



SAPIENZA
UNIVERSITÀ DI ROMA

Linear Programming

Volume 1 — Linear functions, polyhedra and linear programming problems

Linear functions (gradient, Hessian and level sets), half-spaces and polyhedra, linear optimization problems with their optimal solutions, examples and properties, the canonical and standard forms with the graphical method, possible outcomes and equivalences between forms.

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Linear functions

1.1 Linear functions

Definition 1.1: linear function

Given a (column) vector

$$\underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}} \in \mathbb{R}^n$$

the associated (scalar) function

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad f : \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \mapsto \underbrace{\sum_{j=1}^n c_j x_j}_{\mathbf{c}'\mathbf{x}} \quad (1.1)$$

is called a **linear function**.

Definition 1.2: gradient of a linear function

The **gradient** (vector of the partial derivatives) of the linear function $f(\mathbf{x}) = \mathbf{c}'\mathbf{x}$ is:

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} \\ \frac{\partial f(\mathbf{x})}{\partial x_2} \\ \vdots \\ \frac{\partial f(\mathbf{x})}{\partial x_n} \end{pmatrix} = \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}}$$

Definition 1.3: Hessian of a linear function

The **Hessian** (matrix of the second partial derivatives) of the linear function $f(\mathbf{x}) = \mathbf{c}'\mathbf{x}$ is the $n \times n$ matrix:

$$\nabla^2 f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_n} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

since every first partial derivative $\frac{\partial f(\mathbf{x})}{\partial x_j} = c_j$ is a constant.

Definition 1.4: level sets of a linear function

The **level sets** of the linear function $f(\mathbf{x}) = \mathbf{c}'\mathbf{x}$ (the sets of points in which the function is **constant**) are:

$$\left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{c}'\mathbf{x} = k \right\}, \quad \forall k \in \mathbb{R}$$

- The gradient of a linear function is the same vector $\mathbf{c}$ at every point and, accordingly, its **Hessian** (matrix of the second partial derivatives) is the zero matrix everywhere.
- The level sets of a linear function are the **hyperplanes** perpendicular to the gradient.

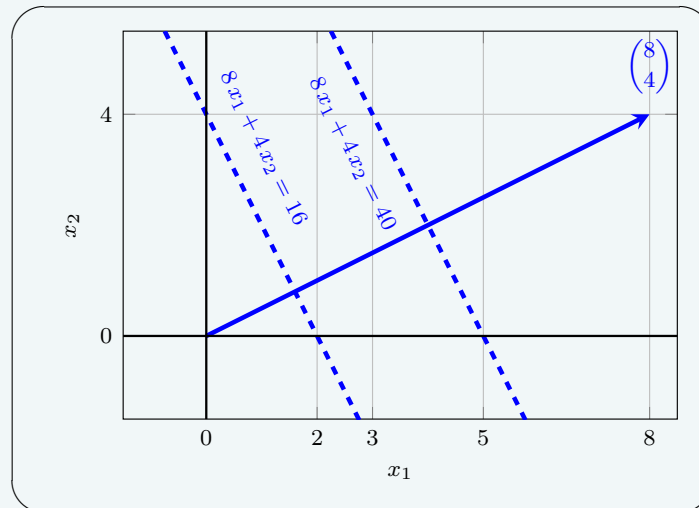
Example 1.1: gradient and level sets of linear functions

With $\mathbf{c} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$, the associated linear function is:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 8 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 8x_1 + 4x_2$$

Gradient:

$$\nabla f \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$



The figure shows:

- The gradient evaluated at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\nabla f\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$ (blue arrow)
- Two level sets (blue dashed lines): $8x_1 + 4x_2 = 16$ and $8x_1 + 4x_2 = 40$

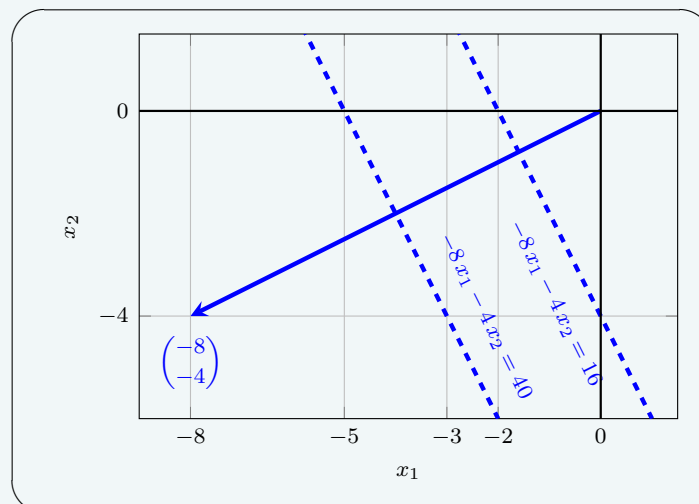
Example 1.2: gradient and level sets of linear functions

With $\mathbf{c} = \begin{pmatrix} -8 \\ -4 \end{pmatrix}$, the associated linear function is:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -8 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -8x_1 - 4x_2$$

Gradient:

$$\nabla f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} -8 \\ -4 \end{pmatrix}$$

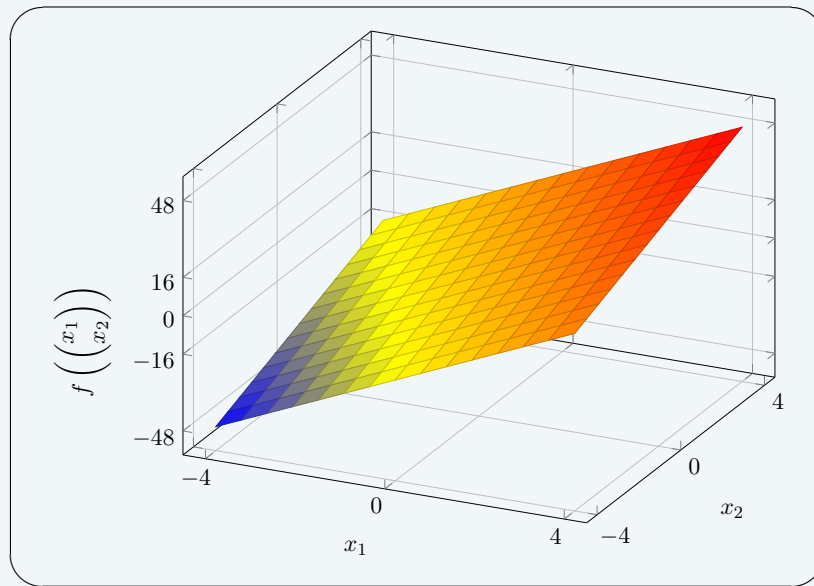


The figure shows:

- The gradient evaluated at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\nabla f \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} -8 \\ -4 \end{pmatrix}$ (blue arrow)
- Two level sets (blue dashed lines): $-8x_1 - 4x_2 = 16$ and $-8x_1 - 4x_2 = 40$

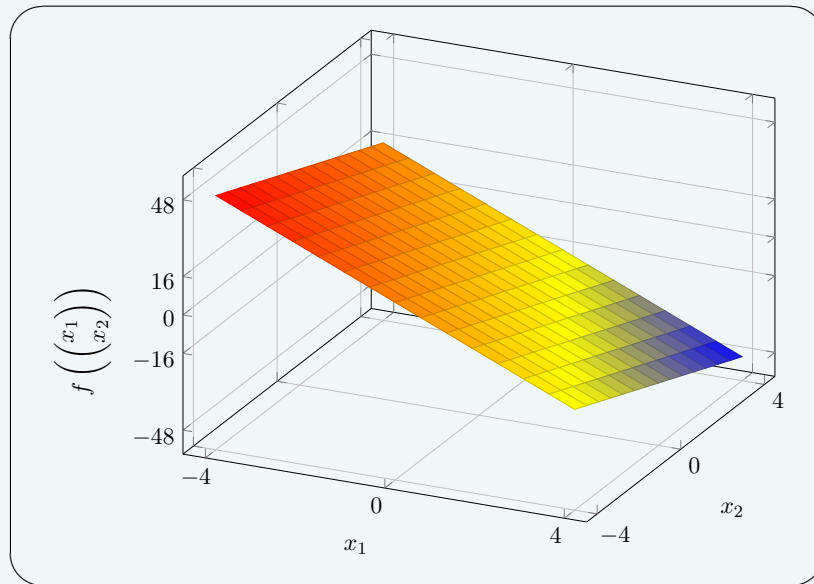
Example 1.3: graphs of linear functions

With $\mathbf{c} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$, we have $f \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = 8x_1 + 4x_2$. Its graph is:



Example 1.4: graphs of linear functions

With $\mathbf{c} = \begin{pmatrix} -8 \\ -4 \end{pmatrix}$, we have $f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = -8x_1 - 4x_2$. Its graph is:



Half-spaces and polyhedra

2.1 Half-spaces and supporting hyperplanes

Definition 2.1: half-space

Given a (row) vector

$$\underbrace{(a_1 \ a_2 \ \dots \ a_n)}_a \in \mathbb{R}^{1 \times n}$$

and a scalar $b \in \mathbb{R}$, the set

$$\left\{ \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_x \in \mathbb{R}^n : \sum_{j=1}^n a_j x_j \leq b \right\}$$

$\underbrace{\hspace{10em}}_{a \ x \leq b}$

is called an **half-space**.

- Given an half-space, the set

$$\left\{ \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_x \in \mathbb{R}^n : \sum_{j=1}^n a_j x_j = b \right\}$$

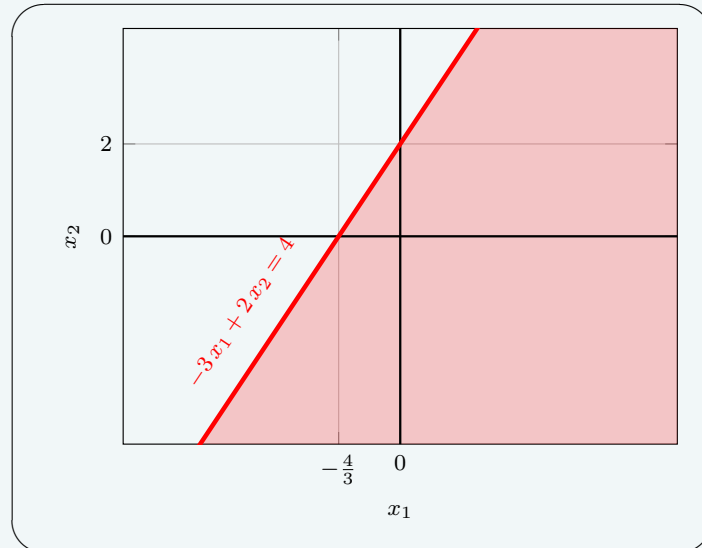
$\underbrace{\hspace{10em}}_{a \ x = b}$

is called the **supporting hyperplane** of the half-space.

