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Linear Programming

Volume 2 — The simplex algorithm

The simplex algorithm on dictionaries, optimality conditions, pivot rules and cycling, Phase 0 with the auxiliary problem. Examples worked out step by step and exercises with complete solutions at the end of each chapter.

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Simplex algorithm

6.1 The standard form of an LP problem in canonical max form

- We consider an LP problem in standard maximization form with $n + m$ non-negative variables and m equality linear constraints obtained from an LP problem in canonical maximization form with n non-negative variables and m less-than-or-equal-to linear constraints by adding m slack non-negative variables:

$$\max \quad \mathbf{c}' \mathbf{x} \quad (6.1a)$$

$$\text{subject to} \quad \mathbf{A} \mathbf{x} = \mathbf{b}, \quad (6.1b)$$

$$\mathbf{x} \geq \mathbf{0}. \quad (6.1c)$$

The input data of the LP problem are:

1. an objective-function column vector $\mathbf{c} \in \mathbb{Z}^{(n+m)}$ of $n + m$ rows;
2. a constraint matrix $\mathbf{A} \in \mathbb{Z}^{m \times (n+m)}$ of m rows and $n + m$ columns;
3. a right-hand side column vector $\mathbf{b} \in \mathbb{Z}^m$ of m rows.

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The $m + n$ non negative variables can be written as a column vector $\mathbf{x}$ of $n + m$ rows.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{pmatrix}$$

Since the LP problem (6.1) is obtained by adding m slack variables to an LP problem in canonical form, we have:

$$\begin{pmatrix} a_{1(n+1)} & a_{1(n+2)} & \cdots & a_{1(n+m)} \\ a_{2(n+1)} & a_{2(n+2)} & \cdots & a_{2(n+m)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m(n+1)} & a_{m(n+2)} & \cdots & a_{m(n+m)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

which is the identity matrix $\mathbf{I}$ of size m . Moreover, for the same reason, we have:

$$\begin{pmatrix} c_{n+1} \\ c_{n+2} \\ \vdots \\ c_{n+m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since the constraint matrix $\mathbf{A} \in \mathbb{Z}^{m \times (n+m)}$ contains the identity matrix $\mathbf{I}$ of size m , we have:

$$\text{rank}(\mathbf{A}) = m$$

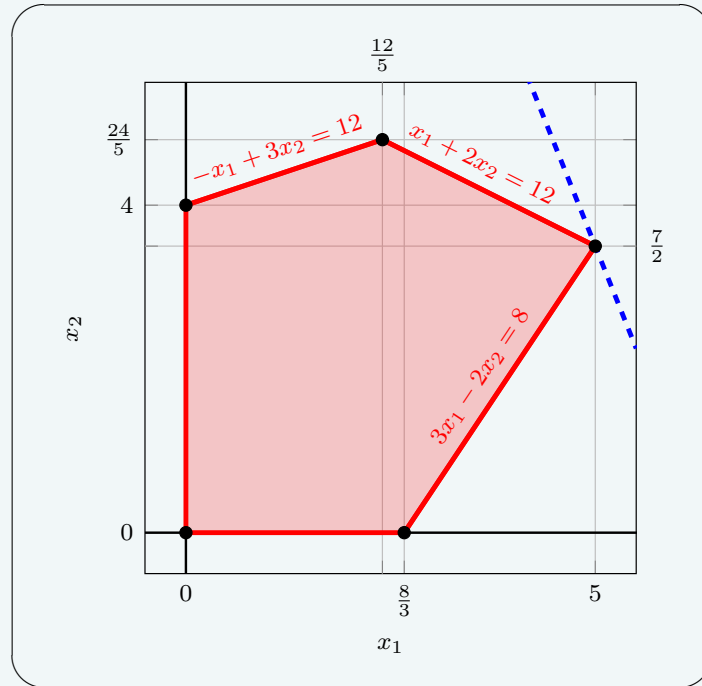
Accordingly there are m **linearly independent rows** and at least one set of m **linearly independent columns**^a in the constraints matrix $\mathbf{A}$.

^aThe row rank and the column rank of a matrix are equal

Example 6.1: standard max form of LP problems in canonical max form

We consider the following LP problem in canonical max form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{array}{llllll} \max & 5 & x_1 & + & 2 & x_2 \\ \text{subject to} & - & x_1 & + & 3 & x_2 & \leq & 12 \\ & & x_1 & + & 2 & x_2 & \leq & 12 \\ & 3 & x_1 & - & 2 & x_2 & \leq & 8 \\ & & x_1, & & x_2 & \geq & 0 \end{array}$$



In dashed blue the level set $5x_1 + 2x_2 = 32$ through the optimal vertex.

An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}$$

and the optimal value is:

$$z(LP) = 32$$

The standard form with $n + m = 5$ variables and $m = 3$ constraints read as follows:

$$\begin{array}{llllllll} \max & 5 & x_1 & + & 2 & x_2 & & & \\ \text{subject to} & - & x_1 & + & 3 & x_2 & + & x_3 & = & 12 \\ & & x_1 & + & 2 & x_2 & & + & x_4 & = & 12 \\ & & 3 & x_1 & - & 2 & x_2 & & + & x_5 & = & 8 \\ & & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0 \end{array}$$

In the LP problem in standard form (called [Example 6.1](#)), we have:

$$\mathbf{c} = \begin{pmatrix} 5 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

An optimal solution $\tilde{\mathbf{x}}(LP)$ of the standard form is: $\tilde{x}_1 = 5$, $\tilde{x}_2 = \frac{7}{2}$, $\tilde{x}_3 = \frac{13}{2}$, $\tilde{x}_4 = 0$, $\tilde{x}_5 = 0$.

6.2 Basis of the LP problem in standard form and dictionaries

Definition 6.1: basis of the LP problem

Given a subset $\mathcal{B} \subseteq \{1, 2, \dots, n + m\}$ of m indices of the $n + m$ variables of the LP problem (6.1), if the matrix $\mathbf{A}_{\mathcal{B}} \in \mathbb{R}^{m \times m}$ containing the m columns of $\mathbf{A}$ with indices $j \in \mathcal{B}$:

$$\mathbf{A}_{\mathcal{B}} = (\mathbf{A}_j)_{j \in \mathcal{B}}$$

is invertible, it is called a **basis** of the LP problem (6.1).

Definition 6.2: partition of the indices, basic and non-basic variables

Given a basis $\mathbf{A}_{\mathcal{B}}$ of the LP problem (6.1), we have the following **partition** of the indices $\{1, 2, \dots, n + m\}$ of the variables:

$$\mathcal{B} \subseteq \{1, 2, \dots, n + m\}, \quad |\mathcal{B}| = m,$$

$$\mathcal{N} = \{1, 2, \dots, n + m\} \setminus \mathcal{B}, \quad |\mathcal{N}| = n.$$

Given a partition $\mathcal{B}$ and $\mathcal{N}$, we define:

1. the column vector of the **basic variables** containing the m variables with indices $i \in \mathcal{B}$:

$$\mathbf{x}_{\mathcal{B}} = (x_i)_{i \in \mathcal{B}}$$

2. the column vector of the **non-basic variables** containing the n variables with indices $i \in \mathcal{N}$:

$$\mathbf{x}_{\mathcal{N}} = (x_i)_{i \in \mathcal{N}}$$

3. the column vector $\mathbf{c}_{\mathcal{B}} \in \mathbb{R}^m$ containing the m values of $\mathbf{c}$ with indices $i \in \mathcal{B}$:

$$\mathbf{c}_{\mathcal{B}} = (c_i)_{i \in \mathcal{B}}$$

4. the matrix $\mathbf{A}_{\mathcal{N}} \in \mathbb{R}^{m \times n}$ containing the n columns of $\mathbf{A}$ with indices $j \in \mathcal{N}$:

$$\mathbf{A}_{\mathcal{N}} = (\mathbf{A}_j)_{j \in \mathcal{N}}$$

5. the column vector $\mathbf{c}_{\mathcal{N}} \in \mathbb{R}^n$ containing the n values of $\mathbf{c}$ with indices $i \in \mathcal{N}$:

$$\mathbf{c}_{\mathcal{N}} = (c_i)_{i \in \mathcal{N}}$$

Example 6.2: Basis of LP problems, basic variables and non-basic variables

For the LP problem of [Example 6.1](#) with $\mathcal{B} = \{1, 3, 4\}$, we have:

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \quad \det(\mathbf{A}_{\mathcal{B}}) = -1 \cdot 0 \cdot 0 + 1 \cdot 1 \cdot 3 + 0 \cdot 1 \cdot 0 - 1 \cdot 1 \cdot 0 - (-1) \cdot 1 \cdot 0 - 0 \cdot 0 \cdot 3 = 3 \neq 0$$

and then $\text{rank}(\mathbf{A}_{\mathcal{B}}) = 3$. Accordingly, the matrix $\mathbf{A}_{\mathcal{B}}$ is invertible and it is a basis of the LP problem. We compute the inverse matrix $\mathbf{A}_{\mathcal{B}}^{-1}$ with the method of Gauss-Jordan:

$$\begin{pmatrix} -1 & 1 & 0 & | & 1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 3 & 0 & 0 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & | & -1 & 0 & 0 \\ 1 & 0 & 1 & | & 0 & 1 & 0 \\ 3 & 0 & 0 & | & 0 & 0 & 1 \end{pmatrix} \quad (R_1 \leftarrow -R_1)$$

$$\begin{pmatrix} 1 & -1 & 0 & | & -1 & 0 & 0 \\ 0 & 1 & 1 & | & 1 & 1 & 0 \\ 0 & 3 & 0 & | & 3 & 0 & 1 \end{pmatrix} \quad (R_2 \leftarrow R_2 - R_1, R_3 \leftarrow R_3 - 3R_1)$$

$$\begin{pmatrix} 1 & 0 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 1 & 1 & 0 \\ 0 & 0 & -3 & | & 0 & -3 & 1 \end{pmatrix} \quad (R_1 \leftarrow R_1 + R_2, R_3 \leftarrow R_3 - 3R_2)$$

$$\begin{pmatrix} 1 & 0 & 1 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & -\frac{1}{3} \end{pmatrix} \quad (R_3 \leftarrow -\frac{1}{3}R_3)$$

$$\begin{pmatrix} 1 & 0 & 0 & | & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & | & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & | & 0 & 1 & -\frac{1}{3} \end{pmatrix} \quad (R_1 \leftarrow R_1 - R_3, R_2 \leftarrow R_2 - R_3)$$

Since $\mathcal{B} = \{1, 3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_{\mathcal{B}}^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \end{pmatrix}$$

accordingly $\mathcal{N} = \{2, 5\}$ and we have:

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 3 & 0 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Example 6.3: Basis of LP problems, basic variables and non-basic variables

For the LP problem of [Example 6.1](#) with $\mathcal{B} = \{1, 2, 4\}$, we have:

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 3 & 0 \\ 1 & 2 & 1 \\ 3 & -2 & 0 \end{pmatrix}$$

and

$$\begin{aligned} \det(\mathbf{A}_{\mathcal{B}}) &= -1 \cdot (2 \cdot 0 - 1 \cdot (-2)) - 3 \cdot (1 \cdot 0 - 1 \cdot 3) + 0 \cdot (1 \cdot (-2) - 2 \cdot 3) \\ &= -1 \cdot 2 + 3 \cdot 3 = 7 \neq 0 \end{aligned}$$

and then $\text{rank}(\mathbf{A}_{\mathcal{B}}) = 3$. Accordingly, the matrix $\mathbf{A}_{\mathcal{B}}$ is invertible and it is a basis of the LP problem. We compute the inverse matrix $\mathbf{A}_{\mathcal{B}}^{-1}$ with the method of Gauss-Jordan:

$$\begin{aligned} &\left(\begin{array}{ccc|ccc} -1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 0 & 1 \end{array} \right) \\ &\left(\begin{array}{ccc|ccc} 1 & -3 & 0 & -1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 0 & 1 \end{array} \right) \quad (R_1 \leftarrow -R_1) \\ &\left(\begin{array}{ccc|ccc} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 5 & 1 & 1 & 1 & 0 \\ 0 & 7 & 0 & 3 & 0 & 1 \end{array} \right) \quad (R_2 \leftarrow R_2 - R_1, R_3 \leftarrow R_3 - 3R_1) \\ &\left(\begin{array}{ccc|ccc} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 7 & 0 & 3 & 0 & 1 \end{array} \right) \quad (R_2 \leftarrow \frac{1}{5}R_2) \\ &\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{5} & -\frac{2}{5} & \frac{3}{5} & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{7}{5} & \frac{8}{5} & -\frac{7}{5} & 1 \end{array} \right) \quad (R_1 \leftarrow R_1 + 3R_2, R_3 \leftarrow R_3 - 7R_2) \\ &\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{5} & -\frac{2}{5} & \frac{3}{5} & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & -\frac{8}{7} & 1 & -\frac{5}{7} \end{array} \right) \quad (R_3 \leftarrow -\frac{5}{7}R_3) \\ &\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{7} & 0 & \frac{3}{7} \\ 0 & 1 & 0 & \frac{3}{7} & 0 & \frac{1}{7} \\ 0 & 0 & 1 & -\frac{8}{7} & 1 & -\frac{5}{7} \end{array} \right) \quad (R_1 \leftarrow R_1 - \frac{3}{5}R_3, R_2 \leftarrow R_2 - \frac{1}{5}R_3) \end{aligned}$$

