



SAPIENZA
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Linear Programming

Exercises 5 — Linear programming models

Worked linear programming models: symbolic models with their instance, a feasible solution and a dual bound, and numerical models.

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Linear Programming modeling techniques

Exercise 16.1 — Production planning (1)

A company produces $r \in \mathbb{Z}_{\geq 1}$ different products using $s \in \mathbb{Z}_{\geq 1}$ resources (for example machine hours and labour hours). For each product $j \in \{1, 2, \dots, r\}$, the value $c_j \in \mathbb{Q}_{>0}$ is the profit in euros of one unit of product and the value $u_j \in \mathbb{Q}_{>0}$ is the maximum number of units that the market can absorb per week. For each resource $i \in \{1, 2, \dots, s\}$ and each product $j \in \{1, 2, \dots, r\}$, the value $a_{ij} \in \mathbb{Q}_{\geq 0}$ is the quantity of resource i consumed by one unit of product j . For each resource $i \in \{1, 2, \dots, s\}$, the value $b_i \in \mathbb{Q}_{>0}$ is the quantity available per week; moreover, the company can buy at most $e_i \in \mathbb{Q}_{\geq 0}$ additional units of the resource per week, at a cost of $g_i \in \mathbb{Q}_{>0}$ euros per unit. The company aims to determine a maximum profit weekly production plan subject to the constraints described above.

Questions

1. Write a Linear Programming (LP) model for the problem. Identify the decisions that must be taken and the corresponding decision variables. Identify and comment the objective function of the model and the constraints. What algorithm can be used to find an optimal solution of the model?
2. Consider now an instance of the problem in which there are $r = 3$ products and $s = 2$ resources. These are the other input data:

	$j = 1$	$j = 2$	$j = 3$		$i = 1$	$i = 2$
c_j	12	18	10	b_i	50	50
u_j	20	15	20	e_i	20	10
a_{1j}	2	3	1	g_i	3	2
a_{2j}	1	2	2			

Write the LP model for this instance.

3. Find a feasible solution of the LP model of the instance and accordingly a lower bound on the optimal solution value of the LP model for the instance.
4. Write the dual model of the LP model and the dual model for the instance. Find a feasible solution of the dual model of the instance and accordingly an upper bound on the optimal solution value of the LP model for the instance.

5. Consider the following solution of the LP model of the instance:

$$\tilde{x}_1 = 20, \quad \tilde{x}_2 = 5, \quad \tilde{x}_3 = 15, \quad \tilde{y}_1 = 20, \quad \tilde{y}_2 = 10,$$

where y_i is the quantity of resource i bought. Prove that it is an optimal solution by finding, with the complementary slackness conditions, a feasible dual solution with the same objective function value. What is the optimal solution value?

Solution

1. The input data of the model are the following ones:

Symbol	Type	Meaning
r	$\mathbb{Z}_{\geq 1}$	number of products
s	$\mathbb{Z}_{\geq 1}$	number of resources
c_j	$\mathbb{Q}_{>0}$	profit of one unit of product j (euros)
u_j	$\mathbb{Q}_{>0}$	maximum weekly demand of product j (units)
a_{ij}	$\mathbb{Q}_{\geq 0}$	quantity of resource i consumed by one unit of product j
b_i	$\mathbb{Q}_{>0}$	weekly availability of resource i
e_i	$\mathbb{Q}_{\geq 0}$	maximum additional quantity of resource i that can be bought
g_i	$\mathbb{Q}_{>0}$	cost of one additional unit of resource i (euros)

We introduce the following $r + s$ non-negative variables:

$$\begin{cases} x_j = & \text{units of product } j \text{ produced per week,} & \forall j \in \{1, 2, \dots, r\}, \\ y_i = & \text{units of resource } i \text{ bought per week,} & \forall i \in \{1, 2, \dots, s\}. \end{cases}$$

Using these variables, an LP model for the problem reads as follows:

$$\max \quad \sum_{j=1}^r c_j x_j - \sum_{i=1}^s g_i y_i \quad (16.1a)$$

$$\text{subject to} \quad \sum_{j=1}^r a_{ij} x_j - y_i \leq b_i, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.1b)$$

$$y_i \leq e_i, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.1c)$$

$$x_j \leq u_j, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.1d)$$

$$x_j \geq 0, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.1e)$$

$$y_i \geq 0, \quad \forall i \in \{1, 2, \dots, s\}. \quad (16.1f)$$

Description of the objective function and constraints of the LP model:

- the linear objective function (16.1a) maximizes the total profit of the production plan, that is, the profit of the products minus the cost of the additional resources bought;
- the linear constraints (16.1b) ensure that, for each resource, the quantity consumed does not exceed the quantity available plus the quantity bought (s linear constraints);
- the linear constraints (16.1c) ensure that, for each resource, the quantity bought does not exceed its maximum (s linear constraints);
- the linear constraints (16.1d) ensure that, for each product, the production does not exceed the demand of the market (r linear constraints);
- the linear constraints (16.1e) and (16.1f) define the variables of the LP model.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. The LP model (16.1) for the instance reads as follows:

$$\begin{aligned}
 \max \quad & 12x_1 + 18x_2 + 10x_3 - 3y_1 - 2y_2 \\
 \text{subject to} \quad & 2x_1 + 3x_2 + x_3 - y_1 \leq 50 \\
 & x_1 + 2x_2 + 2x_3 - y_2 \leq 50 \\
 & y_1 \leq 20 \\
 & y_2 \leq 10 \\
 & x_1 \leq 20 \\
 & x_2 \leq 15 \\
 & x_3 \leq 20 \\
 & x_1, x_2, x_3, y_1, y_2 \geq 0
 \end{aligned}$$

3. For example, without buying additional resources, by setting:

$$\bar{x}_1 = 20, \bar{x}_2 = 0, \bar{x}_3 = 10, \bar{y}_1 = 0, \bar{y}_2 = 0$$

we have a feasible solution: the two resources consume $2 \cdot 20 + 10 = 50 \leq 50$ and $20 + 2 \cdot 10 = 40 \leq 50$ units, and the demands are respected. The objective function value of this feasible solution is:

$$\sum_{j=1}^r c_j \bar{x}_j - \sum_{i=1}^s g_i \bar{y}_i = 12 \cdot 20 + 18 \cdot 0 + 10 \cdot 10 - 3 \cdot 0 - 2 \cdot 0 = 340$$

accordingly 340 is a lower bound on the optimal solution value of the LP model (16.1). So, for this instance, we have:

$$340 \leq z(LP)$$

4. By using

- a non-negative dual variable π_i for each constraint (16.1b) ($i \in \{1, 2, \dots, s\}$)

