



SAPIENZA
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Linear Programming

Volume 3 — Duality and the dual simplex

The dual problem, weak and strong duality, the possible outcomes of a primal-dual pair, complementary slackness; the dual simplex algorithm, infeasibility certificates and re-optimization. Examples worked out step by step and exercises with complete solutions at the end of each chapter.

Fabio Furini

fabiofurini.github.io

Department of Computer, Control and Management Engineering — Sapienza University of Rome

Online version and downloadable material:
fabiofurini.github.io/linear-programming/
github.com/fabiofurini/linear-programming

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Duality theory

9.1 Dual problem of an LP problem in canonical max form

Definition 9.1: dual problem of an LP problem in canonical maximization form

Given a Linear Programming (LP) problem in canonical maximization form with n non-negative variables and m less-than-or-equal-to linear constraints (called the **primal problem** and denoted by LP):

$$\max \sum_{j=1}^n c_j x_j \quad (9.1a)$$

$$\text{subject to } \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \in \{1, 2, \dots, m\}, \quad (9.1b)$$

$$x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n\}, \quad (9.1c)$$

the following LP problem with m non-negative variables and n greater-than-or-equal-to linear constraints

$$\min \sum_{i=1}^m b_i \pi_i \quad (9.2a)$$

$$\text{subject to } \sum_{i=1}^m a_{ij} \pi_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}, \quad (9.2b)$$

$$\pi_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}, \quad (9.2c)$$

is called the **dual problem** and denoted by $D(LP)$.

Example 9.1: pair of primal and dual problems

Let us consider the following primal problem LP in canonical maximization form with $n = 2$ non-negative variables and $m = 3$ less-than-or-equal-to linear constraints ([Example 9.1](#)):

$$\begin{aligned}
 \max \quad & 5 x_1 + 2 x_2 \\
 \text{subject to} \quad & - x_1 + 3 x_2 \leq 12 \\
 & x_1 + 2 x_2 \leq 12 \\
 & 3 x_1 - 2 x_2 \leq 8 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

The dual problem $D(LP)$ with $m = 3$ non-negative variables and $n = 2$ greater-than-or-equal-to linear constraints reads as follows:

$$\begin{aligned}
 \min \quad & 12 \pi_1 + 12 \pi_2 + 8 \pi_3 \\
 \text{subject to} \quad & - \pi_1 + \pi_2 + 3 \pi_3 \geq 5 \\
 & 3 \pi_1 + 2 \pi_2 - 2 \pi_3 \geq 2 \\
 & \pi_1, \pi_2, \pi_3 \geq 0
 \end{aligned}$$

Upper bounds on the optimal value $z(LP)$ of the primal problem can be obtained by taking non-negative linear combinations of the constraints, choosing the multipliers so that the coefficient of each variable in the resulting left-hand side is greater than or equal to its coefficient in the objective function. The right-hand side of the resulting inequality then provides an upper bound on $z(LP)$.

Example 9.2: Computing upper bounds using linear combinations of the constraints

Let us consider the primal problem of [Example 9.1](#).

- For example, by multiplying the second constraint by 5, we get:

$$5 x_1 + 10 x_2 \leq 60$$

Hence, any feasible solution of the primal problem satisfies this inequality, which is an overestimate of the objective function, since we have:

$$5 x_1 + 2 x_2 \leq 5 x_1 + 10 x_2 \leq 60$$

In particular, these inequalities hold for the optimal solution, so $z(LP) \leq 60$.

- For example, by multiplying the second constraint by $\frac{5}{2}$ and then adding it to the third constraint, we get:

$$\frac{5}{2} (x_1 + 2 x_2) + (3 x_1 - 2 x_2) \leq 12 \cdot \frac{5}{2} + 8 \Rightarrow \frac{11}{2} x_1 + 3 x_2 \leq 38$$

Hence, any feasible solution of the primal problem satisfies this inequality, which is an overestimate of the objective function, since we have:

$$5x_1 + 2x_2 \leq \frac{11}{2}x_1 + 3x_2 \leq 38$$

In particular, these inequalities hold for the optimal solution, so $z(LP) \leq 38$.

- For example, by multiplying the first, second, and third constraints by 2, 1, and 2, respectively, and then summing them up, we get:

$$2(-x_1 + 3x_2) + (x_1 + 2x_2) + 2(3x_1 - 2x_2) \leq 12 \cdot 2 + 12 + 8 \cdot 2 \Rightarrow 5x_1 + 4x_2 \leq 52$$

Hence, any feasible solution of the primal problem satisfies this inequality, which is an overestimate of the objective function, since we have:

$$5x_1 + 2x_2 \leq 5x_1 + 4x_2 \leq 52$$

In particular, these inequalities hold for the optimal solution, so $z(LP) \leq 52$.

Example 9.3: Computing upper bounds using linear combinations of the constraints

Let us consider the primal problem of [Example 9.1](#).

Since we have three constraints, we introduce three multipliers: π_1 , π_2 , and π_3 . Multiplying each constraint by its multiplier, we get:

$$(-x_1 + 3x_2) \pi_1 \leq 12 \pi_1$$

$$(x_1 + 2x_2) \pi_2 \leq 12 \pi_2$$

$$(3x_1 - 2x_2) \pi_3 \leq 8 \pi_3$$

Summing the three inequalities, we get:

$$(-x_1 + 3x_2) \pi_1 + (x_1 + 2x_2) \pi_2 + (3x_1 - 2x_2) \pi_3 \leq 12 \pi_1 + 12 \pi_2 + 8 \pi_3$$

Grouping by variable, we get:

$$(-\pi_1 + \pi_2 + 3\pi_3)x_1 + (3\pi_1 + 2\pi_2 - 2\pi_3)x_2 \leq 12\pi_1 + 12\pi_2 + 8\pi_3$$

The right-hand side of this inequality provides a valid upper bound on $z(LP)$ if and only if the coefficient of each variable is at least as large as the corresponding coefficient in the objective function:

$$-\pi_1 + \pi_2 + 3\pi_3 \geq 5$$

$$3\pi_1 + 2\pi_2 - 2\pi_3 \geq 2$$

Each multiplier must be non-negative:

$$\pi_1, \pi_2, \pi_3 \geq 0$$

otherwise, the direction of the corresponding inequality would be reversed. Since we want to obtain the best possible upper bound, i.e., the smallest one, we must minimize the right-hand side of the inequality that provides the upper bound on $z(LP)$.

The dual problem $D(LP)$ is the LP problem of finding the best possible upper bound on the optimal value $z(LP)$ of the primal problem LP by using linear combinations of the constraints.

- An important property of the duality transformation is that it gives back the starting problem if applied twice.

Observation 9.2

The dual problem of the dual problem (9.2) is the primal problem (9.1).

Proof. Starting from the dual problem (9.2),

$$\begin{aligned} & \min \sum_{i=1}^m b_i \pi_i \\ & \text{subject to} \quad \sum_{i=1}^m a_{ij} \pi_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}, \\ & \quad \quad \quad \pi_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}, \end{aligned}$$

consider the equivalent LP problem obtained by multiplying the objective function and each constraint by -1 :

$$\begin{aligned} & \max \sum_{i=1}^m (-b_i) \pi_i \\ & \text{subject to} \quad \sum_{i=1}^m (-a_{ij}) \pi_i \leq -c_j, \quad \forall j \in \{1, 2, \dots, n\}, \\ & \quad \quad \quad \pi_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}. \end{aligned}$$

This problem is now in canonical maximization form. Hence, its dual is

$$\begin{aligned} & \min \sum_{j=1}^n (-c_j) x_j \\ & \text{subject to} \quad \sum_{j=1}^n (-a_{ij}) x_j \geq -b_i, \quad \forall i \in \{1, 2, \dots, m\}, \\ & \quad \quad \quad x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n\}. \end{aligned}$$

Finally, multiplying the objective function and each constraint by -1 , we obtain the equivalent problem

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \in \{1, 2, \dots, m\}, \\ & x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n\}, \end{aligned}$$

which is exactly the primal problem (9.1). Therefore, the dual of the dual problem (9.2) is the primal problem (9.1). $\square$

9.1.1 Weak Duality

- The following theorem, called **Weak Duality**, states that the value of any feasible solution of the primal problem (9.1) is smaller than or equal to the value of any feasible solution of the dual problem (9.2).

Theorem 9.3: Weak Duality

For any feasible solution $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ of the primal problem (9.1) and any feasible solution $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ of the dual problem (9.2), we have

$$\sum_{j=1}^n c_j \bar{x}_j \leq \sum_{i=1}^m b_i \bar{\pi}_i. \quad (9.3)$$

- The idea of the proof is to use the primal and dual feasibility of the two solutions to show that the value of the primal solution is at most equal to the value of the dual solution.

Proof. We have:

$$\begin{aligned} \sum_{j=1}^n c_j \bar{x}_j &\leq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} \bar{\pi}_i \right) \bar{x}_j && \text{(by (9.2b) and (9.1c): } c_j \leq \sum_{i=1}^m a_{ij} \bar{\pi}_i, \bar{x}_j \geq 0, \forall j \in \{1, 2, \dots, n\}) \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} \bar{x}_j \right) \bar{\pi}_i \\ &\leq \sum_{i=1}^m b_i \bar{\pi}_i && \text{(by (9.1b) and (9.2c): } \sum_{j=1}^n a_{ij} \bar{x}_j \leq b_i, \bar{\pi}_i \geq 0, \forall i \in \{1, 2, \dots, m\}) \end{aligned}$$

$\square$

- The Weak Duality theorem states that the value of any feasible solution $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ of a dual problem (9.2) provides a valid **upper bound** on the optimal value $z(LP)$ of the primal problem (9.1):

$$z(LP) \leq \sum_{i=1}^m b_i \bar{\pi}_i$$

Moreover, the value of any feasible solution $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ of a primal problem (9.1) provides a valid **lower bound** on the optimal value $z(D(LP))$ of the dual problem (9.2):

$$\sum_{j=1}^n c_j \bar{x}_j \leq z(D(LP))$$

Example 9.4: Primal and dual feasible solutions

Consider the following pair of primal and dual LP problems:

$$\begin{aligned} \max \quad & 5x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 3x_2 \leq 12 \\ & x_1 + 2x_2 \leq 12 \\ & 3x_1 - 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \\ \\ \min \quad & 12\pi_1 + 12\pi_2 + 8\pi_3 \\ \text{subject to} \quad & -\pi_1 + \pi_2 + 3\pi_3 \geq 5 \\ & 3\pi_1 + 2\pi_2 - 2\pi_3 \geq 2 \\ & \pi_1, \pi_2, \pi_3 \geq 0 \end{aligned}$$

A feasible primal solution is:

$$\bar{x}_1 = \frac{12}{5}, \quad \bar{x}_2 = \frac{24}{5}$$

It provides the following lower bound on the optimal value:

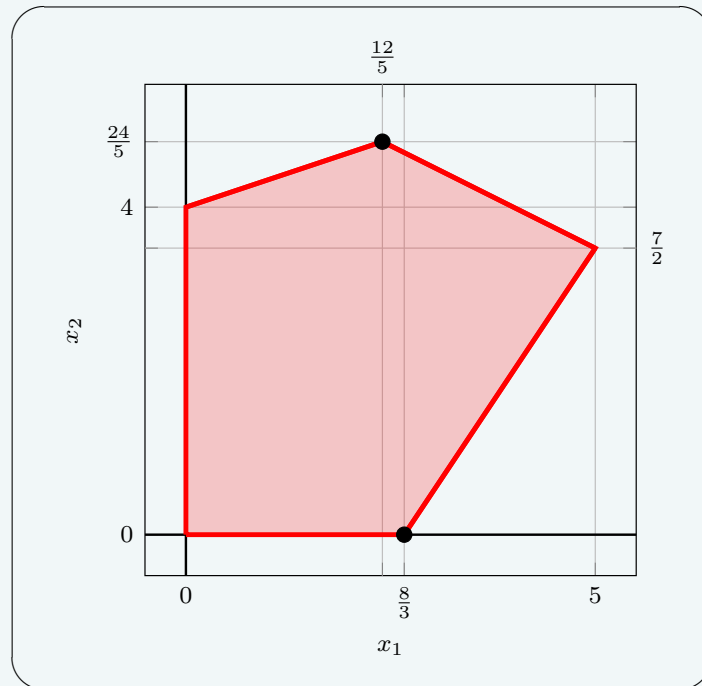
$$LB(\bar{x}) = 5 \cdot \frac{12}{5} + 2 \cdot \frac{24}{5} = \frac{108}{5}$$

Another feasible primal solution is:

$$\bar{x}_1 = \frac{8}{3}, \quad \bar{x}_2 = 0$$

It provides the following lower bound on the optimal value:

$$LB(\bar{x}) = 5 \cdot \frac{8}{3} + 2 \cdot 0 = \frac{40}{3}$$



A feasible dual solution is:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 5, \quad \bar{\pi}_3 = 0$$

It provides the following upper bound on the optimal value:

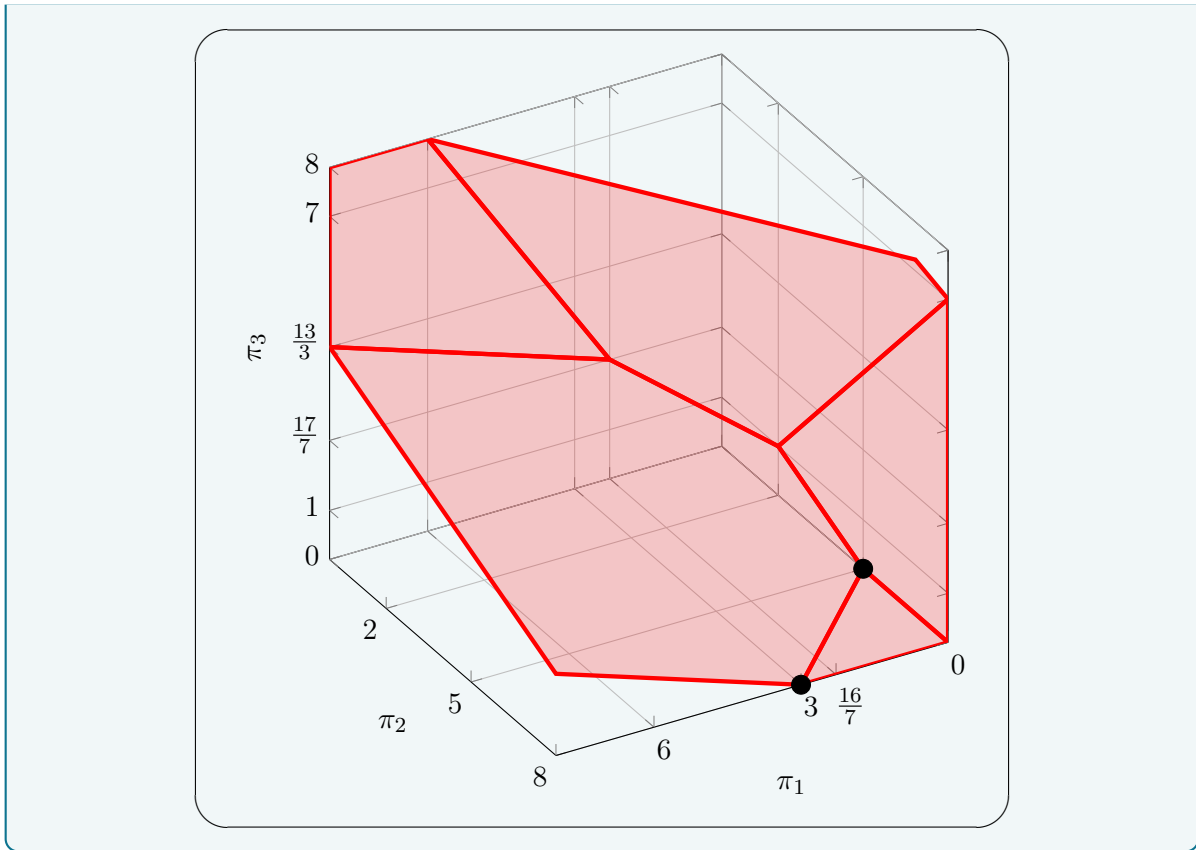
$$UB(\bar{\pi}) = 12 \cdot 0 + 12 \cdot 5 + 8 \cdot 0 = 60$$

Another feasible dual solution is:

$$\bar{\pi}_1 = 3, \quad \bar{\pi}_2 = 8, \quad \bar{\pi}_3 = 0$$

It provides the following upper bound on the optimal value:

$$UB(\bar{\pi}) = 12 \cdot 3 + 12 \cdot 8 + 8 \cdot 0 = 132$$



- Using Weak Duality, we can show that $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ is an optimal solution of the primal problem (9.1) and $(\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m)$ is an optimal solution of the dual problem (9.2) if they are feasible for their respective problems and satisfy

$$\sum_{j=1}^n c_j \tilde{x}_j = \sum_{i=1}^m b_i \tilde{\pi}_i.$$

This follows from Weak Duality as follows:

1. By Weak Duality, for every feasible solution $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ of the primal problem (9.1), and for the feasible dual solution $(\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m)$, we have

$$\sum_{j=1}^n c_j \bar{x}_j \leq \sum_{i=1}^m b_i \tilde{\pi}_i.$$

If, in addition,

$$\sum_{j=1}^n c_j \tilde{x}_j = \sum_{i=1}^m b_i \tilde{\pi}_i,$$

then

$$\sum_{j=1}^n c_j \bar{x}_j \leq \sum_{j=1}^n c_j \tilde{x}_j$$

for every feasible solution $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ of the primal problem. Hence, $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ is an optimal solution of the primal problem (9.1).