Example 2.1: half-space and supporting hyper-plane

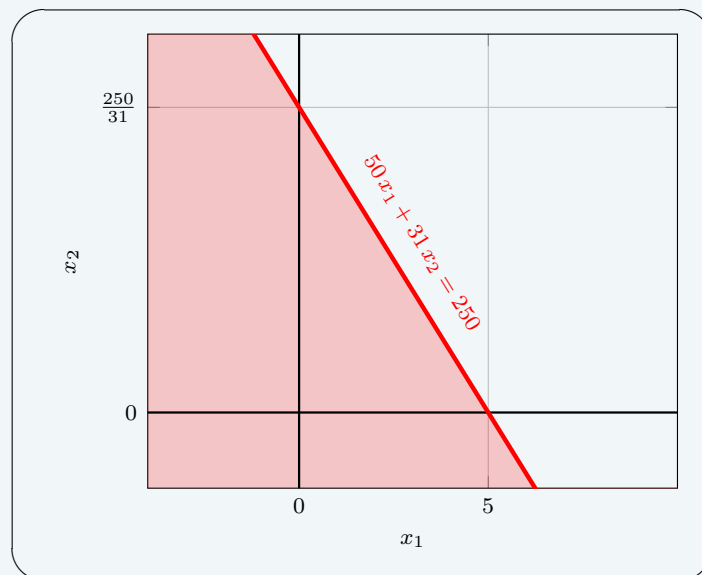
Let us consider the following half-space in $\mathbb{R}^2$:

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -3x_1 + 2x_2 \leq 4 \right\} \quad \text{where } \mathbf{a} = \begin{pmatrix} -3 & 2 \end{pmatrix} \text{ and } b = 4$$



Let us consider the following half-space in $\mathbb{R}^2$:

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : 50x_1 + 31x_2 \leq 250 \right\} \quad \text{where } \mathbf{a} = \begin{pmatrix} 50 & 31 \end{pmatrix} \text{ and } b = 250$$



2.2 Polyhedra

Definition 2.2: polyhedron

Given a matrix

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_A \in \mathbb{R}^{m \times n}$$

and a (column) vector

$$\underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_b \in \mathbb{R}^m$$

the set

$$\left\{ \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_x \in \mathbb{R}^n : \underbrace{\sum_{j=1}^n a_{ij} x_j \leq b_i}_{A x \leq b}, \quad \forall i \in \{1, 2, \dots, m\} \right\}$$

is called a **polyhedron**.

Definition 2.3: bounded polyhedron

A polyhedron $P \subseteq \mathbb{R}^n$ is **bounded** if there exists $\underbrace{\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}}_{=\beta} \in \mathbb{R}_{\geq 0}^n$ such that

$$\underbrace{-\beta_j \leq x_j \leq \beta_j}_{|x_j| \leq \beta_j}, \quad \forall x \in P, \quad \forall j \in \{1, 2, \dots, n\}. \quad (2.1)$$

Here, β_j is an **upper bound**, while $-\beta_j$ is a **lower bound** of P along dimension $j \in \{1, 2, \dots, n\}$.

Definition 2.4: polytope

A bounded polyhedron is called a **polytope**.

Example 2.2: polyhedron

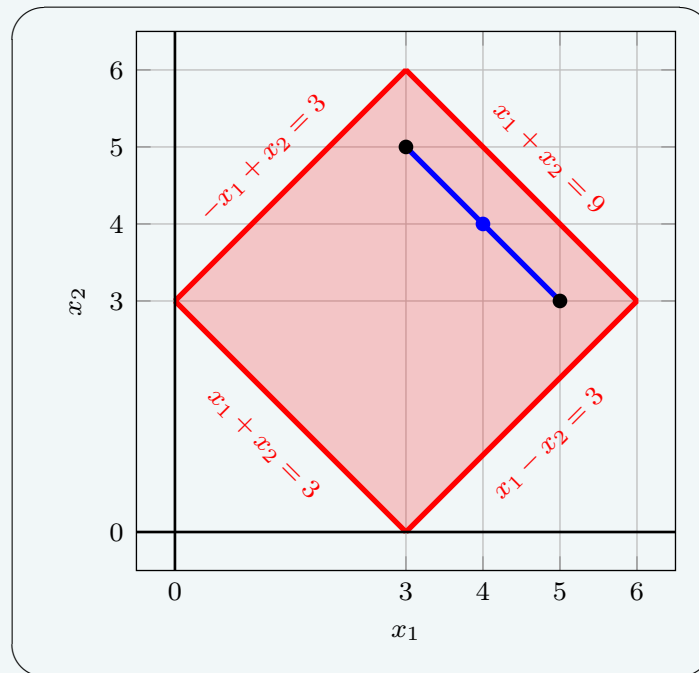
Consider the following polyhedron defined as the intersection of four half-spaces:

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -x_1 - x_2 \leq -3, \quad x_1 + x_2 \leq 9, \quad -x_1 + x_2 \leq 3, \quad x_1 - x_2 \leq 3 \right\}$$

For example, consider two points, $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$, in the set P . All points on the blue segment in the figure, given by:

$$\lambda \begin{pmatrix} 3 \\ 5 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 - 2\lambda \\ 3 + 2\lambda \end{pmatrix} \quad \text{for } \lambda \in [0, 1],$$

also belong to P . Indeed, for every $\lambda \in [0, 1]$ we have $x_1 + x_2 = 8$ and $x_1 - x_2 = 2 - 4\lambda \in [-2, 2]$, so that all four inequalities defining P are satisfied.



For example, with $\lambda = \frac{1}{2}$, the point of the segment is the one shown in blue

$$\begin{pmatrix} \frac{1}{2} \cdot 3 + (1 - \frac{1}{2}) \cdot 5 \\ \frac{1}{2} \cdot 5 + (1 - \frac{1}{2}) \cdot 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

This polyhedron P is also polytope, since it is a bounded polyhedron in $\mathbb{R}^2$. Indeed, by setting $\beta = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$ we have

$$|x_1| \leq 6 \quad \text{and} \quad |x_2| \leq 6 \quad \text{for all} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in P$$

- It is worth noticing that if a polyhedron contains the following two half spaces:

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \mathbf{x} \leq b\} \quad \text{and} \quad \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \mathbf{x} \geq b\}$$

it contains, *de facto*, the following hyperplanes:

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \mathbf{x} = b\}$$

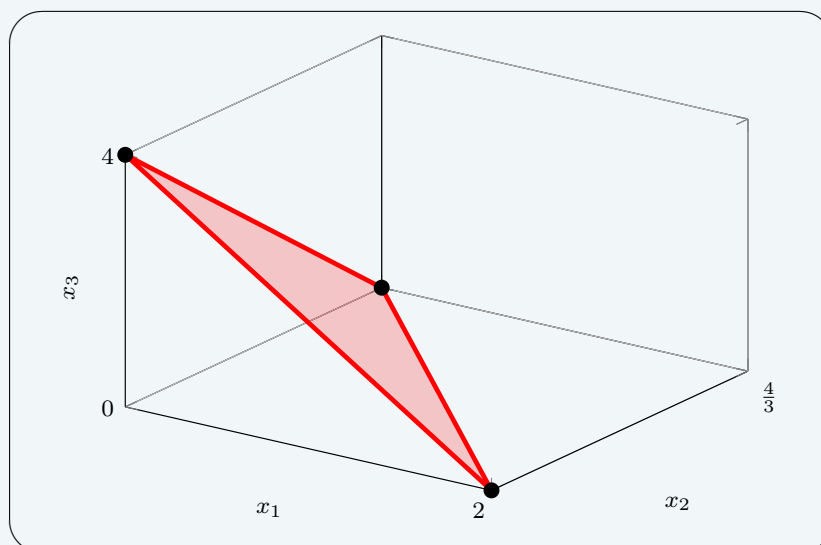
In general, a polyhedron is given by the intersection of half-spaces and hyperplanes.

Example 2.3: polyhedron

Consider the following polyhedron given by the intersection of three half-spaces and one hyperplane:

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, 2x_1 + 3x_2 + x_3 = 4 \right\}$$

Its three-dimensional representation corresponds to the hyperplane shown in the figure.



Linear optimization problems

3.1 Linear Programming problems

Definition 3.1: Linear Programming problem in minimization and maximization form

Given a polyhedron $F \subseteq \mathbb{R}^n$ and a linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the associated **Linear Programming problem** LP consists of,

- in the **minimization form**, finding a point $\tilde{x}(LP) \in F$ such that

$$f(\tilde{x}(LP)) \leq f(x), \quad \forall x \in F;$$

- in the **maximization form**, finding a point $\tilde{x}(LP) \in F$ such that

$$f(\tilde{x}(LP)) \geq f(x), \quad \forall x \in F.$$

3.1.1 Feasible region and objective function

- The polyhedron F is called the **feasible region** of the LP problem; its points $x \in F$ are called the **feasible solutions** of the LP problem.
- The linear function f is the **objective function** of the LP problem; it assigns an objective value to each feasible solution.
- To define the polyhedron of the feasible region of an LP problem, we need the number m of its inequalities/equalities and
 1. a matrix $A \in \mathbb{R}^{m \times n}$ — the **constraint matrix**,
 2. a (column) vector $b \in \mathbb{R}^m$ — the **right-hand-side (RHS) vector**.

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_A \in \mathbb{R}^{m \times n} \quad \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_b \in \mathbb{R}^m$$

Moreover, for each row we need to specify whether it corresponds to a half-space (linear inequality) or a hyperplane (linear equality).

- To define the linear objective function of an LP problem, we need the (column) vector $\mathbf{c} \in \mathbb{R}^n$, called the **objective-function vector**:

$$\underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}} \in \mathbb{R}^n.$$

The **input data** of an LP problem are the constraint matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, the right-hand-side vector $\mathbf{b} \in \mathbb{R}^m$, the objective-function vector $\mathbf{c} \in \mathbb{R}^n$, and, for each row, the information specifying whether the corresponding constraint is an equality or an inequality.

3.1.2 Optimal solutions and optimal value

Definition 3.2: optimal solution and optimal value of an LP problem

- A point $\tilde{\mathbf{x}}(LP) \in F$ as in the definition of the LP problem is called an **optimal solution** of the LP problem.

1. In the minimization form,

$$\tilde{\mathbf{x}}(LP) \in \arg \min \{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \}.$$

2. In the maximization form,

$$\tilde{\mathbf{x}}(LP) \in \arg \max \{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \}.$$

- The value $f(\tilde{\mathbf{x}}(LP))$ is called the **optimal value** of the LP problem, and it is denoted by $z(LP)$.

1. In the minimization form,

$$z(LP) = \underbrace{\min \{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \}}_{=f(\tilde{\mathbf{x}}(LP))}.$$

2. In the maximization form,

$$z(LP) = \underbrace{\max \{ f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F \}}_{=f(\tilde{\mathbf{x}}(LP))}.$$

- An LP problem may admit multiple (possibly infinitely many) optimal solutions, or none at all. In such cases, the operators $\arg \min$ and $\arg \max$ return the full set of optimal solutions when they exist, and return the empty set otherwise.
- With a slight abuse of notation:

1. In the **minimization form**, we write the LP problem as

$$\min\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F\}.$$

2. In the **maximization form**, we write the LP problem as

$$\max\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F\}.$$

Although we use the symbols “min” and “max”, the goal of an LP problem is to **find an optimal solution** $\tilde{\mathbf{x}}(LP)$, whenever it exists, rather than merely determining the optimal value $z(LP)$.

- In the **minimization form**, an optimal solution is a **point of minimum**, and the corresponding optimal value is the **minimum** of the objective function over the feasible region. Similarly, in the **maximization form**, an optimal solution is a **point of maximum**, and the corresponding optimal value is the **maximum** of the objective function over the feasible region.

3.1.3 LP problems without an optimal solution

- An optimal solution of an LP problem may not exist for one of the following two reasons:
 1. If the objective function decreases indefinitely in the minimization form or increases indefinitely in the maximization form, the LP problem is said to be **unbounded**. In these cases, we write $z(LP) = -\infty$ and $z(LP) = +\infty$, respectively.
 2. If the feasible region is empty, meaning that no feasible solutions exist, the LP problem is said to be **infeasible**. It has no optimal solution and $z(LP)$ is not defined.
- There is no third reason: if an LP problem has feasible solutions and it is not unbounded, the infimum (in the minimization form) or the supremum (in the maximization form) of the objective function over the feasible region is always **attained** by a feasible solution. This is due to the special structure of LP problems: the objective function is linear and the feasible region is a polyhedron, a **closed** set defined by linear inequalities " $\leq$ ", " $\geq$ " and linear equalities "=", never by strict inequalities "<", ">". For example, the interval $(0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$ is not a polyhedron, and $\min\{x \text{ subject to } 0 < x \leq 1\}$ is not an LP problem: its infimum 0 is not attained.
- The three possible outcomes of an LP problem (infeasible, unbounded, optimal solution) are formally defined and studied in the chapter «Possible outcomes and equivalent formulations».