- a non-negative dual variable σ_i for each constraint (16.1c) ($i \in \{1, 2, \dots, s\}$)
- a non-negative dual variable ρ_j for each constraint (16.1d) ($j \in \{1, 2, \dots, r\}$)

the dual of the LP model (16.1) reads as follows:

$$\min \sum_{i=1}^s b_i \pi_i + \sum_{i=1}^s e_i \sigma_i + \sum_{j=1}^r u_j \rho_j \quad (16.2a)$$

$$\text{subject to } \sum_{i=1}^s a_{ij} \pi_i + \rho_j \geq c_j, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.2b)$$

$$-\pi_i + \sigma_i \geq -g_i, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.2c)$$

$$\pi_i \geq 0, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.2d)$$

$$\sigma_i \geq 0, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.2e)$$

$$\rho_j \geq 0, \quad \forall j \in \{1, 2, \dots, r\}. \quad (16.2f)$$

The dual model (16.2) for the instance reads as follows:

$$\begin{aligned} \min \quad & 50 \pi_1 + 50 \pi_2 + 20 \sigma_1 + 10 \sigma_2 + 20 \rho_1 + 15 \rho_2 + 20 \rho_3 \\ \text{subject to} \quad & 2 \pi_1 + \pi_2 + \rho_1 \geq 12 \\ & 3 \pi_1 + 2 \pi_2 + \rho_2 \geq 18 \\ & \pi_1 + 2 \pi_2 + \rho_3 \geq 10 \\ & -\pi_1 + \sigma_1 \geq -3 \\ & -\pi_2 + \sigma_2 \geq -2 \\ & \pi_1, \pi_2, \sigma_1, \sigma_2, \rho_1, \rho_2, \rho_3 \geq 0 \end{aligned}$$

For example, by valuing each resource at its purchase price:

$$\bar{\pi}_1 = g_1 = 3, \quad \bar{\pi}_2 = g_2 = 2, \quad \bar{\sigma}_1 = 0, \quad \bar{\sigma}_2 = 0,$$

the constraints (16.2c) are satisfied, and by setting:

$$\begin{aligned} \bar{\rho}_1 &= \max\{0, 12 - 2 \cdot 3 - 2\} = 4, \quad \bar{\rho}_2 = \max\{0, 18 - 3 \cdot 3 - 2 \cdot 2\} = 5, \\ \bar{\rho}_3 &= \max\{0, 10 - 3 - 2 \cdot 2\} = 3 \end{aligned}$$

also the constraints (16.2b) are satisfied: we have a feasible dual solution.

The objective function value of this feasible dual solution is:

$$\begin{aligned} \sum_{i=1}^s b_i \bar{\pi}_i + \sum_{i=1}^s e_i \bar{\sigma}_i + \sum_{j=1}^r u_j \bar{\rho}_j &= 50 \cdot 3 + 50 \cdot 2 + 20 \cdot 0 + 10 \cdot 0 \\ &\quad + 20 \cdot 4 + 15 \cdot 5 + 20 \cdot 3 \\ &= 465 \end{aligned}$$

Accordingly, 465 is an upper bound on the optimal solution value of the LP model. So, for this instance, we have:

$$340 \leq z(LP) \leq 465$$

5. The solution $\tilde{x}, \tilde{y}$ is feasible: the two resources consume $2 \cdot 20 + 3 \cdot 5 + 15 = 70 = 50 + 20$ and $20 + 2 \cdot 5 + 2 \cdot 15 = 60 = 50 + 10$ units, and all the other constraints are satisfied. Its value is $12 \cdot 20 + 18 \cdot 5 + 10 \cdot 15 - 3 \cdot 20 - 2 \cdot 10 = 400$.

The complementary slackness conditions give:

- $\tilde{x}_2 = 5 < 15$ and $\tilde{x}_3 = 15 < 20$: the corresponding constraints (16.1d) are not tight, hence $\tilde{\rho}_2 = 0$ and $\tilde{\rho}_3 = 0$;
- all the five primal variables are positive: the five dual constraints are satisfied at equality:

$$\begin{cases} 2\pi_1 + \pi_2 + \rho_1 = 12 \\ 3\pi_1 + 2\pi_2 = 18 \\ \pi_1 + 2\pi_2 = 10 \\ -\pi_1 + \sigma_1 = -3 \\ -\pi_2 + \sigma_2 = -2 \end{cases}$$

From the second and third equations, $\tilde{\pi}_1 = 4$ and $\tilde{\pi}_2 = 3$; then $\tilde{\rho}_1 = 12 - 8 - 3 = 1$, $\tilde{\sigma}_1 = 4 - 3 = 1$ and $\tilde{\sigma}_2 = 3 - 2 = 1$. All these values are non-negative, hence the dual solution

$$\tilde{\pi}_1 = 4, \quad \tilde{\pi}_2 = 3, \quad \tilde{\sigma}_1 = 1, \quad \tilde{\sigma}_2 = 1, \quad \tilde{\rho}_1 = 1, \quad \tilde{\rho}_2 = 0, \quad \tilde{\rho}_3 = 0$$

is feasible, and its value is:

$$50 \cdot 4 + 50 \cdot 3 + 20 \cdot 1 + 10 \cdot 1 + 20 \cdot 1 + 15 \cdot 0 + 20 \cdot 0 = 400.$$

The primal and the dual solutions have the same value: by the strong duality theorem both are optimal, and

$$z(LP) = 400.$$

In words: it is convenient to buy all the additional resources, since at the optimum one more unit of resource i is worth $\tilde{\pi}_i$ euros, more than its price g_i ($\tilde{\pi}_1 = 4 > 3$ and $\tilde{\pi}_2 = 3 > 2$).

Exercise 16.2 — Production planning (2)

A bicycle workshop assembles city bikes and mountain bikes, which are sold at 200 euros and 400 euros, respectively. Each bike needs one frame of its type. The workshop can weld the frames in house: a city bike frame requires 2 hours of welding and 20 euros of material, while a mountain bike frame requires 3 hours of welding and 40 euros of material; 150 welding hours are available per week. Alternatively, the frames can be bought from a

supplier, at 60 euros (city bike frame) and 100 euros (mountain bike frame). The assembly of a city bike requires 1 hour of work and the assembly of a mountain bike requires 2 hours of work; 120 assembly hours are available per week. The market can absorb at most 30 mountain bikes per week. The goal of the workshop is to determine a weekly production plan that maximizes the total profit, that is, the revenue of the bikes minus the cost of the frames.

Questions

1. Write a Linear Programming (LP) model for the problem. Identify the decisions that must be made and define the corresponding decision variables. Describe the objective function of the model and its constraints. What algorithm can be used to find an optimal solution to the model?
2. Find a feasible solution of the LP model and accordingly a lower bound on its optimal solution value.
3. Write the dual model of the LP model. Find a feasible solution of the dual model and accordingly an upper bound on the optimal solution value of the LP model.
4. Prove that the plan in which the workshop welds 30 city bike frames and 30 mountain bike frames, and buys 30 city bike frames, is optimal, by finding with the complementary slackness conditions a feasible dual solution with the same objective function value. What is the optimal solution value?

Solution

Warning

We assume that the workshop can produce any fractional quantity of bikes (a bike started in a week can be completed in the next one).

