_1 - 3x_2 \\ \text{subject to} \quad & -x_1 - 2x_2 \leq -6 \\ & x_1 - 4x_2 \leq -6 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- By adding $m = 3$ slack variables we obtain its standard maximization form, with

$$\mathbf{c} = \begin{pmatrix} -1 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -1 & -2 & 1 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 \\ 2 & -2 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -6 \\ -6 \\ 3 \end{pmatrix}$$

Since $\mathbf{c} \leq \mathbf{0}$, the partition $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$ is **dual feasible**, with reduced costs $\bar{c}_1 = c_1 = -1$ and $\bar{c}_2 = c_2 = -3$.

Example 12.1: revised dual simplex algorithm – first iteration

With $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$ we have $\mathbf{A}_{\mathcal{B}} = \mathbf{I}$, and the reduced costs are $\bar{c}_1 = -1$ and $\bar{c}_2 = -3$.

First system. We solve $A_B \bar{b} = b$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bar{b} = \begin{pmatrix} -6 \\ -6 \\ 3 \end{pmatrix} \implies \bar{b} = \begin{pmatrix} -6 \\ -6 \\ 3 \end{pmatrix}$$

Since $\bar{b}_3 = -6 < 0$, the basis is not primal feasible. Among x_3 and x_4 , both with value -6 , we choose the one with the smallest index, x_3 , as **pivot row**.

Second system. The basic variable x_3 is the first one of $\mathcal{B}$, and then $e_3 = (1 \ 0 \ 0)'$. We solve $u A_B = e_3'$ and obtain $u = (1 \ 0 \ 0)$. The row of $\bar{A}$ associated with x_3 is then

$$\bar{a}_{31} = u A_1 = -1 \quad \text{and} \quad \bar{a}_{32} = u A_2 = -2,$$

both negative, and then the LP problem is not detected as infeasible.

Ratio test. Both non-basic variables are candidates:

$$\min \left\{ \underbrace{\frac{-1}{-1}}_{=1}, \underbrace{\frac{-3}{-2}}_{=\frac{3}{2}} \right\} = 1,$$

and then the **pivot column** is x_1 .

Update of the reduced costs. With $\frac{\bar{c}_1}{\bar{a}_{31}} = 1$ we obtain

$$\bar{c}_2 \leftarrow \bar{c}_2 - 1 \cdot \bar{a}_{32} = -3 - 1 \cdot (-2) = -1 \quad \text{and} \quad \bar{c}_3 \leftarrow -1 = -1.$$

The new partition is $\mathcal{B} = \{1, \ 4, \ 5\}$ and $\mathcal{N} = \{2, \ 3\}$.

Example 12.2: revised dual simplex algorithm – second iteration

With $\mathcal{B} = \{1, \ 4, \ 5\}$ and $\mathcal{N} = \{2, \ 3\}$ we have

$$A_B = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix},$$

and the reduced costs, updated at the previous iteration, are $\bar{c}_2 = -1$ and $\bar{c}_3 = -1$.

First system. We solve $A_B \bar{b} = b$, that is,

$$\begin{cases} -\bar{b}_1 = -6 \\ \bar{b}_1 + \bar{b}_4 = -6 \\ 2\bar{b}_1 + \bar{b}_5 = 3 \end{cases} \implies \bar{b} = \begin{pmatrix} 6 \\ -12 \\ -9 \end{pmatrix}.$$

Two basic variables have a negative value. We choose the one with the smallest index, x_4 , as **pivot row**.

Second system. The basic variable x_4 is the second one of $\mathcal{B}$, and then $\mathbf{e}_4 = (0 \ 1 \ 0)'$. We solve $\mathbf{u} \mathbf{A}_{\mathcal{B}} = \mathbf{e}_4'$, that is,

$$\begin{cases} -u_1 + u_2 + 2u_3 = 0 \\ u_2 = 1 \\ u_3 = 0 \end{cases} \implies \mathbf{u} = (1 \ 1 \ 0),$$

and the row of $\bar{\mathbf{A}}$ associated with x_4 is

$$\bar{a}_{42} = \mathbf{u} \mathbf{A}_2 = -2 - 4 = -6 \quad \text{and} \quad \bar{a}_{43} = \mathbf{u} \mathbf{A}_3 = 1.$$

Ratio test. The only candidate is x_2 , since $\bar{a}_{42} = -6 < 0$ while $\bar{a}_{43} = 1 \geq 0$:

$$\frac{\bar{c}_2}{\bar{a}_{42}} = \frac{-1}{-6} = \frac{1}{6},$$

and then the **pivot column** is x_2 .

Update of the reduced costs. With $\frac{\bar{c}_2}{\bar{a}_{42}} = \frac{1}{6}$ we obtain

$$\bar{c}_3 \leftarrow \bar{c}_3 - \frac{1}{6} \cdot \bar{a}_{43} = -1 - \frac{1}{6} = -\frac{7}{6} \quad \text{and} \quad \bar{c}_4 \leftarrow -\frac{1}{6}.$$

The new partition is $\mathcal{B} = \{1, \ 2, \ 5\}$ and $\mathcal{N} = \{3, \ 4\}$.

Example 12.3: revised dual simplex algorithm – third iteration

With $\mathcal{B} = \{1, \ 2, \ 5\}$ and $\mathcal{N} = \{3, \ 4\}$ we have

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & -2 & 0 \\ 1 & -4 & 0 \\ 2 & -2 & 1 \end{pmatrix},$$

and the reduced costs are $\bar{c}_3 = -\frac{7}{6}$ and $\bar{c}_4 = -\frac{1}{6}$.

First system. We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$, that is,

$$\begin{cases} -\bar{b}_1 - 2\bar{b}_2 = -6 \\ \bar{b}_1 - 4\bar{b}_2 = -6 \\ 2\bar{b}_1 - 2\bar{b}_2 + \bar{b}_5 = 3 \end{cases} \implies \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}.$$

Now $\bar{\mathbf{b}} \geq \mathbf{0}$: the basis is primal feasible and dual feasible, and then it is **optimal**. No further system has to be solved: the optimal value is the scalar product

$$\bar{z} = \mathbf{c}'_{\mathcal{B}} \bar{\mathbf{b}} = (-1) \cdot 2 + (-3) \cdot 2 + 0 \cdot 3 = -8.$$

An optimal solution of this LP problem is

$$\tilde{\mathbf{x}}(LP) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

and the optimal value is $z(LP) = -8$.

If the optimal solution of the **dual problem** is also required, it is obtained with one further system of linear equations, solved once at the end and not at every iteration:

$$\pi A_B = c'_B \quad \implies \quad \pi = \left(\frac{7}{6} \quad \frac{1}{6} \quad 0 \right),$$

that is, $\tilde{\pi}(D(LP)) = \left(\frac{7}{6} \quad \frac{1}{6} \quad 0 \right)'$, whose objective value is $b'\tilde{\pi} = -8 = z(LP)$, in accordance with the Strong Duality theorem.

The primal-dual method

13.1 The reference pair of primal and dual problems

- The simplex algorithm keeps a **primal feasible** basis and works towards dual feasibility; the dual simplex algorithm keeps a **dual feasible** basis and works towards primal feasibility. The **primal-dual method** does something different: it keeps a dual feasible solution, which is not required to be a basic one, and it uses it to decide which primal variables are allowed to take a positive value.
- The method is naturally stated for an LP problem in **standard minimization form** with non-negative right-hand side, and for its dual:

$$\min \quad \mathbf{c}' \mathbf{x} \quad (13.1a)$$

$$\text{subject to} \quad \mathbf{A} \mathbf{x} = \mathbf{b}, \quad (13.1b)$$

$$\mathbf{x} \geq \mathbf{0} \quad (13.1c)$$

$$\max \quad \mathbf{b}' \boldsymbol{\pi} \quad (13.2a)$$

$$\text{subject to} \quad \mathbf{A}' \boldsymbol{\pi} \leq \mathbf{c}, \quad (13.2b)$$

$$\boldsymbol{\pi} \begin{smallmatrix} \geq \\ \leq \end{smallmatrix} \mathbf{0} \quad (13.2c)$$

with $\mathbf{b} \geq \mathbf{0}$. The n dual constraints (13.2b) read $\mathbf{A}'_j \boldsymbol{\pi} \leq c_j$ for every $j \in \{1, 2, \dots, n\}$, where $\mathbf{A}_j$ is the j -th column of the constraint matrix, and the m dual variables are free, since the primal constraints (13.1b) are equalities.

- Every LP problem can be brought to the form (13.1): a maximization objective function is replaced by the opposite one, the inequality constraints receive slack or surplus variables, the non-positive and free variables are replaced as usual, and every constraint with a negative right-hand side is multiplied by -1 .

Definition 13.1: admissible set

Given a feasible solution π of the dual problem (13.2), the **admissible set** is the set of the indices of the dual constraints satisfied with equality:

$$\mathcal{J}(\pi) = \{j \in \{1, 2, \dots, n\} : \mathbf{A}'_j \pi = c_j\}.$$

By the Complementary Slackness theorem, a feasible solution $\mathbf{x}$ of the primal problem (13.1) and a feasible solution π of the dual problem (13.2) are both optimal if and only if

$$x_j = 0, \quad \forall j \notin \mathcal{J}(\pi),$$

that is, if and only if only the admissible variables take a positive value.

- This is the idea of the method: given a dual feasible π , we look for a primal feasible solution which uses only the variables of $\mathcal{J}(\pi)$. If such a solution exists, the pair is optimal; if it does not, the attempt itself tells us how to improve π .