2. By Weak Duality, for every feasible solution $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ of the dual problem (9.2), and for the feasible primal solution $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$, we have

$$\sum_{j=1}^n c_j \tilde{x}_j \leq \sum_{i=1}^m b_i \bar{\pi}_i.$$

If, in addition,

$$\sum_{j=1}^n c_j \tilde{x}_j = \sum_{i=1}^m b_i \tilde{\pi}_i,$$

then

$$\sum_{i=1}^m b_i \tilde{\pi}_i \leq \sum_{i=1}^m b_i \bar{\pi}_i$$

for every feasible solution $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ of the dual problem. Hence, $(\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m)$ is an optimal solution of the dual problem (9.2).

Example 9.5: Primal and dual optimal solutions

Consider the following pair of primal and dual LP problems:

$$\begin{aligned} \max \quad & 5x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 3x_2 \leq 12 \\ & x_1 + 2x_2 \leq 12 \\ & 3x_1 - 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & 12\pi_1 + 12\pi_2 + 8\pi_3 \\ \text{subject to} \quad & -\pi_1 + \pi_2 + 3\pi_3 \geq 5 \\ & 3\pi_1 + 2\pi_2 - 2\pi_3 \geq 2 \\ & \pi_1, \pi_2, \pi_3 \geq 0 \end{aligned}$$

To show that

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}$$

is an optimal solution of the primal problem (9.1), and that

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 2, \quad \tilde{\pi}_3 = 1$$

is an optimal solution of the dual problem (9.2), we can proceed as follows.

First, we verify that we have a feasible solution for the primal problem:

$$-\tilde{x}_1 + 3\tilde{x}_2 = -5 + 3 \cdot \frac{7}{2} = \frac{11}{2} \leq 12,$$

$$\tilde{x}_1 + 2\tilde{x}_2 = 5 + 2 \cdot \frac{7}{2} = 12 \leq 12,$$

$$3\tilde{x}_1 - 2\tilde{x}_2 = 3 \cdot 5 - 2 \cdot \frac{7}{2} = 8 \leq 8,$$

$$\tilde{x}_1 = 5 \geq 0, \quad \tilde{x}_2 = \frac{7}{2} \geq 0.$$

Second, we verify that we have a feasible solution for the dual problem:

$$-\tilde{\pi}_1 + \tilde{\pi}_2 + 3\tilde{\pi}_3 = -0 + 2 + 3 \cdot 1 = 5 \geq 5,$$

$$3\tilde{\pi}_1 + 2\tilde{\pi}_2 - 2\tilde{\pi}_3 = 3 \cdot 0 + 2 \cdot 2 - 2 \cdot 1 = 2 \geq 2,$$

$$\tilde{\pi}_1 = 0 \geq 0, \quad \tilde{\pi}_2 = 2 \geq 0, \quad \tilde{\pi}_3 = 1 \geq 0.$$

Finally, we compute the objective function values:

$$5\tilde{x}_1 + 2\tilde{x}_2 = 5 \cdot 5 + 2 \cdot \frac{7}{2} = 25 + 7 = 32,$$

and

$$12\tilde{\pi}_1 + 12\tilde{\pi}_2 + 8\tilde{\pi}_3 = 12 \cdot 0 + 12 \cdot 2 + 8 \cdot 1 = 24 + 8 = 32.$$

Since both solutions are feasible for their respective problems and their objective values are equal, by Weak Duality, they are both optimal.

9.1.2 Strong Duality

- The following theorem, called **Strong Duality**, states that the primal problem (9.1) has an optimal solution if and only if the dual problem (9.2) has an optimal solution, and in this case, the optimal values of the two problems coincide.

Theorem 9.4: Strong Duality

The primal problem (9.1) has an optimal solution $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ if and only if the dual problem (9.2) has an optimal solution $(\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m)$, and in this case

$$\sum_{j=1}^n c_j \tilde{x}_j = \sum_{i=1}^m b_i \tilde{\pi}_i. \quad (9.4)$$

- The existence of an optimal solution does not imply its uniqueness. This theorem only states that there exists at least one optimal solution for the primal problem if and only if there exists at least one optimal solution for the dual problem (there may be several optimal solutions on one side or on both sides, but the optimal objective values are equal).

- Strong duality implies that

the primal problem has an optimal solution

$\Updownarrow$

the dual problem has an optimal solution

so “the primal problem has an optimal solution” is a necessary and sufficient condition to “the dual problem has an optimal solution” and vice versa.

- The main steps of the proof of the Strong Duality theorem are as follows:

1. We first show that if the primal problem has an optimal solution, then the dual problem has an optimal solution, and the optimal values of the two problems coincide.

This is done by using an optimal dictionary of the primal problem produced by the simplex algorithm to construct a feasible solution of the dual problem satisfying (9.4).

By Weak Duality, such a dual solution is necessarily optimal, which proves one implication.

2. The converse implication is obtained by applying the previous reasoning to the dual problem (equivalently, by exchanging the roles of primal and dual, since the dual of the dual is the primal problem).

Proof. Assume that the primal problem (9.1) has an optimal solution, and consider an optimal dictionary produced by the simplex algorithm. Its last row can be written as

$$\sum_{j=1}^n c_j x_j = \bar{z} + \sum_{j=1}^{n+m} \hat{c}_j x_j, \quad \text{where} \quad \hat{c}_j = \begin{cases} \bar{c}_j & \text{if } j \in \mathcal{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\hat{c}_j$ is equal to the reduced cost of a variable x_j if $j \in \mathcal{N}$ (the set of indices of the non-basic variables), and it is equal to 0 otherwise.

Rearranging the terms, we get

$$\sum_{j=1}^n c_j x_j = \bar{z} + \sum_{j=1}^n \hat{c}_j x_j + \sum_{i=1}^m \hat{c}_{n+i} x_{n+i},$$

where the sum has been split into the original variables and the slack variables. Recall that, for the slack variables, we have

$$x_{n+i} = b_i - \sum_{j=1}^n a_{ij} x_j, \quad \forall i \in \{1, 2, \dots, m\}.$$

Substituting these expressions into the previous identity, we obtain

$$\begin{aligned}
 \sum_{j=1}^n c_j x_j &= \bar{z} + \sum_{j=1}^n \hat{c}_j x_j + \sum_{i=1}^m \hat{c}_{n+i} \left(b_i - \sum_{j=1}^n a_{ij} x_j \right) \\
 &= \bar{z} + \sum_{j=1}^n \hat{c}_j x_j + \sum_{i=1}^m b_i \hat{c}_{n+i} - \sum_{i=1}^m \hat{c}_{n+i} \sum_{j=1}^n a_{ij} x_j \\
 &= \bar{z} + \sum_{j=1}^n \hat{c}_j x_j + \sum_{i=1}^m b_i \hat{c}_{n+i} - \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} \hat{c}_{n+i} \right) x_j \\
 &= \bar{z} + \sum_{i=1}^m b_i \hat{c}_{n+i} + \sum_{j=1}^n \left(\hat{c}_j - \sum_{i=1}^m a_{ij} \hat{c}_{n+i} \right) x_j. \tag{A}
 \end{aligned}$$

Since (A) is an identity in the variables $x_1, x_2, \dots, x_n$, it holds for every choice of these variables. Therefore, the constant terms on the two sides must coincide, and the coefficients of each variable must also coincide. Hence,

$$\bar{z} + \sum_{i=1}^m b_i \hat{c}_{n+i} = 0 \quad \implies \quad \bar{z} = - \sum_{i=1}^m b_i \hat{c}_{n+i}, \tag{B}$$

and

$$c_j = \hat{c}_j - \sum_{i=1}^m a_{ij} \hat{c}_{n+i}, \quad \forall j \in \{1, 2, \dots, n\}. \tag{C}$$

Now we define

$$\tilde{\pi}_i = -\hat{c}_{n+i}, \quad \forall i \in \{1, 2, \dots, m\}, \tag{D}$$

and we show that $(\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m)$ is an optimal solution of the dual problem (9.2) satisfying (9.4).

Since the dictionary is optimal, we have $\hat{c}_j \leq 0$ for all $j \in \{1, 2, \dots, n+m\}$. Hence, from (D) we obtain

$$\tilde{\pi}_i = -\hat{c}_{n+i} \geq 0, \quad \forall i \in \{1, 2, \dots, m\},$$

and from (C) we get

$$\sum_{i=1}^m a_{ij} \tilde{\pi}_i = c_j - \hat{c}_j \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}.$$

Therefore, $(\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m)$ is feasible for the dual problem (9.2).

Moreover, since $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ is an optimal solution of the primal problem (9.1), its objective value is equal to the optimal value $\bar{z}$ of the primal problem (9.1). Hence, by (B), we have

$$\sum_{j=1}^n c_j \tilde{x}_j = \bar{z} = \sum_{i=1}^m b_i \tilde{\pi}_i.$$

Thus, the feasible dual solution $(\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m)$ satisfies (9.4).

Finally, Weak Duality implies that for every feasible dual solution $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$,

$$\sum_{j=1}^n c_j \tilde{x}_j \leq \sum_{i=1}^m b_i \bar{\pi}_i.$$

Since for the feasible dual solution $(\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m)$ we have

$$\sum_{j=1}^n c_j \tilde{x}_j = \sum_{i=1}^m b_i \tilde{\pi}_i,$$

it follows that

$$\sum_{i=1}^m b_i \tilde{\pi}_i \leq \sum_{i=1}^m b_i \bar{\pi}_i$$

for every feasible dual solution $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$. Hence, $(\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m)$ is an optimal solution of the dual problem (9.2).

The converse implication is obtained by applying the previous reasoning to the dual problem (after rewriting it in the corresponding canonical form, if needed). Since the dual of the dual problem is the primal problem, it follows that if the dual problem (9.2) has an optimal solution, then the primal problem (9.1) also has an optimal solution. $\square$

- The figure below illustrates the relationship between primal and dual bounds for a **maximization** LP problem. As an illustrative example, the figure considers four primal feasible solutions

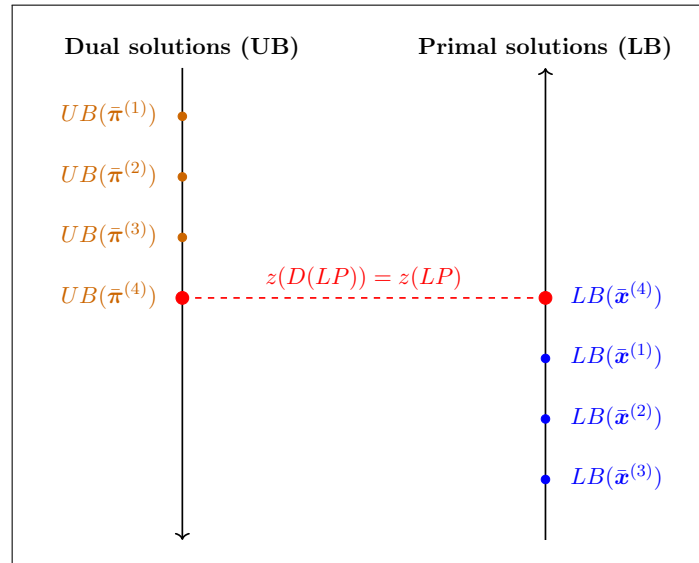
$$\bar{\mathbf{x}}^{(1)}, \bar{\mathbf{x}}^{(2)}, \bar{\mathbf{x}}^{(3)}, \bar{\mathbf{x}}^{(4)},$$

and four dual feasible solutions

$$\bar{\boldsymbol{\pi}}^{(1)}, \bar{\boldsymbol{\pi}}^{(2)}, \bar{\boldsymbol{\pi}}^{(3)}, \bar{\boldsymbol{\pi}}^{(4)}.$$

Primal feasible solutions provide **lower bounds** on the optimal value, while dual feasible solutions provide **upper bounds** on the optimal value by the Weak Duality theorem (9.3).

The fourth solution of each sequence attains the optimal value, and by the Strong Duality theorem (9.4) the two bounds meet: $z(LP) = z(D(LP))$.


Example 9.6: optimal dual solutions from primal optimal dictionaries

Let us consider the primal problem of [Example 9.1](#) (where $n = 2$ and $m = 3$). The standard form of the primal problem is

$$\begin{aligned}
 \max \quad & 5x_1 + 2x_2 \\
 \text{subject to} \quad & -x_1 + 3x_2 + x_3 = 12 \\
 & x_1 + 2x_2 + x_4 = 12 \\
 & 3x_1 - 2x_2 + x_5 = 8 \\
 & x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{aligned}$$

where x_3 , x_4 and x_5 are the slack variables associated with the three constraints. An optimal dictionary reads as follows:

$$\left\{ \begin{array}{l} x_1 = 5 - \frac{1}{4}x_4 - \frac{1}{4}x_5 \\ x_2 = \frac{7}{2} - \frac{3}{8}x_4 + \frac{1}{8}x_5 \\ x_3 = \frac{13}{2} + \frac{7}{8}x_4 - \frac{5}{8}x_5 \\ z = 32 - 2x_4 - x_5 \end{array} \right.$$

where the basic variables are x_1 , x_2 and x_3 , and the non-basic variables are x_4 and x_5 . The optimal primal solution is $\tilde{x}_1 = 5$, $\tilde{x}_2 = \frac{7}{2}$, $\tilde{x}_3 = \frac{13}{2}$, $\tilde{x}_4 = 0$ and $\tilde{x}_5 = 0$, and the optimal value is 32.

The dual problem is

$$\begin{aligned}
 \min \quad & 12 \pi_1 + 12 \pi_2 + 8 \pi_3 \\
 \text{subject to} \quad & -\pi_1 + \pi_2 + 3 \pi_3 \geq 5 \\
 & 3 \pi_1 + 2 \pi_2 - 2 \pi_3 \geq 2 \\
 & \pi_1, \pi_2, \pi_3 \geq 0
 \end{aligned}$$

The values $\hat{c}_j$ with $j \in \{1, 2, 3, 4, 5\}$ used in the proof of the Strong Duality theorem are:

$$\hat{c}_1 = 0, \hat{c}_2 = 0, \hat{c}_3 = 0, \hat{c}_4 = -2, \hat{c}_5 = -1$$

The optimal dual solution can be obtained by assigning the values $\hat{c}_j$ to the corresponding dual variables with reversed signs, as in (D), and it reads as follows:

$$\tilde{\pi}_1 = -\hat{c}_3 = 0, \tilde{\pi}_2 = -\hat{c}_4 = 2, \tilde{\pi}_3 = -\hat{c}_5 = 1$$

The solution is feasible for the dual problem, since $\tilde{\pi}_1, \tilde{\pi}_2$ and $\tilde{\pi}_3$ are non-negative and

$$-\tilde{\pi}_1 + \tilde{\pi}_2 + 3\tilde{\pi}_3 = -0 + 2 + 3 \cdot 1 = 5 \geq 5, \quad 3\tilde{\pi}_1 + 2\tilde{\pi}_2 - 2\tilde{\pi}_3 = 3 \cdot 0 + 2 \cdot 2 - 2 \cdot 1 = 2 \geq 2.$$

The objective function value of this solution is:

$$12 \cdot 0 + 12 \cdot 2 + 8 \cdot 1 = 32$$

9.2 Primal and dual problems: possible outcomes

- The Weak Duality and the Strong Duality theorems relate the optimal values of the primal problem (9.1) and of the dual problem (9.2) whenever both admit an optimal solution. We now describe what happens when one of the two problems is infeasible or unbounded.

Corollary 9.5: unbounded problems and infeasible problems

For a pair of primal (9.1) and dual (9.2) problems:

1. if the primal problem is **unbounded**, then the dual problem is **infeasible**;
2. if the dual problem is **unbounded**, then the primal problem is **infeasible**.

- The idea of the proof is that one single feasible solution of one of the two problems would bound the objective value of the other one, by Weak Duality.

Proof. We prove the first point; the second one is obtained by exchanging the roles of the two problems.

Assume, by contradiction, that the primal problem (9.1) is unbounded and that the dual problem (9.2) admits a feasible solution $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$. By Weak Duality (9.3),

for every feasible solution $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ of the primal problem we have

$$\sum_{j=1}^n c_j \bar{x}_j \leq \underbrace{\sum_{i=1}^m b_i \bar{\pi}_i}_{=M},$$

where M is a fixed real number, since the dual feasible solution is fixed. Accordingly, no feasible solution of the primal problem has an objective value larger than M , which contradicts the assumption that the primal problem is unbounded. $\square$

Theorem 9.6: possible outcomes of a pair of primal and dual problems

For a pair of primal (9.1) and dual (9.2) problems, exactly one of the following four cases occurs:

1. both problems admit an **optimal solution**, and the two optimal values coincide;
2. the primal problem is **unbounded** and the dual problem is **infeasible**;
3. the primal problem is **infeasible** and the dual problem is **unbounded**;
4. both problems are **infeasible**.

Proof. By the Fundamental Theorem of Linear Programming, each of the two problems is infeasible, is unbounded, or admits an optimal solution, and then there are nine possible combinations. We show that five of them cannot occur.