1. The decision is, for each type of bike, how many bikes to assemble with a frame welded in house and how many with a frame bought from the supplier. We introduce the following four non-negative decision variables:

$$\begin{cases} m_1 = \text{city bikes assembled with a frame welded in house,} \\ m_2 = \text{mountain bikes assembled with a frame welded in house,} \\ p_1 = \text{city bikes assembled with a bought frame,} \\ p_2 = \text{mountain bikes assembled with a bought frame.} \end{cases}$$

The profit of a city bike is $200 - 20 = 180$ euros if its frame is welded in house and $200 - 60 = 140$ euros if its frame is bought; the profit of a mountain bike is $400 - 40 = 360$ and $400 - 100 = 300$ euros, respectively. Using these variables, a Linear Programming (LP) model for the problem reads as follows:

$$\begin{aligned}
 \max \quad & 180 m_1 + 360 m_2 + 140 p_1 + 300 p_2 \\
 \text{subject to} \quad & 2 m_1 + 3 m_2 \leq 150 \\
 & m_1 + 2 m_2 + p_1 + 2 p_2 \leq 120 \\
 & m_2 + p_2 \leq 30 \\
 & m_1, m_2, p_1, p_2 \geq 0
 \end{aligned}$$

Description of the objective function and constraints of the LP model:

- the linear objective function maximizes the total profit, that is, the revenue of the bikes minus the cost of the material of the welded frames and of the bought frames;
- the first linear constraint imposes that the welding hours used for the frames made in house do not exceed the available ones;
- the second linear constraint imposes that the assembly hours, needed by all the bikes, do not exceed the available ones;
- the third linear constraint imposes that the mountain bikes produced do not exceed the demand of the market;
- the final constraint defines the variables of the model.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. For example, by welding all the frames in house and buying none of them:

$$\bar{m}_1 = 30, \quad \bar{m}_2 = 30, \quad \bar{p}_1 = 0, \quad \bar{p}_2 = 0$$

we have a feasible solution: it uses $2 \cdot 30 + 3 \cdot 30 = 150$ welding hours and $30 + 2 \cdot 30 = 90 \leq 120$ assembly hours, and it produces 30 mountain bikes. The objective function value of this feasible solution is:

$$180 \cdot 30 + 360 \cdot 30 + 140 \cdot 0 + 300 \cdot 0 = 16200$$

accordingly 16200 is a lower bound on the optimal solution value of the LP model. So, we have:

$$16200 \leq z(LP)$$

3. By using a non-negative dual variable u_1 for the welding constraint, u_2 for the assembly constraint and u_3 for the market constraint, the dual of the LP model reads as follows:

$$\begin{aligned}
 \min \quad & 150 u_1 + 120 u_2 + 30 u_3 \\
 \text{subject to} \quad & 2 u_1 + u_2 \geq 180 \\
 & 3 u_1 + 2 u_2 + u_3 \geq 360 \\
 & u_2 \geq 140 \\
 & 2 u_2 + u_3 \geq 300 \\
 & u_1, u_2, u_3 \geq 0
 \end{aligned}$$

For example, by setting:

$$\bar{u}_1 = 0, \quad \bar{u}_3 = 0,$$

and by setting:

$$\bar{u}_2 = \max \left\{ 180, \frac{360}{2}, 140, \frac{300}{2} \right\} = 180$$

we have a feasible dual solution. The objective function value of this feasible dual solution is:

$$150 \cdot 0 + 120 \cdot 180 + 30 \cdot 0 = 21600$$

Accordingly, 21600 is an upper bound on the optimal solution value of the LP model. So, we have:

$$16200 \leq z(LP) \leq 21600$$

4. The plan of the question is the solution

$$\tilde{m}_1 = 30, \quad \tilde{m}_2 = 30, \quad \tilde{p}_1 = 30, \quad \tilde{p}_2 = 0.$$

It is feasible: it uses $2 \cdot 30 + 3 \cdot 30 = 150$ welding hours and $30 + 2 \cdot 30 + 30 = 120$ assembly hours, and it produces 30 mountain bikes. Its value is $180 \cdot 30 + 360 \cdot 30 + 140 \cdot 30 = 20400$.

The variables $\tilde{m}_1$, $\tilde{m}_2$ and $\tilde{p}_1$ are positive, hence, by the complementary slackness conditions, the first three dual constraints are satisfied at equality:

$$\begin{cases} 2u_1 + u_2 = 180 \\ 3u_1 + 2u_2 + u_3 = 360 \\ u_2 = 140 \end{cases}$$

whose solution is $\tilde{u}_2 = 140$, $\tilde{u}_1 = (180 - 140)/2 = 20$ and $\tilde{u}_3 = 360 - 60 - 280 = 20$. The three values are non-negative and the fourth dual constraint is satisfied, since $2 \cdot 140 + 20 = 300 \geq 300$: the dual solution

$$\tilde{u}_1 = 20, \quad \tilde{u}_2 = 140, \quad \tilde{u}_3 = 20$$

is feasible, and its value is:

$$150 \cdot 20 + 120 \cdot 140 + 30 \cdot 20 = 20400.$$

The primal and the dual solutions have the same value: by the strong duality theorem both are optimal, and

$$z(LP) = 20400.$$

The dual values are the marginal values of the resources: one more welding hour is worth 20 euros, one more assembly hour 140 euros, and one more mountain bike sold on the market 20 euros.

Exercise 16.3 — Blending (1)

A foundry has to produce $d \in \mathbb{Q}_{>0}$ tonnes of an alloy by melting together $r \in \mathbb{Z}_{\geq 1}$ raw materials (ores and metal scraps). For each raw material $j \in \{1, 2, \dots, r\}$, the value $c_j \in \mathbb{Q}_{>0}$ is the cost in thousands of euros of one tonne and the value $a_j \in \mathbb{Q}_{>0}$ is the number of tonnes available. The quality of the alloy depends on $s \in \mathbb{Z}_{\geq 1}$ chemical elements. For each element $k \in \{1, 2, \dots, s\}$ and each raw material $j \in \{1, 2, \dots, r\}$, the value $p_{kj} \in \mathbb{Q}_{\geq 0}$ is the percentage of element k contained in raw material j . The percentage of each element $k \in \{1, 2, \dots, s\}$ in the alloy has to be at least $\ell_k \in \mathbb{Q}_{\geq 0}$ and at most $u_k \in \mathbb{Q}_{>0}$, with $\ell_k \leq u_k$. The foundry aims to determine a minimum cost blend subject to the constraints described above.

Questions

1. Write a Linear Programming (LP) model for the problem. Identify the decisions that must be taken and the corresponding decision variables. Identify and comment the objective function of the model and the constraints. What algorithm can be used to find an optimal solution of the model?
2. Consider now an instance of the problem in which the foundry has to produce $d = 100$ tonnes of alloy from $r = 3$ raw materials, and the quality depends on $s = 2$ elements. These are the other input data:

	$j = 1$	$j = 2$	$j = 3$		$k = 1$	$k = 2$
c_j	5	2	4	ℓ_k	2	2
a_j	40	60	50	u_k	3	3
p_{1j}	4	1	2			
p_{2j}	1	3	2			

Write the LP model for this instance.

3. Find a feasible solution of the LP model of the instance and accordingly an upper bound on the optimal solution value of the LP model for the instance.
4. Write the dual model of the LP model and the dual model for the instance. Find a feasible solution of the dual model of the instance and accordingly a lower bound on the optimal solution value of the LP model for the instance.
5. Prove that the blend with $\tilde{x}_1 = 30$, $\tilde{x}_2 = 60$ and $\tilde{x}_3 = 10$ tonnes of the three raw materials is optimal, by finding with the complementary slackness conditions a feasible dual solution with the same objective function value. What is the optimal solution value?