13.2 The restricted primal problem

- Given a dual feasible solution π with admissible set $\mathcal{J}$, we look for a primal feasible solution using only the variables with index in $\mathcal{J}$. This is done by solving an auxiliary problem with m artificial variables, exactly as in the Phase 0 of the simplex algorithm.

Definition 13.2: restricted primal problem

Given a feasible solution π of the dual problem (13.2) with admissible set $\mathcal{J}$, the **restricted primal problem** $RP(\mathcal{J})$ is:

$$\xi(\mathcal{J}) = \min \sum_{i=1}^m a_i \quad (13.3a)$$

$$\text{subject to } \sum_{j \in \mathcal{J}} \mathbf{A}_j x_j + \mathbf{a} = \mathbf{b}, \quad (13.3b)$$

$$x_j \geq 0, \quad \forall j \in \mathcal{J}, \quad (13.3c)$$

$$\mathbf{a} \geq \mathbf{0}. \quad (13.3d)$$

- The restricted primal problem is never infeasible, since $\mathbf{b} \geq \mathbf{0}$ allows the basic feasible solution $\mathbf{a} = \mathbf{b}$ and $x_j = 0$ for every $j \in \mathcal{J}$, and it is never unbounded, since its objective value is non-negative.

1. If $\xi(\mathcal{J}) = 0$, then the optimal solution of $RP(\mathcal{J})$, completed with $x_j = 0$ for every $j \notin \mathcal{J}$, is a **feasible solution of the primal problem** (13.1) satisfying the Comple-

mentary Slackness conditions together with π : the pair is **optimal** and the method stops.

2. If $\xi(\mathcal{J}) > 0$, then no primal feasible solution uses only the admissible variables, and the dual solution π has to be improved.

Definition 13.3: restricted dual problem

The dual problem of the restricted primal problem $RP(\mathcal{J})$ is called the **restricted dual problem** $DRP(\mathcal{J})$:

$$\begin{aligned} \max \quad & \mathbf{b}' \mathbf{y} \\ \text{subject to} \quad & \mathbf{A}'_j \mathbf{y} \leq 0, \quad \forall j \in \mathcal{J}, \\ & y_i \leq 1, \quad \forall i \in \{1, 2, \dots, m\}. \end{aligned}$$

- Its optimal solution $\mathbf{y}$ is produced by the simplex algorithm together with the optimal solution of $RP(\mathcal{J})$, and by Strong Duality we have $\mathbf{b}' \mathbf{y} = \xi(\mathcal{J}) > 0$.

The vector $\mathbf{y}$ is the **direction** along which the dual solution is improved:

$$\pi \leftarrow \pi + \theta \mathbf{y},$$

since the dual objective value increases, for every $\theta > 0$, by

$$\mathbf{b}'(\pi + \theta \mathbf{y}) - \mathbf{b}'\pi = \theta \underbrace{\mathbf{b}'\mathbf{y}}_{=\xi(\mathcal{J}) > 0} > 0.$$

13.3 The dual step

- The step length θ is the largest value for which the new dual solution remains feasible. For every $j \in \{1, 2, \dots, n\}$ we need

$$\mathbf{A}'_j(\pi + \theta \mathbf{y}) \leq c_j \quad \Longleftrightarrow \quad \theta \mathbf{A}'_j \mathbf{y} \leq \underbrace{c_j - \mathbf{A}'_j \pi}_{\text{dual slack } s_j \geq 0}.$$

- For $j \in \mathcal{J}$ the constraint is satisfied for every $\theta \geq 0$, since $s_j = 0$ and $\mathbf{A}'_j \mathbf{y} \leq 0$ by definition of $DRP(\mathcal{J})$. For $j \notin \mathcal{J}$ we have $s_j > 0$, and the constraint restricts θ only when $\mathbf{A}'_j \mathbf{y} > 0$.

The **step length** is

$$\theta = \min \left\{ \frac{s_j}{\mathbf{A}'_j \mathbf{y}} : j \notin \mathcal{J}, \mathbf{A}'_j \mathbf{y} > 0 \right\}, \quad \text{with} \quad s_j = c_j - \mathbf{A}'_j \pi.$$

If $\mathbf{A}'_j \mathbf{y} \leq 0$ for every $j \notin \mathcal{J}$, then θ can grow without bound: the dual problem (13.2) is

unbounded and, by the theorem on the possible outcomes of a pair of primal and dual problems, the primal problem (13.1) is **infeasible**.

- The index j^* attaining the minimum enters the admissible set: after the step, its dual slack becomes zero. Moreover, every index j of a basic variable of the optimal solution of $RP(\mathcal{J})$ remains in the admissible set, since $\mathbf{A}'_j \mathbf{y} = 0$ by the Complementary Slackness conditions between $RP(\mathcal{J})$ and $DRP(\mathcal{J})$. Only the indices $j \in \mathcal{J}$ with $\mathbf{A}'_j \mathbf{y} < 0$ may leave the admissible set, since their dual slack becomes $-\theta \mathbf{A}'_j \mathbf{y} > 0$.

The initial dual feasible solution is often immediate. If $\mathbf{c} \geq \mathbf{0}$, then $\boldsymbol{\pi} = \mathbf{0}$ is feasible, with $\mathcal{J} = \{j : c_j = 0\}$. Otherwise, a dual feasible solution has to be determined beforehand.

13.4 Pseudo-code of the primal-dual method

Algorithm

Input

An LP problem (13.1) with $\mathbf{b} \geq \mathbf{0}$ and a feasible solution $\boldsymbol{\pi}$ of its dual problem (13.2).

Output

An optimal solution of the LP problem (13.1) and an optimal solution of its dual problem, if they exist; otherwise, a certificate that the LP problem (13.1) is infeasible.

Step 0: Initialization

Consider the given dual feasible solution $\boldsymbol{\pi}$. Go to Step 1.

Step 1: Admissible set

Compute the dual slacks $s_j = c_j - \mathbf{A}'_j \boldsymbol{\pi}$ for every $j \in \{1, 2, \dots, n\}$ and the admissible set $\mathcal{J} = \{j : s_j = 0\}$.

Step 2: Restricted primal problem

Solve $RP(\mathcal{J})$ with the simplex algorithm, obtaining its optimal value $\xi(\mathcal{J})$, an optimal solution x_j with $j \in \mathcal{J}$ and an optimal solution $\mathbf{y}$ of $DRP(\mathcal{J})$.

If $\xi(\mathcal{J}) = 0$, then the point

$$\tilde{x}_j = x_j, \forall j \in \mathcal{J}, \quad \tilde{x}_j = 0, \forall j \notin \mathcal{J},$$

is an optimal solution of the LP problem (13.1), $\boldsymbol{\pi}$ is an optimal solution of its dual problem, and $\mathbf{c}'\tilde{\mathbf{x}} = \mathbf{b}'\boldsymbol{\pi}$. STOP.

Step 3: Dual step

- If $\mathbf{A}'_j \mathbf{y} \leq 0$ for every $j \notin \mathcal{J}$, then the LP problem (13.1) is **infeasible**. STOP.

b. Compute

$$\theta \leftarrow \min \left\{ \frac{s_j}{\mathbf{A}'_j \mathbf{y}} : j \notin \mathcal{J}, \mathbf{A}'_j \mathbf{y} > 0 \right\}.$$

c. Update the dual solution: $\boldsymbol{\pi} \leftarrow \boldsymbol{\pi} + \theta \mathbf{y}$.

Go to Step 1.

At every iteration the dual objective value $\mathbf{b}'\boldsymbol{\pi}$ strictly increases. The finiteness of the method follows instead from the restricted primal problems: the optimal basis of $RP(\mathcal{J})$ is still a feasible basis of the next restricted primal problem, since its indices remain admissible, and the new admissible variable x_{j^*} has a negative reduced cost with respect to it. Accordingly, the next restricted primal problem can be solved starting from that basis, and the optimal values $\xi(\mathcal{J})$ never increase from one iteration to the next. They strictly decrease if the pivoting operation on x_{j^*} is non-degenerate, and in general the method terminates after a finite number of iterations when an **anti-cycling rule** (for instance, Bland's rule) is used in the simplex algorithm applied to the restricted primal problems.

13.5 Execution of the primal-dual method

- We consider the following LP problem in standard minimization form with $n = 4$ non-negative variables and $m = 2$ equality linear constraints, which is solved in the examples of this section:

$$\begin{array}{llllllll} \min & 2 & x_1 & + & 2 & x_2 & + & 5 & x_3 & + & 3 & x_4 \\ \text{subject to} & & x_1 & + & & x_2 & + & 2 & x_3 & + & & x_4 & = & 6 \\ & & x_1 & + & 2 & x_2 & + & & x_3 & + & 3 & x_4 & = & 4 \\ & & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

Its right-hand side is non-negative and its objective-function vector is non-negative, and then $\boldsymbol{\pi} = (0 \ 0)'$ is a feasible solution of the dual problem.

Example 13.1: primal-dual method – first iteration

With $\boldsymbol{\pi} = (0 \ 0)'$ the dual slacks are

$$s_1 = 2, \quad s_2 = 2, \quad s_3 = 5, \quad s_4 = 3,$$

and then the admissible set is $\mathcal{J} = \emptyset$: no primal variable is allowed to be positive.