1. If the primal problem admits an optimal solution, then by Strong Duality (9.4) the dual problem admits an optimal solution as well. This excludes the two combinations in which the primal problem admits an optimal solution and the dual problem is unbounded or infeasible. Symmetrically, the two combinations in which the dual problem admits an optimal solution and the primal problem is unbounded or infeasible are excluded as well.
2. If the primal problem is unbounded, then by Corollary 9.5 the dual problem is infeasible. This excludes the combination in which both problems are unbounded.

The four remaining combinations are the ones of the statement. $\square$

The possible outcomes of a pair of primal and dual problems are summarized in the following table, where the four cases of Theorem 9.6 are marked by $\checkmark$:

primal problem LP	dual problem $D(LP)$		
	optimal solution	unbounded	infeasible
optimal solution	✓		
unbounded			✓
infeasible		✓	✓

In particular, an infeasible primal problem may have either an unbounded or an infeasible dual problem: the infeasibility of the primal problem alone does not determine the outcome of the dual problem.

Example 9.7: unbounded primal problem and infeasible dual problem

Consider the following pair of primal and dual problems, with $n = 2$ and $m = 2$ (called [Example 9.7](#)):

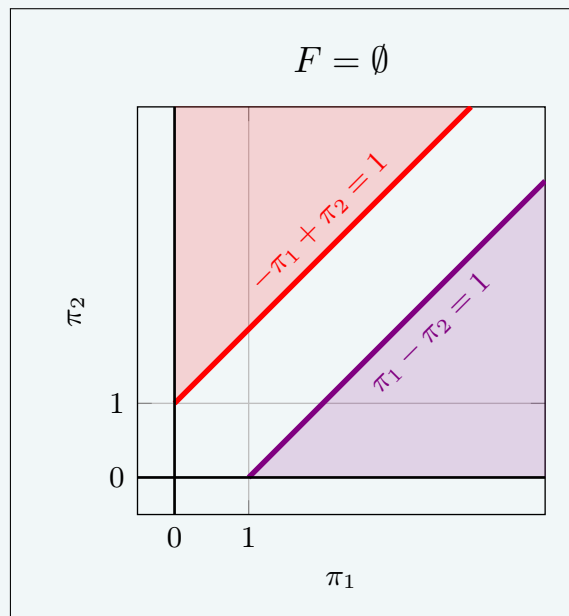
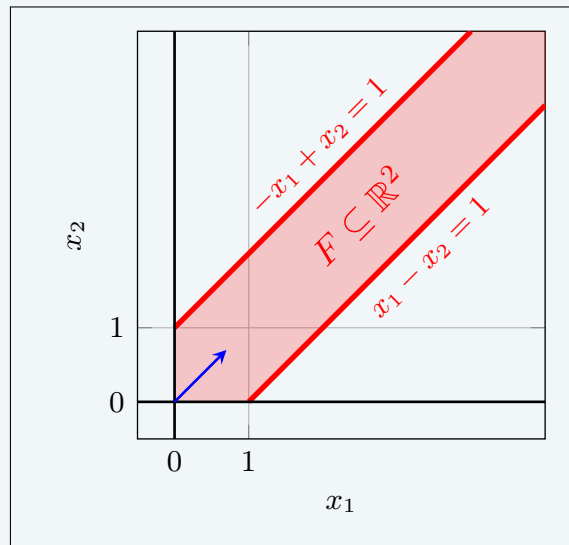
$$\begin{array}{llllll}
 \max & x_1 & + & x_2 & & \\
 \text{subject to} & -x_1 & + & x_2 & \leq & 1 \\
 & x_1 & - & x_2 & \leq & 1 \\
 & x_1, & & x_2 & \geq & 0 \\
 \\
 \min & \pi_1 & + & \pi_2 & & \\
 \text{subject to} & -\pi_1 & + & \pi_2 & \geq & 1 \\
 & \pi_1 & - & \pi_2 & \geq & 1 \\
 & \pi_1, & & \pi_2 & \geq & 0
 \end{array}$$

The primal problem is **unbounded**: for every value $t \geq 0$, the point $\bar{x}_1 = \bar{x}_2 = t$ is a feasible solution of objective value $2t$.

The dual problem is **infeasible**: by summing its two constraints we obtain

$$\underbrace{(-\pi_1 + \pi_2)}_{\geq 1} + \underbrace{(\pi_1 - \pi_2)}_{\geq 1} = 0 \geq 2,$$

which is never satisfied.



In the first figure the feasible region of the primal problem, in the second one the two half-planes defined by the dual constraints, whose intersection is empty.

Example 9.8: infeasible primal problem and unbounded dual problem

Consider the following pair of primal and dual problems, with $n = 2$ and $m = 2$ (called [Example 9.8](#)):

$$\begin{array}{llllll}
 \max & x_1 & + & x_2 & & \\
 \text{subject to} & x_1 & + & x_2 & \leq & 2 \\
 & -x_1 & - & x_2 & \leq & -4 \\
 & x_1, & & x_2 & \geq & 0
 \end{array}$$

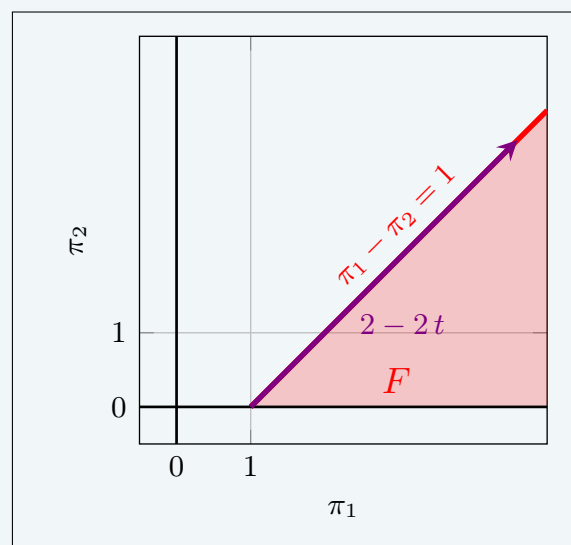
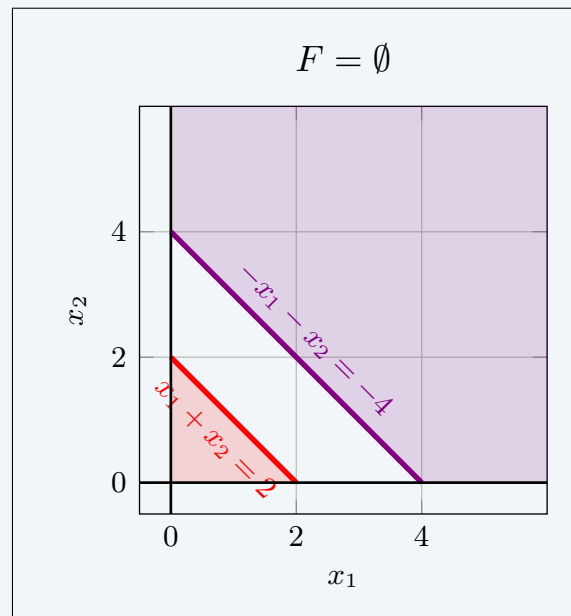
$$\begin{array}{llllll}
 \min & 2 & \pi_1 & - & 4 & \pi_2 \\
 \text{subject to} & & \pi_1 & - & & \pi_2 \geq 1 \\
 & & \pi_1 & - & & \pi_2 \geq 1 \\
 & & \pi_1, & & \pi_2 & \geq 0
 \end{array}$$

The primal problem is **infeasible**: the first constraint imposes $x_1 + x_2 \leq 2$ and the second one imposes $x_1 + x_2 \geq 4$.

The dual problem is **unbounded**: for every value $t \geq 0$, the point $\bar{\pi}_1 = 1 + t$ and $\bar{\pi}_2 = t$ is a feasible solution, since $\bar{\pi}_1 - \bar{\pi}_2 = 1 \geq 1$, and its objective value is

$$2(1+t) - 4t = 2 - 2t,$$

which tends to $-\infty$ as t grows. Since the dual problem is a minimization problem, it is unbounded.



In the first figure the two half-planes defined by the primal constraints, whose intersection is empty; in the second one the feasible region of the dual problem and, in violet, the family of feasible solutions along which its objective value tends to $-\infty$.

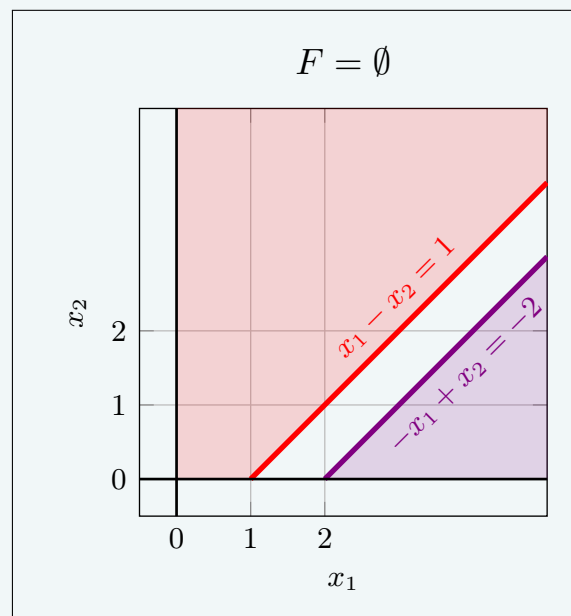
Example 9.9: infeasible primal problem and infeasible dual problem

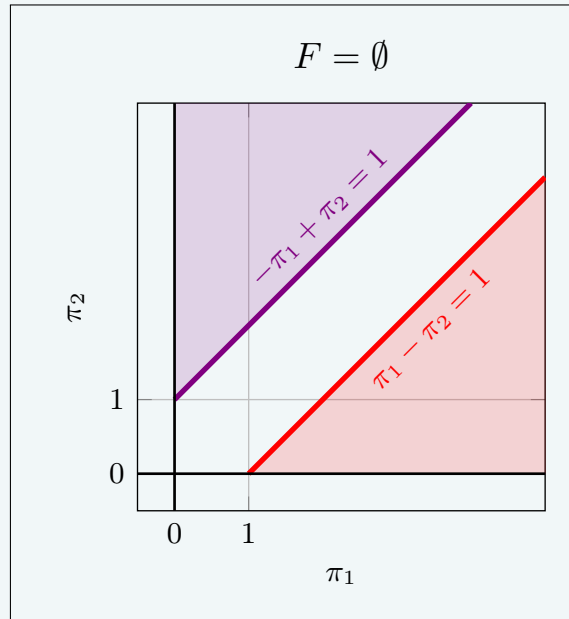
Consider the following pair of primal and dual problems, with $n = 2$ and $m = 2$ (called [Example 9.9](#)):

$$\begin{array}{llllll}
 \max & x_1 & + & x_2 & & \\
 \text{subject to} & x_1 & - & x_2 & \leq & 1 \\
 & - & x_1 & + & x_2 & \leq -2 \\
 & x_1, & & x_2 & \geq & 0 \\
 \\
 \min & \pi_1 & - & 2 & \pi_2 & \\
 \text{subject to} & \pi_1 & - & \pi_2 & \geq & 1 \\
 & - & \pi_1 & + & \pi_2 & \geq 1 \\
 & \pi_1, & & \pi_2 & \geq & 0
 \end{array}$$

The primal problem is **infeasible**: the two constraints impose $x_1 - x_2 \leq 1$ and $x_1 - x_2 \geq 2$.

The dual problem is **infeasible** as well: by summing its two constraints we obtain $0 \geq 2$, which is never satisfied.





In the first figure the primal problem and in the second one the dual problem: both feasible regions are empty.

Theorem 9.6 can be used to **certify** the outcome of an LP problem:

1. one feasible solution of the dual problem proves that the primal problem is not unbounded;
2. one feasible solution of the primal problem proves that the dual problem is not unbounded;
3. one feasible solution of each of the two problems proves that both admit an optimal solution.

9.3 Complementary Slackness

- The following theorem, called **Complementary Slackness**, states that a feasible solution of the primal problem (9.1) and a feasible solution of the dual problem (9.2) are both optimal if and only if the Complementary Slackness Conditions hold.

Theorem 9.7: Complementary Slackness

A feasible solution $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ of the primal problem (9.1) and a feasible solution $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ of the dual problem (9.2) are optimal solutions of their respective problems if and only if the following conditions hold:

$$\sum_{j=1}^n a_{ij} \bar{x}_j = b_i \quad \text{or} \quad \bar{\pi}_i = 0, \quad \forall i \in \{1, \dots, m\}, \quad (9.5a)$$

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i = c_j \quad \text{or} \quad \bar{x}_j = 0, \quad \forall j \in \{1, \dots, n\}. \quad (9.5b)$$

- In (9.5a) and (9.5b), the “or” is inclusive, that is, both conditions may hold simultaneously.

Definition 9.8: Complementary Slackness Conditions

Conditions (9.5a) and (9.5b) are called **Complementary Slackness Conditions** and they can be equivalently rewritten as:

$$\bar{\pi}_i \left(\sum_{j=1}^n a_{ij}\bar{x}_j - b_i \right) = 0, \quad \forall i \in \{1, \dots, m\}, \quad (9.6a)$$

$$\bar{x}_j \left(c_j - \sum_{i=1}^m a_{ij}\bar{\pi}_i \right) = 0, \quad \forall j \in \{1, \dots, n\}. \quad (9.6b)$$

- Complementary Slackness implies that

a pair of primal and dual feasible solutions are optimal



the Complementary Slackness Conditions hold

so “the Complementary Slackness Conditions hold” is a necessary and sufficient condition for “a pair of feasible primal and dual solutions are optimal” and vice versa.

Definition 9.9: active constraints

A primal or a dual constraint is called:

1. **active** if it is satisfied at equality by a given solution (there is no slack);
2. **not active** if it is not satisfied at equality by a given solution (there is a slack).

- Condition (9.5a) (or equivalently (9.6a)) implies that if a primal constraint is not active, i.e., there is a slack, the corresponding dual variable must be zero.
- Condition (9.5b) (or equivalently (9.6b)) implies that if a dual constraint is not active, i.e., there is a slack, the corresponding primal variable must be zero.

Example 9.10: Complementary slackness

Consider the following pair of primal and dual LP problems:

$$\begin{aligned}
 & \max \quad 5x_1 + 2x_2 \\
 & \text{subject to} \quad -x_1 + 3x_2 \leq 12 \\
 & \quad \quad \quad x_1 + 2x_2 \leq 12 \\
 & \quad \quad \quad 3x_1 - 2x_2 \leq 8 \\
 & \quad \quad \quad x_1, \quad x_2 \geq 0 \\
 \\
 & \min \quad 12\pi_1 + 12\pi_2 + 8\pi_3 \\
 & \text{subject to} \quad -\pi_1 + \pi_2 + 3\pi_3 \geq 5 \\
 & \quad \quad \quad 3\pi_1 + 2\pi_2 - 2\pi_3 \geq 2 \\
 & \quad \quad \quad \pi_1, \quad \pi_2, \quad \pi_3 \geq 0
 \end{aligned}$$

An optimal primal solution is

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}.$$

Primal slacks:

$$\begin{aligned}
 12 - (-\tilde{x}_1 + 3\tilde{x}_2) &= 12 - \left(-5 + 3 \cdot \frac{7}{2}\right) = \frac{13}{2} > 0 \Rightarrow \tilde{\pi}_1 = 0, \\
 12 - (\tilde{x}_1 + 2\tilde{x}_2) &= 12 - (5 + 7) = 0 \Rightarrow \tilde{\pi}_2 \text{ may be } > 0, \\
 8 - (3\tilde{x}_1 - 2\tilde{x}_2) &= 8 - (15 - 7) = 0 \Rightarrow \tilde{\pi}_3 \text{ may be } > 0.
 \end{aligned}$$

The first primal constraint is not active, i.e., there is a slack. So the corresponding dual variable $\tilde{\pi}_1$ must be zero.

The second and third primal constraints are active, i.e., there are no slacks. So the corresponding dual variables $\tilde{\pi}_2$ and $\tilde{\pi}_3$ are free to be positive.

An optimal dual solution is

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 2, \quad \tilde{\pi}_3 = 1.$$

Dual slacks:

$$\begin{aligned}
 (-\tilde{\pi}_1 + \tilde{\pi}_2 + 3\tilde{\pi}_3) - 5 &= (-0 + 2 + 3 \cdot 1) - 5 = 0 \Rightarrow \tilde{x}_1 \text{ may be } > 0, \\
 (3\tilde{\pi}_1 + 2\tilde{\pi}_2 - 2\tilde{\pi}_3) - 2 &= (3 \cdot 0 + 2 \cdot 2 - 2 \cdot 1) - 2 = 0 \Rightarrow \tilde{x}_2 \text{ may be } > 0.
 \end{aligned}$$

All dual constraints are active, i.e., there are no slacks. So the corresponding primal variables $\tilde{x}_1$ and $\tilde{x}_2$ are free to be positive.