Example 3.1: LP problems without an optimal solution

- Consider the objective function $f(x) = 2x + 1$ and the feasible region $F = \{x \in \mathbb{R} : x \geq -2\}$, a half-line. In the minimization form, $2x + 1 \geq -3$ for every feasible x , with equality only for $x = -2$: the unique optimal solution is $\tilde{x}(LP) = (-2)$ and $z(LP) = -3$. In the maximization form, for every value $M \geq 0$ the feasible solution $x = M$ has $f(M) = 2M + 1 > M$: the objective function increases indefinitely and the problem is **unbounded**, $z(LP) = +\infty$.
- Consider the feasible region $F = \{x \in \mathbb{R} : x \leq -1, x \geq 1\}$. No real number satisfies both inequalities, so $F = \emptyset$: for every linear objective function, both the minimization and the maximization form are **infeasible**.

3.1.4 Feasible solutions and bounds on the optimal value

- Given

$$\underbrace{\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix}}_{\bar{\mathbf{x}}(LP)} \in F$$

a **feasible solution** of an LP problem, we have

1. in **minimization** form, the following **upper bound**:

$$z(LP) \leq \underbrace{f(\bar{\mathbf{x}}(LP))}_{UB(\bar{\mathbf{x}})}.$$

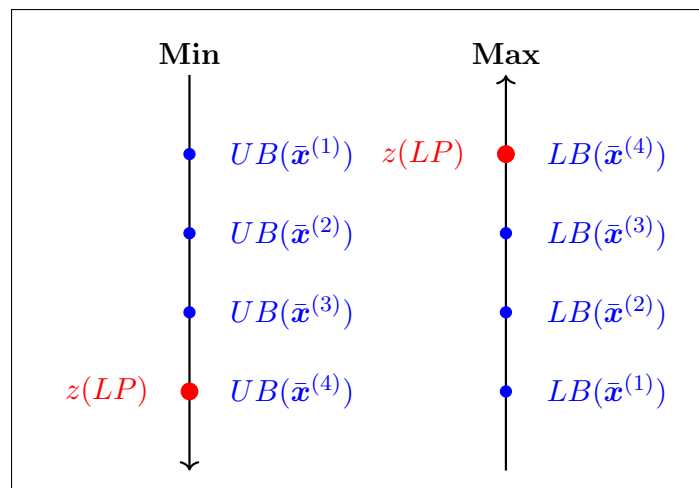
2. in **maximization** form, the following **lower bound**:

$$\underbrace{f(\bar{\mathbf{x}}(LP))}_{LB(\bar{\mathbf{x}})} \leq z(LP).$$

As an illustrative example, the figure below considers four feasible solutions

$$\bar{\mathbf{x}}^{(1)}, \bar{\mathbf{x}}^{(2)}, \bar{\mathbf{x}}^{(3)}, \bar{\mathbf{x}}^{(4)} \in F$$

and shows how they provide lower bounds in the maximization case and upper bounds in the minimization case for the optimal value $z(LP)$ (the fourth one attains the optimal value).



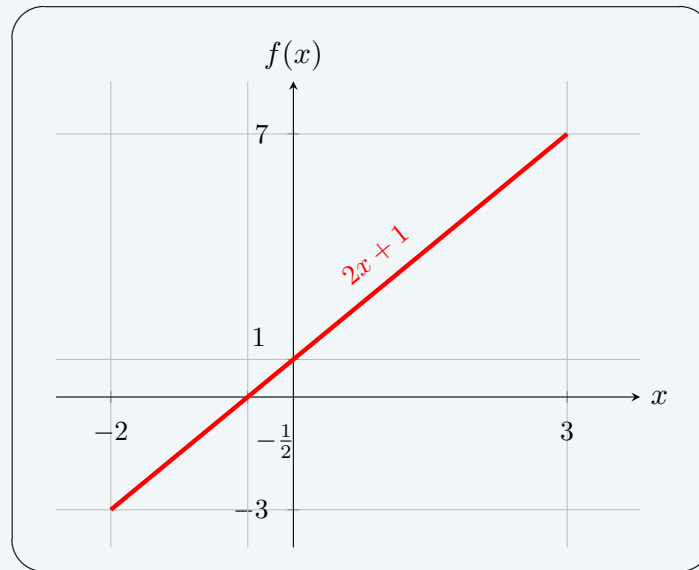
3.2 Examples and properties of linear optimization problems

- The following example is a linear optimization problem in one variable: the feasible region is an interval (a polyhedron in $\mathbb{R}$) and the objective function is linear.

Example 3.2: LP problem in one variable

Let us consider the following objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto 2x + 1 \quad \text{and} \quad F = [-2, 3]$$



The associated LP problems in minimization and maximization forms are:

$$\min \left\{ 2x + 1 \text{ subject to } -2 \leq x \leq 3 \right\}$$

$$\max \left\{ 2x + 1 \text{ subject to } -2 \leq x \leq 3 \right\}$$

- For the minimization form, since the linear function is strictly increasing, the unique optimal solution and the optimal value are:

$$\tilde{x}(LP) = (-2) \quad \text{and} \quad z(LP) = -3.$$

- For the maximization form, since the linear function is strictly increasing, the unique optimal solution and the optimal value are:

$$\tilde{x}(LP) = (3) \quad \text{and} \quad z(LP) = 7.$$

The derivative function is constant:

$$f' : \mathbb{R} \rightarrow \mathbb{R}, f' : x \mapsto 2$$

$$f'(x) = 2 > 0 \quad \forall x \in \mathbb{R},$$

3.2.1 Properties of LP problems

- The following properties hold for every LP problem with feasible region F and objective function $f(\mathbf{x}) = \mathbf{c}'\mathbf{x}$.
- We state them using min and max for LP problems that admit an optimal solution. If one of the two LP problems in an equation is infeasible (unbounded), so is the other one.
- For every LP problem, we have:

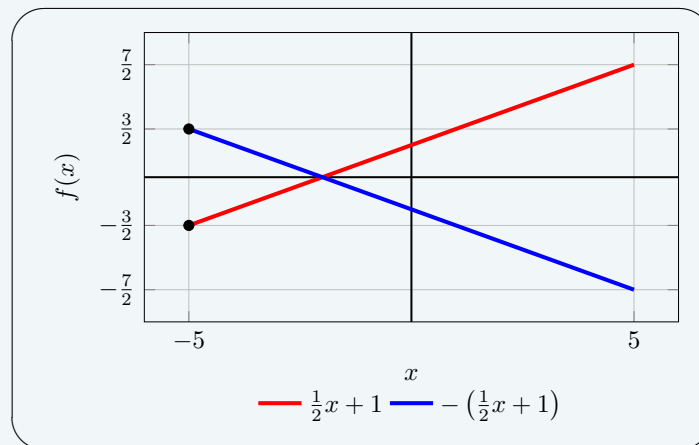
$$\min \left\{ \mathbf{c}'\mathbf{x} \text{ subject to } \mathbf{x} \in F \right\} = - \max \left\{ -\mathbf{c}'\mathbf{x} \text{ subject to } \mathbf{x} \in F \right\} \quad (3.1)$$

- The optimal solutions do not change: a point minimizes $\mathbf{c}'\mathbf{x}$ over F if and only if it maximizes $-\mathbf{c}'\mathbf{x}$ over F .

Example 3.3: property: minimum and maximum

$$\begin{aligned} & \min \left\{ \frac{1}{2}x + 1 \text{ subject to } -5 \leq x \leq 5 \right\} \\ & \quad \quad \quad = -\frac{3}{2} \\ & = - \max \left\{ -\left(\frac{1}{2}x + 1\right) \text{ subject to } -5 \leq x \leq 5 \right\} \\ & \quad \quad \quad = \frac{3}{2} \end{aligned}$$

Both optima are attained at $x = -5$ (black points in the figure).



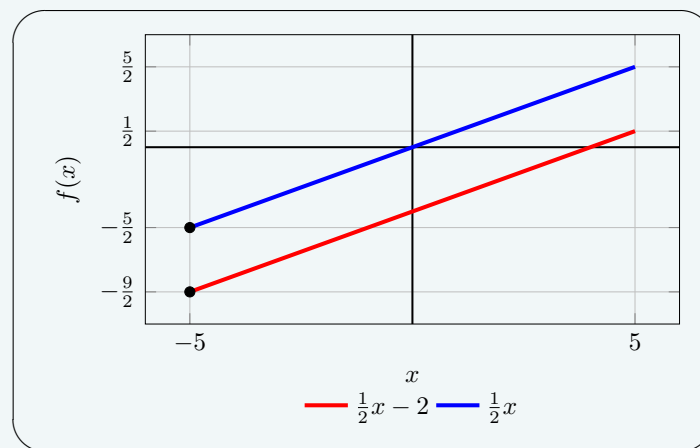
- For every LP problem and every $a \in \mathbb{R}$, we have:

$$\min \left\{ a + \mathbf{c}'\mathbf{x} \text{ subject to } \mathbf{x} \in F \right\} = a + \min \left\{ \mathbf{c}'\mathbf{x} \text{ subject to } \mathbf{x} \in F \right\} \quad (3.2)$$

- Adding a constant moves the graph of the objective function up or down: the optimal solutions do not change.

Example 3.4: property: constant addition

$$\underbrace{\min \left\{ \frac{1}{2}x - 2 \text{ subject to } -5 \leq x \leq 5 \right\}}_{=-\frac{9}{2}} = -2 + \underbrace{\min \left\{ \frac{1}{2}x \text{ subject to } -5 \leq x \leq 5 \right\}}_{=-\frac{5}{2}}$$



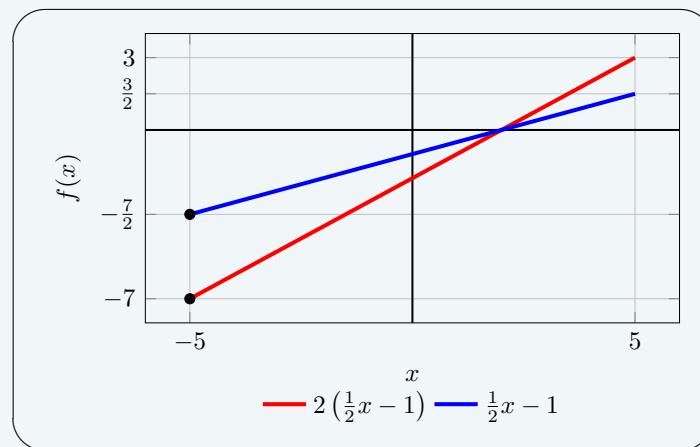
- For every LP problem and every $a \in \mathbb{R}_{>0}$, we have:

$$\min \left\{ a \mathbf{c}'\mathbf{x} \text{ subject to } \mathbf{x} \in F \right\} = a \min \left\{ \mathbf{c}'\mathbf{x} \text{ subject to } \mathbf{x} \in F \right\} \quad (3.3)$$

- Multiplying by a positive constant stretches (or compresses) the graph vertically: the optimal solutions do not change.

Example 3.5: property: multiplication by a positive constant

$$\underbrace{\min \left\{ 2 \left(\frac{1}{2}x - 1 \right) \text{ subject to } -5 \leq x \leq 5 \right\}}_{=-7} = 2 \underbrace{\min \left\{ \frac{1}{2}x - 1 \text{ subject to } -5 \leq x \leq 5 \right\}}_{=-\frac{7}{2}}$$



- For every LP problem and every $a \in \mathbb{R}_{<0}$, we have:

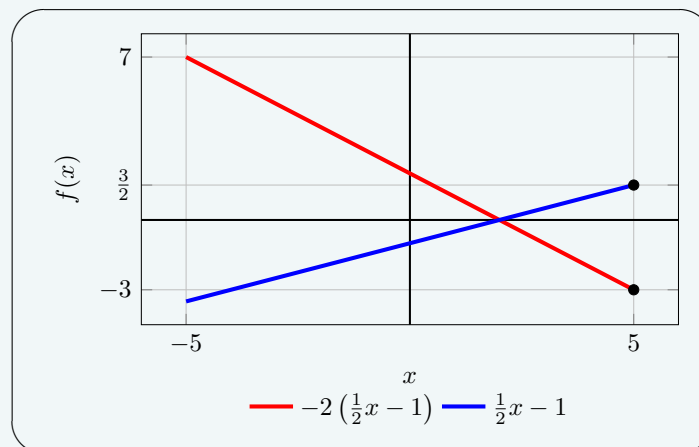
$$\min \left\{ a \mathbf{c}'\mathbf{x} \text{ subject to } \mathbf{x} \in F \right\} = a \max \left\{ \mathbf{c}'\mathbf{x} \text{ subject to } \mathbf{x} \in F \right\} \quad (3.4)$$

- Multiplying by a negative constant turns the graph upside down: the points of minimum of $a \mathbf{c}'\mathbf{x}$ are the points of maximum of $\mathbf{c}'\mathbf{x}$.