The restricted primal problem has only the two artificial variables, its optimal value is

$$\xi(\emptyset) = a_1 + a_2 = 6 + 4 = 10 > 0,$$

and the optimal solution of $DRP(\emptyset)$ is $\mathbf{y} = (1 \ 1)'$.

The step length is determined by the four non-admissible columns:

$$\mathbf{A}'_1 \mathbf{y} = 2, \quad \mathbf{A}'_2 \mathbf{y} = 3, \quad \mathbf{A}'_3 \mathbf{y} = 3, \quad \mathbf{A}'_4 \mathbf{y} = 4,$$

$$\theta = \min \left\{ \underbrace{\frac{2}{2}}_{=1}, \underbrace{\frac{2}{3}}_{=\frac{2}{3}}, \underbrace{\frac{5}{3}}_{=\frac{5}{3}}, \underbrace{\frac{3}{4}}_{=\frac{3}{4}} \right\} = \frac{2}{3},$$

attained by $j^* = 2$. The new dual solution is

$$\pi \leftarrow (0 \ 0)' + \frac{2}{3} (1 \ 1)' = \left(\frac{2}{3} \ \frac{2}{3} \right)',$$

whose dual objective value is $\mathbf{b}'\pi = \frac{20}{3}$.

Example 13.2: primal-dual method – second iteration

With $\pi = \left(\frac{2}{3} \ \frac{2}{3} \right)'$ the dual slacks are

$$s_1 = \frac{2}{3}, \quad s_2 = 0, \quad s_3 = 3, \quad s_4 = \frac{1}{3},$$

and then $\mathcal{J} = \{2\}$.

The restricted primal problem $RP(\{2\})$ has the variable x_2 and the two artificial variables. Its optimal solution is $x_2 = 2$, with

$$\xi(\{2\}) = 4 > 0,$$

and the optimal solution of $DRP(\{2\})$ is $\mathbf{y} = (1 \ -\frac{1}{2})'$.

For the non-admissible columns we have

$$\mathbf{A}'_1 \mathbf{y} = \frac{1}{2}, \quad \mathbf{A}'_3 \mathbf{y} = \frac{3}{2}, \quad \mathbf{A}'_4 \mathbf{y} = -\frac{1}{2},$$

and only the first two are positive:

$$\theta = \min \left\{ \underbrace{\frac{2/3}{1/2}}_{=\frac{4}{3}}, \underbrace{\frac{3}{3/2}}_{=2} \right\} = \frac{4}{3},$$

attained by $j^* = 1$. The new dual solution is

$$\pi \leftarrow \left(\frac{2}{3} \ \frac{2}{3} \right)' + \frac{4}{3} \left(1 \ -\frac{1}{2} \right)' = (2 \ 0)',$$

whose dual objective value is $\mathbf{b}'\pi = 12$.

Example 13.3: primal-dual method – third iteration

With $\pi = (2 \ 0)'$ the dual slacks are

$$s_1 = 0, \quad s_2 = 0, \quad s_3 = 1, \quad s_4 = 1,$$

and then $\mathcal{J} = \{1, 2\}$.

The restricted primal problem $RP(\{1, 2\})$ has the variables x_1 and x_2 and the two artificial variables. Its optimal solution is $x_1 = 4$ and $x_2 = 0$, with

$$\xi(\{1, 2\}) = 2 > 0,$$

and the optimal solution of $DRP(\{1, 2\})$ is $y = (1 \ -1)'$.

For the non-admissible columns we have

$$A'_3 y = 1 \quad \text{and} \quad A'_4 y = -2,$$

and then

$$\theta = \frac{s_3}{A'_3 y} = \frac{1}{1} = 1,$$

attained by $j^* = 3$. The new dual solution is

$$\pi \leftarrow (2 \ 0)' + 1 \cdot (1 \ -1)' = (3 \ -1)',$$

whose dual objective value is $b' \pi = 14$. The second dual variable has become negative: this is allowed, since the dual variables of (13.2) are free.

Example 13.4: primal-dual method – fourth iteration

With $\pi = (3 \ -1)'$ the dual slacks are

$$s_1 = 0, \quad s_2 = 1, \quad s_3 = 0, \quad s_4 = 3,$$

and then $\mathcal{J} = \{1, 3\}$.

The restricted primal problem $RP(\{1, 3\})$ has the variables x_1 and x_3 and the two artificial variables. Its optimal solution is $x_1 = 2$ and $x_3 = 2$, with

$$\xi(\{1, 3\}) = 0.$$

The optimal value of the restricted primal problem is 0, and then the method stops. An optimal solution of this LP problem is

$$\tilde{x}(LP) = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix},$$

an optimal solution of its dual problem is $\tilde{\pi}(D(LP)) = (3 \ -1)'$, and the two optimal values coincide:

$$c' \tilde{x} = 2 \cdot 2 + 5 \cdot 2 = 14 \quad \text{and} \quad b' \tilde{\pi} = 6 \cdot 3 + 4 \cdot (-1) = 14,$$

in accordance with the Strong Duality theorem.

The dual objective value increased at every iteration,

$$0 \longrightarrow \frac{20}{3} \longrightarrow 12 \longrightarrow 14,$$

and the admissible set changed accordingly,

$$\emptyset \longrightarrow \{2\} \longrightarrow \{1, 2\} \longrightarrow \{1, 3\},$$

until it contained enough variables to build a primal feasible solution. The admissible set does not necessarily grow: at the fourth iteration the index 2 left it, since $\mathbf{A}'_2 \mathbf{y} = -1 < 0$ at the third iteration.

13.6 Comparison with the simplex algorithms

	keeps	works towards
simplex algorithm	a primal feasible basis	dual feasibility
dual simplex algorithm	a dual feasible basis	primal feasibility
primal-dual method	a dual feasible <u>solution</u>	primal feasibility

The difference between the dual simplex algorithm and the primal-dual method is that the former keeps a dual feasible **basis**, while the latter keeps a dual feasible **solution**, which need not be basic. For this reason the primal-dual method can start from any dual feasible solution, even one suggested by the structure of the problem.

- The practical interest of the primal-dual method is that the restricted primal problem (13.3) is often much easier than the original LP problem: it has fewer variables, and for several **combinatorial optimization** problems it turns out to be a problem with a known combinatorial algorithm, such as a shortest path or a maximum flow problem. In those cases the primal-dual method becomes a purely combinatorial algorithm, and it is the origin of many classical algorithms and of several approximation algorithms.

Sensitivity analysis

14.1 Optimal bases and their two conditions

- The input data of an LP problem are rarely known exactly: the entries of the objective-function vector $\mathbf{c}$ and of the right-hand-side vector $\mathbf{b}$ are estimates, and they may change after the LP problem has been solved. The **sensitivity analysis** studies how an optimal solution and the optimal value change when the input data change.
- We consider an LP problem in standard maximization form with $n + m$ non-negative variables and m equality linear constraints, obtained from an LP problem in canonical maximization form by adding m slack variables:

$$\max \quad \mathbf{c}' \mathbf{x} \quad (14.1a)$$

$$\text{subject to} \quad \mathbf{A} \mathbf{x} = \mathbf{b}, \quad (14.1b)$$

$$\mathbf{x} \geq \mathbf{0}. \quad (14.1c)$$

Definition 14.1: optimal basis

A basis $\mathbf{A}_B$ of the LP problem (14.1) is called an **optimal basis** if it satisfies the following two conditions, and then its basic feasible solution is an optimal solution:

1. **primal feasibility:** $\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b} \geq \mathbf{0}$;
2. **dual feasibility:** $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}} \leq \mathbf{0}$, where $\boldsymbol{\pi} = \mathbf{c}'_B \mathbf{A}_B^{-1}$.

The two conditions are sufficient for optimality; under degeneracy, an optimal basic feasible solution may also be associated with a basis which is not dual feasible. A change of the input data preserves the optimality of the basis as long as both conditions keep holding.

- The two conditions involve the input data in two different ways:
 1. the right-hand-side vector $\mathbf{b}$ appears only in the primal feasibility condition;

2. the objective-function vector $\mathbf{c}$ appears only in the dual feasibility condition.

Accordingly, a change of $\mathbf{b}$ may only destroy the primal feasibility of the basis, and a change of $\mathbf{c}$ may only destroy its dual feasibility.

Example 14.1: optimal basis

Consider the following LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 3$ less-than-or-equal-to linear constraints:

$$\begin{array}{rcllcl} \max & 5 & x_1 & + & 2 & x_2 \\ \text{subject to} & - & x_1 & + & 3 & x_2 & \leq & 12 \\ & & x_1 & + & 2 & x_2 & \leq & 12 \\ & 3 & x_1 & - & 2 & x_2 & \leq & 8 \\ & & x_1, & & x_2 & \geq & 0 \end{array}$$

Its standard maximization form has $n + m = 5$ non-negative variables, and the optimal basis determined by the simplex algorithm is

$$\mathcal{B} = \{1, 2, 3\} \quad \text{and} \quad \mathcal{N} = \{4, 5\}.$$

The associated dictionary is:

$$\left\{ \begin{array}{rcllcl} x_1 & = & 5 & - & \frac{1}{4} x_4 & - & \frac{1}{4} x_5 \\ x_2 & = & \frac{7}{2} & - & \frac{3}{8} x_4 & + & \frac{1}{8} x_5 \\ x_3 & = & \frac{13}{2} & + & \frac{7}{8} x_4 & - & \frac{5}{8} x_5 \\ z & = & 32 & - & 2 x_4 & - & x_5 \end{array} \right.$$

and we have:

$$\mathbf{A}_{\mathcal{B}}^{-1} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{3}{8} & -\frac{1}{8} \\ 1 & -\frac{7}{8} & \frac{5}{8} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} = (0 \ 2 \ 1).$$

An optimal solution is $\tilde{\mathbf{x}}(LP) = \begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix}$ and the optimal value is $z(LP) = 32$.