- The complementary slackness conditions geometrically mean that, given an optimal solu-

tion $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$ of a primal problem (9.1) and an optimal solution $(\tilde{\pi}_1, \tilde{\pi}_2, \dots, \tilde{\pi}_m)$ of its dual problem (9.2), we have two pairs of perpendicular vectors:

$$\begin{pmatrix} \tilde{\pi}_1 \\ \tilde{\pi}_2 \\ \vdots \\ \tilde{\pi}_m \end{pmatrix}, \begin{pmatrix} \sum_{j=1}^n a_{1j}\tilde{x}_j - b_1 \\ \sum_{j=1}^n a_{2j}\tilde{x}_j - b_2 \\ \vdots \\ \sum_{j=1}^n a_{mj}\tilde{x}_j - b_m \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix}, \begin{pmatrix} c_1 - \sum_{i=1}^m a_{i1}\tilde{\pi}_i \\ c_2 - \sum_{i=1}^m a_{i2}\tilde{\pi}_i \\ \vdots \\ c_n - \sum_{i=1}^m a_{in}\tilde{\pi}_i \end{pmatrix}$$

Example 9.11: Complementary Slackness

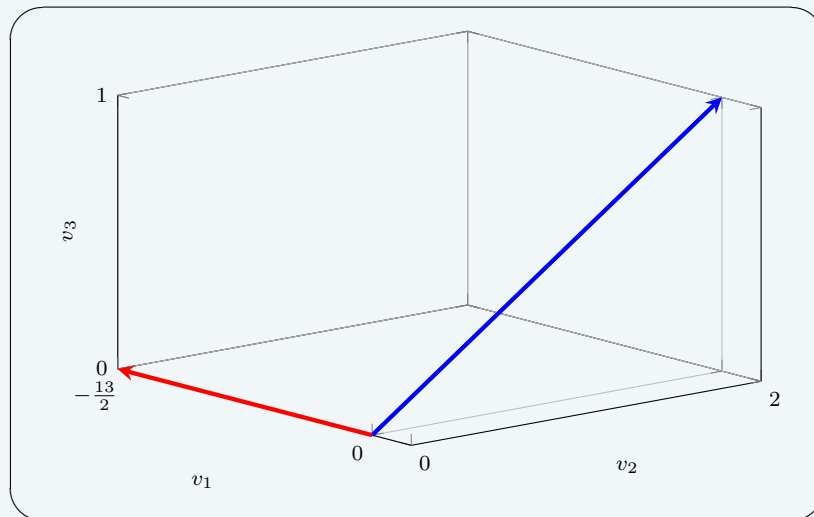
Consider again the previous pair of primal and dual LP problems and their optimal solutions.

The first pair of perpendicular vectors (nonzero case) is

$$\begin{pmatrix} \tilde{\pi}_1 \\ \tilde{\pi}_2 \\ \tilde{\pi}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\tilde{x}_1 + 3\tilde{x}_2 - 12 \\ \tilde{x}_1 + 2\tilde{x}_2 - 12 \\ 3\tilde{x}_1 - 2\tilde{x}_2 - 8 \end{pmatrix} = \begin{pmatrix} -\frac{13}{2} \\ 0 \\ 0 \end{pmatrix}.$$

Orthogonality check:

$$\begin{pmatrix} 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} -\frac{13}{2} \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \left(-\frac{13}{2}\right) + 2 \cdot 0 + 1 \cdot 0 = 0.$$



The two vectors are drawn in the same space $\mathbb{R}^3$ with generic coordinates (v_1, v_2, v_3) : in blue the optimal dual solution $\tilde{\pi}$, in red the vector of the primal constraint values minus the RHS, $(-\frac{13}{2}, 0, 0)$.

The second pair of perpendicular vectors is

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 - (-\tilde{\pi}_1 + \tilde{\pi}_2 + 3\tilde{\pi}_3) \\ 2 - (3\tilde{\pi}_1 + 2\tilde{\pi}_2 - 2\tilde{\pi}_3) \end{pmatrix} = \begin{pmatrix} 5 - (-0 + 2 + 3 \cdot 1) \\ 2 - (3 \cdot 0 + 2 \cdot 2 - 2 \cdot 1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Orthogonality check:

$$\begin{pmatrix} 5 & \frac{7}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0.$$

Proof. Since $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ is a feasible solution of the primal problem (9.1) and $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ is a feasible solution of the dual problem (9.2), by the same argument used in the proof of Weak Duality, we have

$$\sum_{j=1}^n c_j \bar{x}_j \leq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} \bar{\pi}_i \right) \bar{x}_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} \bar{x}_j \right) \bar{\pi}_i \leq \sum_{i=1}^m b_i \bar{\pi}_i. \quad (\text{A})$$

The first inequality of (A) can be rewritten as

$$0 \leq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} \bar{\pi}_i - c_j \right) \bar{x}_j,$$

and by dual feasibility and primal feasibility, for all $j \in \{1, 2, \dots, n\}$, we have

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i - c_j \geq 0 \quad \text{and} \quad \bar{x}_j \geq 0.$$

A sum of non-negative terms equals zero if and only if each term is zero. Therefore, equality holds in the first inequality of (A) if and only if

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{or} \quad \bar{x}_j = 0, \quad \forall j \in \{1, 2, \dots, n\},$$

that is, if and only if condition (9.5b) holds.

Similarly, the second inequality of (A) can be rewritten as

$$0 \leq \sum_{i=1}^m \left(b_i - \sum_{j=1}^n a_{ij} \bar{x}_j \right) \bar{\pi}_i,$$

and by primal feasibility and dual feasibility, for all $i \in \{1, 2, \dots, m\}$, we have

$$b_i - \sum_{j=1}^n a_{ij} \bar{x}_j \geq 0 \quad \text{and} \quad \bar{\pi}_i \geq 0.$$

A sum of non-negative terms equals zero if and only if each term is zero. Therefore, equality holds in the second inequality of (A) if and only if

$$\sum_{j=1}^n a_{ij} \bar{x}_j = b_i \quad \text{or} \quad \bar{\pi}_i = 0, \quad \forall i \in \{1, 2, \dots, m\},$$

that is, if and only if condition (9.5a) holds.

Assume first that $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ and $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ are optimal solutions of their respective problems. Then, by Strong Duality,

$$\sum_{j=1}^n c_j \bar{x}_j = \sum_{i=1}^m b_i \bar{\pi}_i.$$

Therefore, both inequalities in (A) must hold with equality, and hence conditions (9.5a) and (9.5b) hold.

Conversely, assume that conditions (9.5a) and (9.5b) hold. Then both inequalities in (A) hold with equality, and therefore

$$\sum_{j=1}^n c_j \bar{x}_j = \sum_{i=1}^m b_i \bar{\pi}_i.$$

Since $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ and $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ are feasible solutions, Weak Duality implies that they are optimal solutions of their respective problems. $\square$

- The following theorem (derived from the previous one) allows us to check whether a feasible solution of a primal problem (9.1) is optimal by constructing (if possible) a feasible solution of the dual problem (9.2) with the same objective function value, i.e., it is an optimality check.

Corollary 9.10

A feasible solution $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ of the primal problem (9.1) is optimal if and only if there exists $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ such that:

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if } \bar{x}_j > 0, \quad \forall j \in \{1, \dots, n\}, \quad (9.7a)$$

$$\bar{\pi}_i = 0 \quad \text{if } \sum_{j=1}^n a_{ij} \bar{x}_j < b_i, \quad \forall i \in \{1, \dots, m\}, \quad (9.7b)$$

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i \geq c_j, \quad \forall j \in \{1, \dots, n\}, \quad (9.7c)$$

$$\bar{\pi}_i \geq 0, \quad \forall i \in \{1, \dots, m\}. \quad (9.7d)$$

Proof. Assume first that $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ is an optimal solution of problem (9.1).

By Strong Duality, there exists an optimal solution $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ of the dual problem (9.2). Since $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ is feasible for the dual problem, it satisfies (9.7c) and (9.7d). Moreover, since the primal and dual solutions are both optimal, by Complementary Slackness they satisfy (9.7a) and (9.7b).

Hence, there exists $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ satisfying (9.7a)–(9.7d).

Conversely, assume that there exists $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ satisfying (9.7a)–(9.7d).

Since $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ is feasible for the primal problem by hypothesis, and (9.7c)–(9.7d) imply that $(\bar{\pi}_1, \bar{\pi}_2, \dots, \bar{\pi}_m)$ is feasible for the dual problem, both vectors are feasible for their respective problems. Furthermore, (9.7a) implies condition (9.5b). Moreover, primal feasibility together with (9.7b) implies condition (9.5a).

Therefore, the primal and dual feasible solutions satisfy the Complementary Slackness conditions, and by the Complementary Slackness theorem they are optimal for their respective problems. $\square$

Example 9.12: Complementary Slackness

We check if the feasible solution

$$\bar{x}_1 = 5, \quad \bar{x}_2 = \frac{7}{2}$$

of the LP problem of [Example 9.1](#) is optimal.

1. Since

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if } \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of two linear equations with three variables:

$$- \bar{\pi}_1 + \bar{\pi}_2 + 3 \bar{\pi}_3 = 5 \quad (\bar{x}_1 = 5 > 0)$$

$$3 \bar{\pi}_1 + 2 \bar{\pi}_2 - 2 \bar{\pi}_3 = 2 \quad (\bar{x}_2 = \frac{7}{2} > 0)$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij} \bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$-1 \cdot 5 + 3 \cdot \frac{7}{2} = \frac{11}{2} < 12 \quad \text{so} \quad \bar{\pi}_1 = 0$$

$$1 \cdot 5 + 2 \cdot \frac{7}{2} = 12 \not< 12$$

$$3 \cdot 5 - 2 \cdot \frac{7}{2} = 8 \not< 8$$

We then solve the previous system of two equations with two variables and we get:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 2, \quad \bar{\pi}_3 = 1$$

3. We need to check

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$-1 \cdot 0 + 1 \cdot 2 + 3 \cdot 1 = 5 \geq 5$$

$$3 \cdot 0 + 2 \cdot 2 - 2 \cdot 1 = 2 \geq 2$$

4. We need to check

$$\bar{\pi}_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}$$

so since

$$\bar{\pi}_1 = 0 \geq 0, \quad \bar{\pi}_2 = 2 \geq 0, \quad \bar{\pi}_3 = 1 \geq 0$$

the solution is optimal.

Example 9.13: Complementary Slackness

We check if the feasible solution

$$\bar{x}_1 = \frac{12}{5}, \quad \bar{x}_2 = \frac{24}{5}$$

of the LP problem of [Example 9.1](#) is optimal.

1. Since

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of two linear equations with three variables:

$$- \bar{\pi}_1 + \bar{\pi}_2 + 3 \bar{\pi}_3 = 5 \quad (\bar{x}_1 = \frac{12}{5} > 0)$$

$$3 \bar{\pi}_1 + 2 \bar{\pi}_2 - 2 \bar{\pi}_3 = 2 \quad (\bar{x}_2 = \frac{24}{5} > 0)$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij} \bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$-1 \cdot \frac{12}{5} + 3 \cdot \frac{24}{5} = 12 \not< 12$$

$$1 \cdot \frac{12}{5} + 2 \cdot \frac{24}{5} = 12 \not< 12$$

$$3 \cdot \frac{12}{5} - 2 \cdot \frac{24}{5} = -\frac{12}{5} < 8 \quad \text{so} \quad \bar{\pi}_3 = 0$$

We then solve the previous system of two equations with two variables and we get:

$$\bar{\pi}_1 = -\frac{8}{5}, \quad \bar{\pi}_2 = \frac{17}{5}, \quad \bar{\pi}_3 = 0$$

3. We need to check

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$-1 \cdot \left(-\frac{8}{5}\right) + 1 \cdot \frac{17}{5} + 3 \cdot 0 = 5 \geq 5$$

$$3 \cdot \left(-\frac{8}{5}\right) + 2 \cdot \frac{17}{5} - 2 \cdot 0 = 2 \geq 2$$

4. We need to check

$$\bar{\pi}_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}$$

but since

$$\bar{\pi}_1 = -\frac{8}{5} < 0$$

the solution is not optimal.

9.3.1 Dual problem in matrix notation

- The input data of an LP problem in canonical maximization form with n variables and m less-than-or-equal-to inequality constraints can be represented in matrix notation by an **objective-function coefficient vector** $\mathbf{c} \in \mathbb{R}^n$ of n rows, by an **RHS vector** $\mathbf{b} \in \mathbb{R}^m$ of m rows, and by a **constraint matrix** $\mathbf{A} \in \mathbb{R}^{m \times n}$ with m rows and n columns:

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_n \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \vdots & \vdots & \dots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

- The n primal variables can be represented by a column vector $\mathbf{x} \in \mathbb{R}^n$ of n rows containing the n variables, and the m dual variables can be represented by a column vector $\boldsymbol{\pi} \in \mathbb{R}^m$ of m rows containing the m variables:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \quad \boldsymbol{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_m \end{pmatrix}$$

- Using matrix notation, a primal problem in canonical maximization form and its dual problem read as follows:

$$\begin{aligned}
 & \max \quad c' x \\
 & \text{subject to} \quad A x \leq b, \\
 & \quad \quad \quad x \geq 0.
 \end{aligned}$$

$$\begin{aligned}
 & \min \quad b' \pi \\
 & \text{subject to} \quad A' \pi \geq c, \\
 & \quad \quad \quad \pi \geq 0.
 \end{aligned}$$

Example 9.14: matrix notation

For the LP problem of [Example 9.1](#), with $n = 2$ variables and $m = 3$ constraints, the input data are:

$$c = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad A = \begin{pmatrix} -1 & 3 \\ 1 & 2 \\ 3 & -2 \end{pmatrix} \quad A' = \begin{pmatrix} -1 & 1 & 3 \\ 3 & 2 & -2 \end{pmatrix} \quad b = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix}$$

and the variables are:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}$$

9.4 Dual problem of a generic LP problem

Definition 9.11: dual problem of a generic LP problem

Given the following generic LP problem:

$$\max \quad \sum_{j \in N} c_j x_j \tag{9.8a}$$

$$\text{subject to} \quad \sum_{j \in N} a_{ij} x_j \leq b_i, \quad \forall i \in M_{\leq}, \tag{9.8b}$$

$$\sum_{j \in N} a_{ij} x_j = b_i, \quad \forall i \in M_{=}, \tag{9.8c}$$

$$\sum_{j \in N} a_{ij} x_j \geq b_i, \quad \forall i \in M_{\geq}, \tag{9.8d}$$

$$x_j \geq 0, \quad \forall j \in N_{\geq 0}, \quad (9.8e)$$

$$x_j \leq 0, \quad \forall j \in N_{\leq 0}, \quad (9.8f)$$

$$x_j \leq 0, \quad \forall j \in N_{\leq 0}, \quad (9.8g)$$

where

1. $N_{\geq 0}$, $N_{\leq 0}$ and $N_{=0}$ are the disjoint subsets of indices of the non-negative, free and nonpositive variables, respectively, and $N = N_{\geq 0} \cup N_{=0} \cup N_{\leq 0}$;
2. $M_{\leq}$, $M_{=}$ and $M_{\geq}$ are the disjoint subsets of indices of the less-than-or-equal-to, equality and greater-than-or-equal-to constraints, respectively, and $M = M_{\leq} \cup M_{=} \cup M_{\geq}$.

the **dual problem** $D(LP)$ of the LP problem (9.8) is the following LP problem:

$$\min \sum_{i \in M} b_i \pi_i \quad (9.9a)$$

$$\text{subject to } \sum_{i \in M} a_{ij} \pi_i \geq c_j, \quad \forall j \in N_{\geq 0}, \quad (9.9b)$$

$$\sum_{i \in M} a_{ij} \pi_i = c_j, \quad \forall j \in N_{=0}, \quad (9.9c)$$

$$\sum_{i \in M} a_{ij} \pi_i \leq c_j, \quad \forall j \in N_{\leq 0}, \quad (9.9d)$$

$$\pi_i \geq 0, \quad \forall i \in M_{\leq}, \quad (9.9e)$$

$$\pi_i \leq 0, \quad \forall i \in M_{=}, \quad (9.9f)$$

$$\pi_i \leq 0, \quad \forall i \in M_{\geq}. \quad (9.9g)$$

Example 9.15: dual problem of a generic LP problem

Consider the following generic LP problem:

$$\begin{array}{llllllllll} \max & 4 & x_1 & + & & x_2 & + & 5 & x_3 & + & 3 & x_4 \\ \text{subject to} & & x_1 & - & & x_2 & - & & x_3 & + & 3 & x_4 & \leq & 1 \\ & 5 & x_1 & + & & x_2 & + & 3 & x_3 & + & 8 & x_4 & = & 55 \\ & - & x_1 & + & 2 & x_2 & + & 3 & x_3 & - & 5 & x_4 & \geq & 3 \\ & & x_1 & + & 3 & x_2 & - & 2 & x_3 & + & 2 & x_4 & \geq & 6 \\ & & x_1, & & & x_2 & & & & & & & \geq & 0 \\ & & & & & & & & x_3 & & & & \leq & 0 \\ & & & & & & & & & & x_4 & \leq & 0 \end{array}$$

As far as the index of the variables is concerned, we have:

$$N_{\geq 0} = \{1, 2\}, \quad N_{\geq 0} = \{3\}, \quad N_{\leq 0} = \{4\}, \quad N = \{1, 2, 3, 4\}$$

As far as the index of the constraints is concerned, we have:

$$M_{\leq} = \{1\}, \quad M_{=} = \{2\}, \quad M_{\geq} = \{3, 4\}, \quad M = \{1, 2, 3, 4\}$$

The dual problem $D(LP)$ of the LP problem is:

$$\begin{array}{llllllll} \min & \pi_1 & + & 55 & \pi_2 & + & 3 & \pi_3 & + & 6 & \pi_4 \\ \text{subject to} & \pi_1 & + & 5 & \pi_2 & - & & \pi_3 & + & \pi_4 & \geq 4 \\ & - & \pi_1 & + & & \pi_2 & + & 2 & \pi_3 & + & 3 & \pi_4 & \geq 1 \\ & - & \pi_1 & + & 3 & \pi_2 & + & 3 & \pi_3 & - & 2 & \pi_4 & = 5 \\ & 3 & \pi_1 & + & 8 & \pi_2 & - & 5 & \pi_3 & + & 2 & \pi_4 & \leq 3 \\ & & \pi_1 & & & & & & & & & \geq 0 \\ & & & & \pi_2 & & & & & & & \geq 0 \\ & & & & & & \pi_3 & & & & & \leq 0 \\ & & & & & & & & \pi_4 & & & \leq 0 \end{array}$$

Since an equivalent LP problem in canonical maximization form can be obtained from an LP in generic maximization form, Weak Duality, Strong Duality, and the Complementary Slackness theorem hold for the generic case as well.