Example 3.6: property: multiplication by a negative constant

$$\begin{aligned} & \underbrace{\min \left\{ -2 \left(\frac{1}{2}x - 1 \right) \text{ subject to } -5 \leq x \leq 5 \right\}}_{=-3} \\ &= -2 \underbrace{\max \left\{ \frac{1}{2}x - 1 \text{ subject to } -5 \leq x \leq 5 \right\}}_{=\frac{3}{2}} \end{aligned}$$

The minimum of $-2 \left(\frac{1}{2}x - 1 \right)$ and the maximum of $\frac{1}{2}x - 1$ are both attained at $x = 5$.



Linear Programming problems

4.1 Canonical maximization form

Definition 4.1: Linear Programming problems in canonical maximization form

Given:

1. an objective-function vector $\mathbf{c} \in \mathbb{R}^n$
2. a constraint matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$
3. a right-hand-side (RHS) vector $\mathbf{b} \in \mathbb{R}^m$ with m rows

$$\underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}} \in \mathbb{R}^n \quad \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{pmatrix}}_{\mathbf{A}} \in \mathbb{R}^{m \times n} \quad \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\mathbf{b}} \in \mathbb{R}^m$$

a **Linear Programming problem** LP in **canonical maximization form** with n non-negative variables

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \in \mathbb{R}_{\geq 0}^n$$

and m less-than-or-equal-to linear constraints is the following optimization problem:

$$\max \underbrace{\sum_{j=1}^n c_j x_j}_{\mathbf{c}'\mathbf{x}} \quad (4.1a)$$

$$\text{subject to } \underbrace{\sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \in \{1, 2, \dots, m\}}_{\mathbf{A}\mathbf{x} \leq \mathbf{b}}, \quad (4.1b)$$

$$\underbrace{x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n\}}_{\mathbf{x} \geq \mathbf{0}}, \quad (4.1c)$$

where $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a (column) vector of n zeros.

Example 4.1: LP problem in canonical maximization form

Consider the following input data of an LP problem with $n = 2$ and $m = 3$:

$$\mathbf{c} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -1 & 3 \\ 1 & 2 \\ 3 & -2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix}$$

The numerical entries of the two column vectors and the matrix are

$$c_1 = 5, c_2 = 2, a_{11} = -1, a_{12} = 3, a_{21} = 1, a_{22} = 2, \\ a_{31} = 3, a_{32} = -2, b_1 = 12, b_2 = 12, b_3 = 8.$$

The problem involves $n = 2$ non-negative variables $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

The associated LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 3$ less-than-or-equal-to linear constraints reads as follows:

$$\begin{array}{rclclcl} \max & 5 & x_1 & + & 2 & x_2 & \\ \text{subject to} & - & x_1 & + & 3 & x_2 & \leq 12 \\ & & x_1 & + & 2 & x_2 & \leq 12 \\ & 3 & x_1 & - & 2 & x_2 & \leq 8 \\ & & x_1, & & x_2 & \geq & 0 \end{array}$$

Example 4.2: LP problem in canonical maximization form

Consider the following input data of an LP problem with $n = 2$ and $m = 2$:

$$\mathbf{c} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

The numerical entries of the two column vectors and the matrix are

$$c_1 = -2, \quad c_2 = -4, \quad a_{11} = -1, \quad a_{12} = -1, \quad a_{21} = 1, \quad a_{22} = -1, \quad b_1 = -2, \quad b_2 = 0.$$

The problem involves $n = 2$ non-negative decision variables $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

The associated LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 2$ less-than-or-equal-to linear constraints reads as follows:

$$\begin{array}{llllll} \max & -2 & x_1 & - & 4 & x_2 \\ \text{subject to} & - & x_1 & - & & x_2 \leq -2 \\ & & x_1 & - & & x_2 \leq 0 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

4.1.1 Feasible region and objective function

Definition 4.2: feasible region and objective function

- The **feasible region** of an LP problem (4.1) is the **polyhedron** F defined as follows:

$$F = \left\{ \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \in \mathbb{R}^n : \underbrace{\sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \in \{1, 2, \dots, m\}}_{\mathbf{Ax} \leq \mathbf{b}}, \quad \underbrace{x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n\}}_{\mathbf{x} \geq \mathbf{0}} \right\}. \quad (4.2)$$

The feasible region is defined by the less-than-or-equal-to linear constraints (4.1b) and the non-negativity constraints (4.1c).

- The **objective function** (4.1a) of an LP problem (4.1) to be maximized is the

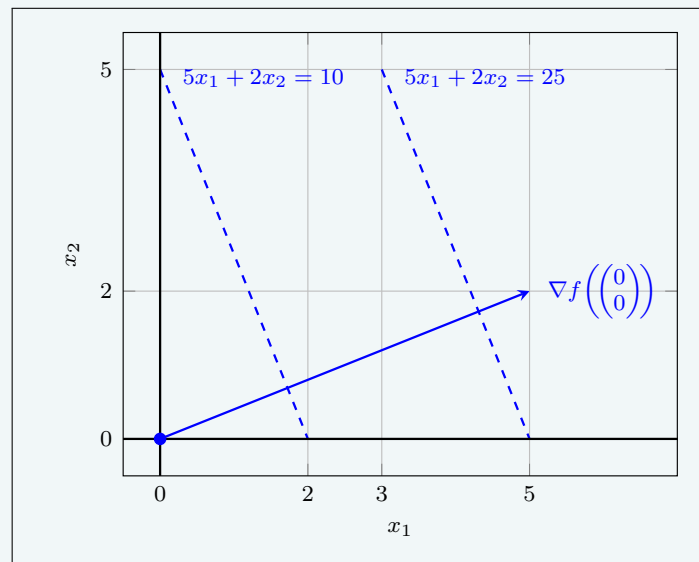
(scalar) **linear function** f defined as follows:

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad f : \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_x \mapsto \underbrace{\sum_{j=1}^n c_j x_j}_{c'x} \quad (4.3)$$

Example 4.3: objective function

The objective function of the LP problem of [Example 4.1](#) is the scalar linear function:

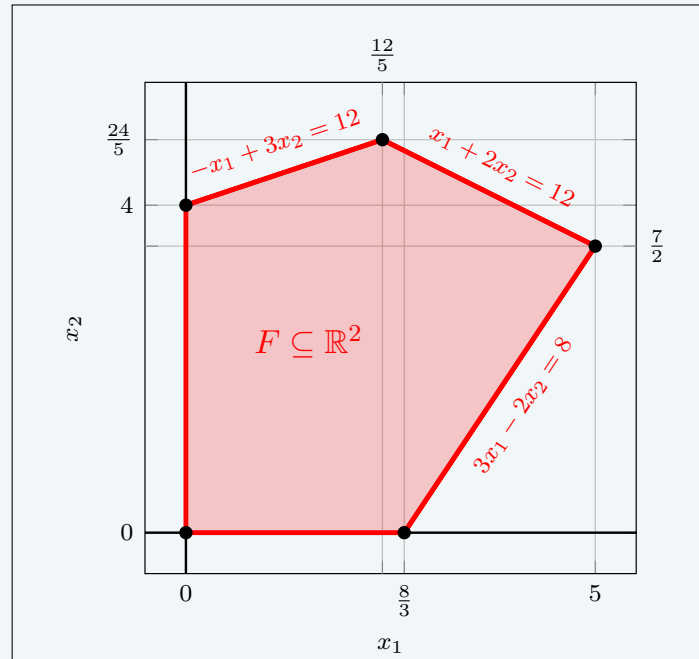
$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto 5x_1 + 2x_2 \quad \text{its gradient is} \quad \nabla f \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$



Example 4.4: feasible region

The feasible region of the LP problem of [Example 4.1](#) is the polyhedron:

$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -x_1 + 3x_2 \leq 12, \quad x_1 + 2x_2 \leq 12, \quad 3x_1 - 2x_2 \leq 8, \quad x_1, x_2 \geq 0 \right\}.$$



4.1.2 Lower bounds

Definition 4.3: feasible solution and lower bound

- Given

$$\underbrace{\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix}}_{\bar{\mathbf{x}}(LP)} \in F$$

a **feasible solution** of the LP problem (4.1) (a point satisfying constraints (4.1b) and (4.1c)), we can compute a **lower bound** $LB(\bar{\mathbf{x}})$ on the optimal value $z(LP)$ as follows:

$$LB(\bar{\mathbf{x}}) = \sum_{j=1}^n c_j \bar{x}_j \quad (4.4)$$

For every feasible solution $\bar{\mathbf{x}}(LP) \in F$ of an LP problem (4.1), we have

$$LB(\bar{\mathbf{x}}) \leq z(LP) \quad (4.5)$$

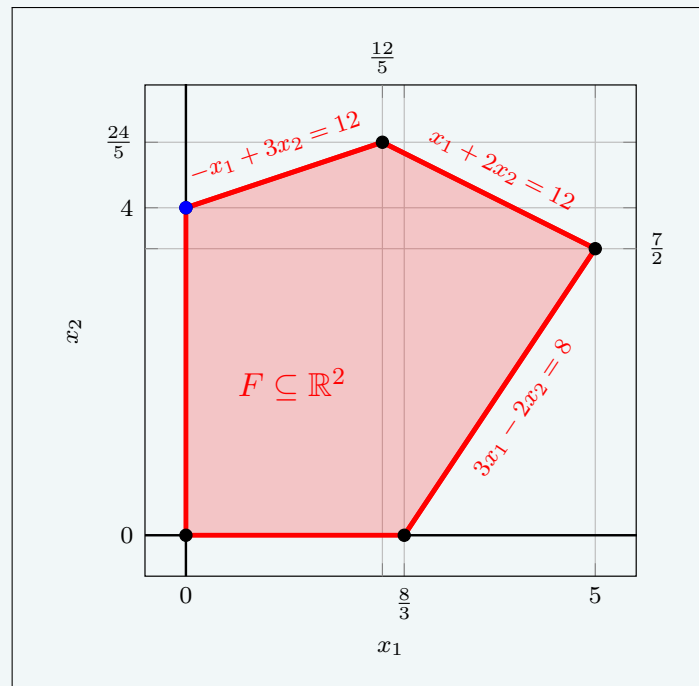
Example 4.5: Feasible solutions and lower bounds

Consider the LP problem of [Example 4.1](#). A feasible solution is for example $\bar{x}(LP) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ (where $\bar{x}_1 = 0$ and $\bar{x}_2 = 4$) since:

$$-0 + 3 \cdot 4 = 12 \leq 12, \quad 0 + 2 \cdot 4 = 8 \leq 12, \quad 3 \cdot 0 - 2 \cdot 4 = -8 \leq 8, \quad 0 \geq 0 \quad \text{and} \quad 4 \geq 0$$

Accordingly, a lower bound $LB(\bar{x})$ on the optimal value $z(LP)$ of the LP problem is:

$$LB(\bar{x}) = 5 \cdot 0 + 2 \cdot 4 = 8$$


4.1.3 Graphical solution method

An optimal solution of an LP problem (4.1) in canonical maximization form with two variables can be graphically determined using the direction of the objective function gradient and the level sets perpendicular to the gradient.