Since $\mathcal{B} = \{1, 2, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 3 & 0 \\ 1 & 2 & 1 \\ 3 & -2 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_{\mathcal{B}}^{-1} = \begin{pmatrix} \frac{2}{7} & 0 & \frac{3}{7} \\ \frac{3}{7} & 0 & \frac{1}{7} \\ -\frac{8}{7} & 1 & -\frac{5}{7} \end{pmatrix}$$

accordingly $\mathcal{N} = \{3, 5\}$ and we have:

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Given a basis $\mathbf{A}_{\mathcal{B}}$ of the LP problem (6.1), we can re-write the equality constraints (6.1b) as follows:

$$\mathbf{A} \mathbf{x} = (\mathbf{A}_{\mathcal{B}} \quad \mathbf{A}_{\mathcal{N}}) \begin{pmatrix} \mathbf{x}_{\mathcal{B}} \\ \mathbf{x}_{\mathcal{N}} \end{pmatrix} = \mathbf{A}_{\mathcal{B}} \mathbf{x}_{\mathcal{B}} + \mathbf{A}_{\mathcal{N}} \mathbf{x}_{\mathcal{N}} = \mathbf{b}$$

and then, by multiplying by $\mathbf{A}_{\mathcal{B}}^{-1}$, we have:

$$\mathbf{A}_{\mathcal{B}}^{-1}(\mathbf{A}_{\mathcal{B}} \mathbf{x}_{\mathcal{B}} + \mathbf{A}_{\mathcal{N}} \mathbf{x}_{\mathcal{N}}) = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} \Rightarrow \underbrace{\mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{B}}}_{=I} \mathbf{x}_{\mathcal{B}} + \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} \mathbf{x}_{\mathcal{N}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$$

The basic variables can be written in function of the non-basic variables as follows:

$$\mathbf{x}_{\mathcal{B}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} - \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} \mathbf{x}_{\mathcal{N}} \quad (6.2)$$

Moreover, we can re-write the objective function (6.1a) value, denoted by z , as follows:

$$z = \mathbf{c}' \mathbf{x} = \begin{pmatrix} \mathbf{c}_{\mathcal{B}} \\ \mathbf{c}_{\mathcal{N}} \end{pmatrix}' \begin{pmatrix} \mathbf{x}_{\mathcal{B}} \\ \mathbf{x}_{\mathcal{N}} \end{pmatrix} = \mathbf{c}'_{\mathcal{B}} \mathbf{x}_{\mathcal{B}} + \mathbf{c}'_{\mathcal{N}} \mathbf{x}_{\mathcal{N}}$$

Using (6.2), we then get:

$$z = \mathbf{c}'_{\mathcal{B}} (\mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} - \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} \mathbf{x}_{\mathcal{N}}) + \mathbf{c}'_{\mathcal{N}} \mathbf{x}_{\mathcal{N}}$$

The objective function can be written in function of the non-basic variables as follows:

$$z = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} + (\mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}}) \mathbf{x}_{\mathcal{N}} \quad (6.3)$$

Using (6.2) and (6.3), the **LP problem** (6.1) can be reformulated in form of a **dictionary**.

Definition 6.3: dictionary

Given a basis $\mathbf{A}_{\mathcal{B}}$ of the LP problem (6.1), the associated **dictionary** is a system of $m + 1$ linear equations in which the basic variables and the objective function

are written in function of the non-basic variables:

$$\begin{cases} x_{\mathcal{B}} = \bar{\mathbf{b}} - \bar{\mathbf{A}} \mathbf{x}_{\mathcal{N}} \\ z = \bar{z} + \bar{\mathbf{c}}' \mathbf{x}_{\mathcal{N}} \end{cases} \quad \text{where} \quad \begin{cases} \bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} \\ \bar{\mathbf{A}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} \\ \bar{z} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} \\ \bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} \end{cases} \quad (6.4)$$

- Given a basis $\mathbf{A}_{\mathcal{B}}$ of the LP problem (6.1), in the associated dictionary there are:
 1. the matrix $\bar{\mathbf{A}} \in \mathbb{R}^{m \times n}$ of m rows associated to the basic variables and n columns associated to the non-basic variables
 2. two columns vectors $\bar{\mathbf{b}} \in \mathbb{R}^m$ and $\bar{\mathbf{c}} \in \mathbb{R}^n$ of m rows associated to the basic variables and of n rows associated to the non-basic variables, respectively:

$$\bar{\mathbf{A}} = (\bar{a}_{ij})_{i \in \mathcal{B}, j \in \mathcal{N}}, \quad \bar{\mathbf{b}} = (\bar{b}_i)_{i \in \mathcal{B}}, \quad \bar{\mathbf{c}} = (\bar{c}_i)_{i \in \mathcal{N}}.$$

A dictionary can be equivalently rewritten using the summations as follows:

$$\begin{cases} x_i = \bar{b}_i - \sum_{j \in \mathcal{N}} \bar{a}_{ij} x_j, \quad \forall i \in \mathcal{B} \\ z = \bar{z} + \sum_{j \in \mathcal{N}} \bar{c}_j x_j \end{cases}$$

Example 6.4: simplex dictionary

For the LP problem of [Example 6.1](#), with $\mathcal{B} = \{1, 3, 4\}$ and $\mathcal{N} = \{2, 5\}$, we have:

$$\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{44}{3} \\ \frac{28}{3} \end{pmatrix}$$

$$\bar{\mathbf{A}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{7}{3} & \frac{1}{3} \\ \frac{8}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\bar{z} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = (5 \quad 0 \quad 0) \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = (0 \quad 0 \quad \frac{5}{3}) \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = \frac{40}{3}$$

$$\begin{aligned}\bar{\mathbf{c}}' &= \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = (2 \ 0) - (5 \ 0 \ 0) \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} \\ &= (2 \ 0) - (0 \ 0 \ \frac{5}{3}) \begin{pmatrix} 3 & 0 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} = (2 \ 0) - (-\frac{10}{3} \ \frac{5}{3}) = (\frac{16}{3} \ -\frac{5}{3})\end{aligned}$$

Accordingly, the associated dictionary is:

$$\begin{cases} \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{44}{3} \\ \frac{28}{3} \end{pmatrix} - \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{7}{3} & \frac{1}{3} \\ \frac{8}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \\ z = \frac{40}{3} + \begin{pmatrix} \frac{16}{3} & -\frac{5}{3} \end{pmatrix} \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \end{cases}$$

or, equivalently:

$$\begin{cases} x_1 &= \frac{8}{3} + \frac{2}{3} x_2 - \frac{1}{3} x_5 \\ x_3 &= \frac{44}{3} - \frac{7}{3} x_2 - \frac{1}{3} x_5 \\ x_4 &= \frac{28}{3} - \frac{8}{3} x_2 + \frac{1}{3} x_5 \\ z &= \frac{40}{3} + \frac{16}{3} x_2 - \frac{5}{3} x_5 \end{cases}$$

This dictionary is associated to the feasible solution $\bar{x}_1 = \frac{8}{3}, \bar{x}_2 = 0, \bar{x}_3 = \frac{44}{3}, \bar{x}_4 = \frac{28}{3}, \bar{x}_5 = 0$ of value $z = \frac{40}{3}$.

Example 6.5: simplex dictionary

For the LP problem of [Example 6.1](#), with $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5\}$, we have:

$$\begin{aligned}\bar{\mathbf{b}} &= \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = \begin{pmatrix} \frac{2}{7} & 0 & \frac{3}{7} \\ \frac{3}{7} & 0 & \frac{1}{7} \\ -\frac{8}{7} & 1 & -\frac{5}{7} \end{pmatrix} \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{48}{7} \\ \frac{44}{7} \\ -\frac{52}{7} \end{pmatrix} \\ \bar{\mathbf{A}} &= \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} \frac{2}{7} & 0 & \frac{3}{7} \\ \frac{3}{7} & 0 & \frac{1}{7} \\ -\frac{8}{7} & 1 & -\frac{5}{7} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{1}{7} \\ -\frac{8}{7} & -\frac{5}{7} \end{pmatrix}\end{aligned}$$

$$\bar{z} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = (5 \ 2 \ 0) \begin{pmatrix} \frac{2}{7} & 0 & \frac{3}{7} \\ \frac{3}{7} & 0 & \frac{1}{7} \\ -\frac{8}{7} & 1 & -\frac{5}{7} \end{pmatrix} \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = (\frac{16}{7} \ 0 \ \frac{17}{7}) \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = \frac{328}{7}$$

$$\begin{aligned} \bar{\mathbf{c}}' &= \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = (0 \ 0) - (5 \ 2 \ 0) \begin{pmatrix} \frac{2}{7} & 0 & \frac{3}{7} \\ \frac{3}{7} & 0 & \frac{1}{7} \\ -\frac{8}{7} & 1 & -\frac{5}{7} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= (0 \ 0) - (\frac{16}{7} \ 0 \ \frac{17}{7}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = (0 \ 0) - (\frac{16}{7} \ \frac{17}{7}) = (-\frac{16}{7} \ -\frac{17}{7}) \end{aligned}$$

Accordingly, the associated dictionary is:

$$\begin{cases} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{48}{7} \\ \frac{44}{7} \\ -\frac{52}{7} \end{pmatrix} - \begin{pmatrix} \frac{2}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{1}{7} \\ -\frac{8}{7} & -\frac{5}{7} \end{pmatrix} \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} \\ z = \frac{328}{7} + \begin{pmatrix} -\frac{16}{7} & -\frac{17}{7} \end{pmatrix} \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} \end{cases}$$

or, equivalently:

$$\begin{cases} x_1 &= \frac{48}{7} &- \frac{2}{7} x_3 &- \frac{3}{7} x_5 \\ x_2 &= \frac{44}{7} &- \frac{3}{7} x_3 &- \frac{1}{7} x_5 \\ x_4 &= -\frac{52}{7} &+ \frac{8}{7} x_3 &+ \frac{5}{7} x_5 \\ z &= \frac{328}{7} &- \frac{16}{7} x_3 &- \frac{17}{7} x_5 \end{cases}$$

This dictionary is not associated to a feasible solution.

6.2.1 Basic feasible solutions and reduced costs

- The basic variable values depend on the non-basic variable values, accordingly:
 1. **the basic variables are the dependent variables**
 2. **the non-basic variables are the independent variables**

Definition 6.4: basic solution

Given a basis $\mathbf{A}_B$ of the LP problem (6.1), by setting all the non-basic variables to zero we have:

$$\mathbf{x}_B = \bar{\mathbf{b}} \quad \text{and} \quad \mathbf{x}_N = \mathbf{0} \quad (6.5)$$

which is called a **basic solution** of the LP problem (6.1).

- The variables are non-negative (6.1c) so not all basic solutions are feasible solutions.

Definition 6.5: basic feasible solutions

Given a basis $\mathbf{A}_B$ of the LP problem (6.1), if $\bar{\mathbf{b}} \geq \mathbf{0}$ then

$$\mathbf{x}_B = \bar{\mathbf{b}} \quad \text{and} \quad \mathbf{x}_N = \mathbf{0} \quad (6.6)$$

is called a **basic feasible solution**.

Definition 6.6: reduced cost vectors

Given a basis $\mathbf{A}_B$ of the LP problem (6.1), the vector $\bar{\mathbf{c}}$ is called the **reduced costs vector**.

- The reduced cost

$$\bar{c}_p = c_p - \mathbf{c}'_B \mathbf{A}_B^{-1} \mathbf{A}_p$$

of a non-basic variable x_p with $p \in \mathcal{N}$ is the change of the objective function per unit increment of the variable.

The increment is proportional to c_p while the decrement (due to the change of the basic variables to preserve the equality constraints (6.1b)) is proportional to $\mathbf{c}'_B \mathbf{A}_B^{-1} \mathbf{A}_p$.