Dual Simplex algorithm

10.1 Pair of primal and dual problems

- We consider the standard maximization form of the dual problem, obtained by adding n surplus non-negative variables to a dual LP problem with m non-negative variables and n greater-than-or-equal-to linear constraints:

$$\max \quad -\mathbf{b}' \boldsymbol{\pi} \quad (10.1a)$$

$$\text{subject to} \quad \mathbf{A}' \boldsymbol{\pi} = \mathbf{c}, \quad (10.1b)$$

$$\boldsymbol{\pi} \geq \mathbf{0}. \quad (10.1c)$$

The n non-negative surplus variables are added *before* the m dual variables in order to shift the indices of the dual variables by n and to obtain the following index correspondence:

- the n surplus variables $\pi_1, \pi_2, \dots, \pi_n$ of the dual problem can be associated to the n primal variables $x_1, \dots, x_n$;
- the m dual variables $\pi_{n+1}, \pi_{n+2}, \dots, \pi_{n+m}$ of the dual problem can be associated to the m slack variables $x_{n+1}, \dots, x_{n+m}$ of the primal problem.

The input data of the dual problem are:

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \\ b_{n+1} \\ b_{n+2} \\ \vdots \\ b_{n+m} \end{pmatrix} \quad \mathbf{A}' = \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} & a'_{1(n+1)} & a'_{1(n+2)} & \cdots & a'_{1(n+m)} \\ a'_{21} & a'_{22} & \cdots & a'_{2n} & a'_{2(n+1)} & a'_{2(n+2)} & \cdots & a'_{2(n+m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a'_{n1} & a'_{n2} & \cdots & a'_{nn} & a'_{n(n+1)} & a'_{n(n+2)} & \cdots & a'_{n(n+m)} \end{pmatrix}$$

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

The $n + m$ non negative dual variables can be written as a column vector $\boldsymbol{\pi}$ of $n + m$ rows.

$$\boldsymbol{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \\ \pi_{n+1} \\ \pi_{n+2} \\ \vdots \\ \pi_{n+m} \end{pmatrix}$$

Since the dual problem (10.1) is obtained by adding n surplus variables to a dual LP problem with n greater-than-or-equal-to constraints, we have:

$$\begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ a'_{21} & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n1} & a'_{n2} & \cdots & a'_{nn} \end{pmatrix} = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}$$

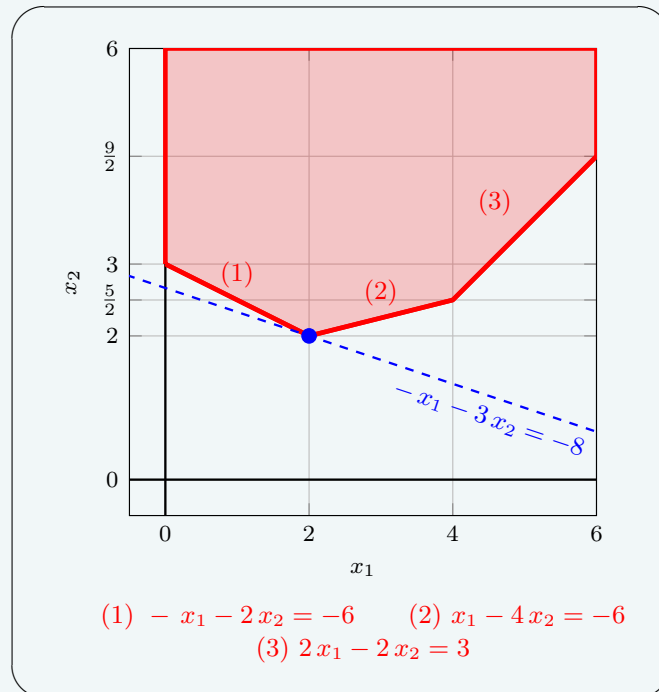
which is the negative identity matrix $-\mathbf{I}$ of size n . Moreover, for the same reason, we have:

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Example 10.1: pair of primal and dual problems

We consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{array}{llllll} \max & - & x_1 & - & 3 & x_2 \\ \text{subject to} & - & x_1 & - & 2 & x_2 \leq -6 \\ & & x_1 & - & 4 & x_2 \leq -6 \\ & 2 & x_1 & - & 2 & x_2 \leq 3 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 2$$

and the optimal value is:

$$z(LP) = -8$$

We call this LP problem [Example 10.1](#).

Example 10.2: pair of primal and dual problems

Adding $m = 3$ slack variables, the standard form of the LP problem of [Example 10.1](#) with $n + m = 5$ variables and $m = 3$ constraints reads as follows:

$$\begin{array}{llllll}
 \max & - & x_1 & - & 3 & x_2 \\
 \text{subject to} & - & x_1 & - & 2 & x_2 & + & x_3 & = & -6 \\
 & & x_1 & - & 4 & x_2 & & + & x_4 & = & -6 \\
 & & 2 & x_1 & - & 2 & x_2 & & + & x_5 & = & 3 \\
 & & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
 \end{array}$$

The dual problem of the LP problem of [Example 10.1](#) is the following LP problem in canonical minimization form with $m = 3$ variables and $n = 2$ constraints:

$$\begin{aligned}
 \min \quad & -6 \pi_1 - 6 \pi_2 + 3 \pi_3 \\
 \text{subject to} \quad & -\pi_1 + \pi_2 + 2 \pi_3 \geq -1 \\
 & -2 \pi_1 - 4 \pi_2 - 2 \pi_3 \geq -3 \\
 & \pi_1, \pi_2, \pi_3 \geq 0
 \end{aligned}$$

Shifting the indices of the dual variables by $n = 2$ and adding $n = 2$ surplus variables, we then obtain the following standard maximization form:

$$\begin{aligned}
 \max \quad & 6 \pi_3 + 6 \pi_4 - 3 \pi_5 \\
 \text{subject to} \quad & -\pi_1 - \pi_3 + \pi_4 + 2 \pi_5 = -1 \\
 & -\pi_2 - 2 \pi_3 - 4 \pi_4 - 2 \pi_5 = -3 \\
 & \pi_1, \pi_2, \pi_3, \pi_4, \pi_5 \geq 0
 \end{aligned}$$

In this dual problem in standard form, we have:

$$\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ -6 \\ -6 \\ 3 \end{pmatrix} \quad \mathbf{A}' = \begin{pmatrix} -1 & 0 & -1 & 1 & 2 \\ 0 & -1 & -2 & -4 & -2 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \end{pmatrix}$$

An optimal solution $\tilde{\pi}(D(LP))$ of this LP problem is: $\tilde{\pi}_1 = 0$, $\tilde{\pi}_2 = 0$, $\tilde{\pi}_3 = \frac{7}{6}$, $\tilde{\pi}_4 = \frac{1}{6}$, $\tilde{\pi}_5 = 0$.

10.2 Primal and dual feasible bases

- The simplex algorithm starts from a basis associated with a **basic feasible solution**, that is, a basis with $\bar{\mathbf{b}} \geq \mathbf{0}$, and it performs pivoting operations until all the reduced costs become non-positive. Throughout the execution, the basic solution is always feasible for the primal problem and the objective value never decreases.
- The **dual simplex algorithm** does the opposite: it starts from a basis whose reduced costs are all non-positive, and it performs pivoting operations until the basic solution becomes feasible.

Definition 10.1: primal feasible and dual feasible bases

Given a basis $\mathbf{A}_B$ of the LP problem in standard maximization form and the associated dictionary, the basis is called:

1. **primal feasible** if $\bar{\mathbf{b}} \geq \mathbf{0}$;
2. **dual feasible** if $\bar{\mathbf{c}} \leq \mathbf{0}$.

If a basis is both primal feasible and dual feasible, i.e.,

$$\bar{\mathbf{b}} \geq \mathbf{0} \quad \text{and} \quad \bar{\mathbf{c}} \leq \mathbf{0},$$

then the associated basic feasible solution is an **optimal solution**. Conversely, if the LP problem has an optimal solution, then there exists a basis which is both primal feasible and dual feasible. A degenerate optimal basic solution may also be associated with a basis which is not dual feasible, i.e., with some $\bar{c}_j > 0$.

10.2.1 The dual solution associated with a basis

- The name “dual feasible” is justified as follows. Given a basis $\mathbf{A}_B$ of the LP problem in standard maximization form, we associate with it the row vector of m components

$$\boldsymbol{\pi} = \mathbf{c}'_B \mathbf{A}_B^{-1}, \quad (10.2)$$

which is exactly the vector appearing in the definition of the reduced costs and of the objective value of the dictionary:

$$\bar{c}_j = c_j - \underbrace{\mathbf{c}'_B \mathbf{A}_B^{-1}}_{=\boldsymbol{\pi}} \mathbf{A}_j, \quad \forall j \in \mathcal{N}, \quad \text{and} \quad \bar{z} = \underbrace{\mathbf{c}'_B \mathbf{A}_B^{-1}}_{=\boldsymbol{\pi}} \mathbf{b} = \boldsymbol{\pi} \mathbf{b}.$$

- The m components of $\boldsymbol{\pi}$ are the values of the m original dual variables, i.e., of the variables $\pi_{n+1}, \pi_{n+2}, \dots, \pi_{n+m}$ of the dual problem (10.1). We now show that $\boldsymbol{\pi}$ is a feasible solution of the dual problem if and only if the basis is dual feasible.
- The reduced costs of the basic variables are zero.** Indeed, by definition of $\boldsymbol{\pi}$ we have

$$\boldsymbol{\pi} \mathbf{A}_B = \mathbf{c}'_B \underbrace{\mathbf{A}_B^{-1} \mathbf{A}_B}_{=\mathbf{I}} = \mathbf{c}'_B,$$

that is, $\boldsymbol{\pi} \mathbf{A}_i = c_i$ for every $i \in \mathcal{B}$, and then $\bar{c}_i = 0$ for every $i \in \mathcal{B}$.

- The reduced costs of the non-basic variables are the dual slacks.** For every $j \in \mathcal{N}$ we have

$$\bar{c}_j \leq 0 \quad \Longleftrightarrow \quad c_j - \boldsymbol{\pi} \mathbf{A}_j \leq 0 \quad \Longleftrightarrow \quad \boldsymbol{\pi} \mathbf{A}_j \geq c_j.$$

Putting together the two previous points, the condition $\bar{\mathbf{c}} \leq \mathbf{0}$ is equivalent to

$$\boldsymbol{\pi} \mathbf{A}_j \geq c_j, \quad \forall j \in \{1, 2, \dots, n+m\},$$

which is precisely the set of constraints of the dual problem written for all the columns of the LP problem in standard maximization form.

- The non-negativity of $\boldsymbol{\pi}$ is contained in these inequalities.** The last m columns of the LP problem in standard maximization form are the columns of the slack variables, i.e., $\mathbf{A}_{n+i} = \mathbf{e}_i$ with $c_{n+i} = 0$ for every $i \in \{1, 2, \dots, m\}$. Accordingly, the inequality written for the column $n+i$ reads

$$\boldsymbol{\pi} \mathbf{e}_i \geq 0 \quad \Longleftrightarrow \quad \pi_i \geq 0.$$

A basis $\mathbf{A}_B$ is **dual feasible** if and only if the associated vector

$$\boldsymbol{\pi} = \mathbf{c}'_B \mathbf{A}_B^{-1}$$

is a **feasible solution of the dual problem**. In this case, by the weak duality theorem, the value

$$\bar{z} = \boldsymbol{\pi} \mathbf{b}$$

is an **upper bound** on the optimal value of the LP problem.

- This is the exact mirror image of what happens in the simplex algorithm: there, every basis visited is primal feasible and then $\bar{z}$ is the value of a feasible solution of the LP problem, i.e., a lower bound on the optimal value, and the algorithm increases it until it becomes also an upper bound. Here, every basis visited is dual feasible and then $\bar{z}$ is an upper bound on the optimal value, and the algorithm decreases it until it becomes also a lower bound.
- Accordingly, the two algorithms can be described as follows:

	simplex algorithm	dual simplex algorithm
starts from a basis which is	primal feasible	dual feasible
maintains at every iteration	primal feasibility	dual feasibility
terminates when the basis is also	dual feasible	primal feasible
or when it detects that the problem is	unbounded	infeasible
the value $\bar{z}$ is	a lower bound	an upper bound
and along the execution it is	non-decreasing	non-increasing

Example 10.3: the dual solutions associated with three bases

For the LP problem of [Example 10.1](#) in standard maximization form, we consider the partition $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$ of the slack variables. We have:

$$\mathbf{A}_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{A}_B^{-1} \quad \text{and} \quad \mathbf{c}'_B = (0 \ 0 \ 0) \quad \implies \quad \boldsymbol{\pi} = (0 \ 0 \ 0)$$

which is a feasible solution of the dual problem, and $\bar{z} = \boldsymbol{\pi} \mathbf{b} = 0$ is an upper bound on the optimal value. Consistently, the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} = (-1 \ -3)$ are all non-positive.

After the first dual pivoting operation we obtain $\mathcal{B} = \{1, 4, 5\}$ and $\mathcal{N} = \{2, 3\}$, with

$$\mathbf{A}_B = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad \mathbf{A}_B^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{c}'_B = (-1 \ 0 \ 0),$$

and then

$$\pi = c'_B A_B^{-1} = (1 \ 0 \ 0) \quad \text{and} \quad \bar{z} = \pi b = (1 \ 0 \ 0) \begin{pmatrix} -6 \\ -6 \\ 3 \end{pmatrix} = -6.$$

After the second dual pivoting operation we obtain $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$, with

$$\pi = \left(\frac{7}{6} \ \frac{1}{6} \ 0\right) \quad \text{and} \quad \bar{z} = \pi b = -8.$$

The three upper bounds $0 \rightarrow -6 \rightarrow -8$ are non-increasing, and the last one is attained by a primal feasible solution: it is then the optimal value.

10.3 Finding an initial dual basic feasible solution

- The dual simplex algorithm requires in input a **dual feasible basis**, i.e., a basis whose dictionary satisfies $\bar{c} \leq 0$. This is the mirror image of the requirement of the simplex algorithm, which needs in input a primal feasible basis and, when none is readily available, resorts to the Phase 0.
- In this section we describe first the situations in which a dual feasible basis is available **for free**, i.e., without any computation, and then what can be done in the general case.

10.3.1 When a dual feasible basis is available for free

- **First case: a non-positive objective-function vector.** For an LP problem in canonical maximization form with $c \leq 0$, the partition of the slack variables is dual feasible. Indeed, with

$$\mathcal{B} = \{n+1, n+2, \dots, n+m\} \quad \text{and} \quad \mathcal{N} = \{1, 2, \dots, n\}$$

we have $A_B = I$ and $c_B = 0$, and then

$$\pi = c'_B A_B^{-1} = 0 \quad \text{and} \quad \bar{c}' = c'_N - \underbrace{c'_B A_B^{-1}}_{=0} A_N = c' \leq 0.$$

For an LP problem in canonical maximization form, if

$$c \leq 0 \quad \text{and} \quad b_i < 0 \text{ for some } i \in \{1, 2, \dots, m\},$$

then the partition $\mathcal{B} = \{n+1, \dots, n+m\}$ and $\mathcal{N} = \{1, \dots, n\}$ of the slack variables is **dual feasible but not primal feasible**: the dual simplex algorithm can be applied directly, while the simplex algorithm would require the Phase 0.

- **Second case: re-optimization after a change of the right-hand-side vector.** Assume that an optimal basis A_B of an LP problem is known and that the right-hand-side vector b is replaced by a new vector $\hat{b}$. The reduced costs

$$\bar{c}_j = c_j - c'_B A_B^{-1} A_j, \quad \forall j \in \mathcal{N},$$

do not depend on $\mathbf{b}$, and then they remain all non-positive: the basis **remains dual feasible**. On the contrary, the values of the basic variables

$$\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \hat{\mathbf{b}}$$

do change and may become negative: the basis may lose primal feasibility. The dual simplex algorithm can then be started from the old optimal basis and typically restores optimality in very few iterations.