Example 4.6: Graphical solution method

For the LP problem of [Example 4.1](#):

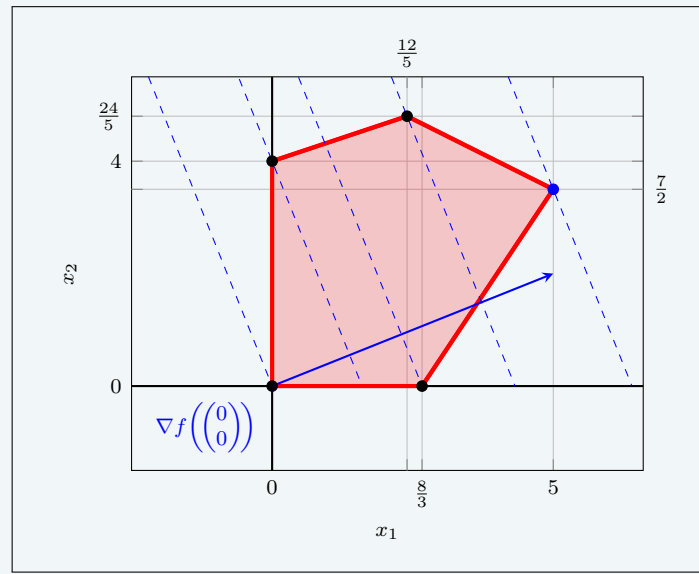
$$\begin{aligned} \max \quad & 5x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 3x_2 \leq 12 \\ & x_1 + 2x_2 \leq 12 \\ & 3x_1 - 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

an optimal solution is:

$$\underbrace{\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}}_{\tilde{\mathbf{x}}(LP)} = \begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix}$$

and the optimal value is:

$$z(LP) = 5\tilde{x}_1 + 2\tilde{x}_2 = 5 \cdot 5 + 2 \cdot \frac{7}{2} = 32$$



In dashed blue, from left to right, the level sets $5x_1 + 2x_2 = 0$, $5x_1 + 2x_2 = 8$, $5x_1 + 2x_2 = \frac{40}{3}$, $5x_1 + 2x_2 = \frac{108}{5}$ and $5x_1 + 2x_2 = 32$ of the objective function; the optimal vertex is depicted in blue.

Example 4.7: Graphical solution method

For the LP problem of [Example 4.2](#):

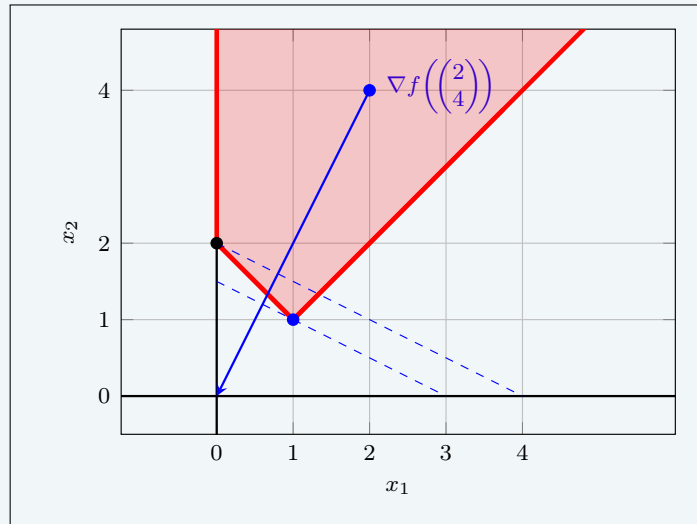
$$\begin{array}{llllll} \max & -2 & x_1 & - & 4 & x_2 \\ \text{subject to} & - & x_1 & - & & x_2 \leq -2 \\ & & x_1 & - & & x_2 \leq 0 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

an optimal solution is:

$$\underbrace{\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}}_{\tilde{\mathbf{x}}(LP)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and the optimal value is:

$$z(LP) = -2\tilde{x}_1 - 4\tilde{x}_2 = -2 \cdot 1 - 4 \cdot 1 = -6$$



In dashed blue, from bottom to top, the level sets $-2x_1 - 4x_2 = -6$ and $-2x_1 - 4x_2 = -8$ of the objective function; the optimal vertex $(1, 1)$ is depicted in blue, and the gradient is drawn from the point $(2, 4)$.

4.2 Standard maximization form

Definition 4.4: Linear Programming problems in standard maximization form

Given:

1. an objective-function vector $\mathbf{c} \in \mathbb{R}^n$
2. a constraint matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$
3. a right-hand-side (RHS) vector $\mathbf{b} \in \mathbb{R}^m$ with m rows

$$\underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}} \in \mathbb{R}^n \quad \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{pmatrix}}_{\mathbf{A}} \in \mathbb{R}^{m \times n} \quad \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\mathbf{b}} \in \mathbb{R}^m$$

a **Linear Programming** problem LP in **standard maximization form** with n non-negative variables

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \in \mathbb{R}_{\geq 0}^n$$

and m equality linear constraints is the following optimization problem:

$$\max \underbrace{\sum_{j=1}^n c_j x_j}_{c'x} \quad (4.6a)$$

$$\text{subject to } \underbrace{\sum_{j=1}^n a_{ij} x_j = b_i}_{Ax=b}, \quad \forall i \in \{1, 2, \dots, m\}, \quad (4.6b)$$

$$\underbrace{x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n\}}_{x \geq 0}, \quad (4.6c)$$

where $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a (column) vector of n zeros.

Example 4.8: LP problem in standard maximization form

Consider the following input data of an LP problem with $n = 3$ and $m = 1$:

$$c = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad A = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \quad b = (4)$$

The numerical entries of the two column vectors and the matrix are

$$c_1 = 2, \quad c_2 = 2, \quad c_3 = 1, \quad a_{11} = 2, \quad a_{12} = 3, \quad a_{13} = 1, \quad b_1 = 4.$$

The problem involves $n = 3$ non-negative variables $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

The associated LP problem in standard maximization form with $n = 3$ non-negative variables and $m = 1$ equality linear constraints reads as follows:

$$\begin{array}{llllll} \max & 2 & x_1 & + & 2 & x_2 & + & x_3 \\ \text{subject to} & 2 & x_1 & + & 3 & x_2 & + & x_3 & = & 4 \\ & & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

4.2.1 Feasible region and objective function

Definition 4.5: feasible region and objective function

- The **feasible region** of an LP problem (4.6) is the **polyhedron** F defined as follows:

$$F = \left\{ \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_x \in \mathbb{R}^n : \underbrace{\sum_{j=1}^n a_{ij} x_j = b_i, \quad \forall i \in \{1, 2, \dots, m\}}_{Ax=b}, \right. \quad (4.7)$$

$$\left. \underbrace{x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n\}}_{x \geq 0} \right\}.$$

The feasible region is defined by the equality linear constraints (4.6b) and the non-negativity constraints (4.6c).

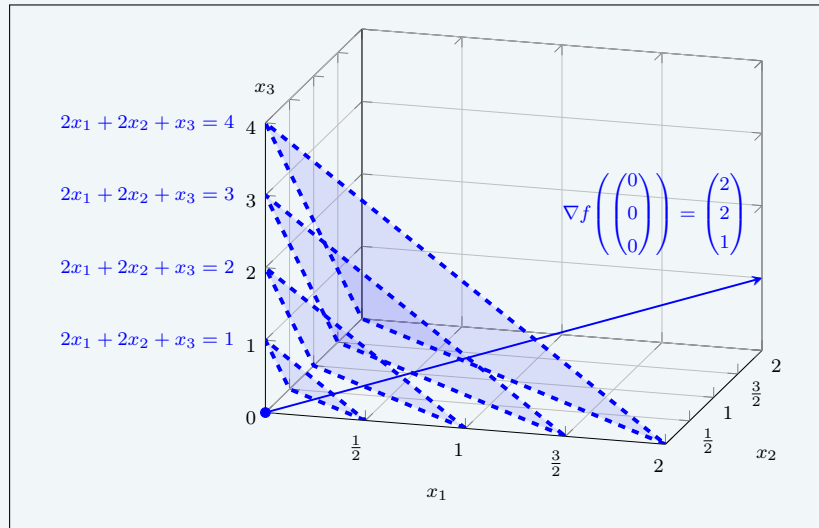
- The **objective function** (4.6a) of an LP problem (4.6) to be maximized is the (scalar) **linear function** f defined as follows:

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, \quad f : \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_x \mapsto \underbrace{\sum_{j=1}^n c_j x_j}_{c'x} \quad (4.8)$$

Example 4.9: objective function

The objective function of the LP problem of [Example 4.8](#) is the scalar linear function:

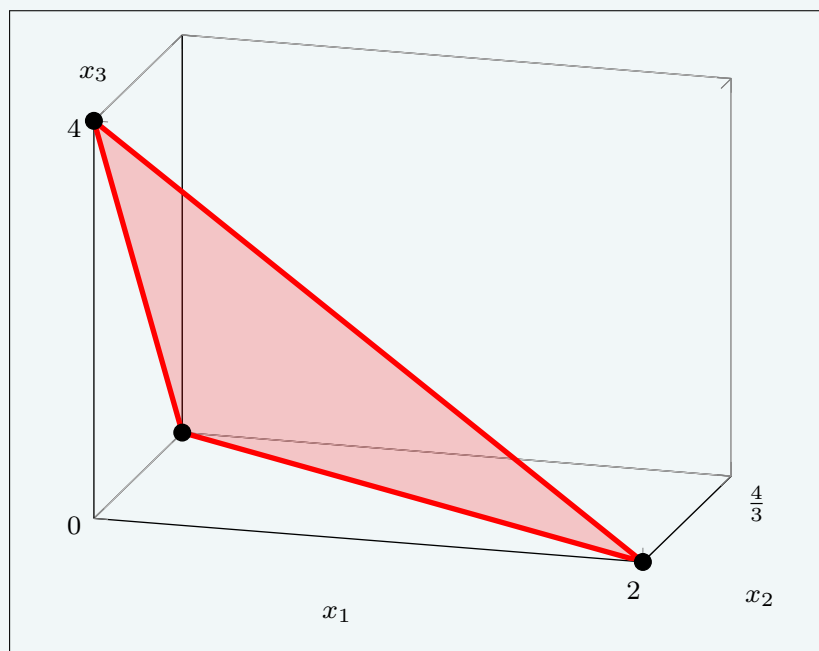
$$f : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto 2x_1 + 2x_2 + x_3 \quad \text{and its gradient is} \quad \nabla f \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$



Example 4.10: feasible region

The feasible region of the LP problem of [Example 4.8](#) is the polyhedron:

$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : 2x_1 + 3x_2 + x_3 = 4, \ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0 \right\}.$$