6.2.2 Finding an initial basic feasible solution

For an LP problem (6.1), if $\mathbf{b} \geq \mathbf{0}$, the partition:

$$\mathcal{B} = \{n+1, n+2, \dots, n+m\} \quad \text{and} \quad \mathcal{N} = \{1, 2, \dots, n\}$$

is associated to a basis $\mathbf{A}_{\mathcal{B}} = \mathbf{I}$ and to the **initial basic feasible solution**:

$$\mathbf{x}_{\mathcal{B}} = \mathbf{b} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \mathbf{0} \quad (6.7)$$

- For an LP problem (6.1), since in $\mathcal{B}$ there are the indices of the m slack variables, we have:

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \mathbf{I} = \mathbf{A}_{\mathcal{B}}^{-1} \quad \text{and} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Accordingly, the matrix $\mathbf{A}_{\mathcal{B}}$ is invertible and then it is a basis. Moreover, we have:

$$\mathbf{A}_{\mathcal{N}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

In these cases, we then have:

$$\begin{cases} \bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = \mathbf{I} \mathbf{b} = \mathbf{b} \\ \bar{\mathbf{A}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = \mathbf{I} \mathbf{A}_{\mathcal{N}} = \mathbf{A}_{\mathcal{N}} \\ \bar{\mathbf{z}} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = \mathbf{c}'_{\mathcal{B}} \mathbf{I} \mathbf{b} = 0 \\ \bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{I} \mathbf{A}_{\mathcal{N}} = \mathbf{c}'_{\mathcal{N}} \end{cases}$$

Example 6.6: initial basic feasible solution

For the LP problem of [Example 6.1](#), since $\begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, with $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$,

we have the basic feasible solution $\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Moreover $\bar{\mathbf{z}} = 0$ and

we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad \mathbf{c}'_{\mathcal{B}} = (0 \quad 0 \quad 0), \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \underbrace{\mathbf{c}'_{\mathcal{N}}}_{=\bar{\mathbf{c}}'} = \begin{pmatrix} 5 & 2 \end{pmatrix}, \quad \underbrace{\mathbf{A}_{\mathcal{N}}}_{=\bar{\mathbf{A}}} = \begin{pmatrix} -1 & 3 \\ 1 & 2 \\ 3 & -2 \end{pmatrix}$$

6.3 Pivoting operations

- Given a basis $\mathbf{A}_{\mathcal{B}}$ and a basic feasible solution $\mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}}$ and $\mathbf{x}_{\mathcal{N}} = \mathbf{0}$ of the LP problem (6.1), if in the associated dictionary there exists $j \in \mathcal{N}$ such that $\bar{c}_j > 0$ (a positive reduced cost), it is necessary to perform a pivoting operation.

Definition 6.7: pivoting operation

A **pivoting operation** consists in determining a **new basic feasible solution** with the goal of increasing the basic feasible solution value by **swapping one non-basic variable and one basic variable**.

- The non-basic variable that becomes basic is called the **pivot column**
- The basic variable that becomes non-basic is called the **pivot row**

6.3.1 Determining a new basic feasible solution

The **pivot column** is a non-basic variable with a positive reduced cost, i.e., a non-basic variable

$$x_p \text{ with } p \in \mathcal{N} \text{ such that } \underbrace{c_p - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_p}_{=\bar{c}_p} > 0$$

- Since only the value of the pivot column x_p with $p \in \mathcal{N}$ is (potentially) increased among the non-basic variables, we have:

$$\mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}} - \underbrace{\mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_p}_{=\bar{\mathbf{A}}_p} x_p$$

where $\bar{\mathbf{A}}_p$ is the column of the matrix $\bar{\mathbf{A}}$ associated to the pivot column x_p . These m equalities determine the change of the values of the basic variables caused by an increment of the value of the pivot column.

The value of the pivot column is increased as much as possible while at the same time maintaining non negative the value of all basic variables. In this way a new basic feasible solution is determined. Accordingly, the value of pivot column

becomes the maximum value t , called t_m , such that:

$$\bar{\mathbf{b}} - t \underbrace{\mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_p}_{=\bar{\mathbf{A}}_p} \geq \mathbf{0} \quad \Rightarrow \quad t_m = \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ip}} : i \in \mathcal{B}, \bar{a}_{ip} > 0 \right\}$$

If $\bar{a}_{ip} \leq 0$ for every $i \in \mathcal{B}$, then the LP problem is unbounded in the direction of x_p . In this case, the value of the pivot column can be increased without bound while maintaining non negative the value of all basic variables.

Certificate of unboundedness. Given a basis $\mathbf{A}_{\mathcal{B}}$ with a basic feasible solution, if there exists a non-basic variable x_p with $p \in \mathcal{N}$ such that

$$\bar{c}_p > 0 \quad \text{and} \quad \bar{a}_{ip} \leq 0, \quad \forall i \in \mathcal{B},$$

then the LP problem (6.1) is **unbounded**.

– Indeed, for every value $t \geq 0$, the point

$$x_p = t, \quad x_i = \bar{b}_i - t \bar{a}_{ip}, \quad \forall i \in \mathcal{B}, \quad x_j = 0, \quad \forall j \in \mathcal{N} \setminus \{p\}, \quad (6.8)$$

is a feasible solution of the LP problem (6.1): it satisfies the equality constraints (6.1b) by construction, and it satisfies the non-negativity constraints (6.1c) since $\bar{b}_i \geq 0$ and $-t \bar{a}_{ip} \geq 0$ for every $i \in \mathcal{B}$.

– The objective value of this feasible solution is

$$\bar{z} + t \bar{c}_p,$$

and then, for every value $M \in \mathbb{R}$, the feasible solution obtained with $t > \frac{M - \bar{z}}{\bar{c}_p}$ has an objective value larger than M .

The pivot row is a basic variable whose value becomes zero by setting the value of the pivot column to t_m , i.e., a basic variable

$$x_r \quad \text{with} \quad r \in \mathcal{B} \quad \text{such that} \quad \bar{a}_{rp} > 0 \quad \text{and} \quad \bar{b}_r - t_m \bar{a}_{rp} = 0 \quad (\text{i.e., } \frac{\bar{b}_r}{\bar{a}_{rp}} = t_m)$$

where $\bar{a}_{rp}$ is the value of the column $\bar{\mathbf{A}}_p$ associated to the basic variable x_r .

- Given t_m and $\bar{\mathbf{A}}_p$, **the new basic feasible solution** (which determines the new vector $\bar{\mathbf{b}}$) is obtained as follows:

$$x_i = \underbrace{\bar{b}_i - t_m \bar{a}_{ip}}_{\text{new } \bar{b}_i}, \quad \forall i \in \mathcal{B} \setminus \{r\}, \quad x_p = \underbrace{t_m}_{\bar{b}_p}, \quad x_r = 0, \quad x_j = 0, \quad \forall j \in \mathcal{N} \setminus \{p\}. \quad (6.9)$$

- Given a pivot column x_p with $p \in \mathcal{N}$ and a pivot row x_r with $r \in \mathcal{B}$, the new partition of the indices of the variables is updated as follows:

$$\mathcal{B} \leftarrow (\mathcal{B} \setminus \{r\}) \cup \{p\} \quad \text{and} \quad \mathcal{N} \leftarrow (\mathcal{N} \setminus \{p\}) \cup \{r\} \quad (6.10)$$

By construction $\mathbf{A}_{\mathcal{B}}$ remains a basis.

Example 6.7: pivoting operations

For the LP problem of Example 6.1, with $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{c}'_{\mathcal{B}} = (0 \quad 0 \quad 0) \quad \bar{\mathbf{b}} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \underbrace{\mathbf{A}_{\mathcal{N}}}_{=\bar{\mathbf{A}}} = \begin{pmatrix} -1 & 3 \\ 1 & 2 \\ 3 & -2 \end{pmatrix} \quad \underbrace{\mathbf{c}'_{\mathcal{N}}}_{=\bar{\mathbf{c}}'} = (5 \quad 2)$$

Accordingly, we can choose x_1 as the pivot column since $\bar{c}_1 = 5 > 0$. We then find the largest t such that $\bar{\mathbf{b}} - t \bar{\mathbf{A}}_1 \geq 0$:

$$\begin{cases} \underbrace{12 - t \cdot (-1)}_{x_3 \geq 0} \geq 0 \\ \underbrace{12 - t \cdot 1}_{x_4 \geq 0} \geq 0 \\ \underbrace{8 - t \cdot 3}_{x_5 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \geq -12 \\ t \leq 12 \\ t \leq \frac{8}{3} \end{cases} \quad \text{then} \quad t_m = \frac{8}{3}$$

The value of the pivot column becomes t_m and $x_1 = \frac{8}{3}$. The value of x_5 becomes zero and x_5 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \bar{\mathbf{A}}_1$, then we have:

$$x_3 = 12 - \frac{8}{3} \cdot (-1) = \frac{44}{3}, \quad x_4 = 12 - \frac{8}{3} \cdot 1 = \frac{28}{3} \quad \text{and accordingly} \quad \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{8}{3} \\ \frac{44}{3} \\ \frac{28}{3} \end{pmatrix}$$

The new partition of the indices of the variables is $\mathcal{B} = \{1, 3, 4\}$ and $\mathcal{N} = \{2, 5\}$.

In the same dictionary also $\bar{c}_2 = 2 > 0$, so x_2 can be chosen as the pivot column as well: the ratio test then gives $t_m = 4$, the pivot row is x_3 and the new partition is $\mathcal{B} = \{2, 4, 5\}$ and $\mathcal{N} = \{1, 3\}$. The execution of the simplex algorithm that starts with this choice is worked out in the exercises of this chapter (*Simplex algorithm with a different pivot column*, problem P10).

Example 6.8: certificate of unboundedness

Consider the following LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 2$ less-than-or-equal-to linear constraints:

$$\begin{array}{llllll} \max & & x_1 & + & x_2 & \\ \text{subject to} & - & x_1 & + & x_2 & \leq 1 \\ & & x_1 & - & x_2 & \leq 1 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

By adding $m = 2$ slack variables, with $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ we obtain the initial dictionary:

$$\begin{cases} x_3 = 1 + x_1 - x_2 \\ x_4 = 1 - x_1 + x_2 \\ z = 0 + x_1 + x_2 \end{cases}$$

We perform one pivoting operation with pivot column x_1 (since $\bar{c}_1 = 1 > 0$). The only positive value in the column of x_1 is $\bar{a}_{41} = 1$, and then $t_m = \frac{1}{1} = 1$ and the pivot row is x_4 . With $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$, the new dictionary is:

$$\begin{cases} x_1 = 1 + x_2 - x_4 \\ x_3 = 2 - x_4 \\ z = 1 + 2x_2 - x_4 \end{cases}$$

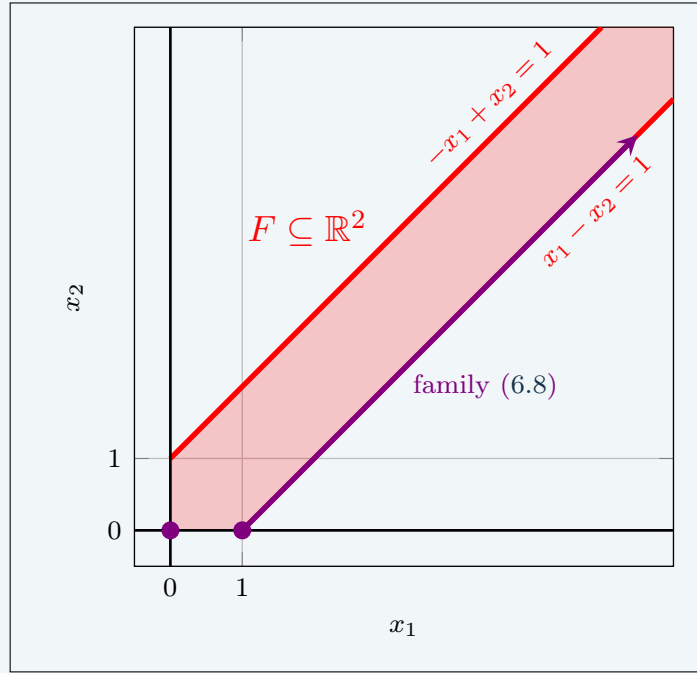
The reduced cost of the non-basic variable x_2 is $\bar{c}_2 = 2 > 0$, and the values of its column are

$$\bar{a}_{12} = -1 \leq 0 \quad \text{and} \quad \bar{a}_{32} = 0 \leq 0.$$

Accordingly, no pivot row can be determined and this LP problem is **unbounded**. The corresponding family of feasible solutions (6.8) is, for every value $t \geq 0$:

$$x_2 = t, \quad x_1 = 1 + t, \quad x_3 = 2, \quad x_4 = 0,$$

whose objective value is $1 + 2t$.



In violet the basic feasible solution reached after the first pivoting operation and the family of feasible solutions (6.8) along which the LP problem is unbounded.

6.3.2 Update of the dictionary

Once the pivot column x_p with $p \in \mathcal{N}$ and the pivot row x_r with $r \in \mathcal{B}$ are determined, the new dictionary can be determined by appropriate row operations to achieve the interchange.