Solution

1. The input data of the model are the following ones:

Symbol	Type	Meaning
d	$\mathbb{Q}_{>0}$	tonnes of alloy to be produced
r	$\mathbb{Z}_{\geq 1}$	number of raw materials
s	$\mathbb{Z}_{\geq 1}$	number of chemical elements
c_j	$\mathbb{Q}_{>0}$	cost of one tonne of raw material j (thousands of euros)
a_j	$\mathbb{Q}_{>0}$	tonnes of raw material j available
p_{kj}	$\mathbb{Q}_{\geq 0}$	percentage of element k in raw material j
ℓ_k	$\mathbb{Q}_{\geq 0}$	minimum percentage of element k in the alloy
u_k	$\mathbb{Q}_{>0}$	maximum percentage of element k in the alloy

We introduce the following r non-negative variables:

$$x_j = \text{tonnes of raw material } j \text{ used in the blend,} \quad \forall j \in \{1, 2, \dots, r\}.$$

The percentage of element k in the alloy is the ratio

$$\frac{\sum_{j=1}^r p_{kj} x_j}{\sum_{j=1}^r x_j},$$

which is not a linear function of the variables. Since its denominator is equal to $d > 0$, the requirement that this ratio is at least ℓ_k is written, by multiplying both sides by the denominator, as $\sum_j p_{kj} x_j \geq \ell_k \sum_j x_j$, that is, $\sum_j (p_{kj} - \ell_k) x_j \geq 0$; the same holds for the maximum percentage u_k . Using these variables, an LP model for the problem reads as follows:

$$\min \sum_{j=1}^r c_j x_j \quad (16.3a)$$

$$\text{subject to } \sum_{j=1}^r x_j = d, \quad (16.3b)$$

$$\sum_{j=1}^r (p_{kj} - \ell_k) x_j \geq 0, \quad \forall k \in \{1, 2, \dots, s\}, \quad (16.3c)$$

$$\sum_{j=1}^r (p_{kj} - u_k) x_j \leq 0, \quad \forall k \in \{1, 2, \dots, s\}, \quad (16.3d)$$

$$x_j \leq a_j, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.3e)$$

$$x_j \geq 0, \quad \forall j \in \{1, 2, \dots, r\}. \quad (16.3f)$$

Description of the objective function and constraints of the LP model:

- the linear objective function (16.3a) minimizes the total cost of the raw materials used;

- the linear constraint (16.3b) ensures that exactly d tonnes of alloy are produced (one linear constraint);
- the linear constraints (16.3c) ensure that, for each element, its percentage in the alloy is at least the minimum one (s linear constraints);
- the linear constraints (16.3d) ensure that, for each element, its percentage in the alloy is at most the maximum one (s linear constraints);
- the linear constraints (16.3e) ensure that, for each raw material, the quantity used does not exceed the quantity available (r linear constraints);
- the linear constraints (16.3f) define the variables of the LP model.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. The coefficients of the constraints (16.3c) and (16.3d) are $p_{1j} - \ell_1 = (2, -1, 0)$, $p_{2j} - \ell_2 = (-1, 1, 0)$, $p_{1j} - u_1 = (1, -2, -1)$ and $p_{2j} - u_2 = (-2, 0, -1)$. The LP model (16.3) for the instance reads as follows:

$$\begin{array}{ll}
 \min & 5x_1 + 2x_2 + 4x_3 \\
 \text{subject to} & x_1 + x_2 + x_3 = 100 \\
 & 2x_1 - x_2 \geq 0 \\
 & -x_1 + x_2 \geq 0 \\
 & x_1 - 2x_2 - x_3 \leq 0 \\
 & -2x_1 - x_3 \leq 0 \\
 & x_1 \leq 40 \\
 & x_2 \leq 60 \\
 & x_3 \leq 50 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

3. For example, by setting:

$$\bar{x}_1 = 40, \quad \bar{x}_2 = 40, \quad \bar{x}_3 = 20$$

we have a feasible solution: the alloy weighs 100 tonnes, it contains $(4 \cdot 40 + 40 + 2 \cdot 20)/100 = 2.4$ percent of element 1 and $(40 + 3 \cdot 40 + 2 \cdot 20)/100 = 2$ percent of element 2, and the availabilities are respected. The objective function value of this feasible solution is:

$$\sum_{j=1}^r c_j \bar{x}_j = 5 \cdot 40 + 2 \cdot 40 + 4 \cdot 20 = 360$$

accordingly 360 is an upper bound on the optimal solution value of the LP model (16.3). So, for this instance, we have:

$$z(LP) \leq 360$$

4. By using

- a free dual variable μ for the constraint (16.3b)
- a non-negative dual variable α_k for each constraint (16.3c) ($k \in \{1, 2, \dots, s\}$)
- a non-positive dual variable β_k for each constraint (16.3d) ($k \in \{1, 2, \dots, s\}$)
- a non-positive dual variable γ_j for each constraint (16.3e) ($j \in \{1, 2, \dots, r\}$)

the dual of the LP model (16.3) reads as follows:

$$\max \quad d \mu + \sum_{j=1}^r a_j \gamma_j \quad (16.4a)$$

$$\begin{aligned} \text{subject to} \quad & \mu + \sum_{k=1}^s (p_{kj} - \ell_k) \alpha_k \\ & + \sum_{k=1}^s (p_{kj} - u_k) \beta_k + \gamma_j \leq c_j, \quad \forall j \in \{1, 2, \dots, r\}, \end{aligned} \quad (16.4b)$$

$$\mu \geq 0, \quad (16.4c)$$

$$\alpha_k \geq 0, \quad \forall k \in \{1, 2, \dots, s\}, \quad (16.4d)$$

$$\beta_k \leq 0, \quad \forall k \in \{1, 2, \dots, s\}, \quad (16.4e)$$

$$\gamma_j \leq 0, \quad \forall j \in \{1, 2, \dots, r\}. \quad (16.4f)$$

The right-hand sides of the constraints (16.3c) and (16.3d) are equal to 0, hence the variables α_k and β_k do not appear in the objective function. The dual model (16.4) for the instance reads as follows:

$$\begin{aligned} \max \quad & 100 \mu + 40 \gamma_1 + 60 \gamma_2 + 50 \gamma_3 \\ \text{subject to} \quad & \mu + 2 \alpha_1 - \alpha_2 + \beta_1 - 2 \beta_2 + \gamma_1 \leq 5 \\ & \mu - \alpha_1 + \alpha_2 - 2 \beta_1 + \gamma_2 \leq 2 \\ & \mu - \beta_1 - \beta_2 + \gamma_3 \leq 4 \\ & \mu \geq 0 \\ & \alpha_1, \alpha_2 \geq 0 \\ & \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3 \leq 0 \end{aligned}$$

For example, by setting:

$$\bar{\alpha}_1 = \bar{\alpha}_2 = 0, \quad \bar{\beta}_1 = \bar{\beta}_2 = 0, \quad \bar{\gamma}_1 = \bar{\gamma}_2 = \bar{\gamma}_3 = 0,$$

and by setting:

$$\bar{\mu} = \min\{5, 2, 4\} = 2$$

we have a feasible dual solution. The objective function value of this feasible dual solution is:

$$d \bar{\mu} + \sum_{j=1}^r a_j \bar{\gamma}_j = 100 \cdot 2 + 40 \cdot 0 + 60 \cdot 0 + 50 \cdot 0 = 200$$

Accordingly, 200 is a lower bound on the optimal solution value of the LP model. So, for this instance, we have:

$$200 \leq z(LP) \leq 360$$

This lower bound is the cost of an alloy made only with the cheapest raw material, which ignores the quality and availability constraints.