4.2.2 Lower bounds

Definition 4.6: feasible solution and lower bound

- Given

$$\underbrace{\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix}}_{\bar{\mathbf{x}}(LP)} \in F$$

a **feasible solution** of the LP problem (4.6) (a point satisfying constraints (4.6b) and (4.6c)), we can compute a **lower bound** $LB(\bar{\mathbf{x}})$ on the optimal value $z(LP)$ as follows:

$$LB(\bar{\mathbf{x}}) = \sum_{j=1}^n c_j \bar{x}_j \quad (4.9)$$

For every feasible solution $\bar{\mathbf{x}}(LP) \in F$ of an LP problem (4.6), we have

$$LB(\bar{\mathbf{x}}) \leq z(LP) \quad (4.10)$$

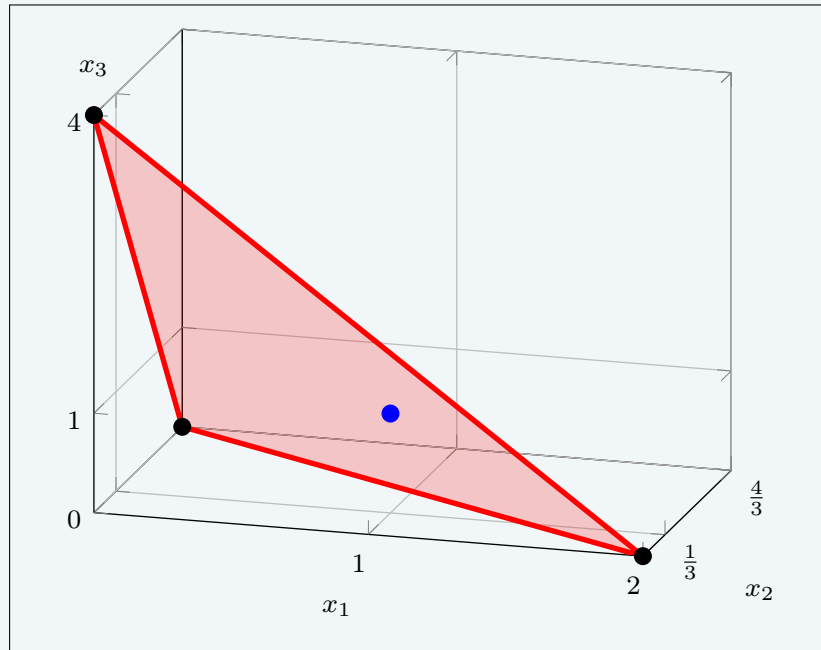
Example 4.11: Feasible solutions and lower bounds

Consider the LP problem of [Example 4.8](#). A feasible solution is for example $\bar{\mathbf{x}}(LP) = \begin{pmatrix} 1 \\ \frac{1}{3} \\ 1 \end{pmatrix}$ (where $\bar{x}_1 = 1$, $\bar{x}_2 = \frac{1}{3}$ and $\bar{x}_3 = 1$) since:

$$2 \cdot 1 + 3 \cdot \frac{1}{3} + 1 = 4, \quad 1 \geq 0, \quad \frac{1}{3} \geq 0, \quad 1 \geq 0.$$

Accordingly, a lower bound $LB(\bar{\mathbf{x}})$ on the optimal value $z(LP)$ of the LP problem is:

$$LB(\bar{\mathbf{x}}) = 2 \cdot 1 + 2 \cdot \frac{1}{3} + 1 = \frac{11}{3}.$$



4.2.3 Graphical solution method

An optimal solution of an LP problem (4.6) in standard maximization form with two or three variables can be graphically determined using the direction of the objective function gradient and the level sets perpendicular to the gradient.

Example 4.12: Graphical solution method

For the LP problem of [Example 4.8](#):

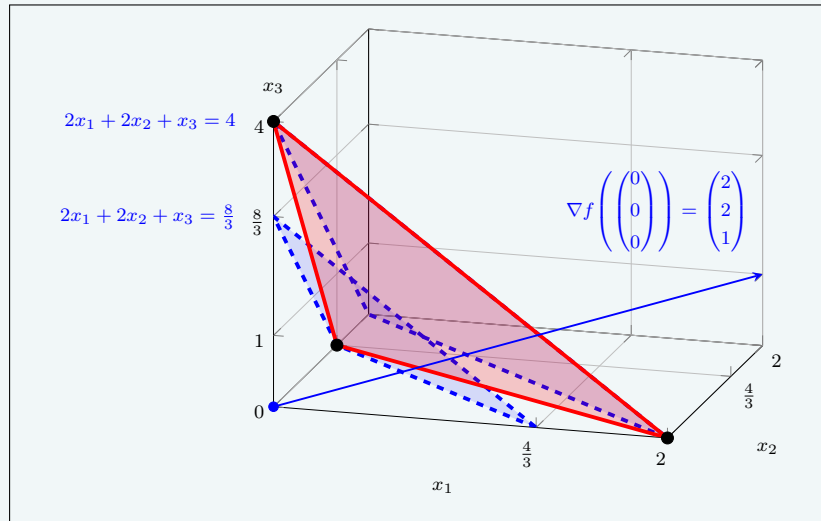
$$\begin{aligned} \max \quad & 2x_1 + 2x_2 + x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 + x_3 = 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

All optimal solutions lie on the line segment

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_2 = 0, 2x_1 + x_3 = 4, x_1 \geq 0, x_3 \geq 0 \right\},$$

i.e., on the segment joining the points $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$. Taking for instance the optimal

solution $\tilde{\mathbf{x}}(LP) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ (where $\tilde{x}_1 = 2$, $\tilde{x}_2 = 0$ and $\tilde{x}_3 = 0$), the optimal value is:
 $z(LP) = 2\tilde{x}_1 + 2\tilde{x}_2 + \tilde{x}_3 = 2 \cdot 2 + 2 \cdot 0 + 0 = 4.$



Possible outcomes and equivalent formulations

5.1 The two reference forms

- Throughout this chapter we refer to the two forms of an LP problem introduced in the previous one. Given an objective-function vector $\mathbf{c} \in \mathbb{R}^n$, a constraint matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and a right-hand-side vector $\mathbf{b} \in \mathbb{R}^m$, they read as follows.

An LP problem in **canonical maximization form** with n non-negative variables and m less-than-or-equal-to linear constraints:

$$\max \underbrace{\sum_{j=1}^n c_j x_j}_{\mathbf{c}'\mathbf{x}} \quad (5.1a)$$

$$\text{subject to } \underbrace{\sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \in \{1, 2, \dots, m\}}_{\mathbf{A}\mathbf{x} \leq \mathbf{b}}, \quad (5.1b)$$

$$\underbrace{x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n\}}_{\mathbf{x} \geq \mathbf{0}}. \quad (5.1c)$$

An LP problem in **standard maximization form** with n non-negative variables and m equality linear constraints:

$$\max \underbrace{\sum_{j=1}^n c_j x_j}_{\mathbf{c}'\mathbf{x}} \quad (5.2a)$$

$$\text{subject to } \underbrace{\sum_{j=1}^n a_{ij} x_j = b_i, \quad \forall i \in \{1, 2, \dots, m\}}_{\mathbf{A}\mathbf{x} = \mathbf{b}}, \quad (5.2b)$$

$$\underbrace{x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n\}}_{x \geq 0}. \quad (5.2c)$$

- We also refer to the three LP problems used as guiding examples in the previous chapter, which we recall in the following three examples.

Example 5.1: guiding LP problem in canonical maximization form

The first guiding example is the following LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 3$ less-than-or-equal-to linear constraints:

$$\begin{array}{llllll} \max & 5 & x_1 & + & 2 & x_2 \\ \text{subject to} & - & x_1 & + & 3 & x_2 & \leq & 12 \\ & & x_1 & + & 2 & x_2 & \leq & 12 \\ & 3 & x_1 & - & 2 & x_2 & \leq & 8 \\ & x_1, & & & x_2 & \geq & 0 \end{array}$$

An optimal solution is $\tilde{x}(LP) = \begin{pmatrix} 5 \\ 7/2 \end{pmatrix}$ and the optimal value is $z(LP) = 32$.

Example 5.2: guiding LP problem in canonical maximization form

The second guiding example is the following LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 2$ less-than-or-equal-to linear constraints:

$$\begin{array}{llllll} \max & -2 & x_1 & - & 4 & x_2 \\ \text{subject to} & - & x_1 & - & & x_2 & \leq & -2 \\ & & x_1 & - & & x_2 & \leq & 0 \\ & x_1, & & & x_2 & \geq & 0 \end{array}$$

An optimal solution is $\tilde{x}(LP) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the optimal value is $z(LP) = -6$.

Example 5.3: guiding LP problem in standard maximization form

The third guiding example is the following LP problem in standard maximization form with $n = 3$ non-negative variables and $m = 1$ equality linear constraint:

$$\begin{array}{llllll} \max & 2 & x_1 & + & 2 & x_2 & + & x_3 \\ \text{subject to} & 2 & x_1 & + & 3 & x_2 & + & x_3 & = & 4 \\ & x_1, & & & x_2, & & & x_3 & \geq & 0 \end{array}$$

Its optimal value is $z(LP) = 4$, attained by every point of the segment joining $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

5.2 Infeasible and unbounded LP problems

- The LP problems of [Example 5.1](#), [Example 5.2](#) and [Example 5.3](#) admit an optimal solution. This is not always the case: an LP problem may have no feasible solution at all, or it may have feasible solutions of arbitrarily large objective value.

Definition 5.1: infeasible LP problem

An LP problem is called **infeasible** if its feasible region is empty:

$$F = \emptyset.$$

Definition 5.2: unbounded LP problem

An LP problem in **maximization form** is called **unbounded** if, for every value $M \in \mathbb{R}$, there exists a feasible solution $\bar{x}(LP) \in F$ such that

$$f(\bar{x}(LP)) > M.$$

An LP problem in **minimization form** is called **unbounded** if, for every value $M \in \mathbb{R}$, there exists a feasible solution $\bar{x}(LP) \in F$ such that $f(\bar{x}(LP)) < M$.

An LP problem satisfies exactly one of the following three conditions:

1. it is **infeasible**;
2. it is **unbounded**;
3. it admits an **optimal solution** $\tilde{x}(LP)$, and the optimal value $z(LP)$ is a finite number.

- An infeasible LP problem has no feasible solution, so it has no optimal solution and the optimal value $z(LP)$ is not defined. An unbounded LP problem has feasible solutions, but none of them is optimal, and the optimal value $z(LP)$ is not defined either.
- In the two cases, with a slight abuse of notation, we write $z(LP) = -\infty$ for an infeasible LP problem in maximization form and $z(LP) = +\infty$ for an unbounded LP problem in maximization form.

Example 5.4: infeasible LP problem

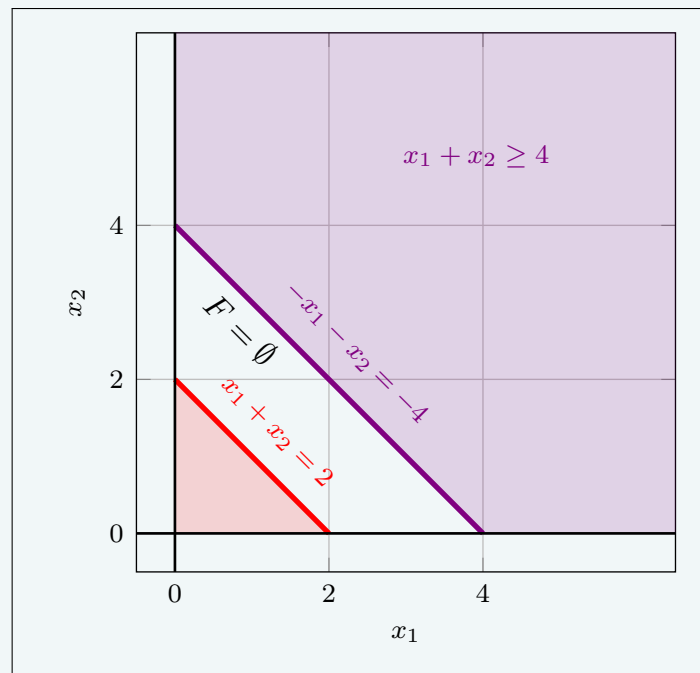
Consider the following LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 2$ less-than-or-equal-to linear constraints:

$$\begin{array}{llllll} \max & x_1 & + & x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \leq & 2 \\ & -x_1 & - & x_2 & \leq & -4 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

The first constraint imposes $x_1 + x_2 \leq 2$ and the second one imposes $x_1 + x_2 \geq 4$. No point of $\mathbb{R}^2$ satisfies both of them, and then we have:

$$F = \emptyset.$$

Accordingly, this LP problem is **infeasible**.



In red the half-plane $x_1 + x_2 \leq 2$ (with the non-negativity constraints) and in violet the half-plane $x_1 + x_2 \geq 4$: their intersection is empty.