- Let x_p be the pivot column and x_r the pivot row. From the equation of the pivot row, we have:

$$x_r = \bar{b}_r - \bar{a}_{rp}x_p - \sum_{j \in \mathcal{N} \setminus \{p\}} \bar{a}_{rj}x_j.$$

Solving for x_p gives the equation of the pivot column in the new dictionary:

$$x_p = \frac{\bar{b}_r}{\bar{a}_{rp}} - \sum_{j \in \mathcal{N} \setminus \{p\}} \frac{\bar{a}_{rj}}{\bar{a}_{rp}}x_j - \frac{1}{\bar{a}_{rp}}x_r. \quad (6.11)$$

For each remaining basic variable $i \in \mathcal{B} \setminus \{r\}$, substituting with (6.11) gives their equations in the new dictionary:

$$x_i = \bar{b}_i - \sum_{j \in \mathcal{N} \setminus \{p\}} \bar{a}_{ij}x_j - \bar{a}_{ip} \left(\frac{\bar{b}_r}{\bar{a}_{rp}} - \sum_{j \in \mathcal{N} \setminus \{p\}} \frac{\bar{a}_{rj}}{\bar{a}_{rp}}x_j - \frac{1}{\bar{a}_{rp}}x_r \right). \quad (6.12)$$

For the objective function, substituting with (6.11) gives the equation of the objective function in the new dictionary:

$$z = \bar{z} + \sum_{j \in \mathcal{N} \setminus \{p\}} \bar{c}_jx_j + \bar{c}_p \left(\frac{\bar{b}_r}{\bar{a}_{rp}} - \sum_{j \in \mathcal{N} \setminus \{p\}} \frac{\bar{a}_{rj}}{\bar{a}_{rp}}x_j - \frac{1}{\bar{a}_{rp}}x_r \right). \quad (6.13)$$

Example 6.9: update of the dictionary

For the LP problem of [Example 6.1](#), with $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$, we have the dictionary:

$$\begin{cases} x_3 = 12 + x_1 - 3x_2 \\ x_4 = 12 - x_1 - 2x_2 \\ x_5 = 8 - 3x_1 + 2x_2 \\ z = 0 + 5x_1 + 2x_2 \end{cases}$$

Performing the pivot operation with x_1 as the pivot column and x_5 as the pivot row, we have:

$$x_5 = 8 - 3x_1 + 2x_2 \quad \text{then} \quad x_1 = \frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5$$

Substituting in the other equations of the dictionary, we get:

$$x_3 = 12 + x_1 - 3x_2 = 12 + \left(\frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5\right) - 3x_2 = \frac{44}{3} - \frac{7}{3}x_2 - \frac{1}{3}x_5$$

$$x_4 = 12 - x_1 - 2x_2 = 12 - \left(\frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5\right) - 2x_2 = \frac{28}{3} - \frac{8}{3}x_2 + \frac{1}{3}x_5$$

$$z = 5x_1 + 2x_2 = 5\left(\frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5\right) + 2x_2 = \frac{40}{3} + \frac{16}{3}x_2 - \frac{5}{3}x_5$$

Accordingly the new dictionary reads as follows:

$$\begin{cases} x_1 = \frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5 \\ x_3 = \frac{44}{3} - \frac{7}{3}x_2 - \frac{1}{3}x_5 \\ x_4 = \frac{28}{3} - \frac{8}{3}x_2 + \frac{1}{3}x_5 \\ z = \frac{40}{3} + \frac{16}{3}x_2 - \frac{5}{3}x_5 \end{cases}$$

This dictionary is associated to the feasible solution $\bar{x}_1 = \frac{8}{3}$, $\bar{x}_2 = 0$, $\bar{x}_3 = \frac{44}{3}$, $\bar{x}_4 = \frac{28}{3}$, $\bar{x}_5 = 0$ of value $z = \frac{40}{3}$.

6.4 High-level pseudo-code of the simplex algorithm

- The **Phase 0 of the simplex algorithm** consists in a procedure to determine an initial basic feasible solution and the associated initial dictionary. The Phase 0 is also able to prove that no basic feasible solution exists for an LP problem (6.1).

Algorithm

Input

An LP problem (6.1).

Output

An optimal solution of the LP problem (6.1), if one exists; otherwise, a certificate that the LP problem is infeasible or unbounded.

Step 0: Initialization

If $b_i \geq 0$ for all $i \in \{1, 2, \dots, m\}$, then set

$$\mathcal{B} \leftarrow \{n+1, n+2, \dots, n+m\}, \quad \mathcal{N} \leftarrow \{1, 2, \dots, n\},$$

and consider the basic feasible solution

$$x_i = \bar{b}_i, \quad \forall i \in \mathcal{B}, \quad x_j = 0, \quad \forall j \in \mathcal{N},$$

of the LP problem (6.1), together with the associated initial dictionary. Otherwise, execute Phase 0 of the simplex algorithm.

If Phase 0 proves that no basic feasible solution exists, then the LP problem is **infeasible** and **STOP**. Otherwise, consider the basic feasible solution found by Phase 0 together with its dictionary.

Step 1: Optimality check

With respect to the current dictionary

$$\begin{cases} x_i = \bar{b}_i - \sum_{j \in \mathcal{N}} \bar{a}_{ij} x_j, & \forall i \in \mathcal{B}, \\ z = \bar{z} + \sum_{j \in \mathcal{N}} \bar{c}_j x_j, \end{cases}$$

check the reduced costs. If

$$\bar{c}_j \leq 0, \quad \forall j \in \mathcal{N},$$

then the basic feasible solution

$$x_i = \bar{b}_i, \quad \forall i \in \mathcal{B}, \quad x_j = 0, \quad \forall j \in \mathcal{N},$$

is an optimal solution and $\bar{z}$ is the optimal value of the LP problem (6.1). In this case **STOP**.

Step 2: Pivoting operations

The **pivot column** is a non-basic variable x_p with $p \in \mathcal{N}$ such that:

$$\bar{c}_p > 0$$

a. If

$$\bar{a}_{ip} \leq 0, \quad \forall i \in \mathcal{B},$$

then the LP problem (6.1) is **unbounded. STOP.**

b. Find the largest value t (called t_m) such that:

$$\bar{b}_i - t \bar{a}_{ip} \geq 0, \quad \forall i \in \mathcal{B} \quad \Rightarrow \quad t_m \leftarrow \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ip}} : i \in \mathcal{B}, \bar{a}_{ip} > 0 \right\}$$

The **pivot row** is a basic variable x_r with $r \in \mathcal{B}$ such that:

$$\bar{a}_{rp} > 0 \quad \text{and} \quad \bar{b}_r - t_m \bar{a}_{rp} = 0$$

c. Update the dictionary with the new basis by performing the following operations:

– The new equation of the pivot column x_p is obtained by isolating x_p in the pivot row:

$$x_p = \frac{\bar{b}_r}{\bar{a}_{rp}} - \sum_{j \in \mathcal{N} \setminus \{p\}} \frac{\bar{a}_{rj}}{\bar{a}_{rp}} x_j - \frac{1}{\bar{a}_{rp}} x_r.$$

– The new equations of the basic variables x_i with $i \in \mathcal{B} \setminus \{r\}$ are obtained by substituting the new equation of x_p in the old equations of x_i :

$$x_i = \bar{b}_i - \sum_{j \in \mathcal{N} \setminus \{p\}} \bar{a}_{ij} x_j - \bar{a}_{ip} \left(\frac{\bar{b}_r}{\bar{a}_{rp}} - \sum_{j \in \mathcal{N} \setminus \{p\}} \frac{\bar{a}_{rj}}{\bar{a}_{rp}} x_j - \frac{1}{\bar{a}_{rp}} x_r \right).$$

– The new equation of the objective function z is obtained by substituting the new equation of x_p in the old equation of z :

$$z = \bar{z} + \sum_{j \in \mathcal{N} \setminus \{p\}} \bar{c}_j x_j + \bar{c}_p \left(\frac{\bar{b}_r}{\bar{a}_{rp}} - \sum_{j \in \mathcal{N} \setminus \{p\}} \frac{\bar{a}_{rj}}{\bar{a}_{rp}} x_j - \frac{1}{\bar{a}_{rp}} x_r \right).$$

d. Update the basis with the new partition of the indices of the variables:

$$\mathcal{B} \leftarrow (\mathcal{B} \setminus \{r\}) \cup \{p\} \quad \text{and} \quad \mathcal{N} \leftarrow (\mathcal{N} \setminus \{p\}) \cup \{r\}$$

go to step 1.

Example 6.10: execution of the simplex algorithm

We execute the simplex algorithm on the LP problem of [Example 6.1](#). Since $\mathbf{b} \geq \mathbf{0}$, the initial basic feasible solution is the one of [Example 6.6](#), associated with $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$.

- **Initialization**

Consider the initial dictionary:

$$\begin{cases} x_3 = 12 + x_1 - 3x_2 \\ x_4 = 12 - x_1 - 2x_2 \\ x_5 = 8 - 3x_1 + 2x_2 \\ z = 0 + 5x_1 + 2x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 12, \quad \bar{x}_4 = 12, \quad \bar{x}_5 = 8, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

- **Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_3 = 12 + x_1 \\ x_4 = 12 - x_1 \\ x_5 = 8 - 3x_1 \end{cases} \quad \begin{cases} \underbrace{12 + x_1 \geq 0}_{x_3 \geq 0} \\ \underbrace{12 - x_1 \geq 0}_{x_4 \geq 0} \\ \underbrace{8 - 3x_1 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_1 \geq -12 \\ x_1 \leq 12 \\ x_1 \leq \frac{8}{3} \end{cases}$$

By increasing the value of x_1 to $\min \left\{ 12, \frac{8}{3} \right\} = \frac{8}{3}$, the value of x_5 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = 8 - 3x_1 + 2x_2 \quad \text{then} \quad x_1 = \frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5.$$

Substituting in the other equations of the dictionary, we get:

$$x_3 = 12 + \left(\frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5 \right) - 3x_2 = \frac{44}{3} - \frac{7}{3}x_2 - \frac{1}{3}x_5$$

$$x_4 = 12 - \left(\frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5 \right) - 2x_2 = \frac{28}{3} - \frac{8}{3}x_2 + \frac{1}{3}x_5$$

$$z = 5 \left(\frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5 \right) + 2x_2 = \frac{40}{3} + \frac{16}{3}x_2 - \frac{5}{3}x_5$$

• **Iteration 2**

Consider the dictionary:

$$\begin{cases} x_1 = \frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5 \\ x_3 = \frac{44}{3} - \frac{7}{3}x_2 - \frac{1}{3}x_5 \\ x_4 = \frac{28}{3} - \frac{8}{3}x_2 + \frac{1}{3}x_5 \\ z = \frac{40}{3} + \frac{16}{3}x_2 - \frac{5}{3}x_5 \end{cases}$$

where $\mathcal{B} = \{1, 3, 4\}$ and $\mathcal{N} = \{2, 5\}$. The basic feasible solution is:

$$\bar{x}_1 = \frac{8}{3}, \quad \bar{x}_3 = \frac{44}{3}, \quad \bar{x}_4 = \frac{28}{3}, \quad \bar{x}_2 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{40}{3}$$

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_1 = \frac{8}{3} + \frac{2}{3}x_2 \\ x_3 = \frac{44}{3} - \frac{7}{3}x_2 \\ x_4 = \frac{28}{3} - \frac{8}{3}x_2 \end{cases} \quad \begin{cases} \overbrace{\frac{8}{3} + \frac{2}{3}x_2 \geq 0}^{x_1 \geq 0} \\ \frac{44}{3} - \frac{7}{3}x_2 \geq 0 \\ \overbrace{\frac{28}{3} - \frac{8}{3}x_2 \geq 0}^{x_3 \geq 0} \\ \overbrace{\phantom{\frac{28}{3} - \frac{8}{3}x_2 \geq 0}}^{x_4 \geq 0} \end{cases} \quad \begin{cases} x_2 \geq -4 \\ x_2 \leq \frac{44}{7} \\ x_2 \leq \frac{7}{2} \end{cases}$$

By increasing the value of x_2 to $\min \left\{ \frac{44}{7}, \frac{7}{2} \right\} = \frac{7}{2}$, the value of x_4 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = \frac{28}{3} - \frac{8}{3}x_2 + \frac{1}{3}x_5 \quad \text{then} \quad x_2 = \frac{7}{2} - \frac{3}{8}x_4 + \frac{1}{8}x_5.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned}
 x_1 &= \frac{8}{3} + \frac{2}{3} \left(\frac{7}{2} - \frac{3}{8} x_4 + \frac{1}{8} x_5 \right) - \frac{1}{3} x_5 = 5 - \frac{1}{4} x_4 - \frac{1}{4} x_5 \\
 x_3 &= \frac{44}{3} - \frac{7}{3} \left(\frac{7}{2} - \frac{3}{8} x_4 + \frac{1}{8} x_5 \right) - \frac{1}{3} x_5 = \frac{13}{2} + \frac{7}{8} x_4 - \frac{5}{8} x_5 \\
 z &= \frac{40}{3} + \frac{16}{3} \left(\frac{7}{2} - \frac{3}{8} x_4 + \frac{1}{8} x_5 \right) - \frac{5}{3} x_5 = 32 - 2 x_4 - x_5
 \end{aligned}$$