5. The blend $\tilde{x}$ is feasible: the alloy weighs 100 tonnes, it contains $(4 \cdot 30 + 60 + 2 \cdot 10)/100 = 2$ percent of element 1 and $(30 + 3 \cdot 60 + 2 \cdot 10)/100 = 2.3$ percent of element 2, and the availabilities are respected. Its value is $5 \cdot 30 + 2 \cdot 60 + 4 \cdot 10 = 310$.

The complementary slackness conditions give:

- the constraints (16.3c) for $k = 2$ and (16.3d) for $k = 1$ and $k = 2$ are not tight (their left-hand sides are 30, -100 and -70), hence $\tilde{\alpha}_2 = 0$, $\tilde{\beta}_1 = 0$ and $\tilde{\beta}_2 = 0$;
- the constraints (16.3e) for $j = 1$ and $j = 3$ are not tight ($30 < 40$ and $10 < 50$), hence $\tilde{\gamma}_1 = 0$ and $\tilde{\gamma}_3 = 0$;
- the three primal variables are positive: the three dual constraints (16.4b) are satisfied at equality:

$$\begin{cases} \mu + 2\alpha_1 = 5 \\ \mu - \alpha_1 + \gamma_2 = 2 \\ \mu = 4 \end{cases}$$

The solution of this system is $\tilde{\mu} = 4$, $\tilde{\alpha}_1 = \frac{1}{2}$ and $\tilde{\gamma}_2 = 2 - 4 + \frac{1}{2} = -\frac{3}{2}$. Since $\tilde{\alpha}_1 \geq 0$ and $\tilde{\gamma}_2 \leq 0$, the dual solution

$$\tilde{\mu} = 4, \quad \tilde{\alpha}_1 = \frac{1}{2}, \quad \tilde{\alpha}_2 = 0, \quad \tilde{\beta}_1 = \tilde{\beta}_2 = 0, \quad \tilde{\gamma}_1 = 0, \quad \tilde{\gamma}_2 = -\frac{3}{2}, \quad \tilde{\gamma}_3 = 0$$

is feasible, and its value is:

$$100 \cdot 4 + 40 \cdot 0 + 60 \cdot \left(-\frac{3}{2}\right) + 50 \cdot 0 = 310.$$

The primal and the dual solutions have the same value: by the strong duality theorem both are optimal, and

$$z(LP) = 310.$$

Exercise 16.4 — Transportation (1)

A company produces a liquid detergent in $s \in \mathbb{Z}_{\geq 1}$ plants and delivers it to $r \in \mathbb{Z}_{\geq 1}$ clients. The product cannot be shipped directly from a plant to a client: it has to pass through one

of $q \in \mathbb{Z}_{\geq 1}$ depots. For each plant $i \in \{1, 2, \dots, s\}$, the value $a_i \in \mathbb{Q}_{>0}$ is the quantity in thousands of liters that the plant can produce per week. For each depot $k \in \{1, 2, \dots, q\}$, the value $h_k \in \mathbb{Q}_{>0}$ is the maximum quantity in thousands of liters that the depot can handle per week. For each client $j \in \{1, 2, \dots, r\}$, the value $d_j \in \mathbb{Q}_{>0}$ is the weekly demand in thousands of liters. For each plant $i \in \{1, 2, \dots, s\}$ and each depot $k \in \{1, 2, \dots, q\}$, the value $c_{ik} \in \mathbb{Q}_{\geq 0}$ is the cost in euros of transporting one thousand liters from plant i to depot k ; for each depot $k \in \{1, 2, \dots, q\}$ and each client $j \in \{1, 2, \dots, r\}$, the value $g_{kj} \in \mathbb{Q}_{\geq 0}$ is the cost in euros of transporting one thousand liters from depot k to client j . The depots do not keep any stock. The company aims to determine a weekly minimum-cost transportation plan subject to the constraints described above.

Questions

1. Write a Linear Programming (LP) model for the problem. Identify the decisions that must be taken and the corresponding decision variables. Identify and comment the objective function of the model and the constraints. What algorithm can be used to find an optimal solution of the model?
2. Consider now an instance of the problem with $s = 2$ plants, $q = 2$ depots and $r = 2$ clients. These are the other input data:

c_{ik}	$k = 1$	$k = 2$	g_{kj}	$j = 1$	$j = 2$		1	2
$i = 1$	2	4	$k = 1$	3	5	a_i	50	40
$i = 2$	3	1	$k = 2$	4	2	h_k	50	40
						d_j	30	50

Write the LP model for this instance.

3. Find a feasible solution of the LP model of the instance and accordingly an upper bound on the optimal solution value of the LP model for the instance.
4. Write the dual model of the LP model and the dual model for the instance. Find a feasible solution of the dual model of the instance and accordingly a lower bound on the optimal solution value of the LP model for the instance.
5. Consider the following transportation plan: plant 1 sends 40 thousand liters to depot 1 and plant 2 sends 40 thousand liters to depot 2; depot 1 sends 30 thousand liters to client 1 and 10 thousand liters to client 2, and depot 2 sends 40 thousand liters to client 2. Prove that it is optimal by finding, with the complementary slackness conditions, a feasible dual solution with the same objective function value. What is the optimal solution value?

Solution

1. The input data of the model are the following ones:

Symbol	Type	Meaning
s	$\mathbb{Z}_{\geq 1}$	number of plants
q	$\mathbb{Z}_{\geq 1}$	number of depots
r	$\mathbb{Z}_{\geq 1}$	number of clients
a_i	$\mathbb{Q}_{>0}$	weekly production capacity of plant i
h_k	$\mathbb{Q}_{>0}$	weekly handling capacity of depot k
d_j	$\mathbb{Q}_{>0}$	weekly demand of client j
c_{ik}	$\mathbb{Q}_{\geq 0}$	unit transportation cost from plant i to depot k
g_{kj}	$\mathbb{Q}_{\geq 0}$	unit transportation cost from depot k to client j

We introduce the following $s q + q r$ non-negative variables:

$$x_{ik} = \text{thousands of liters sent from plant } i \text{ to depot } k, \\ \forall i \in \{1, 2, \dots, s\}, \forall k \in \{1, 2, \dots, q\},$$

$$y_{kj} = \text{thousands of liters sent from depot } k \text{ to client } j, \\ \forall k \in \{1, 2, \dots, q\}, \forall j \in \{1, 2, \dots, r\}.$$