Example 5.5: unbounded LP problem

Consider the following LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 2$ less-than-or-equal-to linear constraints:

$$\begin{array}{llllll}
 \max & x_1 & + & x_2 & & \\
 \text{subject to} & -x_1 & + & x_2 & \leq & 1 \\
 & x_1 & - & x_2 & \leq & 1 \\
 & x_1, & & x_2 & \geq & 0
 \end{array}$$

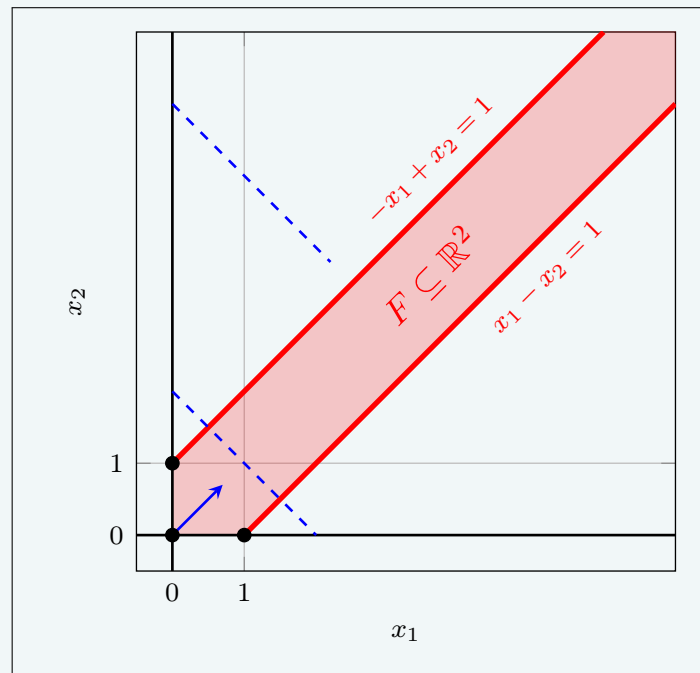
For every value $t \geq 0$, the point $\bar{x}(LP) = \begin{pmatrix} t \\ t \end{pmatrix}$ is a feasible solution, since

$$-t + t = 0 \leq 1, \quad t - t = 0 \leq 1, \quad t \geq 0 \quad \text{and} \quad t \geq 0,$$

and its objective value is

$$LB(\bar{x}) = t + t = 2t.$$

Accordingly, for every value $M \in \mathbb{R}$, the feasible solution obtained with $t > \frac{M}{2}$ has an objective value larger than M , and then this LP problem is **unbounded**.



In dashed blue the level sets $x_1 + x_2 = 2$ and $x_1 + x_2 = 6$ of the objective function: the gradient points along the unbounded feasible region.

An LP problem whose feasible region F is unbounded is not necessarily an unbounded LP problem: the objective function may well attain a finite optimal value on F .

Example 5.6: unbounded feasible region and multiple optimal solutions

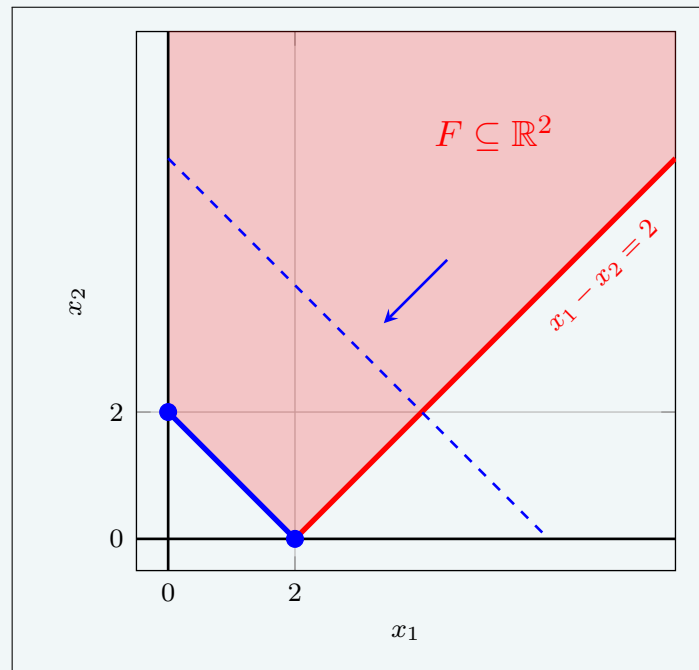
Consider the following LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 2$ less-than-or-equal-to linear constraints:

$$\begin{array}{llllll} \max & - & x_1 & - & x_2 & \\ \text{subject to} & - & x_1 & - & x_2 & \leq -2 \\ & & x_1 & - & x_2 & \leq 2 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

The feasible region F is unbounded. Nevertheless, the first constraint imposes $x_1 + x_2 \geq 2$ for every feasible solution, and then

$$-x_1 - x_2 \leq -2, \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in F.$$

The feasible solutions $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ attain this value, and then they are two **optimal solutions** and the optimal value is $z(LP) = -2$.



In dashed blue the level set $-x_1 - x_2 = -6$ and in solid blue the level set $-x_1 - x_2 = -2$, which contains the segment of optimal solutions joining the two blue vertices.

Observation 5.3: multiple optimal solutions

If $\tilde{\mathbf{x}}^{(1)}(LP)$ and $\tilde{\mathbf{x}}^{(2)}(LP)$ are two optimal solutions of an LP problem, then every point

$$\lambda \tilde{\mathbf{x}}^{(1)}(LP) + (1 - \lambda) \tilde{\mathbf{x}}^{(2)}(LP), \quad \lambda \in [0, 1],$$

is an optimal solution as well. Accordingly, an LP problem that admits an optimal solution admits either exactly one optimal solution or infinitely many optimal solutions.

Proof. Let $\mathbf{x}^\lambda = \lambda \tilde{\mathbf{x}}^{(1)}(LP) + (1 - \lambda) \tilde{\mathbf{x}}^{(2)}(LP)$ with $\lambda \in [0, 1]$. First, $\mathbf{x}^\lambda \in F$. Indeed, the feasible region F is a polyhedron and, for each of its inequalities $\mathbf{a} \mathbf{x} \leq b$, since $\lambda \geq 0$, $1 - \lambda \geq 0$ and both solutions satisfy it, we have:

$$\mathbf{a} \mathbf{x}^\lambda = \lambda \mathbf{a} \tilde{\mathbf{x}}^{(1)}(LP) + (1 - \lambda) \mathbf{a} \tilde{\mathbf{x}}^{(2)}(LP) \leq \lambda b + (1 - \lambda) b = b;$$

the same computation, with $=$ in place of $\leq$, holds for each of its equalities. Moreover, since the objective function f is linear, we have:

$$\begin{aligned} f(\mathbf{x}^\lambda) &= \lambda f(\tilde{\mathbf{x}}^{(1)}(LP)) + (1 - \lambda) f(\tilde{\mathbf{x}}^{(2)}(LP)) \\ &= \lambda z(LP) + (1 - \lambda) z(LP) = z(LP), \end{aligned}$$

where the second equality holds because both solutions are optimal. Accordingly, $\mathbf{x}^\lambda$ is a feasible solution whose objective value is the optimal value, and then it is an optimal solution. $\square$

5.3 Equivalent reformulations

- A **generic LP problem** may be written in a form which is neither the canonical maximization form (5.1) nor the standard maximization form (5.2): its objective function may have to be minimized, its constraints may be of the three types “ $\leq$ ”, “ $\geq$ ” and “ $=$ ”, and its variables may be non-positive or free.
- In this section we show that every generic LP problem can be rewritten in canonical maximization form and in standard maximization form.

Definition 5.4: equivalent LP problems

Two LP problems are called **equivalent** if, from an optimal solution of one of them, an optimal solution of the other one can be directly obtained, and the two problems are both infeasible, both unbounded, or both admit an optimal solution.

5.3.1 Minimization and maximization form

- For every linear function f and every polyhedron F , we have:

$$\min\{f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F\} = -\max\{-f(\mathbf{x}) \text{ subject to } \mathbf{x} \in F\}.$$

- Accordingly, an LP problem in **minimization form** with objective-function vector $\mathbf{c}$ is equivalent to the LP problem in **maximization form** with the same feasible region and objective-function vector $-\mathbf{c}$:

$$\underbrace{\sum_{j=1}^n c_j x_j}_{\mathbf{c}'\mathbf{x}} \quad \longrightarrow \quad \underbrace{\sum_{j=1}^n (-c_j) x_j}_{-\mathbf{c}'\mathbf{x}}.$$

The two equivalent LP problems have the **same optimal solutions**, while their optimal values are opposite numbers:

$$\tilde{\mathbf{x}}(LP_{\min}) = \tilde{\mathbf{x}}(LP_{\max}) \quad \text{and} \quad z(LP_{\min}) = -z(LP_{\max}).$$

Example 5.7: from the minimization to the maximization form

Consider the following LP problem in minimization form with $n = 2$ non-negative variables:

$$\begin{array}{llllll} \min & 2 & x_1 & + & 4 & x_2 \\ \text{subject to} & - & x_1 & - & & x_2 \leq -2 \\ & & x_1 & - & & x_2 \leq 0 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

By changing the sign of the objective-function vector, we obtain the equivalent LP problem in canonical maximization form of [Example 5.2](#):

$$\begin{array}{llllll} \max & -2 & x_1 & - & 4 & x_2 \\ \text{subject to} & - & x_1 & - & & x_2 \leq -2 \\ & & x_1 & - & & x_2 \leq 0 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

The optimal solution of both problems is $\tilde{\mathbf{x}}(LP) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, while the optimal value of the minimization form is 6 and the optimal value of the maximization form is -6 .

5.3.2 Greater-than-or-equal-to and equality constraints

- A **greater-than-or-equal-to** linear constraint is equivalent to a less-than-or-equal-to linear constraint, since, by multiplying both sides by -1 , we have:

$$\sum_{j=1}^n a_{ij} x_j \geq b_i \quad \Longleftrightarrow \quad \sum_{j=1}^n (-a_{ij}) x_j \leq -b_i.$$

- An **equality** linear constraint is equivalent to two less-than-or-equal-to linear constraints, since:

$$\sum_{j=1}^n a_{ij} x_j = b_i \quad \Longleftrightarrow \quad \underbrace{\sum_{j=1}^n a_{ij} x_j \leq b_i}_{\text{first constraint}} \quad \text{and} \quad \underbrace{\sum_{j=1}^n (-a_{ij}) x_j \leq -b_i}_{\text{second constraint}}.$$

An LP problem (5.2) in standard maximization form with n non-negative variables and m equality linear constraints is equivalent to an LP problem (5.1) in canonical maximization form with the same n non-negative variables and $2m$ less-than-or-equal-to linear constraints:

$$\begin{aligned} & \max \quad \sum_{j=1}^n c_j x_j \\ & \text{subject to} \quad \sum_{j=1}^n a_{ij} x_j \leq b_i, & \forall i \in \{1, 2, \dots, m\}, \\ & \quad \quad \quad \sum_{j=1}^n (-a_{ij}) x_j \leq -b_i, & \forall i \in \{1, 2, \dots, m\}, \\ & \quad \quad \quad x_j \geq 0, & \forall j \in \{1, 2, \dots, n\}. \end{aligned}$$

The two LP problems have the **same feasible region**, and then the same optimal solutions and the same optimal value.