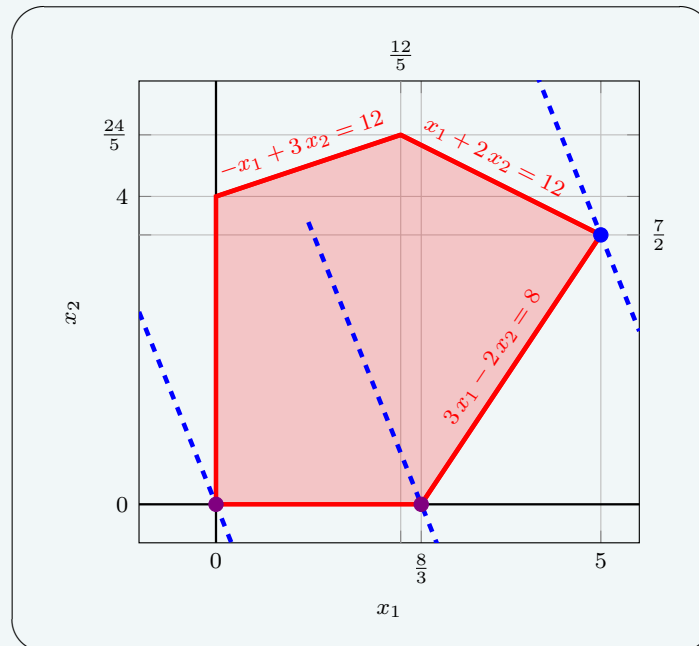
Accordingly, we obtained an optimal solution and this is the optimal dictionary:

$$\begin{cases}
 x_1 = 5 - \frac{1}{4} x_4 - \frac{1}{4} x_5 \\
 x_2 = \frac{7}{2} - \frac{3}{8} x_4 + \frac{1}{8} x_5 \\
 x_3 = \frac{13}{2} + \frac{7}{8} x_4 - \frac{5}{8} x_5 \\
 z = 32 - 2 x_4 - x_5
 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$. The optimal solution is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}, \quad \tilde{x}_3 = \frac{13}{2}, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad 32$$

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



In violet the first and the second vertex explored by this execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines $5x_1 + 2x_2 = 0$, $5x_1 + 2x_2 = \frac{40}{3}$ and $5x_1 + 2x_2 = 32$ of the objective function associated to the values taken in the three vertices.

At the first iteration the pivot column x_1 was chosen among the two non-basic variables with a positive reduced cost. Choosing x_2 instead, the simplex algorithm reaches the same optimal solution after three pivoting operations, visiting the vertices $(0, 4)$ and $(\frac{12}{5}, \frac{24}{5})$: the choice of the pivot column changes the path on the polyhedron and the number of iterations (see the exercise *Simplex algorithm with a different pivot column*, problem P10).

Optimality conditions and pivot rules

7.1 Sufficient conditions for optimality

- We consider an LP problem in standard maximization form with $n + m$ non-negative variables and m equality linear constraints, obtained from an LP problem in canonical maximization form with n non-negative variables and m less-than-or-equal-to linear constraints by adding m slack non-negative variables:

$$\max \quad \mathbf{c}' \mathbf{x} \quad (7.1a)$$

$$\text{subject to} \quad \mathbf{A} \mathbf{x} = \mathbf{b}, \quad (7.1b)$$

$$\mathbf{x} \geq \mathbf{0}. \quad (7.1c)$$

Theorem 7.1: sufficient conditions for optimality

Given a basis $\mathbf{A}_B$ of the LP problem (7.1), if $\bar{\mathbf{b}} \geq \mathbf{0}$ and $\bar{\mathbf{c}} \leq \mathbf{0}$ then $\mathbf{x}_B = \bar{\mathbf{b}}$ and $\mathbf{x}_N = \mathbf{0}$ is an optimal solution and $\mathbf{c}'_B \bar{\mathbf{b}}$ is the optimal solution value.

Proof. Given a basis $\mathbf{A}_B$ of the LP problem (7.1), if $\bar{\mathbf{b}} \geq \mathbf{0}$ then $\mathbf{x}_B = \bar{\mathbf{b}}$ and $\mathbf{x}_N = \mathbf{0}$ is a feasible solution. If $\bar{\mathbf{c}} \leq \mathbf{0}$, for all feasible solutions $\mathbf{x}$ we have:

$$\begin{aligned} \mathbf{c}' \mathbf{x} &= \mathbf{c}'_B \mathbf{x}_B + \mathbf{c}'_N \mathbf{x}_N = \mathbf{c}'_B \mathbf{A}_B^{-1} \mathbf{b} + \underbrace{\bar{\mathbf{c}}'}_{\leq \mathbf{0}} \underbrace{\mathbf{x}_N}_{\geq \mathbf{0}} \\ &\leq \mathbf{c}'_B \mathbf{A}_B^{-1} \mathbf{b} = \mathbf{c}'_B \bar{\mathbf{b}} + \mathbf{c}'_N \mathbf{0} \end{aligned}$$

Then the objective function value of all feasible solutions is smaller or equal to the objective function value of the basic feasible solution $\mathbf{x}_B = \bar{\mathbf{b}}$ and $\mathbf{x}_N = \mathbf{0}$, and accordingly, the latter is an optimal solution of value $\mathbf{c}'_B \bar{\mathbf{b}}$. $\square$

- We have proven that:

$$\bar{\mathbf{b}} \geq \mathbf{0} \text{ and } \bar{\mathbf{c}} \leq \mathbf{0} \implies \mathbf{x}_B = \bar{\mathbf{b}} \text{ and } \mathbf{x}_N = \mathbf{0}, \text{ optimal solution}$$

so “ $\bar{\mathbf{b}} \geq \mathbf{0}$ and $\bar{\mathbf{c}} \leq \mathbf{0}$ ” is a sufficient condition to “ $\mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}}$ and $\mathbf{x}_{\mathcal{N}} = \mathbf{0}$, optimal solution” and “ $\mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}}$ and $\mathbf{x}_{\mathcal{N}} = \mathbf{0}$, optimal solution” is a necessary condition to “ $\bar{\mathbf{b}} \geq \mathbf{0}$ and $\bar{\mathbf{c}} \leq \mathbf{0}$ ”. The implication does not work in the other direction (we will see a counterexample). Accordingly:

$$\mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}} \text{ and } \mathbf{x}_{\mathcal{N}} = \mathbf{0}, \text{ optimal solution} \not\Rightarrow \bar{\mathbf{c}} \leq \mathbf{0}$$

Corollary 7.2

Given a basis $\mathbf{A}_{\mathcal{B}}$ of the LP problem (7.1), if $\bar{\mathbf{b}} \geq \mathbf{0}$ and $\bar{\mathbf{c}} < \mathbf{0}$ then $\mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}}$ and $\mathbf{x}_{\mathcal{N}} = \mathbf{0}$ is the unique optimal solution.

Proof. It follows from the previous Theorem that $\mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}}$ and $\mathbf{x}_{\mathcal{N}} = \mathbf{0}$ is an optimal solution. If $\bar{\mathbf{c}} < \mathbf{0}$ and since any feasible solutions $\mathbf{x}$ different from $\mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}}$ and $\mathbf{x}_{\mathcal{N}} = \mathbf{0}$ must have $\mathbf{x}_{\mathcal{N}} \neq \mathbf{0}$, we have:

$$\begin{aligned} \mathbf{c}' \mathbf{x} &= \mathbf{c}'_{\mathcal{B}} \mathbf{x}_{\mathcal{B}} + \mathbf{c}'_{\mathcal{N}} \mathbf{x}_{\mathcal{N}} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} + \underbrace{\bar{\mathbf{c}}'}_{< \mathbf{0}} \underbrace{\mathbf{x}_{\mathcal{N}}}_{\geq \mathbf{0} \text{ and } \neq \mathbf{0}} \\ &< \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = \mathbf{c}'_{\mathcal{B}} \bar{\mathbf{b}} + \mathbf{c}'_{\mathcal{N}} \mathbf{0} \end{aligned}$$

Accordingly, $\mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}}$ and $\mathbf{x}_{\mathcal{N}} = \mathbf{0}$ is the unique optimal solution. $\square$

- Accordingly, we have shown that:

$$\bar{\mathbf{b}} \geq \mathbf{0} \text{ and } \bar{\mathbf{c}} < \mathbf{0} \implies \mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}} \text{ and } \mathbf{x}_{\mathcal{N}} = \mathbf{0}, \text{ unique optimal solution}$$

so “ $\bar{\mathbf{b}} \geq \mathbf{0}$ and $\bar{\mathbf{c}} < \mathbf{0}$ ” is a sufficient condition to “ $\mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}}$ and $\mathbf{x}_{\mathcal{N}} = \mathbf{0}$, unique optimal solution”. The implication does not work in the other direction (we will give a counterexample). Accordingly

$$\mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}} \text{ and } \mathbf{x}_{\mathcal{N}} = \mathbf{0}, \text{ unique optimal solution} \not\Rightarrow \bar{\mathbf{c}} < \mathbf{0}$$

Example 7.1: standard max form of LP problems in canonical max form

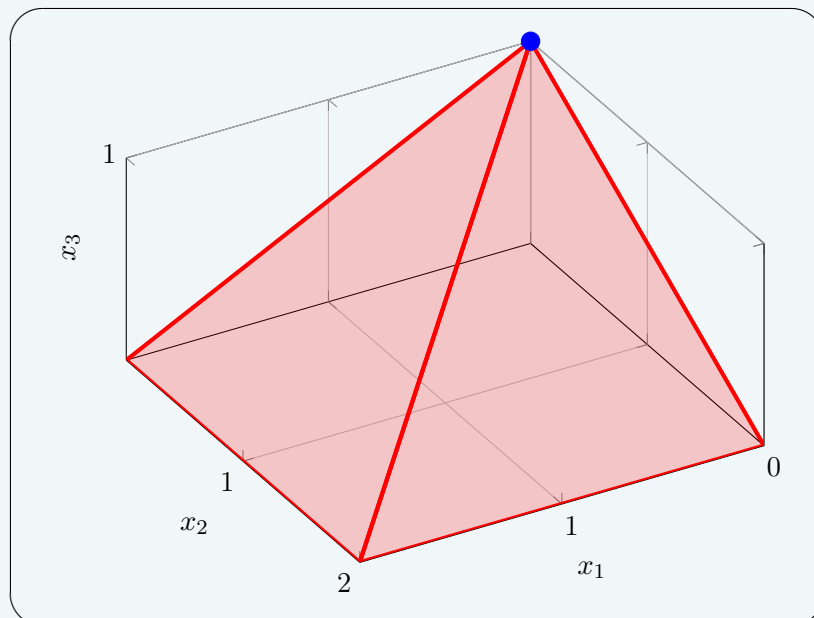
We introduce the following LP problem in canonical max form with $n = 3$ variables and $m = 2$ constraints and its standard form with $n + m = 5$ variables and $m = 2$ constraints:

$$\begin{aligned}
 & \max \quad x_1 + 2x_2 + 7x_3 \\
 & \text{subject to} \quad x_1 + 2x_3 \leq 2 \\
 & \quad \quad \quad x_2 + 2x_3 \leq 2 \\
 & \quad \quad \quad x_1, x_2, x_3 \geq 0
 \end{aligned}$$

$$\begin{aligned}
 & \max \quad x_1 + 2x_2 + 7x_3 \\
 & \text{subject to} \quad x_1 + 2x_3 + x_4 = 2 \\
 & \quad \quad \quad x_2 + 2x_3 + x_5 = 2 \\
 & \quad \quad \quad x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{aligned}$$

In the LP problem in standard form, we have:

$$\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 7 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$



An optimal solution $\tilde{\mathbf{x}}(\text{LP})$ of the canonical form is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 1$$

and the optimal solution value is:

$$z(\text{LP}) = 7$$

An optimal solution $\tilde{\mathbf{x}}(\text{LP})$ of the standard form is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 1, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0$$

with the same optimal solution value.

Example 7.2: optimality conditions

For the LP problem of [Example 7.1](#), with $\mathcal{B} = \{2, 3\}$ and $\mathcal{N} = \{1, 4, 5\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}}^{-1} = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Accordingly, we have:

$$\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar{\mathbf{A}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$\bar{z} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = (2 \quad 7) \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \left(\frac{3}{2} \quad 2\right) \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 7$$

$$\begin{aligned} \bar{\mathbf{c}}' &= \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = (1 \quad 0 \quad 0) - (2 \quad 7) \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= (1 \quad 0 \quad 0) - \left(\frac{3}{2} \quad 2\right) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (1 \quad 0 \quad 0) - \left(\frac{3}{2} \quad \frac{3}{2} \quad 2\right) = \left(-\frac{1}{2} \quad -\frac{3}{2} \quad -2\right) \end{aligned}$$

Since $\bar{\mathbf{b}} \geq \mathbf{0}$ and $\bar{\mathbf{c}} \leq \mathbf{0}$, then the basic feasible solution

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is the unique optimal solution and $\bar{z} = 7$ is the optimal solution value.