14.2 Changes in the objective-function vector

- We replace one entry c_k of the objective-function vector with $c_k + \delta$, where $\delta \in \mathbb{R}$, and we look for the values of δ for which the optimal basis remains optimal. Since $\mathbf{b}$ does not change, the primal feasibility condition keeps holding, and only the dual feasibility condition has to be checked.

14.2.1 Objective-function coefficient of a non-basic variable

If $k \in \mathcal{N}$, then $\mathbf{c}_\mathcal{B}$ does not change, and then $\boldsymbol{\pi}$ does not change either. Only the reduced cost of x_k changes:

$$\bar{c}_k \longrightarrow \bar{c}_k + \delta,$$

and the optimal basis remains optimal if and only if

$$\delta \leq -\bar{c}_k.$$

- The optimal solution and the optimal value do not change for these values of δ : the dictionary is the same, except for one reduced cost.

14.2.2 Objective-function coefficient of a basic variable

If $k \in \mathcal{B}$, then $\mathbf{c}_\mathcal{B}$ changes, and then all the reduced costs change. Denoting by $\bar{a}_{kj}$ the entry of the matrix $\bar{\mathbf{A}}$ in the row of x_k and in the column of x_j , we have

$$\bar{c}_j \longrightarrow \bar{c}_j - \delta \bar{a}_{kj}, \quad \forall j \in \mathcal{N},$$

and the optimal basis remains optimal if and only if $\bar{c}_j - \delta \bar{a}_{kj} \leq 0$ for every $j \in \mathcal{N}$, that is, if and only if

$$\max \left\{ \frac{\bar{c}_j}{\bar{a}_{kj}} : j \in \mathcal{N}, \bar{a}_{kj} > 0 \right\} \leq \delta \leq \min \left\{ \frac{\bar{c}_j}{\bar{a}_{kj}} : j \in \mathcal{N}, \bar{a}_{kj} < 0 \right\}.$$

- The optimal solution does not change for these values of δ , since $\bar{\mathbf{b}}$ is unchanged, but the optimal value becomes

$$z(LP) + \delta \tilde{x}_k.$$

Example 14.2: changes in the objective-function vector

Consider the optimal basis of [Example 14.1](#), with $\mathcal{N} = \{4, 5\}$, $\bar{c}_4 = -2$, $\bar{c}_5 = -1$ and

$$\bar{\mathbf{A}} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & -\frac{1}{8} \\ -\frac{7}{8} & \frac{5}{8} \end{pmatrix}.$$

Coefficient of the basic variable x_1 . We replace $c_1 = 5$ with $5 + \delta$. The row of x_1 in $\bar{\mathbf{A}}$ is $(\frac{1}{4} \quad \frac{1}{4})$, whose two entries are both positive, and then both of them give a lower bound

on δ :

$$\delta \geq \max \left\{ \underbrace{\frac{-2}{1/4}}_{=-8}, \underbrace{\frac{-1}{1/4}}_{=-4} \right\} = -4,$$

and there is no upper bound, since no entry of the row is negative. Accordingly,

$$\delta \geq -4 \quad \Rightarrow \quad c_1 \in [1, +\infty).$$

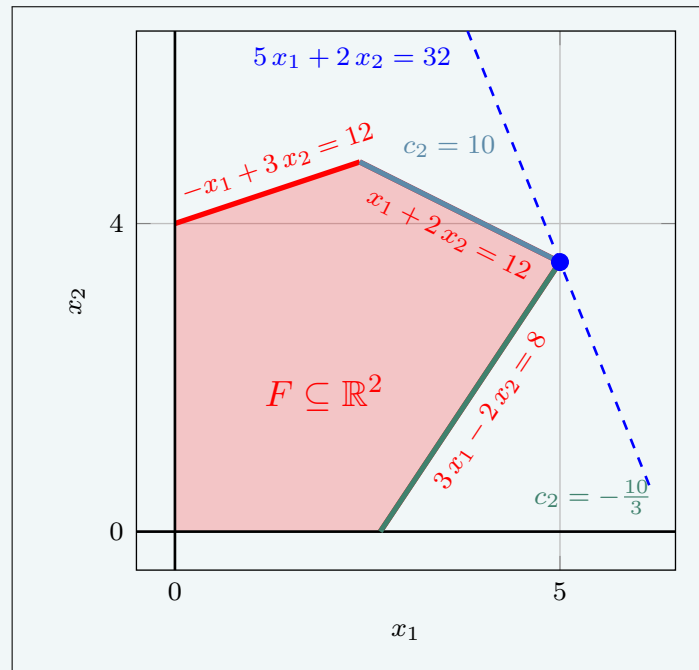
Coefficient of the basic variable x_2 . We replace $c_2 = 2$ with $2 + \delta$. The row of x_2 in $\bar{A}$ is $(\frac{3}{8} \quad -\frac{1}{8})$, with one positive entry, which gives a lower bound, and one negative entry, which gives an upper bound:

$$\underbrace{\frac{-2}{3/8}}_{=-\frac{16}{3}} \leq \delta \leq \underbrace{\frac{-1}{-1/8}}_{=8},$$

that is,

$$-\frac{16}{3} \leq \delta \leq 8 \quad \Rightarrow \quad c_2 \in \left[-\frac{10}{3}, 10 \right].$$

For every value of c_2 in this interval, the optimal solution remains $\hat{x}(LP) = \begin{pmatrix} 5 \\ 7/2 \end{pmatrix}$, while the optimal value becomes $32 + \delta \cdot \frac{7}{2}$.



The level sets of the objective function **rotate** as c_2 varies, while the optimal solution remains the vertex $\begin{pmatrix} 5 \\ 7/2 \end{pmatrix}$. The two endpoints of the interval have a precise geometric meaning: they are the values of c_2 for which the level sets become parallel to one of the two edges

meeting at that vertex. For $c_2 = 10$ they are parallel to the edge on $x_1 + 2x_2 = 12$, drawn in blue, and for $c_2 = -\frac{10}{3}$ they are parallel to the edge on $3x_1 - 2x_2 = 8$, drawn in green. Beyond those values the optimal solution moves to an adjacent vertex.

14.3 Changes in the right-hand-side vector

- We replace one entry b_k of the right-hand-side vector with $b_k + \delta$, where $\delta \in \mathbb{R}$. Since $\mathbf{c}$ does not change, the dual feasibility condition keeps holding, and only the primal feasibility condition has to be checked.

The values of the basic variables become

$$\bar{\mathbf{b}} \rightarrow \mathbf{A}_B^{-1} (\mathbf{b} + \delta \mathbf{e}_k) = \bar{\mathbf{b}} + \delta \mathbf{A}_B^{-1} \mathbf{e}_k,$$

where $\mathbf{A}_B^{-1} \mathbf{e}_k$ is the k -th column of the matrix $\mathbf{A}_B^{-1}$, whose entries we denote by β_{ik} . The optimal basis remains optimal if and only if

$$\max \left\{ -\frac{\bar{b}_i}{\beta_{ik}} : i \in \mathcal{B}, \beta_{ik} > 0 \right\} \leq \delta \leq \min \left\{ -\frac{\bar{b}_i}{\beta_{ik}} : i \in \mathcal{B}, \beta_{ik} < 0 \right\}.$$

Definition 14.2: shadow prices

Given an optimal basis $\mathbf{A}_B$ of the LP problem (14.1), the entries of the row vector

$$\boldsymbol{\pi} = \mathbf{c}'_B \mathbf{A}_B^{-1}$$

are called the **shadow prices** of the m linear constraints.

As long as the optimal basis remains optimal, the optimal value changes linearly with δ :

$$z(LP) \rightarrow z(LP) + \delta \pi_k.$$

The shadow price π_k is then the **marginal value** of the k -th constraint: it measures the increase of the optimal value per unit increase of b_k .

- By the Strong Duality theorem, the vector $\boldsymbol{\pi}$ of the shadow prices is an optimal solution of the dual problem, and then

$$z(LP) = \boldsymbol{\pi} \mathbf{b} = \sum_{i=1}^m \pi_i b_i.$$

- If the k -th constraint is not binding at the optimal solution, i.e., if its slack variable is basic with a strictly positive value, then $\pi_k = 0$: a small change of b_k does not change the optimal value.

Example 14.3: changes in the right-hand-side vector

Consider the optimal basis of [Example 14.1](#), with $\bar{\mathbf{b}} = (5 \quad \frac{7}{2} \quad \frac{13}{2})'$ and $\boldsymbol{\pi} = (0 \quad 2 \quad 1)$.