- **Third case: adding a constraint to an optimal basis.** Assume that an optimal basis $\mathbf{A}_{\mathcal{B}}$ of an LP problem is known, with optimal solution $\tilde{\mathbf{x}}$, and that a new constraint

$$\mathbf{a}' \mathbf{x} \leq \beta$$

is added to the problem, together with its slack variable x_{n+m+1} . We consider the new partition

$$\hat{\mathcal{B}} = \mathcal{B} \cup \{n+m+1\} \quad \text{and} \quad \hat{\mathcal{N}} = \mathcal{N}.$$

- The new basis matrix and its inverse are

$$\mathbf{A}_{\hat{\mathcal{B}}} = \begin{pmatrix} \mathbf{A}_{\mathcal{B}} & \mathbf{0} \\ \mathbf{a}'_{\mathcal{B}} & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_{\hat{\mathcal{B}}}^{-1} = \begin{pmatrix} \mathbf{A}_{\mathcal{B}}^{-1} & \mathbf{0} \\ -\mathbf{a}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} & 1 \end{pmatrix},$$

where $\mathbf{a}'_{\mathcal{B}}$ collects the components of $\mathbf{a}'$ associated with the basic variables. In particular, $\mathbf{A}_{\hat{\mathcal{B}}}$ is invertible and then it is a basis.

- Since the new slack variable has objective-function coefficient equal to zero, we have $\mathbf{c}'_{\hat{\mathcal{B}}} = (\mathbf{c}'_{\mathcal{B}} \ 0)$ and then

$$\hat{\boldsymbol{\pi}} = \mathbf{c}'_{\hat{\mathcal{B}}} \mathbf{A}_{\hat{\mathcal{B}}}^{-1} = (\mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \ 0) = (\boldsymbol{\pi} \ 0),$$

that is, the dual variable of the new constraint is set to zero and all the other dual variables are unchanged. Accordingly, for every $j \in \hat{\mathcal{N}}$,

$$\hat{c}_j = c_j - (\boldsymbol{\pi} \ 0) \begin{pmatrix} \mathbf{A}_j \\ a_j \end{pmatrix} = c_j - \boldsymbol{\pi} \mathbf{A}_j = \bar{c}_j \leq 0:$$

the reduced costs are unchanged and the new basis is **dual feasible**.

- The value of the new basic variable is

$$\bar{b}_{n+m+1} = \beta - \mathbf{a}' \tilde{\mathbf{x}},$$

which is negative if and only if the old optimal solution $\tilde{\mathbf{x}}$ violates the new constraint. In this case the dual simplex algorithm is started from $\hat{\mathcal{B}}$.

- **Fourth case: branch-and-bound and cutting planes.** The third case is exactly what happens when an integer linear programming problem is solved: at every node of the branch-and-bound tree a bound such as

$$x_j \leq \lfloor \tilde{x}_j \rfloor \quad \text{or} \quad x_j \geq \lceil \tilde{x}_j \rceil$$

is added to the LP relaxation solved at the parent node, and the same happens every time a cutting plane is added. The optimal basis of the parent node is dual feasible for the child node, and the dual simplex algorithm re-optimizes it with a handful of pivoting operations instead of solving the new LP problem from scratch. This is one of the main reasons why the dual simplex algorithm is among the most used algorithms in modern LP solvers, and the one typically used to re-optimize inside branch-and-bound.

Example 10.4: re-optimization after a change of b

We consider the LP problem of [Example 10.1](#) and its optimal basis $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$, with the optimal dictionary:

$$\begin{cases} x_1 = 2 + \frac{2}{3}x_3 - \frac{1}{3}x_4 \\ x_2 = 2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \\ x_5 = 3 - x_3 + x_4 \\ z = -8 - \frac{7}{6}x_3 - \frac{1}{6}x_4 \end{cases}$$

Assume now that the right-hand side of the first constraint changes from -6 to -12 , i.e., that

$$b = \begin{pmatrix} -6 \\ -6 \\ 3 \end{pmatrix} \longrightarrow \hat{b} = \begin{pmatrix} -12 \\ -6 \\ 3 \end{pmatrix}.$$

The reduced costs do not depend on the right-hand-side vector, and then $\bar{c}' = (-\frac{7}{6} \quad -\frac{1}{6})$ is unchanged and the basis remains **dual feasible**. The values of the basic variables, instead, become

$$\bar{b} = A_B^{-1} \hat{b} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} & 0 \\ -\frac{1}{6} & -\frac{1}{6} & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -12 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix},$$

and the dictionary becomes:

$$\begin{cases} x_1 = 6 + \frac{2}{3}x_3 - \frac{1}{3}x_4 \\ x_2 = 3 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \\ x_5 = -3 - x_3 + x_4 \\ z = -15 - \frac{7}{6}x_3 - \frac{1}{6}x_4 \end{cases}$$

The basis is no longer primal feasible, since $\bar{b}_5 = -3 < 0$. We then perform a dual pivoting operation with pivot row x_5 . The values of its row are $\bar{a}_{53} = 1 \geq 0$ and $\bar{a}_{54} = -1 < 0$, and

then x_4 is the only candidate and it is the pivot column, with

$$s_m = \frac{\bar{c}_4}{\bar{a}_{54}} = \frac{-\frac{1}{6}}{-1} = \frac{1}{6}.$$

Updating the dictionary, with $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5\}$, we obtain:

$$\begin{cases} x_1 = 5 + \frac{1}{3}x_3 - \frac{1}{3}x_5 \\ x_2 = \frac{7}{2} + \frac{1}{3}x_3 + \frac{1}{6}x_5 \\ x_4 = 3 + x_3 + x_5 \\ z = -\frac{31}{2} - \frac{4}{3}x_3 - \frac{1}{6}x_5 \end{cases}$$

Now $\bar{\mathbf{b}} \geq \mathbf{0}$ and $\bar{\mathbf{c}} \leq \mathbf{0}$: after a single dual pivoting operation the new optimal solution is

$$\tilde{\mathbf{x}}(LP) = \begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix} \quad \text{and} \quad z(LP) = -\frac{31}{2}.$$

Example 10.5: adding a constraint to an optimal basis

We consider again the LP problem of [Example 10.1](#) with its optimal basis $\mathcal{B} = \{1, 2, 5\}$ and its optimal solution $\tilde{x}_1 = 2$ and $\tilde{x}_2 = 2$, and we add the new constraint

$$x_1 + x_2 \leq 3,$$

together with its slack variable x_6 . The new basis is $\hat{\mathcal{B}} = \{1, 2, 5, 6\}$ and $\hat{\mathcal{N}} = \{3, 4\}$.

The reduced costs are unchanged, and then the new basis is dual feasible. The value of the new basic variable is

$$\bar{b}_6 = 3 - (\tilde{x}_1 + \tilde{x}_2) = 3 - 4 = -1 < 0,$$

since the old optimal solution violates the new constraint. The equation of x_6 in the new dictionary is obtained by substituting the equations of x_1 and x_2 :

$$x_6 = 3 - x_1 - x_2 = 3 - \left(2 + \frac{2}{3}x_3 - \frac{1}{3}x_4\right) - \left(2 + \frac{1}{6}x_3 + \frac{1}{6}x_4\right) = -1 - \frac{5}{6}x_3 + \frac{1}{6}x_4,$$

and the new dictionary is:

$$\begin{cases} x_1 = 2 + \frac{2}{3}x_3 - \frac{1}{3}x_4 \\ x_2 = 2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \\ x_5 = 3 - x_3 + x_4 \\ x_6 = -1 - \frac{5}{6}x_3 + \frac{1}{6}x_4 \\ z = -8 - \frac{7}{6}x_3 - \frac{1}{6}x_4 \end{cases}$$

We perform a dual pivoting operation with pivot row x_6 . The values of its row are $\bar{a}_{63} = \frac{5}{6} \geq 0$ and $\bar{a}_{64} = -\frac{1}{6} < 0$, and then x_4 is the pivot column, with

$$s_m = \frac{\bar{c}_4}{\bar{a}_{64}} = \frac{-\frac{1}{6}}{-\frac{1}{6}} = 1.$$

Updating the dictionary, with $\mathcal{B} = \{1, 2, 4, 5\}$ and $\mathcal{N} = \{3, 6\}$, we obtain:

$$\begin{cases} x_1 = 0 - x_3 - 2x_6 \\ x_2 = 3 + x_3 + x_6 \\ x_4 = 6 + 5x_3 + 6x_6 \\ x_5 = 9 + 4x_3 + 6x_6 \\ z = -9 - 2x_3 - x_6 \end{cases}$$

Again a single dual pivoting operation restores optimality, and the new optimal solution is

$$\tilde{\mathbf{x}}(LP) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad \text{and} \quad z(LP) = -9.$$

The new optimal value -9 is not larger than the old one -8 , as it must be, since adding a constraint to a maximization problem can only reduce the optimal value.

10.3.2 The general case

- When neither $\mathbf{b} \geq \mathbf{0}$ nor $\mathbf{c} \leq \mathbf{0}$, the partition of the slack variables is neither primal feasible nor dual feasible, and no basis is available for free. Besides the Phase 0, the dual simplex algorithm offers the following alternative.

Finding a primal feasible basis with the dual simplex algorithm.

- Replace the objective-function vector $\mathbf{c}$ by any vector $\hat{\mathbf{c}} \leq \mathbf{0}$, for instance $\hat{\mathbf{c}} = \mathbf{0}$ or $\hat{c}_j = \min\{c_j, 0\}$. The feasible region is unchanged.
- The partition of the slack variables is dual feasible for the modified problem. Run the dual simplex algorithm on it, with the smallest-index rule (Bland's rule) of Section 10.4.3.
- If it terminates with the certificate of infeasibility, the modified problem is infeasible and then the original problem is infeasible too, since they have the same feasible region. **STOP**.
- Otherwise it terminates with a basis $\mathcal{B}$ which is primal feasible for the modified problem, and then also for the original problem.
- Restore the original vector $\mathbf{c}$, recompute the reduced costs $\bar{\mathbf{c}}$ and the value $\bar{z}$ of the basis $\mathcal{B}$, and run the simplex algorithm starting from it.

- This is a genuine alternative to the Phase 0: it plays exactly the same role, i.e., it provides an initial primal feasible basis or a proof that none exists, but it does so without introducing the auxiliary variable x_0 and without solving an auxiliary problem.
- With $\hat{\mathbf{c}} = \mathbf{0}$ all the reduced costs are equal to zero at every iteration, all the ratios of the ratio test are equal to zero and the pivot column is always chosen by the tie-breaking part of Bland's rule alone, i.e., as the candidate with the smallest index. Since Bland's rule prevents cycling, the procedure terminates.

Example 10.6: finding a primal feasible basis with the dual simplex algorithm

We consider the following LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 3$ less-than-or-equal-to linear constraints (called [Example 10.6](#)):

$$\begin{array}{rcllcl} \max & 3 & x_1 & + & x_2 & \\ \text{subject to} & - & x_1 & - & x_2 & \leq -2 \\ & & x_1 & + & 2x_2 & \leq 8 \\ & 2 & x_1 & + & x_2 & \leq 10 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

Here $\mathbf{c} \not\leq \mathbf{0}$ and $\mathbf{b} \not\geq \mathbf{0}$: the partition $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$ of the slack variables is neither primal feasible nor dual feasible.

Step a. We replace the objective-function vector by $\hat{\mathbf{c}} = \mathbf{0}$. The dictionary of the modified problem is:

$$\left\{ \begin{array}{rcllcl} x_3 & = & -2 & + & x_1 & + & x_2 \\ x_4 & = & 8 & - & x_1 & - & 2x_2 \\ x_5 & = & 10 & - & 2x_1 & - & x_2 \\ \hat{z} & = & 0 & + & 0x_1 & + & 0x_2 \end{array} \right.$$

All the reduced costs are non-positive, and then the basis is dual feasible.

Step b. The pivot row is x_3 , since $\bar{b}_3 = -2 < 0$. The values of its row are $\bar{a}_{31} = -1 < 0$ and $\bar{a}_{32} = -1 < 0$, and then both non-basic variables are candidates:

$$s_m = \min \left\{ \underbrace{\frac{0}{-1}}_{=0}, \underbrace{\frac{0}{-1}}_{=0} \right\} = 0,$$

and the two ratios are equal. By Bland's rule we choose the smallest index, i.e., x_1 is the pivot column. Updating the dictionary, with $\mathcal{B} = \{1, 4, 5\}$ and $\mathcal{N} = \{2, 3\}$, we obtain:

$$\begin{cases} x_1 = 2 - x_2 + x_3 \\ x_4 = 6 - x_2 - x_3 \\ x_5 = 6 + x_2 - 2x_3 \\ \hat{z} = 0 + 0x_2 + 0x_3 \end{cases}$$

Step d. Now $\bar{\mathbf{b}} = (2 \ 6 \ 6)' \geq \mathbf{0}$: the basis $\mathcal{B} = \{1, 4, 5\}$ is **primal feasible**, both for the modified problem and for the original one.

Step e. We restore the original objective function and we recompute the objective-function row by substituting the equation of x_1 :

$$z = 3x_1 + x_2 = 3(2 - x_2 + x_3) + x_2 = 6 - 2x_2 + 3x_3,$$

which gives the dictionary:

$$\begin{cases} x_1 = 2 - x_2 + x_3 \\ x_4 = 6 - x_2 - x_3 \\ x_5 = 6 + x_2 - 2x_3 \\ z = 6 - 2x_2 + 3x_3 \end{cases}$$

The reduced cost $\bar{c}_3 = 3 > 0$, and then we perform a (primal) pivoting operation with pivot column x_3 . We find the largest t such that $\bar{\mathbf{b}} - t \bar{\mathbf{A}}_3 \geq \mathbf{0}$:

$$\begin{cases} \underbrace{2 - t \cdot (-1)}_{x_1 \geq 0} \geq 0 \\ \underbrace{6 - t \cdot 1}_{x_4 \geq 0} \geq 0 \\ \underbrace{6 - t \cdot 2}_{x_5 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \geq -2 \\ t \leq 6 \\ t \leq 3 \end{cases} \quad \text{then } t_m = 3$$

and the pivot row is x_5 . Updating the dictionary, with $\mathcal{B} = \{1, 3, 4\}$ and $\mathcal{N} = \{2, 5\}$, we obtain:

$$\begin{cases} x_1 = 5 - \frac{1}{2}x_2 - \frac{1}{2}x_5 \\ x_3 = 3 + \frac{1}{2}x_2 - \frac{1}{2}x_5 \\ x_4 = 3 - \frac{3}{2}x_2 + \frac{1}{2}x_5 \\ z = 15 - \frac{1}{2}x_2 - \frac{3}{2}x_5 \end{cases}$$

All the reduced costs are non-positive: the basis is optimal and

$$\tilde{\mathbf{x}}(LP) = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad \text{and} \quad z(LP) = 15.$$

10.4 Dual pivoting operations

- Given a dual feasible basis A_B of the LP problem in standard maximization form, if in the associated dictionary there exists $r \in \mathcal{B}$ such that $\bar{b}_r < 0$ (a negative basic value), it is necessary to perform a dual pivoting operation.
- The dual simplex algorithm works on the primal dictionary, exactly the same object used by the simplex algorithm. What changes is the **order** of the two choices.

Definition 10.2: dual pivoting operation

A **dual pivoting operation** consists in determining a **new dual feasible basis** with the goal of restoring the non-negativity of one basic variable, by **swapping one basic variable and one non-basic variable**.

- The basic variable that becomes non-basic is called the **pivot row**, and it is chosen first;
- The non-basic variable that becomes basic is called the **pivot column**, and it is chosen second.

- The order of the two choices is the opposite of the one of the simplex algorithm.

10.4.1 Determining the pivot row

The **pivot row** is a **basic variable with a negative value**, i.e., a basic variable

$$x_r \text{ with } r \in \mathcal{B} \text{ such that } \bar{b}_r < 0.$$

- The variable x_r leaves the basis and becomes non-basic, and then it takes the value 0: this is precisely what restores its non-negativity. If several basic variables have a negative value, any one of them preserves dual feasibility. In these notes we always choose the one with the **smallest index**:

$$r = \min \{ i \in \mathcal{B} : \bar{b}_i < 0 \}.$$

LP solvers often choose instead the most negative value $\bar{b}_r$, or more refined criteria; the smallest-index choice is part of Bland's rule (Section 10.4.3), which guarantees that the algorithm terminates.