The associated dictionary is:

$$\begin{cases} x_2 = 0 + x_1 + x_4 - x_5 \\ x_3 = 1 - \frac{1}{2}x_1 - \frac{1}{2}x_4 \\ z = 7 - \frac{1}{2}x_1 - \frac{3}{2}x_4 - 2x_5 \end{cases}$$

Example 7.3: optimality conditions

For the LP problem of [Example 7.1](#), with $\mathcal{B} = \{3, 5\}$ and $\mathcal{N} = \{1, 2, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Accordingly, we have:

$$\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\bar{\mathbf{A}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 1 & -1 \end{pmatrix}$$

$$\bar{z} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = (7 \ 0) \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = (\frac{7}{2} \ 0) \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 7$$

$$\begin{aligned} \bar{\mathbf{c}}' &= \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = (1 \ 2 \ 0) - (7 \ 0) \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= (1 \ 2 \ 0) - (\frac{7}{2} \ 0) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = (1 \ 2 \ 0) - (\frac{7}{2} \ 0 \ \frac{7}{2}) = (-\frac{5}{2} \ 2 \ -\frac{7}{2}) \end{aligned}$$

Since $\bar{\mathbf{b}} \geq \mathbf{0}$ and $\bar{z} = 7$ (the optimal solution value), then the basic feasible solution

$$\begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is an optimal solution but $\bar{\mathbf{c}} \not\leq \mathbf{0}$. It is a **counterexample** to the fact that a basic optimal solution implies non-positive reduced costs. Since this optimal solution is unique (see

Example 7.2), it is also a **counterexample** to the fact that a unique basic optimal solution implies negative reduced costs.

The associated dictionary is:

$$\begin{cases} x_3 = 1 - \frac{1}{2} x_1 & - \frac{1}{2} x_4 \\ x_5 = 0 + x_1 - x_2 + x_4 \\ z = 7 - \frac{5}{2} x_1 + 2 x_2 - \frac{7}{2} x_4 \end{cases}$$

7.2 Necessary conditions for optimality

7.2.1 Degenerate basic feasible solutions

Definition 7.3: degenerate basic feasible solutions

Given a basis $\mathbf{A}_B$ of the LP problem (7.1), if $\bar{\mathbf{b}} \geq \mathbf{0}$ and for some $i \in \mathcal{B}$ we have $\bar{b}_i = 0$ then $\mathbf{x}_B = \bar{\mathbf{b}}$ and $\mathbf{x}_N = \mathbf{0}$ is called a **degenerate basic feasible solution**.

- In a degenerate basic feasible solution there are more than n variables at zero (also some basic variables are at zero and not only all the non-basic variables).

Theorem 7.4: necessary conditions for optimality

Given a basis $\mathbf{A}_B$ of the LP problem (7.1), if $\mathbf{x}_B = \bar{\mathbf{b}}$ and $\mathbf{x}_N = \mathbf{0}$ is a non degenerate optimal solution then $\bar{\mathbf{c}} \leq \mathbf{0}$.

Proof. Suppose by contradiction that $\mathbf{x}_B = \bar{\mathbf{b}}$ and $\mathbf{x}_N = \mathbf{0}$ is a non degenerate optimal solution and that $\bar{c}_p > 0$ for some $p \in \mathcal{N}$. Since the solution is non degenerate and feasible we have $\bar{\mathbf{b}} > \mathbf{0}$ but then there is a positive value $\vartheta > 0$ such that:

$$\mathbf{x}_B = \bar{\mathbf{b}} - \underbrace{\mathbf{A}_B^{-1} \mathbf{A}_p}_{=\bar{\mathbf{A}}_p} \vartheta \geq \mathbf{0}$$

Then by setting $x_p = \vartheta$ and $\mathbf{x}_B = \bar{\mathbf{b}} - \bar{\mathbf{A}}_p \vartheta$ (all other variables at zero), we get another feasible solution with a strictly better objective function value. This is a contradiction with $\mathbf{x}_B = \bar{\mathbf{b}}$ and $\mathbf{x}_N = \mathbf{0}$ being optimal, so we must have $\bar{\mathbf{c}} \leq \mathbf{0}$. $\square$

- Combined with the sufficient conditions for optimality, we have shown that:

$$\mathbf{x}_B = \bar{\mathbf{b}} \text{ and } \mathbf{x}_N = \mathbf{0}, \text{ non degenerate optimal solution} \iff \bar{\mathbf{b}} > \mathbf{0} \text{ and } \bar{\mathbf{c}} \leq \mathbf{0}$$

so “ $\bar{\mathbf{b}} > \mathbf{0}$ and $\bar{\mathbf{c}} \leq \mathbf{0}$ ” is a necessary and sufficient condition to “ $\mathbf{x}_B = \bar{\mathbf{b}}$ and $\mathbf{x}_N = \mathbf{0}$, non-degenerate optimal solution” and vice-versa.

Example 7.4: degenerate basic feasible solutions

For the LP problem of [Example 7.1](#), with $\mathcal{B} = \{3, 5\}$ and $\mathcal{N} = \{1, 2, 4\}$, we have the degenerate basic feasible solution:

$$\begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Definition 7.5: degenerate pivoting operation

A **degenerate pivoting operation** occurs when $t_m = 0$, i.e., when the pivot row x_r has $\bar{b}_r = 0$; this can happen only if the basic feasible solution is degenerate.

The **value of the objective function increases in each non degenerate pivoting operation and can remain unchanged only in a degenerate one.**

7.3 Pivot rules

- In pivoting operations, a **pivot rule** is a way of answering to the two questions:
 1. How to determine the pivot column in cases where there are multiple non-basic variables with positive reduced costs?
 2. How to determine the pivot row in cases where there are multiple basic variables taking a value of zero when setting the value of the pivot column to its maximum possible value t_m ?

7.3.1 Anti-cycling pivot rules

Theorem 7.6

If the simplex algorithm fails to terminate, then it must cycle.

Proof. There are at most $\binom{n+m}{m}$ bases $\mathbf{A}_B$ of the LP problem (7.1) and, accordingly, dictionaries. Then, if the simplex algorithm fails to terminate, it must visit some of these dictionaries more than once. Hence, the algorithm cycles. $\square$

- It is very rare that the simplex algorithm cycles, but instances in which it cycles exist.
- Cycling can occur only within a sequence of degenerate basic feasible solutions when the objective function remains unchanged.

7.3.1.1 Bland's pivot rule

- The **Bland's pivot rule** (smallest-index pivot rule) selects the pivot column and the pivot row as follows:

1. the **pivot column** is the non-basic variable x_p with the **smallest index** $p \in \mathcal{N}$ such that:

$$\bar{c}_p > 0 \quad \text{or equivalently} \quad c_p > \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_p$$

2. the **pivot row** is the basic variable x_r with the **smallest index** $r \in \mathcal{B}$ such that:

$$\bar{a}_{rp} > 0 \quad \text{and} \quad \frac{\bar{b}_r}{\bar{a}_{rp}} = t_m$$

Theorem 7.7

The simplex algorithm terminates as long as the pivot column and pivot row are selected by Bland's pivot rule in each pivoting operation.

- According to the previous Theorem, the proof only needs to show that cycling is impossible when the Bland's pivot rule is used.

Proof. We proceed by contradiction and we assume that cycling does occur. Accordingly, let

$$D_0, D_1, D_2, \dots, D_{k-1}, D_0, D_1, D_2, \dots$$

be a sequence of dictionaries such that the dictionary D_0 is repeated after k degenerate pivoting operations with constant objective function value.

We say that a variable is *fickle* if it is a basic variable in some dictionaries and a non-basic variable in some others.

Let x_t be the fickle variable having the largest index t and let D denote a dictionary

$$\begin{cases} x_i = \bar{b}_i - \sum_{j \in \mathcal{N}} \bar{a}_{ij} x_j, \quad \forall i \in \mathcal{B} \\ z = \bar{z} + \sum_{j \in \mathcal{N}} \bar{c}_j x_j \end{cases}$$

of the cycling sequence $D_0, D_1, D_2, \dots, D_{k-1}$ in which x_t is the pivot row. Let x_s denote the pivot column of the dictionary D . W.l.o.g, we assume that $D = D_0$.

Let D^* denote a dictionary

$$\begin{cases} x_i = \bar{b}_i^* - \sum_{j \in \mathcal{N}^*} \bar{a}_{ij}^* x_j, \quad \forall i \in \mathcal{B}^* \\ z = \bar{z}^* + \sum_{j \in \mathcal{N}^*} \bar{c}_j^* x_j \end{cases}$$

of the sequence $D_1, D_2, \dots, D_{k-1}$ in which x_t is the pivot column.

Since all the pivoting operations in the cycle are degenerate (the objective function value remains constant), we have that $\bar{z} = \bar{z}^* = v$, and therefore we can write the objective function of the dictionary D^* as

$$z = v + \sum_{j=1}^{n+m} \hat{c}_j^* x_j \quad \text{where} \quad \hat{c}_j^* = \begin{cases} \bar{c}_j^* & \text{if } j \in \mathcal{N}^* \\ 0 & \text{otherwise} \end{cases} \quad (7.2)$$

We consider the solution obtained by increasing the value of x_s to ϑ :

$$x_s = \vartheta, \quad x_i = 0, \quad \forall i \in \mathcal{N} \setminus \{s\}, \quad x_i = \bar{b}_i - \bar{a}_{is} \vartheta, \quad \forall i \in \mathcal{B}$$

Accordingly, for the objective function z of the dictionaries D and D^* , we have:

$$z = v + \bar{c}_s \vartheta \quad \text{and} \quad z = v + \hat{c}_s^* \vartheta + \sum_{i \in \mathcal{B}} \hat{c}_i^* (\bar{b}_i - \bar{a}_{is} \vartheta)$$

Equating these two expressions for z , we have:

$$\left(\bar{c}_s - \hat{c}_s^* + \sum_{i \in \mathcal{B}} \hat{c}_i^* \bar{a}_{is} \right) \vartheta = \sum_{i \in \mathcal{B}} \hat{c}_i^* \bar{b}_i \Rightarrow \bar{c}_s - \hat{c}_s^* + \sum_{i \in \mathcal{B}} \hat{c}_i^* \bar{a}_{is} = 0$$

The implication is due to the fact that the equation must hold for every ϑ (accordingly, we also have $\sum_{i \in \mathcal{B}} \hat{c}_i^* \bar{b}_i = 0$).

The fact that x_s is the pivot column in D implies that $\bar{c}_s > 0$. Since x_s is fickle and $s \neq t$ (x_s is non-basic and x_t is basic in D), we have $s < t$. Moreover, we have $\hat{c}_s^* \leq 0$: if $s \in \mathcal{B}^*$, then $\hat{c}_s^* = 0$; if $s \in \mathcal{N}^*$ and $\hat{c}_s^* = \bar{c}_s^* > 0$, then, according to Bland's rule and since $s < t$, x_s and not x_t would be the pivot column in D^* . Accordingly, we have:

$$\sum_{i \in \mathcal{B}} \hat{c}_i^* \bar{a}_{is} = \hat{c}_s^* - \bar{c}_s < 0 \Rightarrow \text{there exists } r \in \mathcal{B} : \hat{c}_r^* \bar{a}_{rs} < 0 \text{ and then } \hat{c}_r^* \neq 0 \text{ and } r \in \mathcal{N}^*$$

Hence x_r is fickle and therefore $r \leq t$. In fact, $r < t$, since $\hat{c}_t^* \bar{a}_{ts} > 0$: both factors are positive, since $\hat{c}_t^* = \bar{c}_t^* > 0$ (x_t is the pivot column in D^*) and $\bar{a}_{ts} > 0$ (x_t is the pivot row in D).

The fact that $r < t$ implies that $\hat{c}_r^* \leq 0$ (otherwise, according to Bland's rule, x_r would be the pivot column in D^*). Then we have:

$$\hat{c}_r^* \bar{a}_{rs} < 0 \text{ and } \hat{c}_r^* \leq 0 \Rightarrow \bar{a}_{rs} > 0$$

Since each of the dictionaries in the cycle describes the same feasible solution (all pivoting operations are degenerate), every fickle variable is zero in all these dictionaries (it is zero in a dictionary in which it is non-basic).

In particular, $x_r = 0$. But in D , x_r is basic. Hence $\bar{b}_r = 0$ and, since $\bar{a}_{rs} > 0$, we have $\frac{\bar{b}_r}{\bar{a}_{rs}} = 0 = t_m$. So x_r was a candidate to be the pivot row in D , and since $r < t$,

according to Bland's rule it should have been chosen instead of x_t .