First constraint. Its shadow price is $\pi_1 = 0$, since its slack variable x_3 is basic with value $\frac{13}{2} > 0$: the first constraint is not binding at the optimal solution, and a small change of b_1 does not change the optimal value. The first column of $\mathbf{A}_B^{-1}$ is $(0 \quad 0 \quad 1)'$, and then only the value of x_3 changes, giving

$$\frac{13}{2} + \delta \geq 0 \quad \Rightarrow \quad \delta \geq -\frac{13}{2} \quad \Rightarrow \quad b_1 \in \left[\frac{11}{2}, +\infty \right).$$

Second constraint. Its shadow price is $\pi_2 = 2$. The second column of $\mathbf{A}_B^{-1}$ is $(\frac{1}{4} \quad \frac{3}{8} \quad -\frac{7}{8})'$, and then

$$\begin{cases} 5 + \frac{1}{4}\delta \geq 0 \\ \frac{7}{2} + \frac{3}{8}\delta \geq 0 \\ \frac{13}{2} - \frac{7}{8}\delta \geq 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \delta \geq -20 \\ \delta \geq -\frac{28}{3} \\ \delta \leq \frac{52}{7} \end{cases} \quad \Rightarrow \quad -\frac{28}{3} \leq \delta \leq \frac{52}{7},$$

that is,

$$b_2 \in \left[\frac{8}{3}, \frac{136}{7} \right].$$

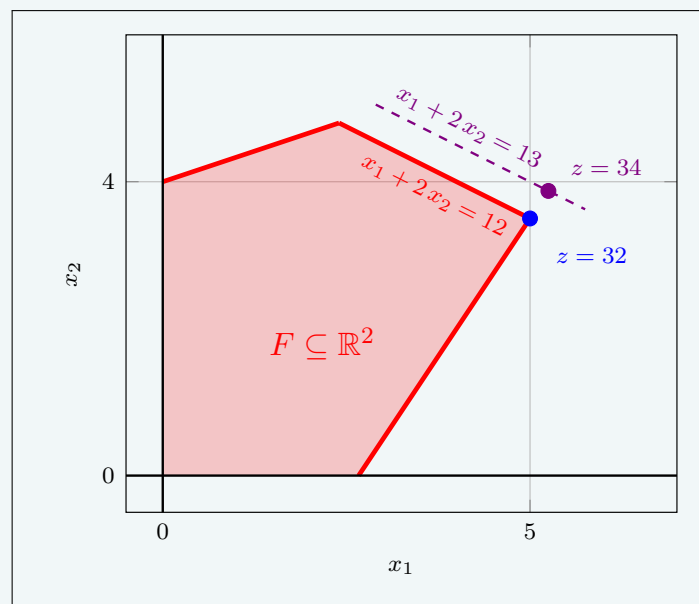
For every value of b_2 in this interval, the optimal value is

$$z(LP) = 32 + 2\delta.$$

For instance, with $b_2 = 13$ we have $\delta = 1$ and the optimal value becomes 34; with $b_2 = 11$ we have $\delta = -1$ and the optimal value becomes 30.

Third constraint. Its shadow price is $\pi_3 = 1$, and its column is $(\frac{1}{4} \quad -\frac{1}{8} \quad \frac{5}{8})'$, which gives

$$b_3 \in \left[-\frac{12}{5}, 36 \right].$$



By increasing b_2 from 12 to 13, the second constraint translates outwards, the optimal solution moves from $\begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 21 \\ 4 \\ 31 \\ 8 \end{pmatrix}$ and the optimal value increases by $\pi_2 = 2$.

14.4 Adding a new variable

- We add one non-negative variable x_* to the LP problem (14.1), with objective-function coefficient c_* and column $\mathbf{A}_* \in \mathbb{R}^m$. The optimal basis of the original LP problem is still a basis of the new one, and it is still primal feasible, since neither $\mathbf{A}_B$ nor $\mathbf{b}$ has changed.

The new variable x_* is non-basic, and its reduced cost is

$$\bar{c}_* = c_* - \boldsymbol{\pi} \mathbf{A}_*.$$

1. If $\bar{c}_* \leq 0$, then the basis is still dual feasible and the optimal solution does not change: the new variable takes the value $\tilde{x}_* = 0$.
2. If $\bar{c}_* > 0$, then the basis is no longer dual feasible: the **simplex algorithm** is restarted from it, with x_* as the first pivot column.

Example 14.4: adding a new variable

Consider the LP problem of [Example 14.1](#) and add a new non-negative variable x_* with objective-function coefficient $c_* = 4$ and column $\mathbf{A}_* = (1 \ 1 \ 1)'$, that is, the LP problem becomes:

$$\begin{array}{llllll} \max & 5 & x_1 & + & 2 & x_2 & + & 4 & x_* \\ \text{subject to} & - & x_1 & + & 3 & x_2 & + & & x_* & \leq & 12 \\ & & x_1 & + & 2 & x_2 & + & & x_* & \leq & 12 \\ & & 3 & x_1 & - & 2 & x_2 & + & & x_* & \leq & 8 \\ & & x_1, & & x_2, & & & & x_* & \geq & 0 \end{array}$$

The reduced cost of the new variable is

$$\bar{c}_* = c_* - \boldsymbol{\pi} \mathbf{A}_* = 4 - (0 \ 2 \ 1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 4 - 3 = 1 > 0,$$

and then the basis $\mathcal{B} = \{1, 2, 3\}$ is no longer dual feasible: it is convenient to produce the new variable.

We restart the simplex algorithm from this basis, with x_* as pivot column. Its column in the dictionary is

$$\bar{\mathbf{A}}_* = \mathbf{A}_B^{-1} \mathbf{A}_* = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{3}{8} & -\frac{1}{8} \\ 1 & -\frac{7}{8} & \frac{5}{8} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{3}{4} \end{pmatrix},$$

and the ratio test gives

$$t_m = \min \left\{ \frac{5}{1/2}, \frac{7/2}{1/4}, \frac{13/2}{3/4} \right\} = \min \left\{ 10, 14, \frac{26}{3} \right\} = \frac{26}{3},$$

attained by the basic variable x_3 , which leaves the basis. After this single pivoting operation all the reduced costs become non-positive, and the new optimal solution is

$$\tilde{x}_1 = \frac{2}{3}, \quad \tilde{x}_2 = \frac{4}{3}, \quad \tilde{x}_3 = \frac{26}{3},$$

with optimal value

$$z(LP) = 5 \cdot \frac{2}{3} + 2 \cdot \frac{4}{3} + 4 \cdot \frac{26}{3} = \frac{122}{3}.$$

14.5 Adding a new constraint

- We add one linear constraint

$$\sum_{j=1}^{n+m} a_{*j} x_j \leq b_*$$

to the LP problem (14.1), together with its slack variable $x_{n+m+1} \geq 0$. The optimal basis of the original LP problem, extended with x_{n+m+1} , is a basis of the new LP problem.

The extended basis is still dual feasible, since the reduced costs do not change and the reduced cost of the new slack variable is 0. Its primal feasibility, instead, depends on the optimal solution $\tilde{x}(LP)$ of the original LP problem:

1. if $\tilde{x}(LP)$ **satisfies** the new constraint, then the extended basis is still optimal and the optimal solution does not change;
2. if $\tilde{x}(LP)$ **violates** the new constraint, then the new slack variable takes a negative value and the **dual simplex algorithm** is restarted from the extended basis, with x_{n+m+1} as the first pivot row.

Example 14.5: adding a new constraint

Consider the LP problem of [Example 14.1](#) and add the new linear constraint

$$x_1 + x_2 \leq 6,$$

with its slack variable x_6 . The optimal solution of the original LP problem violates it, since

$$\tilde{x}_1 + \tilde{x}_2 = 5 + \frac{7}{2} = \frac{17}{2} > 6,$$

and then the value of the new slack variable is

$$\bar{b}_6 = 6 - \frac{17}{2} = -\frac{5}{2} < 0.$$

The extended basis $\mathcal{B} = \{1, 2, 3, 6\}$ is dual feasible, since the reduced costs are still $\bar{c}_4 = -2$ and $\bar{c}_5 = -1$, but it is not primal feasible. We apply the **dual simplex algorithm**,

with x_6 as pivot row. The row of x_6 in the dictionary is obtained by substituting the equations of x_1 and x_2 in the new constraint:

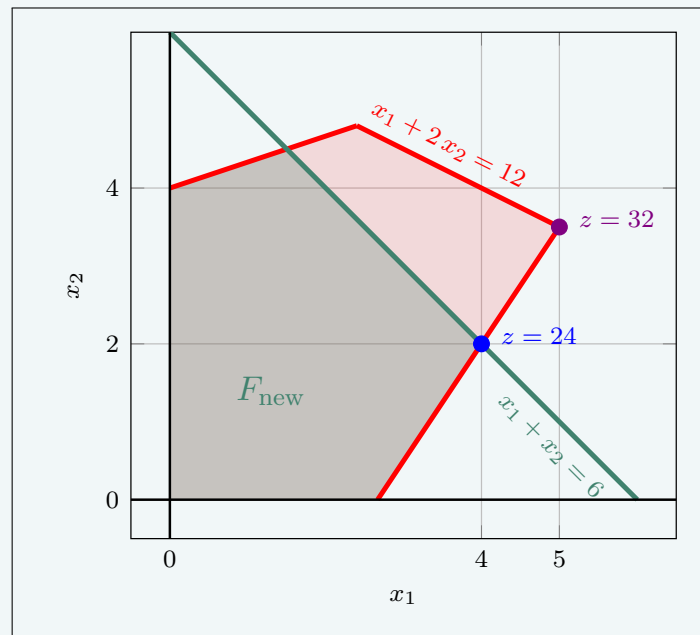
$$x_6 = 6 - x_1 - x_2 = 6 - \left(5 - \frac{1}{4}x_4 - \frac{1}{4}x_5\right) - \left(\frac{7}{2} - \frac{3}{8}x_4 + \frac{1}{8}x_5\right) = -\frac{5}{2} + \frac{5}{8}x_4 + \frac{1}{8}x_5,$$

that is, $\bar{a}_{64} = -\frac{5}{8}$ and $\bar{a}_{65} = -\frac{1}{8}$. The ratio test gives

$$\min \left\{ \underbrace{\frac{-2}{-5/8}}_{=\frac{16}{5}}, \underbrace{\frac{-1}{-1/8}}_{=8} \right\} = \frac{16}{5},$$

and then the pivot column is x_4 . After this single dual pivoting operation the basis becomes primal feasible, and the new optimal solution is

$$\tilde{\mathbf{x}}(LP) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad \text{with} \quad z(LP) = 5 \cdot 4 + 2 \cdot 2 = 24.$$



In violet the optimal solution of the original LP problem, which violates the new constraint, and in blue the optimal solution of the new LP problem, reached with one dual pivoting operation.