10.4.2 Determining the pivot column

- The pivot column is chosen so that dual feasibility is preserved, that is, so that all the reduced costs remain non-positive after the dual pivoting operation. We now derive the rule.
- The equation of the pivot row in the dictionary and the equation of the objective function are

$$x_r = \bar{b}_r - \sum_{j \in \mathcal{N}} \bar{a}_{rj} x_j \quad \text{and} \quad z = \bar{z} + \sum_{j \in \mathcal{N}} \bar{c}_j x_j.$$

The first one can be rewritten as the identity

$$0 = \bar{b}_r - \sum_{j \in \mathcal{N}} \bar{a}_{rj} x_j - x_r,$$

which holds for every point satisfying the equality constraints. Accordingly, for every value $s \in \mathbb{R}$, we can add s times this identity to the equation of the objective function without changing it:

$$z = \underbrace{\bar{z} + s \bar{b}_r}_{\text{new } \bar{z}} + \sum_{j \in \mathcal{N}} \underbrace{(\bar{c}_j - s \bar{a}_{rj})}_{\text{new } \bar{c}_j} x_j + \underbrace{(-s)}_{\text{new } \bar{c}_r} x_r. \quad (10.3)$$

- These $|\mathcal{N}| + 1$ coefficients are the reduced costs of the new dictionary, in which x_r has become non-basic. **The value of s is increased as much as possible while at the same time maintaining non-positive all the reduced costs.** In this way a new dual feasible basis is determined. Imposing the non-positivity of all the coefficients of (10.3), we obtain:

$$\begin{cases} -s \leq 0 \\ \bar{c}_j - s \bar{a}_{rj} \leq 0, \end{cases} \quad \forall j \in \mathcal{N} \quad \begin{cases} s \geq 0 \\ s \leq \frac{\bar{c}_j}{\bar{a}_{rj}}, \end{cases} \quad \forall j \in \mathcal{N} \text{ with } \bar{a}_{rj} < 0$$

where the inequalities with $\bar{a}_{rj} > 0$ are satisfied for every $s \geq 0$, since $\bar{c}_j \leq 0$, and those with $\bar{a}_{rj} = 0$ reduce to $\bar{c}_j \leq 0$, which holds by hypothesis. The direction of the inequality is reversed when dividing by $\bar{a}_{rj} < 0$. Accordingly, the value of s becomes the maximum value, called s_m , such that:

$$s_m = \min \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} : j \in \mathcal{N}, \bar{a}_{rj} < 0 \right\} \geq 0.$$

- With this value of s , the new reduced cost of the variable x_p attaining the minimum becomes

$$\bar{c}_p - s_m \bar{a}_{rp} = \bar{c}_p - \frac{\bar{c}_p}{\bar{a}_{rp}} \bar{a}_{rp} = 0,$$

and a variable whose reduced cost is zero can enter the basis: x_p is the pivot column.

The pivot column is a non-basic variable whose reduced cost becomes zero by setting the dual step to s_m , i.e., a non-basic variable

$$x_p \text{ with } p \in \mathcal{N} \text{ such that } \frac{\bar{c}_p}{\bar{a}_{rp}} = s_m = \min \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} : j \in \mathcal{N}, \bar{a}_{rj} < 0 \right\},$$

where $\bar{a}_{rp}$ is the value of the row $\bar{\mathbf{a}}_r$ associated with the pivot column x_p .

- Given s_m and the row $\bar{\mathbf{a}}_r$, the new reduced costs (which determine the new vector $\bar{\mathbf{c}}$) and the new objective value are obtained as follows:

$$\bar{c}_j \leftarrow \bar{c}_j - s_m \bar{a}_{rj}, \quad \forall j \in \mathcal{N} \setminus \{p\}, \quad \bar{c}_r \leftarrow -s_m, \quad \bar{c}_p \leftarrow 0, \quad \bar{z} \leftarrow \bar{z} + s_m \bar{b}_r. \quad (10.4)$$

- Since $s_m \geq 0$ and $\bar{b}_r < 0$, we have

$$\bar{z} + s_m \bar{b}_r \leq \bar{z} :$$

the objective value never increases, which is the mirror image of what happens in the simplex algorithm, where the objective value never decreases. By the previous section, $\bar{z}$ is at every iteration an upper bound on the optimal value, and the dual simplex algorithm decreases it.

- Given a pivot row x_r with $r \in \mathcal{B}$ and a pivot column x_p with $p \in \mathcal{N}$, the new partition of the indices of the variables is updated as follows:

$$\mathcal{B} \leftarrow (\mathcal{B} \setminus \{r\}) \cup \{p\} \quad \text{and} \quad \mathcal{N} \leftarrow (\mathcal{N} \setminus \{p\}) \cup \{r\} \quad (10.5)$$

By construction $\mathbf{A}_{\mathcal{B}}$ remains a basis, since $\bar{a}_{rp} \neq 0$.

10.4.3 The smallest-index rule (Bland's rule)

- The pivot row and the pivot column are not always unique: several basic variables may have a negative value, and several non-basic variables may attain the minimum s_m . If $s_m = 0$ (a **degenerate** dual pivoting operation) the value $\bar{z}$ does not change and, with an arbitrary choice, the algorithm could cycle. In these notes we always break the ties as follows.

Bland's rule for the dual simplex algorithm.

- The pivot row is the basic variable with negative value having the **smallest index**:

$$r = \min \{ i \in \mathcal{B} : \bar{b}_i < 0 \}.$$

- The pivot column is the candidate attaining s_m having the **smallest index**:

$$p = \min \left\{ j \in \mathcal{N} : \bar{a}_{rj} < 0, \frac{\bar{c}_j}{\bar{a}_{rj}} = s_m \right\}.$$

By the correspondence $\pi_j \leftrightarrow x_j$, this is exactly Bland's rule of the simplex algorithm applied to the dual problem, and then the dual simplex algorithm with this rule always terminates.

10.4.4 Certificate of infeasibility

- If $\bar{a}_{rj} \geq 0$ for every $j \in \mathcal{N}$, then the ratio test has no candidate: the value of s can be increased without bound while maintaining non-positive all the reduced costs. In this case the LP problem is infeasible.

Certificate of infeasibility. Given a dual feasible basis $\mathbf{A}_B$, if there exists a basic variable x_r with $r \in \mathcal{B}$ such that

$$\bar{b}_r < 0 \quad \text{and} \quad \bar{a}_{rj} \geq 0, \quad \forall j \in \mathcal{N},$$

then the LP problem in standard maximization form is **infeasible**.

- **First proof (on the primal).** The equation of the pivot row in the dictionary is

$$x_r = \bar{b}_r - \sum_{j \in \mathcal{N}} \bar{a}_{rj} x_j,$$

and every feasible solution satisfies $x_j \geq 0$ for every $j \in \mathcal{N}$. Since $\bar{a}_{rj} \geq 0$ for every $j \in \mathcal{N}$, every term of the sum is non-negative and we obtain

$$x_r \leq \bar{b}_r < 0,$$

which contradicts the non-negativity of x_r . Accordingly, no feasible solution exists.

- **Second proof (on the dual).** For every value $s \geq 0$, the coefficients of (10.3) are all non-positive, and then they are the reduced costs of a dual feasible solution of value

$$\bar{z} + s \bar{b}_r.$$

Since $\bar{b}_r < 0$, this value tends to $-\infty$ as s grows: the dual problem is **unbounded**. By the weak duality theorem, the primal problem is then **infeasible**. This is the exact mirror image of the certificate of unboundedness of the simplex algorithm.

10.4.5 Update of the dictionary

Once the pivot row x_r with $r \in \mathcal{B}$ and the pivot column x_p with $p \in \mathcal{N}$ are determined, the new dictionary is obtained with **exactly the same row operations as in the simplex algorithm**.

- Let x_p be the entering variable and x_r the leaving variable. From the equation of the pivot row, we have:

$$x_r = \bar{b}_r - \bar{a}_{rp}x_p - \sum_{j \in \mathcal{N} \setminus \{p\}} \bar{a}_{rj}x_j.$$

Solving for x_p gives the equation of the pivot column in the new dictionary:

$$x_p = \frac{\bar{b}_r}{\bar{a}_{rp}} - \sum_{j \in \mathcal{N} \setminus \{p\}} \frac{\bar{a}_{rj}}{\bar{a}_{rp}}x_j - \frac{1}{\bar{a}_{rp}}x_r. \quad (10.6)$$

For each remaining basic variable $i \in \mathcal{B} \setminus \{r\}$, substituting with (10.6) gives their equations in the new dictionary, and substituting in the equation of the objective function gives the new objective-function row, whose coefficients are exactly those of (10.4).

- The new value of the pivot column is

$$\bar{b}_p = \frac{\bar{b}_r}{\bar{a}_{rp}} \geq 0,$$

since $\bar{b}_r < 0$ and $\bar{a}_{rp} < 0$: the variable entering the basis takes a non-negative value. The values of the other basic variables, instead, may well be negative, and further dual pivoting operations are then needed.

Example 10.7: dual pivoting operations

For the LP problem of [Example 10.1](#), with $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$, we have the dictionary:

$$\begin{cases} x_3 = -6 + x_1 + 2x_2 \\ x_4 = -6 - x_1 + 4x_2 \\ x_5 = 3 - 2x_1 + 2x_2 \\ z = 0 - x_1 - 3x_2 \end{cases}$$

All the reduced costs are non-positive, and then the basis is dual feasible. Two basic variables have a negative value, $\bar{b}_3 = -6 < 0$ and $\bar{b}_4 = -6 < 0$, and by the smallest-index rule we choose x_3 as the pivot row.

We then find the largest s such that all the reduced costs of (10.3) remain non-positive:

$$\left\{ \begin{array}{l} -s \leq 0 \\ \bar{c}_3 \leq 0 \\ -1 - s \cdot (-1) \leq 0 \\ \bar{c}_1 \leq 0 \\ -3 - s \cdot (-2) \leq 0 \\ \bar{c}_2 \leq 0 \end{array} \right. \quad \left\{ \begin{array}{l} s \geq 0 \\ s \leq 1 \\ s \leq \frac{3}{2} \end{array} \right. \quad \text{then} \quad s_m = \min \left\{ \underbrace{\frac{-1}{-1}}_{=1}, \underbrace{\frac{-3}{-2}}_{=\frac{3}{2}} \right\} = 1$$

The dual step becomes $s_m = 1$, the reduced cost of x_1 becomes zero and x_1 is the pivot column. The other reduced costs and the objective value become $\bar{c}_j - s_m \bar{a}_{3j}$ and $\bar{z} + s_m \bar{b}_3$, then we have:

$$\bar{c}_2 = -3 - 1 \cdot (-2) = -1, \quad \bar{c}_3 = -1, \quad \bar{z} = 0 + 1 \cdot (-6) = -6,$$

and accordingly

$$\begin{pmatrix} \bar{c}_2 \\ \bar{c}_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The new partition of the indices of the variables is $\mathcal{B} = \{1, 4, 5\}$ and $\mathcal{N} = \{2, 3\}$.

Example 10.8: update of the dictionary

For the LP problem of [Example 10.1](#), performing the dual pivoting operation of [Example 10.7](#), with x_3 as the pivot row and x_1 as the pivot column, we have:

$$x_3 = -6 + x_1 + 2x_2 \quad \text{then} \quad x_1 = 6 - 2x_2 + x_3$$

Substituting in the other equations of the dictionary, we get:

$$x_4 = -6 - x_1 + 4x_2 = -6 - (6 - 2x_2 + x_3) + 4x_2 = -12 + 6x_2 - x_3$$

$$x_5 = 3 - 2x_1 + 2x_2 = 3 - 2(6 - 2x_2 + x_3) + 2x_2 = -9 + 6x_2 - 2x_3$$

$$z = -x_1 - 3x_2 = -(6 - 2x_2 + x_3) - 3x_2 = -6 - x_2 - x_3$$

and then the new dictionary, with $\mathcal{B} = \{1, 4, 5\}$ and $\mathcal{N} = \{2, 3\}$, is:

$$\left\{ \begin{array}{lclcl} x_1 & = & 6 & - & 2x_2 & + & x_3 \\ x_4 & = & -12 & + & 6x_2 & - & x_3 \\ x_5 & = & -9 & + & 6x_2 & - & 2x_3 \\ z & = & -6 & - & x_2 & - & x_3 \end{array} \right.$$

The reduced costs and the objective value coincide with those computed in [Example 10.7](#), as expected, and the value of the entering variable is $\bar{b}_1 = \frac{\bar{b}_3}{\bar{a}_{31}} = \frac{-6}{-1} = 6 \geq 0$.

10.5 Pseudo-code of the dual simplex algorithm

Algorithm

Input

An LP problem in standard maximization form and a **dual feasible** basis, i.e., a basis whose dictionary satisfies $\bar{c} \leq 0$.

Output

An optimal solution of the LP problem, if one exists; otherwise, a certificate that the LP problem is infeasible.

Step 0: Initialization

Consider the given dual feasible basis and the associated dictionary. Go to Step 1.

Step 1: Feasibility check

With respect to the current dictionary, if

$$\bar{b}_i \geq 0, \quad \forall i \in \mathcal{B},$$

then the basis is primal feasible and dual feasible, the basic feasible solution $x_i = \bar{b}_i$, $\forall i \in \mathcal{B}$, and $x_j = 0$, $\forall j \in \mathcal{N}$, is an optimal solution and $\bar{z}$ is the optimal value. In this case **STOP**.

Step 2: Dual pivoting operations

The **pivot row** is the basic variable x_r with negative value having the smallest index:

$$r = \min \{ i \in \mathcal{B} : \bar{b}_i < 0 \}.$$

a. If

$$\bar{a}_{rj} \geq 0, \quad \forall j \in \mathcal{N},$$

then the LP problem is **infeasible**. **STOP**.

b. Determine the **pivot column**, i.e., a non-basic variable x_p with $p \in \mathcal{N}$ such that

$$\frac{\bar{c}_p}{\bar{a}_{rp}} = s_m = \min \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} : j \in \mathcal{N}, \bar{a}_{rj} < 0 \right\};$$

in case of ties, choose the candidate with the smallest index p .

c. Update the dictionary with the same row operations as in the simplex algorithm. In particular, $\bar{z} \leftarrow \bar{z} + s_m \bar{b}_r$.

d. Update the basis:

$$\mathcal{B} \leftarrow (\mathcal{B} \setminus \{r\}) \cup \{p\} \quad \text{and} \quad \mathcal{N} \leftarrow (\mathcal{N} \setminus \{p\}) \cup \{r\}.$$

Go to Step 1.

Example 10.9: execution of the dual simplex algorithm

Consider the LP problem of [Example 10.1](#) in standard maximization form. Its objective-function vector satisfies $\mathbf{c} \leq \mathbf{0}$, and then the partition $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$ of the slack variables is **dual feasible**. The associated dictionary is:

$$\begin{cases} x_3 = -6 + x_1 + 2x_2 \\ x_4 = -6 - x_1 + 4x_2 \\ x_5 = 3 - 2x_1 + 2x_2 \\ z = 0 - x_1 - 3x_2 \end{cases}$$

All the reduced costs are non-positive, but the basic solution is not feasible, since $\bar{b}_3 = -6 < 0$ and $\bar{b}_4 = -6 < 0$.

First dual pivoting operation. The pivot row is x_3 , the basic variable with negative value having the smallest index. The values of its row are $\bar{a}_{31} = -1 < 0$ and $\bar{a}_{32} = -2 < 0$, and then both non-basic variables are candidates:

$$\min \left\{ \underbrace{\frac{-1}{-1}}_{=1}, \underbrace{\frac{-3}{-2}}_{=\frac{3}{2}} \right\} = s_m = 1$$

and the pivot column is x_1 . The new objective value is $\bar{z} + s_m \bar{b}_3 = 0 + 1 \cdot (-6) = -6$. The new dictionary, with $\mathcal{B} = \{1, 4, 5\}$ and $\mathcal{N} = \{2, 3\}$, is:

$$\begin{cases} x_1 = 6 - 2x_2 + x_3 \\ x_4 = -12 + 6x_2 - x_3 \\ x_5 = -9 + 6x_2 - 2x_3 \\ z = -6 - x_2 - x_3 \end{cases}$$

The reduced costs are still all non-positive, as expected, but the basic solution is still not feasible.

Second dual pivoting operation. The pivot row is x_4 , the basic variable with negative value having the smallest index ($\bar{b}_4 = -12$, $\bar{b}_5 = -9$). The only negative value of its row is $\bar{a}_{42} = -6$, and then the pivot column is x_2 , with dual step $s_m = \frac{\bar{c}_2}{\bar{a}_{42}} = \frac{-1}{-6} = \frac{1}{6}$ and new objective value $\bar{z} + s_m \bar{b}_4 = -6 + \frac{1}{6} \cdot (-12) = -8$. The new dictionary, with $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$, is:

$$\begin{cases} x_1 = 2 + \frac{2}{3}x_3 - \frac{1}{3}x_4 \\ x_2 = 2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \\ x_5 = 3 - x_3 + x_4 \\ z = -8 - \frac{7}{6}x_3 - \frac{1}{6}x_4 \end{cases}$$

Now $\bar{\mathbf{b}} \geq \mathbf{0}$ and $\bar{\mathbf{c}} \leq \mathbf{0}$: the basis is primal feasible and dual feasible, and then it is optimal. An optimal solution of the LP problem of [Example 10.1](#) is

$$\tilde{\mathbf{x}}(LP) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

and the optimal value is $z(LP) = -8$, in accordance with the execution reported in the previous pages.