This is the contradiction that proves that a cycling sequence of dictionaries cannot exist using Bland's rule. $\square$

Phase 0 of the simplex algorithm

8.1 The simplex auxiliary problem

- We consider an LP problem in standard maximization form with $n + m$ non-negative variables and m equality linear constraints, obtained from an LP problem in canonical maximization form with n non-negative variables and m less-than-or-equal-to linear constraints by adding m slack non-negative variables:

$$\max \quad \mathbf{c}' \mathbf{x} \quad (8.1a)$$

$$\text{subject to} \quad \mathbf{A} \mathbf{x} = \mathbf{b}, \quad (8.1b)$$

$$\mathbf{x} \geq \mathbf{0}. \quad (8.1c)$$

- The Step 0 of the simplex algorithm requires an initial basic feasible solution. If $\mathbf{b} \geq \mathbf{0}$, the partition

$$\mathcal{B} = \{n + 1, n + 2, \dots, n + m\} \quad \text{and} \quad \mathcal{N} = \{1, 2, \dots, n\}$$

is associated with the basis $\mathbf{A}_{\mathcal{B}} = \mathbf{I}$ and with the initial basic feasible solution $\mathbf{x}_{\mathcal{B}} = \mathbf{b}$ and $\mathbf{x}_{\mathcal{N}} = \mathbf{0}$.

- If instead $b_i < 0$ for some $i \in \{1, 2, \dots, m\}$, this partition is not associated with a basic feasible solution, and the simplex algorithm cannot be started. In this case we execute the **Phase 0** of the simplex algorithm.

Definition 8.1: Phase 0 of the simplex algorithm

The **Phase 0** of the simplex algorithm is a procedure which determines a basic feasible solution of the LP problem (8.1), together with the associated dictionary, or proves that the LP problem (8.1) is **infeasible**.

Definition 8.2: simplex auxiliary problem

Given an LP problem (8.1) in standard maximization form, the associated **simplex auxiliary problem** is the LP problem in standard maximization form obtained by adding one non-negative variable x_0 , called the **auxiliary variable**, subtracting it from the left-hand side of every linear constraint, and replacing the objective function:

$$\max \quad -x_0 \quad (8.2a)$$

$$\text{subject to} \quad \sum_{j=1}^{n+m} a_{ij} x_j - x_0 = b_i, \quad \forall i \in \{1, 2, \dots, m\} \quad (8.2b)$$

$$x_0, x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n+m\} \quad (8.2c)$$

- The auxiliary variable x_0 is placed in the first position, and then the input data of the simplex auxiliary problem (8.2) are the column vector $\mathbf{x} \in \mathbb{R}^{n+m+1}$ of the variables, the objective-function vector $\mathbf{c} \in \mathbb{Z}^{n+m+1}$, whose first entry is -1 and whose other entries are 0 , the constraint matrix $\mathbf{A} \in \mathbb{Z}^{m \times (n+m+1)}$, obtained by prepending to the original constraint matrix a column of m entries all equal to -1 , and the right-hand-side vector $\mathbf{b} \in \mathbb{Z}^m$:

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -1 & a_{11} & \dots & a_{1n} & 1 & \dots & 0 \\ -1 & a_{21} & \dots & a_{2n} & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ -1 & a_{m1} & \dots & a_{mn} & 0 & \dots & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The simplex auxiliary problem (8.2) is never infeasible and it is never unbounded:

1. it always admits a feasible solution, obtained by setting $x_j = 0$ for every $j \in \{1, 2, \dots, n\}$, $x_0 = \max\{0, -\min_i b_i\}$ and the slack variables $x_{n+i} = b_i + x_0 \geq 0$ for every $i \in \{1, 2, \dots, m\}$;
2. its objective value is $-x_0 \leq 0$ for every feasible solution.

Accordingly, by the Fundamental Theorem of Linear Programming, it always admits an optimal solution.

Theorem 8.3: feasibility of an LP problem

The LP problem (8.1) admits a feasible solution if and only if the optimal value of the associated simplex auxiliary problem (8.2) is equal to 0.

- The idea of the proof is that a feasible solution of the LP problem (8.1) is exactly a feasible solution of the simplex auxiliary problem (8.2) in which the auxiliary variable takes the value $x_0 = 0$.

Proof. We prove the two implications separately.

($\Rightarrow$) Let $\bar{\mathbf{x}}$ be a feasible solution of the LP problem (8.1). The point obtained by adding $\bar{x}_0 = 0$ is a feasible solution of the simplex auxiliary problem (8.2), since

$$\sum_{j=1}^{n+m} a_{ij} \bar{x}_j - \underbrace{\bar{x}_0}_{=0} = b_i, \quad \forall i \in \{1, 2, \dots, m\},$$

and its objective value is $-\bar{x}_0 = 0$. Since the objective value of every feasible solution of (8.2) is $-x_0 \leq 0$, this solution is optimal and the optimal value is 0.

($\Leftarrow$) Let the optimal value of the simplex auxiliary problem (8.2) be equal to 0 and let $\tilde{\mathbf{x}}$ be an optimal solution. Then $\tilde{x}_0 = 0$ and, by (8.2b),

$$\sum_{j=1}^{n+m} a_{ij} \tilde{x}_j = b_i + \underbrace{\tilde{x}_0}_{=0} = b_i, \quad \forall i \in \{1, 2, \dots, m\}.$$

Accordingly, the point obtained by removing the auxiliary variable satisfies the equality constraints (8.1b) and the non-negativity constraints (8.1c), and then it is a feasible solution of the LP problem (8.1). $\square$

If the optimal value of the simplex auxiliary problem (8.2) is strictly negative, then the LP problem (8.1) is **infeasible**.

Example 8.1: simplex auxiliary problem

Consider the following LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 3$ less-than-or-equal-to linear constraints (called [Example 8.1](#)):

$$\begin{array}{llllll} \max & x_1 & + & 2 & x_2 & \\ \text{subject to} & - & x_1 & - & x_2 & \leq -3 \\ & & - & x_1 & + & x_2 \leq 2 \\ & & & x_1 & + & x_2 \leq 6 \\ & & & x_1, & & x_2 \geq 0 \end{array}$$

By adding $m = 3$ slack variables, we obtain the following LP problem in standard maximization form with $n + m = 5$ non-negative variables:

$$\begin{array}{llllllll}
 \max & x_1 & + & 2 & x_2 & & & \\
 \text{subject to} & -x_1 & - & & x_2 & + & x_3 & = & -3 \\
 & -x_1 & + & & x_2 & & + & x_4 & = & 2 \\
 & x_1 & + & & x_2 & & & + & x_5 & = & 6 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
 \end{array}$$

Since $b_1 = -3 < 0$, the partition $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$ is not associated with a basic feasible solution, and we have to execute the Phase 0.

The associated simplex auxiliary problem with $n + m + 1 = 6$ non-negative variables reads as follows:

$$\begin{array}{llllllllll}
 \max & - & x_0 & & & & & & & \\
 \text{subject to} & - & x_0 & - & x_1 & - & x_2 & + & x_3 & & = & -3 \\
 & - & x_0 & - & x_1 & + & x_2 & & & + & x_4 & = & 2 \\
 & - & x_0 & + & x_1 & + & x_2 & & & & + & x_5 & = & 6 \\
 & x_0, & & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
 \end{array}$$

8.2 Solving the simplex auxiliary problem

- The simplex auxiliary problem (8.2) is solved with the simplex algorithm. Its m slack variables give the partition

$$\mathcal{B} = \{n+1, n+2, \dots, n+m\} \quad \text{and} \quad \mathcal{N} = \{0, 1, 2, \dots, n\},$$

which is associated with the basis $\mathbf{A}_{\mathcal{B}} = \mathbf{I}$ and with the dictionary

$$\begin{cases} x_i = b_i - \sum_{j \in \mathcal{N}} \bar{a}_{ij} x_j, & \forall i \in \mathcal{B} \\ -w = 0 - x_0, \end{cases}$$

where $-w = -x_0$ denotes the objective function of the simplex auxiliary problem (i.e., $w = x_0$). This dictionary is not associated with a basic feasible solution, since $b_i < 0$ for some $i \in \mathcal{B}$.

- A single **special pivoting operation** is enough to obtain a basic feasible solution of the simplex auxiliary problem: the auxiliary variable x_0 enters the basis, and the basic variable leaving the basis is the one with the most negative value.

Special pivoting operation. The pivot column is the auxiliary variable x_0 and the pivot row is the basic variable x_r with

$$r \in \arg \min \{ b_i : i \in \mathcal{B} \}.$$

After this pivoting operation, the values of the variables are

$$x_0 = -b_r \quad \text{and} \quad x_i = b_i - b_r, \quad \forall i \in \mathcal{B} \setminus \{r\},$$

and they are all non-negative, since $b_r \leq b_i$ for every $i \in \mathcal{B}$.

- Starting from the basic feasible solution obtained with the special pivoting operation, the simplex algorithm is executed on the simplex auxiliary problem until an optimal solution is found. Since the simplex auxiliary problem is never unbounded, the algorithm always terminates with an optimal dictionary.

At the end of the Phase 0, exactly one of the following three cases occurs:

1. the optimal value of the simplex auxiliary problem is **strictly negative**: the LP problem (8.1) is **infeasible**;
2. the optimal value is 0 and the auxiliary variable x_0 is **non-basic**: the optimal basis of the simplex auxiliary problem, after removing x_0 from $\mathcal{N}$, is a basis of the LP problem (8.1) associated with a basic feasible solution;
3. the optimal value is 0 and the auxiliary variable x_0 is **basic**, necessarily with value $x_0 = 0$: one further pivoting operation makes x_0 leave the basis, bringing back to the previous case.

- In the third case, the basic feasible solution of the simplex auxiliary problem is **degenerate**, since the basic variable x_0 takes the value 0. A pivoting operation is performed on the row of x_0 , choosing as pivot column any non-basic variable x_j with $j \in \mathcal{N}$ such that $\bar{a}_{0j} \neq 0$: the value of the other basic variables does not change, since the value of x_0 is 0. Such a variable always exists: otherwise, the row of x_0 in $\mathbf{A}_B^{-1}\mathbf{A}$ (without the column of x_0) would be zero, contradicting $\text{rank}(\mathbf{A}) = m$, which holds since $\mathbf{A}$ contains the identity matrix of the slack variables.

Example 8.2: Phase 0 of the simplex algorithm

Consider the simplex auxiliary problem of [Example 8.1](#). With $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{0, 1, 2\}$, the initial dictionary is:

$$\begin{cases} x_3 = -3 + x_0 + x_1 + x_2 \\ x_4 = 2 + x_0 + x_1 - x_2 \\ x_5 = 6 + x_0 - x_1 - x_2 \\ -w = 0 - x_0 \end{cases}$$

The associated basic solution is not feasible, since $\bar{b}_3 = -3 < 0$.

Special pivoting operation. The pivot column is x_0 and the pivot row is x_3 , the basic variable with the most negative value. By isolating x_0 in the first equation we obtain

$x_0 = 3 + x_3 - x_1 - x_2$, and by substituting it in the other equations we obtain the dictionary with $\mathcal{B} = \{0, 4, 5\}$ and $\mathcal{N} = \{1, 2, 3\}$:

$$\begin{cases} x_0 = 3 - x_1 - x_2 + x_3 \\ x_4 = 5 - 2x_2 + x_3 \\ x_5 = 9 - 2x_1 - 2x_2 + x_3 \\ -w = -3 + x_1 + x_2 - x_3 \end{cases}$$

The associated basic feasible solution is $\bar{x}_0 = 3$, $\bar{x}_4 = 5$, $\bar{x}_5 = 9$ and $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 0$, of value $-w = -3$.

First pivoting operation. The reduced cost of x_1 is $\bar{c}_1 = 1 > 0$, and then the pivot column is x_1 . The values of its column are $\bar{a}_{01} = 1$ and $\bar{a}_{51} = 2$, and then

$$t_m = \min \left\{ \frac{3}{1}, \frac{9}{2} \right\} = 3$$

and the pivot row is x_0 . The new dictionary, with $\mathcal{B} = \{1, 4, 5\}$ and $\mathcal{N} = \{0, 2, 3\}$, is:

$$\begin{cases} x_1 = 3 - x_0 - x_2 + x_3 \\ x_4 = 5 - 2x_2 + x_3 \\ x_5 = 3 + 2x_0 - x_3 \\ -w = 0 - x_0 \end{cases}$$

All the reduced costs are non-positive, and then this dictionary is optimal: the optimal value of the simplex auxiliary problem is 0 and the auxiliary variable x_0 is non-basic. Accordingly, the LP problem of [Example 8.1](#) is **feasible** and

$$\mathcal{B} = \{1, 4, 5\}$$

is a basis associated with a basic feasible solution.

8.3 From the Phase 0 to the simplex algorithm

- At the end of the Phase 0, the optimal dictionary of the simplex auxiliary problem gives the equations of the basic variables of the LP problem (8.1) in function of its non-basic variables, once the auxiliary variable x_0 is removed from $\mathcal{N}$.
- The last equation of that dictionary, however, is the one of the objective function $-w$ of the simplex auxiliary problem, and it has to be replaced by the equation of the original objective function z .

The equation of the original objective function is obtained by **substituting**, in

$$z = \sum_{j=1}^{n+m} c_j x_j,$$

the equation of every basic variable x_i with $i \in \mathcal{B}$, so that z is written in function of the non-basic variables only. Equivalently, it is computed as

$$\bar{z} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} \quad \text{and} \quad \bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}}.$$

- Once the initial dictionary of the LP problem (8.1) is available, the simplex algorithm is executed as usual, starting from its Step 1.