Using these variables, an LP model for the problem reads as follows:

$$\min \sum_{i=1}^s \sum_{k=1}^q c_{ik} x_{ik} + \sum_{k=1}^q \sum_{j=1}^r g_{kj} y_{kj} \quad (16.5a)$$

$$\text{subject to } \sum_{k=1}^q x_{ik} \leq a_i, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.5b)$$

$$\sum_{i=1}^s x_{ik} - \sum_{j=1}^r y_{kj} = 0, \quad \forall k \in \{1, 2, \dots, q\}, \quad (16.5c)$$

$$\sum_{i=1}^s x_{ik} \leq h_k, \quad \forall k \in \{1, 2, \dots, q\}, \quad (16.5d)$$

$$\sum_{k=1}^q y_{kj} \geq d_j, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.5e)$$

$$x_{ik} \geq 0, \quad \forall i \in \{1, 2, \dots, s\}, \\ \forall k \in \{1, 2, \dots, q\}, \quad (16.5f)$$

$$y_{kj} \geq 0, \quad \forall k \in \{1, 2, \dots, q\}, \\ \forall j \in \{1, 2, \dots, r\}. \quad (16.5g)$$

Description of the objective function and constraints of the LP model:

- the linear objective function (16.5a) minimizes the total transportation cost, from the plants to the depots and from the depots to the clients;

- the linear constraints (16.5b) ensure that each plant does not ship more than its production capacity (s linear constraints);
- the linear constraints (16.5c) are the *flow conservation constraints*: each depot ships to the clients exactly what it receives from the plants, since it does not keep any stock (q linear constraints);
- the linear constraints (16.5d) ensure that each depot does not handle more than its capacity (q linear constraints);
- the linear constraints (16.5e) ensure that each client receives at least its demand (r linear constraints);
- the linear constraints (16.5f) and (16.5g) define the variables of the LP model.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. The LP model (16.5) for the instance reads as follows:

$$\begin{aligned}
 \min \quad & 2x_{11} + 4x_{12} + 3x_{21} + x_{22} + 3y_{11} + 5y_{12} + 4y_{21} + 2y_{22} \\
 \text{subject to} \quad & x_{11} + x_{12} \leq 50 \\
 & x_{21} + x_{22} \leq 40 \\
 & x_{11} + x_{21} - y_{11} - y_{12} = 0 \\
 & x_{12} + x_{22} - y_{21} - y_{22} = 0 \\
 & x_{11} + x_{21} \leq 50 \\
 & x_{12} + x_{22} \leq 40 \\
 & y_{11} + y_{21} \geq 30 \\
 & y_{12} + y_{22} \geq 50 \\
 & x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22} \geq 0
 \end{aligned}$$

3. For example, plant 1 can send its whole production to depot 1, which serves client 1 and part of client 2, and plant 2 can send the rest of the demand of client 2 through depot 2:

$$\begin{aligned}
 \bar{x}_{11} &= 50, \quad \bar{x}_{12} = 0, \quad \bar{x}_{21} = 0, \quad \bar{x}_{22} = 30, \\
 \bar{y}_{11} &= 30, \quad \bar{y}_{12} = 20, \quad \bar{y}_{21} = 0, \quad \bar{y}_{22} = 30
 \end{aligned}$$

This is a feasible solution: the plants ship $50 \leq 50$ and $30 \leq 40$, the depots handle $50 \leq 50$ and $30 \leq 40$ and ship what they receive, and the clients receive 30 and $20 + 30 = 50$. The objective function value of this feasible solution is:

$$2 \cdot 50 + 1 \cdot 30 + 3 \cdot 30 + 5 \cdot 20 + 2 \cdot 30 = 380$$

accordingly 380 is an upper bound on the optimal solution value of the LP model (16.5). So, for this instance, we have:

$$z(LP) \leq 380$$

4. By using

- a non-positive dual variable λ_i for each constraint (16.5b) ($i \in \{1, 2, \dots, s\}$)
- a free dual variable μ_k for each constraint (16.5c) ($k \in \{1, 2, \dots, q\}$)
- a non-positive dual variable θ_k for each constraint (16.5d) ($k \in \{1, 2, \dots, q\}$)
- a non-negative dual variable π_j for each constraint (16.5e) ($j \in \{1, 2, \dots, r\}$)

the dual of the LP model (16.5) reads as follows:

$$\max \sum_{i=1}^s a_i \lambda_i + \sum_{k=1}^q h_k \theta_k + \sum_{j=1}^r d_j \pi_j \quad (16.6a)$$

$$\text{subject to } \lambda_i + \mu_k + \theta_k \leq c_{ik}, \quad \forall i \in \{1, 2, \dots, s\}, \quad \forall k \in \{1, 2, \dots, q\}, \quad (16.6b)$$

$$-\mu_k + \pi_j \leq g_{kj}, \quad \forall k \in \{1, 2, \dots, q\}, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.6c)$$

$$\lambda_i \leq 0, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.6d)$$

$$\mu_k \geq 0, \quad \forall k \in \{1, 2, \dots, q\}, \quad (16.6e)$$

$$\theta_k \leq 0, \quad \forall k \in \{1, 2, \dots, q\}, \quad (16.6f)$$

$$\pi_j \geq 0, \quad \forall j \in \{1, 2, \dots, r\}. \quad (16.6g)$$

The dual model (16.6) for the instance reads as follows:

$$\begin{aligned} \max \quad & 50 \lambda_1 + 40 \lambda_2 + 50 \theta_1 + 40 \theta_2 + 30 \pi_1 + 50 \pi_2 \\ \text{subject to} \quad & \lambda_1 + \mu_1 + \theta_1 \leq 2 \\ & \lambda_1 + \mu_2 + \theta_2 \leq 4 \\ & \lambda_2 + \mu_1 + \theta_1 \leq 3 \\ & \lambda_2 + \mu_2 + \theta_2 \leq 1 \\ & -\mu_1 + \pi_1 \leq 3 \\ & -\mu_1 + \pi_2 \leq 5 \\ & -\mu_2 + \pi_1 \leq 4 \\ & -\mu_2 + \pi_2 \leq 2 \\ & \lambda_1, \lambda_2, \theta_1, \theta_2 \leq 0 \\ & \mu_1, \mu_2 \geq 0 \\ & \pi_1, \pi_2 \geq 0 \end{aligned}$$

For example, by setting:

$$\bar{\lambda}_1 = \bar{\lambda}_2 = 0, \quad \bar{\theta}_1 = \bar{\theta}_2 = 0,$$

the constraints (16.6b) read $\mu_k \leq c_{ik}$, and we can set μ_k equal to the cheapest cost of reaching depot k from a plant:

$$\bar{\mu}_1 = \min\{2, 3\} = 2, \quad \bar{\mu}_2 = \min\{4, 1\} = 1.$$

Then the constraints (16.6c) read $\pi_j \leq \mu_k + g_{kj}$, and we can set π_j equal to the cheapest cost of reaching client j from a plant through a depot:

$$\bar{\pi}_1 = \min\{2 + 3, 1 + 4\} = 5, \quad \bar{\pi}_2 = \min\{2 + 5, 1 + 2\} = 3.$$

In this way we have a feasible dual solution. The objective function value of this feasible dual solution is:

$$50 \cdot 0 + 40 \cdot 0 + 50 \cdot 0 + 40 \cdot 0 + 30 \cdot 5 + 50 \cdot 3 = 300$$

Accordingly, 300 is a lower bound on the optimal solution value of the LP model: it is the cost of sending each client its demand along the cheapest route, ignoring the capacities. So, for this instance, we have:

$$300 \leq z(LP) \leq 380$$

5. The plan of the question is the solution

$$\begin{aligned} \tilde{x}_{11} &= 40, \quad \tilde{x}_{12} = 0, \quad \tilde{x}_{21} = 0, \quad \tilde{x}_{22} = 40, \\ \tilde{y}_{11} &= 30, \quad \tilde{y}_{12} = 10, \quad \tilde{y}_{21} = 0, \quad \tilde{y}_{22} = 40 \end{aligned}$$

which is feasible (the plants ship $40 \leq 50$ and $40 \leq 40$, the depots handle $40 \leq 50$ and $40 \leq 40$, and the clients receive 30 and 50), with value $2 \cdot 40 + 1 \cdot 40 + 3 \cdot 30 + 5 \cdot 10 + 2 \cdot 40 = 340$.