14.6 Summary

Given an optimal basis $\mathbf{A}_B$ of an LP problem, a change of its input data is handled as follows:

change	condition to check	how to restart
c_k with $k \in \mathcal{N}$	dual feasibility	simplex algorithm
c_k with $k \in \mathcal{B}$	dual feasibility	simplex algorithm
b_k	primal feasibility	dual simplex algorithm
new variable $x_\star$	dual feasibility	simplex algorithm
new constraint	primal feasibility	dual simplex algorithm

In all the cases the previous optimal basis is used as a **warm start**: the two algorithms restart from it, and usually a small number of pivoting operations is enough to restore optimality.

The geometry of Linear Programming

15.1 The dimension of a polyhedron

- The feasible region of an LP problem is a polyhedron $F \subseteq \mathbb{R}^n$. So far we have described it through its linear constraints; in this document we describe its **geometric** structure, and we relate it to the algebraic notions of basis and basic feasible solution.

Definition 15.1: affine hull and dimension

Given a polyhedron $F \subseteq \mathbb{R}^n$, its **affine hull** $\text{aff}(F)$ is the smallest affine subspace of $\mathbb{R}^n$ containing F . The **dimension** of F , denoted by $\dim(F)$, is the dimension of $\text{aff}(F)$. Equivalently, $\dim(F) = k$ if and only if F contains $k + 1$ points $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}$ such that the k vectors $\mathbf{x}^{(1)} - \mathbf{x}^{(0)}, \dots, \mathbf{x}^{(k)} - \mathbf{x}^{(0)}$ are linearly independent, and it does not contain $k + 2$ points with this property.

Definition 15.2: full-dimensional polyhedron

A polyhedron $F \subseteq \mathbb{R}^n$ is called **full-dimensional** if $\dim(F) = n$. By convention, $\dim(\emptyset) = -1$, and a polyhedron consisting of a single point has dimension 0.

- The dimension of a polyhedron is not the number of variables: it is the number of independent directions along which one can move inside the polyhedron. Every equality constraint satisfied by all the points of F , and linearly independent of the other ones, reduces its dimension by one.

The feasible region of an LP problem in **standard maximization form**,

$$F = \{ \mathbf{x} \in \mathbb{R}^n : \mathbf{A} \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \},$$

is never full-dimensional when $F \neq \emptyset$ and $\text{rank}(\mathbf{A}) \geq 1$: all its points satisfy the m equality constraints, and then

$$\dim(F) \leq n - \text{rank}(\mathbf{A}).$$

The feasible region of an LP problem in **canonical maximization form** is instead full-dimensional whenever it contains a point satisfying all the constraints strictly.

Example 15.1: dimension of a polyhedron

- The feasible region of the LP problem in canonical maximization form

$$\begin{array}{rcll} \max & 5 & x_1 & + & 2 & x_2 \\ \text{subject to} & - & x_1 & + & 3 & x_2 & \leq & 12 \\ & & x_1 & + & 2 & x_2 & \leq & 12 \\ & 3 & x_1 & - & 2 & x_2 & \leq & 8 \\ & x_1, & & & x_2 & \geq & 0 \end{array}$$

is **full-dimensional**: $\dim(F) = 2$, since for instance the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ satisfies all five constraints strictly, and then a small disc around it is contained in F .

- The feasible region of the LP problem in standard maximization form

$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : 2x_1 + 3x_2 + x_3 = 4, \quad \mathbf{x} \geq \mathbf{0} \right\}$$

has dimension $\dim(F) = 2$, and then it is not full-dimensional: all its points lie on the plane $2x_1 + 3x_2 + x_3 = 4$. Here $n = 3$, $\text{rank}(\mathbf{A}) = 1$ and indeed $\dim(F) = 2 = n - \text{rank}(\mathbf{A})$.

15.2 Faces, facets and vertices

Definition 15.3: valid inequality

A linear inequality $\mathbf{a}'\mathbf{x} \leq \beta$, with $\mathbf{a} \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, is **valid** for a polyhedron $F \subseteq \mathbb{R}^n$ if it is satisfied by all its points:

$$\mathbf{a}'\mathbf{x} \leq \beta, \quad \forall \mathbf{x} \in F.$$

Definition 15.4: face

Given a polyhedron $F \subseteq \mathbb{R}^n$ and a valid inequality $\mathbf{a}'\mathbf{x} \leq \beta$ for F , the set

$$G = \{ \mathbf{x} \in F : \mathbf{a}'\mathbf{x} = \beta \}$$

is called a **face** of F . A face is **proper** if $G \neq \emptyset$ and $G \neq F$.

- A face of a polyhedron is obtained by requiring that one valid inequality is satisfied with equality, and it is itself a polyhedron. Both F and $\emptyset$ are faces of F : the first one is obtained with the valid inequality $\mathbf{0}'\mathbf{x} \leq 0$, the second one with any valid inequality which is never satisfied with equality.

- Every linear constraint $\sum_j a_{ij}x_j \leq b_i$ of the LP problem is a valid inequality for its feasible region, and then it defines a face: the set of the feasible solutions for which the constraint is **binding**. Together with the Complementary Slackness conditions, this is the link between the geometry of F and the dual problem.

Definition 15.5: facet and vertex

Given a polyhedron $F \subseteq \mathbb{R}^n$ and a non-empty face G of F :

1. G is a **facet** of F if $\dim(G) = \dim(F) - 1$;
2. G is a **vertex** of F if $\dim(G) = 0$, that is, if G consists of a single point.

A facet is always a proper face. A vertex is a proper face, except when F itself consists of a single point: in that case $G = F$ is the only vertex of F .

The facets are the **largest** proper faces and, when the polyhedron has at least one vertex (as the feasible region of an LP problem with non-negative variables), the vertices are the **smallest** non-empty ones. The facets are important because they are exactly the constraints that cannot be removed: a full-dimensional polyhedron is described by its facet-defining inequalities, and by those only. The vertices are important because, as we show below, they are exactly the basic feasible solutions.

Definition 15.6: extreme point

A point $\mathbf{x} \in F$ is an **extreme point** of the polyhedron F if it does not lie strictly between two distinct points of F , that is, if there are no $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in F$ with $\mathbf{x}^{(1)} \neq \mathbf{x}^{(2)}$ and $\lambda \in (0, 1)$ such that

$$\mathbf{x} = \lambda \mathbf{x}^{(1)} + (1 - \lambda) \mathbf{x}^{(2)}.$$

Theorem 15.7: vertices and extreme points

Given a polyhedron $F \subseteq \mathbb{R}^n$ and a point $\mathbf{x} \in F$, the following two statements are equivalent:

1. $\{\mathbf{x}\}$ is a vertex of F ;
2. $\mathbf{x}$ is an extreme point of F .

Proof. ($1 \Rightarrow 2$) Let $\{\mathbf{x}\}$ be the face defined by the valid inequality $\mathbf{a}'\mathbf{y} \leq \beta$, and assume $\mathbf{x} = \lambda \mathbf{x}^{(1)} + (1 - \lambda) \mathbf{x}^{(2)}$ with $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in F$ and $\lambda \in (0, 1)$. Then

$$\beta = \mathbf{a}'\mathbf{x} = \lambda \underbrace{\mathbf{a}'\mathbf{x}^{(1)}}_{\leq \beta} + (1 - \lambda) \underbrace{\mathbf{a}'\mathbf{x}^{(2)}}_{\leq \beta} \leq \beta,$$

and the inequality can hold with equality only if $\mathbf{a}'\mathbf{x}^{(1)} = \mathbf{a}'\mathbf{x}^{(2)} = \beta$. Accordingly both points belong to the face $\{\mathbf{x}\}$, and then $\mathbf{x}^{(1)} = \mathbf{x}^{(2)} = \mathbf{x}$: the two points are not distinct.