Example 8.3: from the Phase 0 to the simplex algorithm

Consider the optimal dictionary of the simplex auxiliary problem of [Example 8.1](#), with $\mathcal{B} = \{1, 4, 5\}$. By removing the auxiliary variable x_0 , the equations of the basic variables of the LP problem are:

$$\begin{cases} x_1 = 3 - x_2 + x_3 \\ x_4 = 5 - 2x_2 + x_3 \\ x_5 = 3 - x_3 \end{cases}$$

The objective function of the LP problem of [Example 8.1](#) is $z = x_1 + 2x_2$, and the basic variable x_1 appears in it. By substituting its equation, we obtain:

$$z = \underbrace{3 - x_2 + x_3}_{= x_1} + 2x_2 = 3 + x_2 + x_3.$$

Accordingly, with $\mathcal{B} = \{1, 4, 5\}$ and $\mathcal{N} = \{2, 3\}$, the initial dictionary of the LP problem of [Example 8.1](#) is:

$$\begin{cases} x_1 = 3 - x_2 + x_3 \\ x_4 = 5 - 2x_2 + x_3 \\ x_5 = 3 - x_3 \\ z = 3 + x_2 + x_3 \end{cases}$$

and the associated basic feasible solution is $\bar{x}_1 = 3$, $\bar{x}_4 = 5$, $\bar{x}_5 = 3$ and $\bar{x}_2 = \bar{x}_3 = 0$, of value 3.

Example 8.4: execution of the simplex algorithm after the Phase 0

Starting from the initial dictionary of [Example 8.1](#) determined by the Phase 0, we execute the simplex algorithm.

First pivoting operation. The reduced cost of x_2 is $\bar{c}_2 = 1 > 0$, and then the pivot column is x_2 . The values of its column are $\bar{a}_{12} = 1$ and $\bar{a}_{42} = 2$, and then

$$t_m = \min \left\{ \frac{3}{1}, \frac{5}{2} \right\} = \frac{5}{2}$$

and the pivot row is x_4 . The new dictionary, with $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$, is:

$$\begin{cases} x_1 = \frac{1}{2} + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_2 = \frac{5}{2} + \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_5 = 3 - x_3 \\ z = \frac{11}{2} + \frac{3}{2}x_3 - \frac{1}{2}x_4 \end{cases}$$

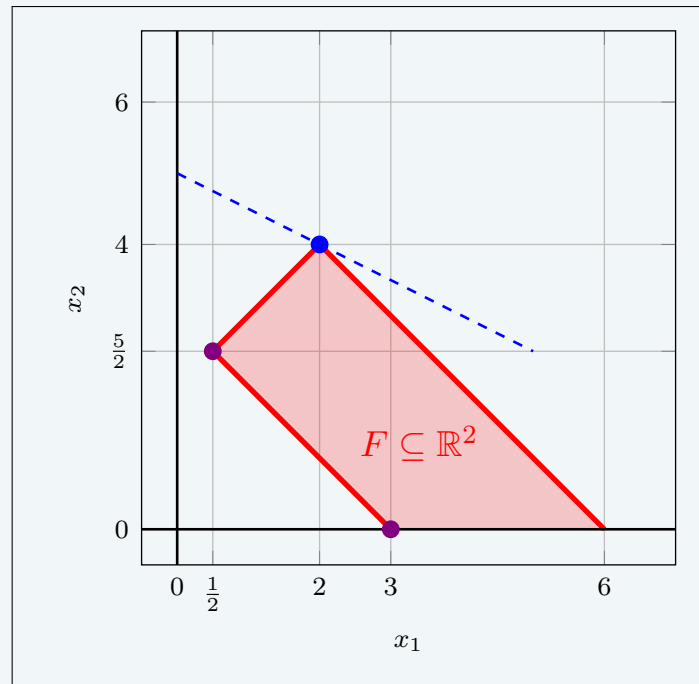
Second pivoting operation. The reduced cost of x_3 is $\bar{c}_3 = \frac{3}{2} > 0$, and then the pivot column is x_3 . The only positive value in its column is $\bar{a}_{53} = 1$, and then $t_m = \frac{3}{1} = 3$ and the pivot row is x_5 . The new dictionary, with $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$, is:

$$\begin{cases} x_1 = 2 + \frac{1}{2}x_4 - \frac{1}{2}x_5 \\ x_2 = 4 - \frac{1}{2}x_4 - \frac{1}{2}x_5 \\ x_3 = 3 - x_5 \\ z = 10 - \frac{1}{2}x_4 - \frac{3}{2}x_5 \end{cases}$$

All the reduced costs are non-positive, and then this dictionary is optimal. An optimal solution of the LP problem of [Example 8.1](#) is

$$\tilde{x}(LP) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

and the optimal value is $z(LP) = 10$.



The feasible region F is bounded by the lines $-x_1 - x_2 = -3$ (bottom left), $-x_1 + x_2 = 2$ (top left), $x_1 + x_2 = 6$ (right) and the x_1 -axis; in dashed blue the level set $x_1 + 2x_2 = 10$ through the optimal vertex. In violet the basic feasible solution $(3,0)$ determined by the Phase 0 and the vertex $(\frac{1}{2}, \frac{5}{2})$ reached after the first pivoting operation, in blue the optimal solution reached by the simplex algorithm.

8.4 Detecting infeasible LP problems

- If the optimal value of the simplex auxiliary problem (8.2) is strictly negative, then, by Theorem 8.3, the LP problem (8.1) is infeasible. In this case the optimal dictionary of the simplex auxiliary problem is a **certificate of infeasibility**.

Example 8.5: infeasible LP problem

Consider the following LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 2$ less-than-or-equal-to linear constraints (called [Example 8.5](#)):

$$\begin{array}{llllll} \max & x_1 & + & x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \leq & 2 \\ & -x_1 & - & x_2 & \leq & -4 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

Since $b_2 = -4 < 0$, we execute the Phase 0. With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{0, 1, 2\}$, the initial dictionary of the simplex auxiliary problem is:

$$\begin{cases} x_3 = 2 + x_0 - x_1 - x_2 \\ x_4 = -4 + x_0 + x_1 + x_2 \\ -w = 0 - x_0 \end{cases}$$

Special pivoting operation. The pivot column is x_0 and the pivot row is x_4 , the basic variable with the most negative value. The new dictionary, with $\mathcal{B} = \{0, 3\}$ and $\mathcal{N} = \{1, 2, 4\}$, is:

$$\begin{cases} x_0 = 4 - x_1 - x_2 + x_4 \\ x_3 = 6 - 2x_1 - 2x_2 + x_4 \\ -w = -4 + x_1 + x_2 - x_4 \end{cases}$$

First pivoting operation. The pivot column is x_1 , since $\bar{c}_1 = 1 > 0$. The values of its column are $\bar{a}_{01} = 1$ and $\bar{a}_{31} = 2$, and then $t_m = \min \left\{ \frac{4}{1}, \frac{6}{2} \right\} = 3$ and the pivot row is x_3 . The new dictionary, with $\mathcal{B} = \{0, 1\}$ and $\mathcal{N} = \{2, 3, 4\}$, is:

$$\begin{cases} x_0 = 1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_1 = 3 - x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ -w = -1 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \end{cases}$$

All the reduced costs are non-positive, and then this dictionary is optimal. The optimal value of the simplex auxiliary problem is $-1 < 0$, that is, the auxiliary variable takes the value $\tilde{x}_0 = 1$ in every optimal solution. Accordingly, the LP problem of [Example 8.5](#) is **infeasible**.

8.5 Pseudo-code of the simplex algorithm with the Phase 0

Algorithm

Input

An LP problem (8.1).

Output

An optimal solution of the LP problem (8.1), if one exists; otherwise, a certificate that the LP problem is infeasible or unbounded.

Step 0: Initialization

If $b_i \geq 0$ for all $i \in \{1, 2, \dots, m\}$, then set

$$\mathcal{B} \leftarrow \{n+1, n+2, \dots, n+m\}, \quad \mathcal{N} \leftarrow \{1, 2, \dots, n\},$$

together with the associated dictionary, and go to Step 1.

Otherwise, execute the **Phase 0**:

- a. build the simplex auxiliary problem (8.2) and its initial dictionary with $\mathcal{B} \leftarrow \{n+1, \dots, n+m\}$ and $\mathcal{N} \leftarrow \{0, 1, \dots, n\}$;
- b. perform the **special pivoting operation**: the pivot column is x_0 and the pivot row is x_r with $r \in \arg \min \{b_i : i \in \mathcal{B}\}$;
- c. execute Step 1 and Step 2 on the simplex auxiliary problem until an optimal dictionary is found;
- d. if the optimal value is strictly negative, then the LP problem (8.1) is **infeasible** and **STOP**;
- e. if the auxiliary variable x_0 is basic, perform one pivoting operation on its row to make it leave the basis;
- f. remove x_0 from $\mathcal{N}$ and replace the equation of $-w$ with the equation of the original objective function z , written in function of the non-basic variables.

Go to Step 1.

Step 1: Optimality check

With respect to the current dictionary, if $\bar{c}_j \leq 0$ for all $j \in \mathcal{N}$, then the basic feasible solution $x_i = \bar{b}_i, \forall i \in \mathcal{B}$, and $x_j = 0, \forall j \in \mathcal{N}$, is an optimal solution and $\bar{z}$ is the optimal value. In this case **STOP**.

Step 2: Pivoting operations

The **pivot column** is a non-basic variable x_p with $p \in \mathcal{N}$ such that $\bar{c}_p > 0$. If $\bar{a}_{ip} \leq 0$ for all $i \in \mathcal{B}$, then the LP problem (8.1) is **unbounded** and **STOP**. Otherwise compute

$$t_m \leftarrow \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ip}} : i \in \mathcal{B}, \bar{a}_{ip} > 0 \right\},$$

determine the **pivot row** x_r with $\bar{a}_{rp} > 0$ and $\bar{b}_r - t_m \bar{a}_{rp} = 0$, update the dictionary and set $\mathcal{B} \leftarrow (\mathcal{B} \setminus \{r\}) \cup \{p\}$ and $\mathcal{N} \leftarrow (\mathcal{N} \setminus \{p\}) \cup \{r\}$. Go to Step 1.

8.6 The Fundamental Theorem of Linear Programming

- The Phase 0 and the simplex algorithm, executed one after the other, prove the following theorem, which describes all the possible outcomes of an LP problem and shows that, whenever an optimal solution exists, one of them is a basic feasible solution.

Theorem 8.4: Fundamental Theorem of Linear Programming

For every LP problem (8.1) in standard maximization form:

1. if it admits a feasible solution, then it admits a **basic** feasible solution;
2. if it admits an optimal solution, then it admits a **basic** feasible solution which is optimal;
3. it is **infeasible**, it is **unbounded**, or it admits an **optimal solution**.

- The idea of the proof is that the Phase 0 and the simplex algorithm, executed with an anti-cycling pivot rule, terminate after a finite number of pivoting operations, and that each of their possible outcomes is one of the three cases of the statement.

Proof. We execute the Phase 0 on the LP problem (8.1), and then the simplex algorithm with Bland's pivot rule. Both terminate after a finite number of pivoting operations, since the number of bases is finite and Bland's pivot rule prevents cycling.

Point 1. If the LP problem (8.1) admits a feasible solution, then, by Theorem 8.3, the optimal value of the simplex auxiliary problem is 0. The Phase 0 terminates with an optimal dictionary of the simplex auxiliary problem in which, possibly after one further pivoting operation, the auxiliary variable x_0 is non-basic. The associated partition $\mathcal{B}$ and $\mathcal{N}$ is then a basis of the LP problem (8.1) whose basic solution $\mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}} \geq \mathbf{0}$ is feasible.

Point 3. If the LP problem (8.1) is not infeasible, by Point 1 the simplex algorithm can be started from a basic feasible solution. It terminates in one of two ways:

- (i) with a dictionary in which $\bar{c}_j \leq 0$ for every $j \in \mathcal{N}$: by the sufficient conditions for optimality, the associated basic feasible solution is an optimal solution;
- (ii) with a non-basic variable x_p such that $\bar{c}_p > 0$ and $\bar{a}_{ip} \leq 0$ for every $i \in \mathcal{B}$: by the certificate of unboundedness, the LP problem is unbounded.

Accordingly, the LP problem is infeasible, unbounded, or it admits an optimal solution, and the three cases are mutually exclusive.

Point 2. If the LP problem (8.1) admits an optimal solution, then it is neither infeasible nor unbounded, and then the simplex algorithm terminates in case (i) above, with an optimal solution which is a basic feasible solution. $\square$

The Fundamental Theorem of Linear Programming has an important algorithmic consequence: to solve an LP problem it is enough to look for an optimal solution among its **basic** feasible solutions, whose number is finite and at most

$$\binom{n+m}{m}.$$

The simplex algorithm explores a small subset of them, moving from one basic feasible solution to another one whose objective value is not smaller.