The three bases visited by the algorithm correspond to three feasible solutions of the **dual** problem, given by $\boldsymbol{\pi} = \mathbf{c}'_B \mathbf{A}_B^{-1}$:

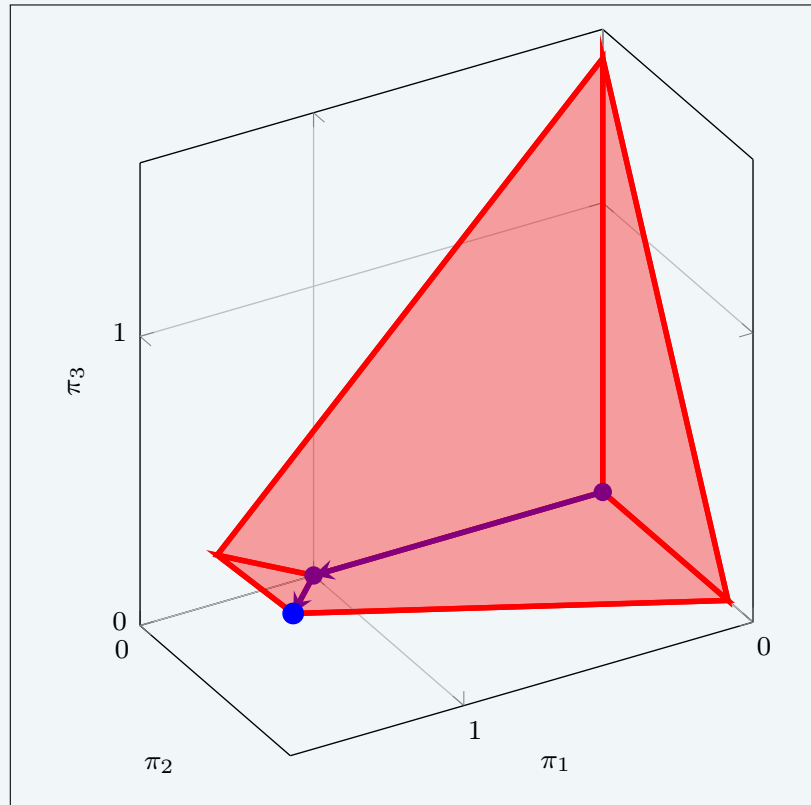
$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{7}{6} \\ \frac{1}{6} \\ 0 \end{pmatrix},$$

whose dual objective values $\mathbf{b}'\boldsymbol{\pi}$ are

$$0 \rightarrow -6 \rightarrow -8,$$

non-increasing, as it must be for a minimization problem (here they even decrease strictly, but a degenerate dual pivoting operation with $s_m = 0$ leaves the value unchanged). They are vertices of the dual polyhedron, and the dual simplex algorithm moves from one of them to an adjacent one: it is the simplex algorithm executed on the dual problem.

In the following figure the axes are the three original dual variables, called π_1, π_2, π_3 in the dual problem of [Example 10.1](#); recalling that in the standard maximization form their indices are shifted by $n = 2$, they correspond to π_3, π_4, π_5 .



In violet the first two dual vertices visited by the algorithm, in blue the optimal one. The corresponding primal basic solutions are instead not feasible until the last iteration.

Example 10.10: certificate of infeasibility

Consider the following LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 3$ less-than-or-equal-to linear constraints (called [Example 10.10](#)):

$$\begin{array}{llllll}
 \max & -2 & x_1 & - & x_2 & \\
 \text{subject to} & - & x_1 & - & 2 & x_2 \leq -6 \\
 & -2 & x_1 & - & x_2 & \leq -6 \\
 & & x_1 & + & x_2 & \leq 2 \\
 & & x_1, & & x_2 & \geq 0
 \end{array}$$

Since $\mathbf{c} \leq \mathbf{0}$, the partition $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$ is dual feasible. The initial dictionary is:

$$\left\{ \begin{array}{llllll}
 x_3 & = & -6 & + & x_1 & + & 2x_2 \\
 x_4 & = & -6 & + & 2x_1 & + & x_2 \\
 x_5 & = & 2 & - & x_1 & - & x_2 \\
 z & = & 0 & - & 2x_1 & - & x_2
 \end{array} \right.$$

First dual pivoting operation. By the smallest-index rule the pivot row is x_3 , whose value is $\bar{b}_3 = -6 < 0$. The values of its row are $\bar{a}_{31} = -1 < 0$ and $\bar{a}_{32} = -2 < 0$, and then both non-basic variables are candidates:

$$\min \left\{ \underbrace{\frac{-2}{-1}}_{=2}, \underbrace{\frac{-1}{-2}}_{=\frac{1}{2}} \right\} = \frac{1}{2}$$

and the pivot column is x_2 . The new dictionary, with $\mathcal{B} = \{2, 4, 5\}$ and $\mathcal{N} = \{1, 3\}$, is:

$$\begin{cases} x_2 = 3 - \frac{1}{2}x_1 + \frac{1}{2}x_3 \\ x_4 = -3 + \frac{3}{2}x_1 + \frac{1}{2}x_3 \\ x_5 = -1 - \frac{1}{2}x_1 - \frac{1}{2}x_3 \\ z = -3 - \frac{3}{2}x_1 - \frac{1}{2}x_3 \end{cases}$$

Second dual pivoting operation. Two basic variables have a negative value, x_4 and x_5 : by the smallest-index rule the pivot row is x_4 . The values of its row are $\bar{a}_{41} = -\frac{3}{2} < 0$ and $\bar{a}_{43} = -\frac{1}{2} < 0$, and then both non-basic variables are candidates:

$$\min \left\{ \underbrace{\frac{-\frac{3}{2}}{-\frac{3}{2}}}_{=1}, \underbrace{\frac{-\frac{1}{2}}{-\frac{1}{2}}}_{=1} \right\} = 1.$$

The two ratios are equal and, by the smallest-index rule, the pivot column is x_1 . We have:

$$x_4 = -3 + \frac{3}{2}x_1 + \frac{1}{2}x_3 \quad \text{then} \quad x_1 = 2 - \frac{1}{3}x_3 + \frac{2}{3}x_4,$$

and, substituting in the other equations of the dictionary:

$$\begin{aligned} x_2 &= 3 - \frac{1}{2} \left(2 - \frac{1}{3}x_3 + \frac{2}{3}x_4 \right) + \frac{1}{2}x_3 = 2 + \frac{2}{3}x_3 - \frac{1}{3}x_4 \\ x_5 &= -1 - \frac{1}{2} \left(2 - \frac{1}{3}x_3 + \frac{2}{3}x_4 \right) - \frac{1}{2}x_3 = -2 - \frac{1}{3}x_3 - \frac{1}{3}x_4 \\ z &= -3 - \frac{3}{2} \left(2 - \frac{1}{3}x_3 + \frac{2}{3}x_4 \right) - \frac{1}{2}x_3 = -6 + 0x_3 - x_4. \end{aligned}$$

The new dictionary, with $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$, is:

$$\begin{cases} x_1 = 2 - \frac{1}{3}x_3 + \frac{2}{3}x_4 \\ x_2 = 2 + \frac{2}{3}x_3 - \frac{1}{3}x_4 \\ x_5 = -2 - \frac{1}{3}x_3 - \frac{1}{3}x_4 \\ z = -6 + 0x_3 - x_4 \end{cases}$$

Because of the tie in the ratio test, the reduced cost $\bar{c}_3$ becomes 0 too: it is still non-positive and the basis is still dual feasible, with $\bar{z} = -3 + 1 \cdot (-3) = -6$.

Third dual pivoting operation. The only basic variable with a negative value is x_5 , whose row has values

$$\bar{a}_{53} = \frac{1}{3} \geq 0 \quad \text{and} \quad \bar{a}_{54} = \frac{1}{3} \geq 0,$$

that is, all non-negative. By the certificate of infeasibility, the LP problem of [Example 10.10](#) is **infeasible**.

Indeed, the equation of x_5 in this dictionary reads

$$x_5 = -2 - \frac{1}{3}x_3 - \frac{1}{3}x_4,$$

and every feasible solution satisfies $x_3 \geq 0$ and $x_4 \geq 0$, which gives

$$x_5 \leq -2 < 0,$$

in contradiction with the non-negativity of x_5 . Accordingly, no feasible solution exists.

The certificate was already available in the previous dictionary, where the row of x_5 was $x_5 = -1 - \frac{1}{2}x_1 - \frac{1}{2}x_3$ with all non-negative values $\bar{a}_{51}, \bar{a}_{53}$: the smallest-index rule selected x_4 first. A certificate of infeasibility can be read in any row with $\bar{b}_r < 0$ and $\bar{a}_{rj} \geq 0$ for every $j \in \mathcal{N}$, even if it is not the pivot row.

10.6 Solving the dual problem with the simplex algorithm

- In this section we execute the **simplex algorithm on the dual problem** of the LP problem of [Example 10.1](#) and, in parallel, we report at every iteration the corresponding dictionary of the primal problem. The two executions are two different ways of describing the same sequence of bases: the dual simplex algorithm applied to the primal problem is exactly the simplex algorithm applied to the dual problem.

Example 10.11: the simplex algorithm on the dual problem

- **Initialization**

We shift the indices of the original dual variables by $n = 2$ and we add $n = 2$ surplus variables in front. We then obtain the following standard maximization form:

$$\begin{array}{llllllllll} \max & & & & 6 & \pi_3 & + & 6 & \pi_4 & - & 3 & \pi_5 \\ \text{subject to} & - & \pi_1 & & & & - & \pi_3 & + & \pi_4 & + & 2 & \pi_5 & = & -1 \\ & & & - & \pi_2 & - & 2 & \pi_3 & - & 4 & \pi_4 & - & 2 & \pi_5 & = & -3 \\ & & \pi_1, & \pi_2, & \pi_3, & \pi_4, & \pi_5 & \geq & 0 \end{array}$$

The dual initial dictionary is:

$$\begin{cases} \pi_1 = 1 - \pi_3 + \pi_4 + 2\pi_5 \\ \pi_2 = 3 - 2\pi_3 - 4\pi_4 - 2\pi_5 \\ -w = 0 + 6\pi_3 + 6\pi_4 - 3\pi_5 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4, 5\}$. The basic dual feasible solution is:

$$\bar{\pi}_1 = 1, \quad \bar{\pi}_2 = 3, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0$$

with $-w = 0$, i.e., $w = 0$.

After having added the slack variables to the primal problem, we obtain the following standard form:

$$\begin{aligned} \max \quad & -x_1 - 3x_2 \\ \text{subject to} \quad & -x_1 - 2x_2 + x_3 = -6 \\ & x_1 - 4x_2 + x_4 = -6 \\ & 2x_1 - 2x_2 + x_5 = 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

The primal initial dictionary is:

$$\begin{cases} x_3 = -6 + x_1 + 2x_2 \\ x_4 = -6 - x_1 + 4x_2 \\ x_5 = 3 - 2x_1 + 2x_2 \\ z = 0 - x_1 - 3x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$. The basic primal non-feasible solution is:

$$\bar{x}_3 = -6, \quad \bar{x}_4 = -6, \quad \bar{x}_5 = 3, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

• Iteration 1

If we increase the value of the variable π_3 , we can get a better dual solution. We have:

$$\begin{cases} \pi_1 = 1 - \pi_3 \\ \pi_2 = 3 - 2\pi_3 \end{cases} \quad \begin{cases} \underbrace{1 - \pi_3}_{\pi_1 \geq 0} \geq 0 \\ \underbrace{3 - 2\pi_3}_{\pi_2 \geq 0} \geq 0 \end{cases} \quad \begin{cases} \pi_3 \leq 1 \\ \pi_3 \leq \frac{3}{2} \end{cases}$$

By increasing the value of π_3 to 1, the value of π_1 decreases to 0. Then we have that:

π_3 becomes dual basic and, accordingly, x_3 becomes primal non-basic;
 π_1 becomes dual non-basic and, accordingly, x_1 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_1 = 1 - \pi_3 + \pi_4 + 2\pi_5 \quad \text{then} \quad \pi_3 = 1 - \pi_1 + \pi_4 + 2\pi_5.$$

Substituting in the other equations of the dual dictionary, we get:

$$\begin{aligned} \pi_2 &= 3 - 2(1 - \pi_1 + \pi_4 + 2\pi_5) - 4\pi_4 - 2\pi_5 = 1 + 2\pi_1 - 6\pi_4 - 6\pi_5 \\ -w &= 0 + 6(1 - \pi_1 + \pi_4 + 2\pi_5) + 6\pi_4 - 3\pi_5 = 6 - 6\pi_1 + 12\pi_4 + 9\pi_5 \end{aligned}$$

Accordingly, we now have the following dual dictionary:

$$\left\{ \begin{array}{l} \pi_2 = 1 + 2\pi_1 - 6\pi_4 - 6\pi_5 \\ \pi_3 = 1 - \pi_1 + \pi_4 + 2\pi_5 \\ -w = 6 - 6\pi_1 + 12\pi_4 + 9\pi_5 \end{array} \right.$$

where $\mathcal{B} = \{2, 3\}$ and $\mathcal{N} = \{1, 4, 5\}$. The basic dual feasible solution is:

$$\bar{\pi}_2 = 1, \quad \bar{\pi}_3 = 1, \quad \bar{\pi}_1 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0, \quad -w = 6, \quad w = -6.$$

– For the primal LP problem, we have:

$$x_3 = -6 + x_1 + 2x_2 \quad \text{then} \quad x_1 = 6 - 2x_2 + x_3.$$

Substituting in the other equations of the primal dictionary, we get:

$$\begin{aligned} x_4 &= -6 - (6 - 2x_2 + x_3) + 4x_2 = -12 + 6x_2 - x_3 \\ x_5 &= 3 - 2(6 - 2x_2 + x_3) + 2x_2 = -9 + 6x_2 - 2x_3 \\ z &= 0 - (6 - 2x_2 + x_3) - 3x_2 = -6 - x_2 - x_3 \end{aligned}$$

Accordingly, we now have the following primal dictionary:

$$\left\{ \begin{array}{l} x_1 = 6 - 2x_2 + x_3 \\ x_4 = -12 + 6x_2 - x_3 \\ x_5 = -9 + 6x_2 - 2x_3 \\ z = -6 - x_2 - x_3 \end{array} \right.$$

where $\mathcal{B} = \{1, 4, 5\}$ and $\mathcal{N} = \{2, 3\}$. The basic primal non-feasible solution is:

$$\bar{x}_1 = 6, \quad \bar{x}_4 = -12, \quad \bar{x}_5 = -9, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad -6$$

• **Iteration 2**

If we increase the value of the variable π_4 , we can get a better dual solution. We have:

$$\begin{cases} \pi_2 = 1 - 6\pi_4 \\ \pi_3 = 1 + \pi_4 \end{cases} \quad \begin{cases} \underbrace{1 - 6\pi_4 \geq 0}_{\pi_2 \geq 0} \\ \underbrace{1 + \pi_4 \geq 0}_{\pi_3 \geq 0} \end{cases} \quad \begin{cases} \pi_4 \leq \frac{1}{6} \\ \pi_4 \geq -1 \end{cases}$$

By increasing the value of π_4 to $\frac{1}{6}$, the value of π_2 decreases to 0. Then we have that:

π_4 becomes dual basic and, accordingly, x_4 becomes primal non-basic;
 π_2 becomes dual non-basic and, accordingly, x_2 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_2 = 1 + 2\pi_1 - 6\pi_4 - 6\pi_5 \quad \text{then} \quad \pi_4 = \frac{1}{6} + \frac{1}{3}\pi_1 - \frac{1}{6}\pi_2 - \pi_5.$$

Substituting in the other equations of the dual dictionary, we get:

$$\begin{aligned} \pi_3 &= 1 - \pi_1 + \left(\frac{1}{6} + \frac{1}{3}\pi_1 - \frac{1}{6}\pi_2 - \pi_5 \right) + 2\pi_5 = \frac{7}{6} - \frac{2}{3}\pi_1 - \frac{1}{6}\pi_2 + \pi_5 \\ -w &= 6 - 6\pi_1 + 12 \left(\frac{1}{6} + \frac{1}{3}\pi_1 - \frac{1}{6}\pi_2 - \pi_5 \right) + 9\pi_5 = 8 - 2\pi_1 - 2\pi_2 - 3\pi_5. \end{aligned}$$

Accordingly, we now have the following optimal dual dictionary:

$$\begin{cases} \pi_3 = \frac{7}{6} - \frac{2}{3}\pi_1 - \frac{1}{6}\pi_2 + \pi_5 \\ \pi_4 = \frac{1}{6} + \frac{1}{3}\pi_1 - \frac{1}{6}\pi_2 - \pi_5 \\ -w = 8 - 2\pi_1 - 2\pi_2 - 3\pi_5 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2, 5\}$. The dual basic optimal solution is:

$$\tilde{\pi}_3 = \frac{7}{6}, \quad \tilde{\pi}_4 = \frac{1}{6}, \quad \tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_5 = 0, \quad -w = 8, \quad w = -8.$$

– For the primal LP problem, we have:

$$x_4 = -12 + 6x_2 - x_3 \quad \text{then} \quad x_2 = 2 + \frac{1}{6}x_3 + \frac{1}{6}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$x_1 = 6 - 2 \left(2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \right) + x_3 = 2 + \frac{2}{3}x_3 - \frac{1}{3}x_4$$

$$x_5 = -9 + 6 \left(2 + \frac{1}{6} x_3 + \frac{1}{6} x_4 \right) - 2 x_3 = 3 - x_3 + x_4$$

$$z = -6 - \left(2 + \frac{1}{6} x_3 + \frac{1}{6} x_4 \right) - x_3 = -8 - \frac{7}{6} x_3 - \frac{1}{6} x_4$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = 2 + \frac{2}{3} x_3 - \frac{1}{3} x_4 \\ x_2 = 2 + \frac{1}{6} x_3 + \frac{1}{6} x_4 \\ x_5 = 3 - x_3 + x_4 \\ z = -8 - \frac{7}{6} x_3 - \frac{1}{6} x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$. The primal basic optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 2, \quad \tilde{x}_5 = 3, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad -8$$