Example 5.8: from the standard to the canonical maximization form

Consider the LP problem in standard maximization form of [Example 5.3](#), with $n = 3$ non-negative variables and $m = 1$ equality linear constraint:

$$\begin{aligned} & \max \quad 2x_1 + 2x_2 + x_3 \\ & \text{subject to} \quad 2x_1 + 3x_2 + x_3 = 4 \\ & \quad \quad \quad x_1, \quad x_2, \quad x_3 \geq 0 \end{aligned}$$

By replacing the equality linear constraint with two less-than-or-equal-to linear constraints, we obtain the equivalent LP problem in canonical maximization form with $n = 3$ non-negative variables and $2m = 2$ less-than-or-equal-to linear constraints:

$$\begin{array}{llllllll}
 \max & 2 & x_1 & + & 2 & x_2 & + & x_3 \\
 \text{subject to} & 2 & x_1 & + & 3 & x_2 & + & x_3 \leq 4 \\
 & -2 & x_1 & - & 3 & x_2 & - & x_3 \leq -4 \\
 & & x_1, & & x_2, & & x_3 & \geq 0
 \end{array}$$

5.3.3 Slack and surplus variables

- A **less-than-or-equal-to** linear constraint is equivalent to an equality linear constraint and a non-negativity constraint, obtained by adding one non-negative variable x_{n+i} :

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \quad \Longleftrightarrow \quad \sum_{j=1}^n a_{ij} x_j + x_{n+i} = b_i \quad \text{and} \quad x_{n+i} \geq 0.$$

- A **greater-than-or-equal-to** linear constraint is equivalent to an equality linear constraint and a non-negativity constraint, obtained by subtracting one non-negative variable x_{n+i} :

$$\sum_{j=1}^n a_{ij} x_j \geq b_i \quad \Longleftrightarrow \quad \sum_{j=1}^n a_{ij} x_j - x_{n+i} = b_i \quad \text{and} \quad x_{n+i} \geq 0.$$

Definition 5.5: slack and surplus variables

The additional non-negative variable x_{n+i} is called the **slack variable** of a less-than-or-equal-to linear constraint and the **surplus variable** of a greater-than-or-equal-to linear constraint. Its value measures, for a given feasible solution, the difference between the two sides of the constraint.

An LP problem (5.1) in canonical maximization form with n non-negative variables and m less-than-or-equal-to linear constraints is equivalent to an LP problem (5.2) in standard maximization form with $n + m$ non-negative variables and the same m linear constraints, now equalities:

$$\begin{array}{ll}
 \max & \sum_{j=1}^n c_j x_j \\
 \text{subject to} & \sum_{j=1}^n a_{ij} x_j + x_{n+i} = b_i, \quad \forall i \in \{1, 2, \dots, m\}, \\
 & x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n + m\}.
 \end{array}$$

- In matrix notation, the m slack variables form a column vector $\mathbf{x}_S \in \mathbb{R}_{\geq 0}^m$ and the constraint matrix of the standard maximization form is obtained by appending to $\mathbf{A}$ the

identity matrix I of size m :

$$\begin{aligned} & \max \quad \mathbf{c}' \mathbf{x} \\ & \text{subject to} \quad \underbrace{\mathbf{A} \mathbf{x} + \mathbf{I} \mathbf{x}_S}_{(\mathbf{A} \quad \mathbf{I}) \begin{pmatrix} \mathbf{x} \\ \mathbf{x}_S \end{pmatrix}} = \mathbf{b}, \\ & \mathbf{x}, \mathbf{x}_S \geq \mathbf{0}, \end{aligned}$$

where

$$\mathbf{x}_S = \begin{pmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{pmatrix} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Example 5.9: from the canonical to the standard maximization form

By adding $m = 3$ slack variables, the LP problem of [Example 5.1](#) becomes the following equivalent LP problem in standard maximization form with $n + m = 5$ non-negative variables:

$$\begin{aligned} & \max \quad 5x_1 + 2x_2 \\ & \text{subject to} \quad -x_1 + 3x_2 + x_3 = 12 \\ & \quad \quad \quad x_1 + 2x_2 + x_4 = 12 \\ & \quad \quad \quad 3x_1 - 2x_2 + x_5 = 8 \\ & \quad \quad \quad x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

The optimal solution $\tilde{\mathbf{x}}(LP)$ of [Example 5.1](#) satisfies the first constraint strictly and the other two with equality, and then the values of the three slack variables are $\tilde{x}_3 = \frac{13}{2}$, $\tilde{x}_4 = 0$ and $\tilde{x}_5 = 0$.

5.3.4 Non-positive and free variables

- A **non-positive** variable $x_j \leq 0$ is equivalent to one non-negative variable $x_j^+ \geq 0$, by setting:

$$x_j = -x_j^+.$$

- A **free** variable $x_j \geq 0$ is equivalent to two non-negative variables $x_j^+ \geq 0$ and $x_j^- \geq 0$, by setting:

$$x_j = x_j^+ - x_j^-.$$

In both cases, the substitution is performed in the objective function and in every linear constraint of the LP problem. A free variable is replaced by two non-negative variables, since every real number is the difference of two non-negative numbers.

Example 5.10: equivalent LP problem with only non-negative variables

Consider the following LP problem with one non-negative variable $x_1 \geq 0$, one non-positive variable $x_2 \leq 0$ and one free variable $x_3 \geq 0$:

$$\begin{array}{llllllll} \max & 2 & x_1 & + & 4 & x_2 & + & 2 & x_3 \\ \text{subject to} & - & x_1 & + & & x_2 & + & & x_3 = 2 \\ & & & & 8 & x_1 & + & 2 & x_2 = 19 \\ & & & & & & & & x_1 \geq 0 \\ & & & & & & & & & x_2 \leq 0 \\ & & & & & & & & & & x_3 \geq 0 \end{array}$$

By setting $x_2 = -x_2^+$ and $x_3 = x_3^+ - x_3^-$, we obtain the following equivalent LP problem with only non-negative variables:

$$\begin{array}{llllllllll} \max & 2 & x_1 & - & 4 & x_2^+ & + & 2 & x_3^+ & - & 2 & x_3^- \\ \text{subject to} & - & x_1 & - & & x_2^+ & + & & x_3^+ & - & & x_3^- = 2 \\ & & & & 8 & x_1 & - & 2 & x_2^+ & & & = 19 \\ & & & & & & & & x_1, & & x_2^+, & x_3^+, & x_3^- \geq 0 \end{array}$$

From an optimal solution of the second LP problem, an optimal solution of the first one is obtained by computing $\tilde{x}_2 = -\tilde{x}_2^+$ and $\tilde{x}_3 = \tilde{x}_3^+ - \tilde{x}_3^-$.

5.3.5 From a generic LP problem to the canonical and standard maximization form

Every generic LP problem can be rewritten in canonical maximization form and in standard maximization form by applying the following transformations:

	canonical max form	standard max form
$\min \mathbf{c}'\mathbf{x}$	$-\max(-\mathbf{c})'\mathbf{x}$	$-\max(-\mathbf{c})'\mathbf{x}$
$\sum_j a_{ij}x_j \leq b_i$	unchanged	add a slack variable $x_{n+i} \geq 0$
$\sum_j a_{ij}x_j \geq b_i$	multiply by -1	subtract a surplus variable $x_{n+i} \geq 0$
$\sum_j a_{ij}x_j = b_i$	split into two " $\leq$ ", the second one multiplied by -1	unchanged
$x_j \geq 0$	unchanged	unchanged
$x_j \leq 0$	$x_j = -x_j^+$	$x_j = -x_j^+$
$x_j \geq 0$	$x_j = x_j^+ - x_j^-$	$x_j = x_j^+ - x_j^-$

Example 5.11: from a generic LP problem to the canonical maximization form

Consider the following generic LP problem with 3 variables:

$$\begin{aligned}
 \min \quad & 3x_1 - 2x_2 + x_3 \\
 \text{subject to} \quad & x_1 + x_2 - x_3 \geq 4 \\
 & 2x_1 - x_2 = 6 \\
 & x_1 \geq 0 \\
 & x_2 \leq 0 \\
 & x_3 \geq 0
 \end{aligned}$$

We apply the transformations of the table above:

1. the objective function is minimized, and then we maximize the opposite objective function;
2. the first constraint is a greater-than-or-equal-to constraint, and then we multiply it by -1 ;
3. the second constraint is an equality constraint, and then we split it into two less-than-or-equal-to constraints;
4. the variable x_2 is non-positive, and then we set $x_2 = -x_2^+$ with $x_2^+ \geq 0$;
5. the variable x_3 is free, and then we set $x_3 = x_3^+ - x_3^-$ with $x_3^+, x_3^- \geq 0$.

We obtain the following equivalent LP problem in canonical maximization form with $n = 4$ non-negative variables and $m = 3$ less-than-or-equal-to linear constraints:

$$\begin{aligned}
 & - \max \quad -3 x_1 - 2 x_2^+ - x_3^+ + x_3^- \\
 & \text{subject to} \quad - x_1 + x_2^+ + x_3^+ - x_3^- \leq -4 \\
 & \quad \quad \quad 2 x_1 + x_2^+ \leq 6 \\
 & \quad \quad \quad -2 x_1 - x_2^+ \leq -6 \\
 & \quad \quad \quad x_1, x_2^+, x_3^+, x_3^- \geq 0
 \end{aligned}$$

Finally, by adding $m = 3$ slack variables x_5, x_6, x_7 , we obtain the equivalent LP problem in standard maximization form with $n + m = 7$ non-negative variables and $m = 3$ equality linear constraints.

5.3.6 Rational input data

Definition 5.6: Least Common Denominator

Given a vector $\mathbf{v} \in \mathbb{Q}^k$ of k rational numbers $v_j = \frac{p_j}{q_j}$ with $p_j \in \mathbb{Z}$ and $q_j \in \mathbb{N}$ coprime, $\forall j \in \{1, 2, \dots, k\}$, the **Least Common Denominator**^a, denoted by $lcd(\mathbf{v})$, is the smallest positive integer number divisible by all the denominators q_j , with $j \in \{1, 2, \dots, k\}$.

^aor the Least Common Multiple of the denominators.

- Multiplying a linear constraint by a positive number does not change the set of points satisfying it, and multiplying the objective function by a positive number does not change the set of optimal solutions. Accordingly, an LP problem with **rational input data** is equivalent to an LP problem with **integer input data**.

Given an LP problem (5.1) with rational input data, the equivalent LP problem with integer input data reads as follows:

$$\begin{aligned}
 & \frac{1}{lcd(\mathbf{c})} \max \quad \sum_{j=1}^n lcd(\mathbf{c}) c_j x_j \\
 & \text{subject to} \quad \sum_{j=1}^n lcd([\mathbf{a}_i \ b_i]) a_{ij} x_j \leq lcd([\mathbf{a}_i \ b_i]) b_i, \quad \forall i \in \{1, 2, \dots, m\}, \\
 & \quad \quad \quad x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n\},
 \end{aligned}$$

where $\mathbf{a}_i$ denotes the i -th row of the constraint matrix $\mathbf{A}$.

Example 5.12: equivalent LP problem with integer input data

For the following LP problem (5.1) with $n = 5$ non-negative variables, $m = 1$ less-than-or-equal-to linear constraint and rational input data:

$$\begin{aligned} \max \quad & \frac{1}{3} x_1 - \frac{11}{27} x_2 + \frac{7}{10} x_3 + \frac{7}{5} x_4 + \frac{2}{9} x_5 \\ \text{subject to} \quad & \frac{1}{2} x_1 + \frac{3}{4} x_2 + \frac{11}{6} x_3 - \frac{1}{4} x_4 + \frac{5}{2} x_5 \leq \frac{25}{12} \\ & x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5 \geq 0 \end{aligned}$$

we have:

$$\underbrace{\left(\begin{pmatrix} \frac{1}{3} \\ -\frac{11}{27} \\ \frac{7}{10} \\ \frac{7}{5} \\ \frac{2}{9} \end{pmatrix} \right)}_{=lcd(c)} = 270 \quad \text{and} \quad \underbrace{\left(\begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ \frac{11}{6} \\ -\frac{1}{4} \\ \frac{5}{2} \\ \frac{25}{12} \end{pmatrix} \right)}_{=lcd\left(\begin{bmatrix} a_1 & \frac{25}{12} \end{bmatrix}\right)} = 12$$

and, accordingly, the equivalent LP problem with the same variables and constraints and with integer input data reads as follows:

$$\begin{aligned} \frac{1}{270} \max \quad & 90 x_1 - 110 x_2 + 189 x_3 + 378 x_4 + 60 x_5 \\ \text{subject to} \quad & 6 x_1 + 9 x_2 + 22 x_3 - 3 x_4 + 30 x_5 \leq 25 \\ & x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5 \geq 0 \end{aligned}$$