(2 $\Rightarrow$ 1) Let $\mathbf{x}$ be an extreme point and let I be the set of the indices of the constraints of F binding at $\mathbf{x}$. If the rows $\{\mathbf{a}_i\}_{i \in I}$ did not span $\mathbb{R}^n$, there would be a direction $\mathbf{d} \neq \mathbf{0}$ with $\mathbf{a}_i'\mathbf{d} = 0$ for every $i \in I$, and then, for $\varepsilon > 0$ small enough, both $\mathbf{x} \pm \varepsilon\mathbf{d}$ would belong to F , contradicting the fact that $\mathbf{x}$ is an extreme point, since $\mathbf{x}$ is their midpoint. Accordingly the binding rows span $\mathbb{R}^n$, the system $\{\mathbf{a}_i'\mathbf{y} = b_i\}_{i \in I}$ has $\mathbf{x}$ as its unique solution, and the valid inequality obtained by summing the binding constraints defines the face $\{\mathbf{x}\}$. $\square$

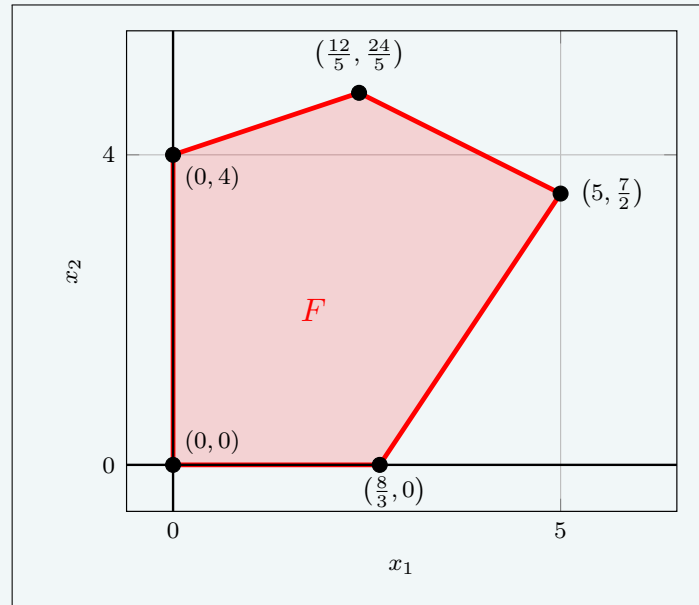
Example 15.2: faces, facets and vertices

Consider the feasible region F of the first LP problem of [Example 15.1](#). It is full-dimensional, $\dim(F) = 2$, and then:

1. its **facets** are its faces of dimension 1, that is, its five edges;
2. its **vertices** are its faces of dimension 0, that is, its five corner points.

Each of the five linear constraints defines one facet, and then all of them are necessary to describe F :

constraint	facet it defines	vertices it contains
$-x_1 + 3x_2 \leq 12$	edge from $(0, 4)$ to $(\frac{12}{5}, \frac{24}{5})$	2
$x_1 + 2x_2 \leq 12$	edge from $(\frac{12}{5}, \frac{24}{5})$ to $(5, \frac{7}{2})$	2
$3x_1 - 2x_2 \leq 8$	edge from $(5, \frac{7}{2})$ to $(\frac{8}{3}, 0)$	2
$x_2 \geq 0$	edge from $(\frac{8}{3}, 0)$ to $(0, 0)$	2
$x_1 \geq 0$	edge from $(0, 0)$ to $(0, 4)$	2



The five points marked in black are the vertices of F , and the five red segments joining them are its facets.

15.3 Basic feasible solutions and vertices

- We now relate the geometric notion of vertex to the algebraic notion of basic feasible solution, which is the one used by the simplex algorithm. We consider an LP problem in standard maximization form with $n + m$ non-negative variables and m equality linear constraints, with $\text{rank}(\mathbf{A}) = m$.

Theorem 15.8: basic feasible solutions and vertices

Given an LP problem in standard maximization form and a point $\bar{\mathbf{x}} \in F$, the following two statements are equivalent:

1. $\bar{\mathbf{x}}$ is a **basic feasible solution**, that is, there exists a basis $\mathbf{A}_{\mathcal{B}}$ such that $\bar{\mathbf{x}}_{\mathcal{B}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} \geq \mathbf{0}$ and $\bar{\mathbf{x}}_{\mathcal{N}} = \mathbf{0}$;
2. $\bar{\mathbf{x}}$ is a **vertex** of F .

- The idea of the proof is that the columns associated with the strictly positive entries of $\bar{\mathbf{x}}$ are linearly independent if and only if $\bar{\mathbf{x}}$ cannot be moved in two opposite directions while staying in F .

Proof. Let $P = \{j : \bar{x}_j > 0\}$.

(1 $\Rightarrow$ 2) Let $\bar{\mathbf{x}}$ be a basic feasible solution with basis $\mathcal{B}$. Then $P \subseteq \mathcal{B}$, and then the columns $\{\mathbf{A}_j\}_{j \in P}$ are linearly independent, being a subset of the columns of the invertible matrix $\mathbf{A}_{\mathcal{B}}$. Assume $\bar{\mathbf{x}} = \lambda \mathbf{x}^{(1)} + (1 - \lambda) \mathbf{x}^{(2)}$ with $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in F$ and

$\lambda \in (0, 1)$. For every $j \notin P$ we have $\bar{x}_j = 0$ and, since both points are non-negative, $x_j^{(1)} = x_j^{(2)} = 0$. Setting $\mathbf{d} = \mathbf{x}^{(1)} - \mathbf{x}^{(2)}$, we obtain $d_j = 0$ for every $j \notin P$ and

$$\mathbf{A}\mathbf{d} = \mathbf{A}\mathbf{x}^{(1)} - \mathbf{A}\mathbf{x}^{(2)} = \mathbf{b} - \mathbf{b} = \mathbf{0} \quad \implies \quad \sum_{j \in P} d_j \mathbf{A}_j = \mathbf{0}.$$

By the linear independence of $\{\mathbf{A}_j\}_{j \in P}$ we get $\mathbf{d} = \mathbf{0}$, that is, $\mathbf{x}^{(1)} = \mathbf{x}^{(2)}$. Accordingly $\bar{\mathbf{x}}$ is an extreme point and, by Theorem 15.7, a vertex of F .

(2 $\Rightarrow$ 1) Let $\bar{\mathbf{x}}$ be a vertex of F and assume, by contradiction, that the columns $\{\mathbf{A}_j\}_{j \in P}$ are linearly dependent: there exists $\mathbf{d} \neq \mathbf{0}$ with $d_j = 0$ for every $j \notin P$ and $\sum_{j \in P} d_j \mathbf{A}_j = \mathbf{0}$. Since $\bar{x}_j > 0$ for every $j \in P$, for $\varepsilon > 0$ small enough both points $\bar{\mathbf{x}} \pm \varepsilon \mathbf{d}$ are non-negative, and they satisfy $\mathbf{A}(\bar{\mathbf{x}} \pm \varepsilon \mathbf{d}) = \mathbf{b}$: they belong to F . But $\bar{\mathbf{x}}$ is their midpoint, which contradicts Theorem 15.7. Accordingly $\{\mathbf{A}_j\}_{j \in P}$ are linearly independent; since $\text{rank}(\mathbf{A}) = m$, they can be completed to m linearly independent columns, whose indices form a basis $\mathcal{B} \supseteq P$. With respect to it, $\bar{\mathbf{x}}_{\mathcal{N}} = \mathbf{0}$ and $\bar{\mathbf{x}}_{\mathcal{B}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} \geq \mathbf{0}$: $\bar{\mathbf{x}}$ is a basic feasible solution. $\square$

The correspondence between bases and vertices is **onto** but not necessarily **one-to-one**: every basic feasible solution is a vertex, but the same vertex may be associated with several different bases. This happens only if the vertex is **degenerate**, that is, if some basic variable takes the value 0: a non-degenerate vertex is associated with exactly one basis, formed by the indices of its strictly positive entries, while a degenerate vertex is usually, but not always, associated with several bases.

Example 15.3: bases and vertices

Consider the first LP problem of [Example 15.1](#) in standard maximization form, with $n+m = 5$ non-negative variables and $m = 3$ equality linear constraints. A basis is a subset of 3 indices out of 5, and there are $\binom{5}{3} = 10$ such subsets. All of them give an invertible matrix $\mathbf{A}_{\mathcal{B}}$, and then all of them are bases:

$\mathcal{B}$	$\mathbf{x}_{\mathcal{B}} = \mathbf{A}_{\mathcal{B}}^{-1}\mathbf{b}$	feasible	vertex (x_1, x_2)
$\{1, 2, 3\}$	$(5, \frac{7}{2}, \frac{13}{2})$	✓	$(5, \frac{7}{2})$
$\{1, 2, 4\}$	$(\frac{48}{7}, \frac{44}{7}, -\frac{52}{7})$		
$\{1, 2, 5\}$	$(\frac{12}{5}, \frac{24}{5}, \frac{52}{5})$	✓	$(\frac{12}{5}, \frac{24}{5})$
$\{1, 3, 4\}$	$(\frac{8}{3}, \frac{44}{3}, \frac{28}{3})$	✓	$(\frac{8}{3}, 0)$
$\{1, 3, 5\}$	$(12, 24, -28)$		
$\{1, 4, 5\}$	$(-12, 24, 44)$		
$\{2, 3, 4\}$	$(-4, 24, 20)$		
$\{2, 3, 5\}$	$(6, -6, 20)$		
$\{2, 4, 5\}$	$(4, 4, 16)$	✓	$(0, 4)$
$\{3, 4, 5\}$	$(12, 12, 8)$	✓	$(0, 0)$

Exactly 5 of the 10 bases are feasible, and they correspond to the 5 vertices of F determined in the previous example. In this LP problem the correspondence is **one-to-one**, since no basic feasible solution is degenerate: in every feasible row of the table all three basic variables are strictly positive.

Example 15.4: a degenerate vertex

Consider instead the following LP problem in canonical maximization form, whose feasible region consists of a single point:

$$\begin{array}{llllll}
 \max & x_1 & + & 2x_2 & & \\
 \text{subject to} & x_1 & + & x_2 & \leq & 2 \\
 & -x_1 & - & x_2 & \leq & -2 \\
 & x_1 & - & x_2 & \leq & 0 \\
 & -x_1 & + & x_2 & \leq & 0 \\
 & x_1, & & x_2 & \geq & 0
 \end{array}$$

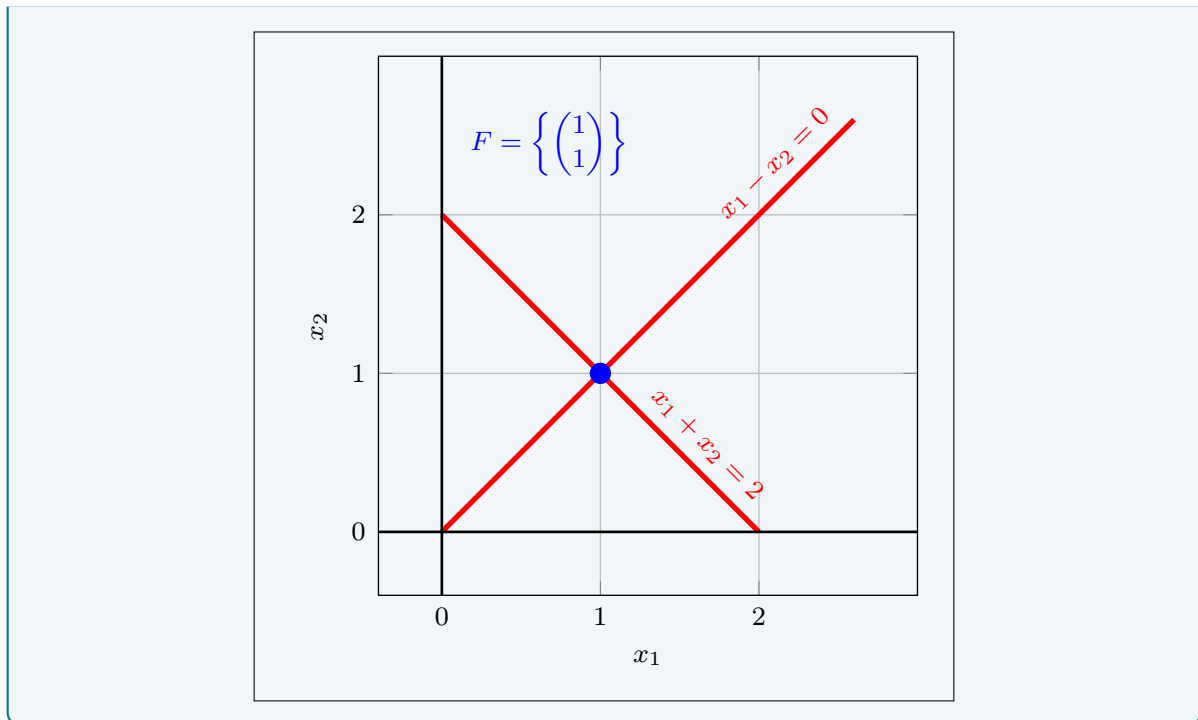
The first two constraints impose $x_1 + x_2 = 2$ and the last two impose $x_1 = x_2$, and then

$$F = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad \dim(F) = 0.$$

In standard maximization form the problem has 6 variables and 4 constraints, and the single vertex $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is associated with **four** different bases,

$$\{1, 2, 3, 5\}, \quad \{1, 2, 3, 6\}, \quad \{1, 2, 4, 5\}, \quad \{1, 2, 4, 6\},$$

in each of which two basic variables take the value 0. This vertex is therefore **degenerate**, and the correspondence between bases and vertices is not one-to-one.



15.4 The Fundamental Theorem, geometrically

- The Fundamental Theorem of Linear Programming states that, whenever an LP problem admits an optimal solution, it admits a basic feasible solution which is optimal. By Theorem 15.8, this statement has an immediate geometric reading.

Corollary 15.9: optimal vertices

If an LP problem admits an optimal solution, then at least one **vertex** of its feasible region is an optimal solution.

This is the result that justifies the **graphical solution method**: to solve an LP problem with two variables it is enough to compute the objective value at each vertex of the feasible region and to keep the best one. It is also the result that makes the simplex algorithm possible: the search for an optimal solution, which is a priori a search over infinitely many points, becomes a search over the finitely many vertices.

- The set of the optimal solutions of an LP problem is itself a **face** of the feasible region: it is the face defined by the valid inequality $\mathbf{c}'\mathbf{x} \leq z(\text{LP})$. Accordingly:
 - if that face is a vertex, the optimal solution is unique;
 - if that face has dimension 1 or more, the LP problem admits infinitely many optimal solutions, and among them at least one vertex.
- A vertex is an optimal solution if and only if no edge leaving it, bounded or unbounded, improves the objective function. The **optimality check** of the simplex algorithm is

based on this property: the non-basic variables correspond to the directions leaving the vertex, and their effect on the objective function is measured by the reduced costs. The check is a sufficient condition for optimality but, at a degenerate vertex, not a necessary one: a degenerate vertex may be optimal even if some reduced cost is positive, since the corresponding direction may immediately leave F .

Definition 15.10: adjacent vertices

Two vertices of a polyhedron are called **adjacent** if the segment joining them is a face, that is, an edge.

In terms of bases, two non-degenerate basic feasible solutions are adjacent if and only if their bases differ by exactly one index.

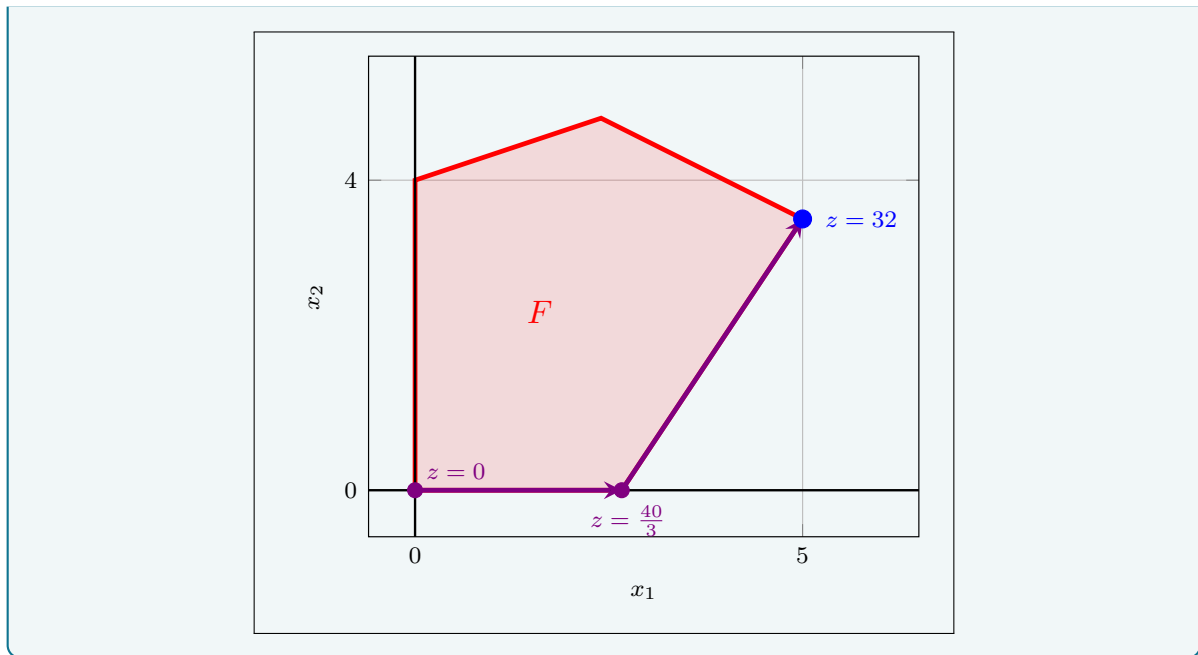
A **pivoting operation** of the simplex algorithm replaces one index of the basis with another one. If it is non-degenerate, it moves from one vertex to an **adjacent** one, along an edge of the feasible region, strictly increasing the objective value; if it is degenerate ($t_m = 0$), it changes the basis but remains at the same vertex. The simplex algorithm is, geometrically, a walk on the edges of the polyhedron.

Example 15.5: the simplex algorithm as a walk on the edges

The execution of the simplex algorithm on the first LP problem of [Example 15.1](#), started from the basis $\mathcal{B} = \{3, 4, 5\}$, visits the following vertices:

$$(0, 0) \longrightarrow \left(\frac{8}{3}, 0\right) \longrightarrow \left(5, \frac{7}{2}\right),$$

with objective values 0, $\frac{40}{3}$ and 32. Each step moves along one edge of F , to an adjacent vertex, and the objective value strictly increases. In the figure, in violet the two vertices visited before the last one, in blue the optimal vertex.



15.5 Summary

geometric notion	algebraic counterpart
polyhedron F	feasible region of the LP problem
$\dim(F)$	n minus the independent equalities valid on F
face	solutions making a valid inequality binding
facet	a constraint which cannot be removed
vertex = extreme point	basic feasible solution
degenerate vertex	several bases, one single point
edge between two vertices	pivoting operation
optimal face	set of the optimal solutions

- The number of vertices of a polyhedron can be **exponential** in the number of its constraints and variables: for an LP problem in standard maximization form it is bounded by $\binom{n+m}{m}$. This is why the simplex algorithm, which visits one vertex at a time, cannot be shown to be efficient by simply counting the vertices, and why its behaviour in practice is much better than its worst case.