The complementary slackness conditions give:

- plant 1 and depot 1 are not saturated ($40 < 50$), hence $\tilde{\lambda}_1 = 0$ and $\tilde{\theta}_1 = 0$;
- the variables $\tilde{x}_{11}$, $\tilde{x}_{22}$, $\tilde{y}_{11}$, $\tilde{y}_{12}$ and $\tilde{y}_{22}$ are positive, hence the corresponding dual constraints are satisfied at equality:

$$\begin{cases} \lambda_1 + \mu_1 + \theta_1 = 2 \\ \lambda_2 + \mu_2 + \theta_2 = 1 \\ -\mu_1 + \pi_1 = 3 \\ -\mu_1 + \pi_2 = 5 \\ -\mu_2 + \pi_2 = 2 \end{cases}$$

We obtain $\tilde{\mu}_1 = 2$, $\tilde{\pi}_1 = 5$, $\tilde{\pi}_2 = 7$, $\tilde{\mu}_2 = 5$ and $\lambda_2 + \theta_2 = -4$. The last equation has many solutions: we have to choose one that satisfies also the dual constraints of x_{12} and x_{21} , that is, $\theta_2 \leq 4 - 5 = -1$ and $\lambda_2 \leq 3 - 2 = 1$, and the sign conditions. For example, $\tilde{\theta}_2 = -1$ and $\tilde{\lambda}_2 = -3$. The remaining dual constraint, of y_{21} , is satisfied: $-5 + 5 = 0 \leq 4$. The dual solution

$$\tilde{\lambda}_1 = 0, \quad \tilde{\lambda}_2 = -3, \quad \tilde{\mu}_1 = 2, \quad \tilde{\mu}_2 = 5, \quad \tilde{\theta}_1 = 0, \quad \tilde{\theta}_2 = -1, \quad \tilde{\pi}_1 = 5, \quad \tilde{\pi}_2 = 7$$

is then feasible, and its value is:

$$40 \cdot (-3) + 40 \cdot (-1) + 30 \cdot 5 + 50 \cdot 7 = 340.$$

The primal and the dual solutions have the same value: by the strong duality theorem both are optimal, and

$$z(LP) = 340.$$

Exercise 16.5 — Multi-period planning (1)

A company plans the production of a fertilizer over a time horizon of $n \in \mathbb{Z}_{\geq 2}$ weeks. For each week $t \in \{1, 2, \dots, n\}$, the value $d_t \in \mathbb{Q}_{\geq 0}$ is the demand in tonnes, the value $p_t \in \mathbb{Q}_{>0}$ is the maximum quantity in tonnes that can be produced, and the value $c_t \in \mathbb{Q}_{>0}$ is the production cost in euros of one tonne. The product left at the end of a week can be stored and sold in the following weeks: for each week $t \in \{1, 2, \dots, n-1\}$, storing one tonne from week t to week $t+1$ costs $h_t \in \mathbb{Q}_{>0}$ euros. Moreover, the demand of a week can be satisfied late (*backlogging*): for each week $t \in \{1, 2, \dots, n-1\}$, each tonne of demand not yet satisfied at the end of week t costs a penalty of $g_t \in \mathbb{Q}_{>0}$ euros. There is no stock at the beginning of the horizon, and all the demand has to be satisfied by the end of week n . The company aims to determine a minimum-cost production plan subject to the constraints described above.

Questions

1. Write a Linear Programming (LP) model for the problem. Identify the decisions that must be taken and the corresponding decision variables. Identify and comment the objective function of the model and the constraints. What algorithm can be used to find an optimal solution of the model?
2. Consider now an instance of the problem with $n = 3$ weeks and the following input data:

	$t = 1$	$t = 2$	$t = 3$
d_t	30	50	20
p_t	40	30	40
c_t	10	12	11
h_t	2	2	—
g_t	3	3	—

Write the LP model for this instance.

3. Find a feasible solution of the LP model of the instance and accordingly an upper bound on the optimal solution value of the LP model for the instance.
4. Write the dual model of the LP model and the dual model for the instance. Find a feasible solution of the dual model of the instance and accordingly a lower bound on the optimal solution value of the LP model for the instance.

5. Consider the plan that produces 40, 30 and 30 tonnes in the three weeks, stores 10 tonnes at the end of week 1 and satisfies 10 tonnes of the demand of week 2 in week 3. Prove that it is optimal by finding, with the complementary slackness conditions, a feasible dual solution with the same objective function value. What is the optimal solution value?

Solution

1. The input data of the model are the following ones:

Symbol	Type	Meaning
n	$\mathbb{Z}_{\geq 2}$	number of weeks
d_t	$\mathbb{Q}_{\geq 0}$	demand of week t (tonnes)
p_t	$\mathbb{Q}_{> 0}$	production capacity of week t (tonnes)
c_t	$\mathbb{Q}_{> 0}$	production cost of one tonne in week t (euros)
h_t	$\mathbb{Q}_{> 0}$	cost of storing one tonne from week t to week $t + 1$ (euros)
g_t	$\mathbb{Q}_{> 0}$	penalty of one tonne of demand not satisfied at the end of week t (euros)

We introduce the following $n + 2(n - 1)$ non-negative variables:

$$\begin{aligned}
 x_t &= \text{tonnes produced in week } t, & \forall t \in \{1, 2, \dots, n\}, \\
 \begin{cases} s_t = & \text{tonnes in stock at the end of week } t \\ b_t = & \text{tonnes backlogged at the end of week } t \end{cases} & \forall t \in \{1, 2, \dots, n - 1\}.
 \end{aligned}$$

In each week, the product available (production, plus stock of the previous week) is used for the demand of the week and for the demand backlogged from the previous week, and what is left goes in stock; with the variables above, the *inventory balance* of week t is: $x_t + s_{t-1} - b_{t-1} - s_t + b_t = d_t$. In the first week there is no stock and no backlog from the previous week, and in the last week nothing is left in stock and no demand is backlogged. Using these variables, an LP model for the problem reads as follows:

$$\min \sum_{t=1}^n c_t x_t + \sum_{t=1}^{n-1} h_t s_t + \sum_{t=1}^{n-1} g_t b_t \quad (16.7a)$$

$$\text{subject to } x_1 - s_1 + b_1 = d_1, \quad (16.7b)$$

$$x_t + s_{t-1} - b_{t-1} - s_t + b_t = d_t, \quad \forall t \in \{2, 3, \dots, n-1\}, \quad (16.7c)$$

$$x_n + s_{n-1} - b_{n-1} = d_n, \quad (16.7d)$$

$$x_t \leq p_t, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.7e)$$

$$x_t \geq 0, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.7f)$$

$$s_t \geq 0, \quad \forall t \in \{1, 2, \dots, n-1\}, \quad (16.7g)$$

$$b_t \geq 0, \quad \forall t \in \{1, 2, \dots, n-1\}. \quad (16.7h)$$

Description of the objective function and constraints of the LP model:

- the linear objective function (16.7a) minimizes the total cost of production, storage and backlogging;
- the linear constraint (16.7b) is the inventory balance of the first week (one linear constraint);
- the linear constraints (16.7c) are the inventory balances of the intermediate weeks: they link each week to the previous one ($n-2$ linear constraints);
- the linear constraint (16.7d) is the inventory balance of the last week, in which all the backlogged demand is satisfied (one linear constraint);
- the linear constraints (16.7e) ensure that the production does not exceed the capacity (n linear constraints);
- the linear constraints (16.7f), (16.7g) and (16.7h) define the variables of the LP model.

A solution with both $s_t > 0$ and $b_t > 0$ in the same week is feasible but never convenient: decreasing both by the same amount leaves the balances unchanged and decreases the cost, since $h_t > 0$ and $g_t > 0$.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. The LP model (16.7) for the instance reads as follows:

$$\begin{aligned}
 \min \quad & 10x_1 + 12x_2 + 11x_3 + 2s_1 + 2s_2 + 3b_1 + 3b_2 \\
 \text{subject to} \quad & x_1 - s_1 + b_1 = 30 \\
 & x_2 + s_1 - s_2 - b_1 + b_2 = 50 \\
 & x_3 + s_2 - b_2 = 20 \\
 & x_1 \leq 40 \\
 & x_2 \leq 30 \\
 & x_3 \leq 40 \\
 & x_1, x_2, x_3, s_1, s_2, b_1, b_2 \geq 0
 \end{aligned}$$

3. For example, by producing in each week its demand, up to the capacity, and by satisfying in week 3 the part of the demand of week 2 that exceeds the capacity:

$$\bar{x}_1 = 30, \bar{x}_2 = 30, \bar{x}_3 = 40, \bar{s}_1 = 0, \bar{s}_2 = 0, \bar{b}_1 = 0, \bar{b}_2 = 20$$

we have a feasible solution: the three balances read $30 = 30$, $30 + 20 = 50$ and $40 - 20 = 20$, and the capacities are respected. The objective function value of this feasible solution is:

$$10 \cdot 30 + 12 \cdot 30 + 11 \cdot 40 + 2 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 + 3 \cdot 20 = 1160$$

accordingly 1160 is an upper bound on the optimal solution value of the LP model (16.7). So, for this instance, we have:

$$z(LP) \leq 1160$$

4. By using

- a free dual variable μ_1 for the constraint (16.7b)
- a free dual variable μ_t for each constraint (16.7c) ($t \in \{2, 3, \dots, n-1\}$)
- a free dual variable μ_n for the constraint (16.7d)
- a non-positive dual variable π_t for each constraint (16.7e) ($t \in \{1, 2, \dots, n\}$)

the dual of the LP model (16.7) reads as follows:

$$\max \sum_{t=1}^n d_t \mu_t + \sum_{t=1}^n p_t \pi_t \quad (16.8a)$$

$$\text{subject to } \mu_t + \pi_t \leq c_t, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.8b)$$

$$-\mu_t + \mu_{t+1} \leq h_t, \quad \forall t \in \{1, 2, \dots, n-1\}, \quad (16.8c)$$

$$\mu_t - \mu_{t+1} \leq g_t, \quad \forall t \in \{1, 2, \dots, n-1\}, \quad (16.8d)$$

$$\mu_t \geq 0, \quad \forall t \in \{1, 2, \dots, n\}, \quad (16.8e)$$

$$\pi_t \leq 0, \quad \forall t \in \{1, 2, \dots, n\}. \quad (16.8f)$$

The dual model (16.8) for the instance reads as follows:

$$\begin{aligned}
 \max \quad & 30 \mu_1 + 50 \mu_2 + 20 \mu_3 + 40 \pi_1 + 30 \pi_2 + 40 \pi_3 \\
 \text{subject to} \quad & \mu_1 + \pi_1 \leq 10 \\
 & \mu_2 + \pi_2 \leq 12 \\
 & \mu_3 + \pi_3 \leq 11 \\
 & -\mu_1 + \mu_2 \leq 2 \\
 & -\mu_2 + \mu_3 \leq 2 \\
 & \mu_1 - \mu_2 \leq 3 \\
 & \mu_2 - \mu_3 \leq 3 \\
 & \mu_1, \mu_2, \mu_3 \geq 0 \\
 & \pi_1, \pi_2, \pi_3 \leq 0
 \end{aligned}$$

For example, by setting:

$$\bar{\pi}_1 = \bar{\pi}_2 = \bar{\pi}_3 = 0,$$

and by setting μ_t equal to the production cost of week t :

$$\bar{\mu}_1 = 10, \quad \bar{\mu}_2 = 12, \quad \bar{\mu}_3 = 11,$$

the constraints (16.8b) are satisfied at equality, and also the constraints (16.8c) and (16.8d) are satisfied, since $-10 + 12 = 2 \leq 2$, $-12 + 11 = -1 \leq 2$, $10 - 12 = -2 \leq 3$ and $12 - 11 = 1 \leq 3$. Hence we have a feasible dual solution. The objective function value of this feasible dual solution is:

$$30 \cdot 10 + 50 \cdot 12 + 20 \cdot 11 + 40 \cdot 0 + 30 \cdot 0 + 40 \cdot 0 = 1120$$

Accordingly, 1120 is a lower bound on the optimal solution value of the LP model: it is the cost of producing the demand of each week in the same week, ignoring the capacities. So, for this instance, we have:

$$1120 \leq z(LP) \leq 1160$$

5. The plan of the question is the solution

$$\tilde{x}_1 = 40, \quad \tilde{x}_2 = 30, \quad \tilde{x}_3 = 30, \quad \tilde{s}_1 = 10, \quad \tilde{s}_2 = 0, \quad \tilde{b}_1 = 0, \quad \tilde{b}_2 = 10,$$

which is feasible: the three balances read $40 - 10 = 30$, $30 + 10 + 10 = 50$ and $30 - 10 = 20$, and the capacities are respected. Its value is $10 \cdot 40 + 12 \cdot 30 + 11 \cdot 30 + 2 \cdot 10 + 3 \cdot 10 = 1140$.

The complementary slackness conditions give:

- the capacity of week 3 is not saturated ($30 < 40$), hence $\tilde{\pi}_3 = 0$;

- the variables $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{s}_1$ and $\tilde{b}_2$ are positive, hence the corresponding dual constraints are satisfied at equality:

$$\begin{cases} \mu_1 + \pi_1 = 10 \\ \mu_2 + \pi_2 = 12 \\ \mu_3 = 11 \\ -\mu_1 + \mu_2 = 2 \\ \mu_2 - \mu_3 = 3 \end{cases}$$

We obtain $\tilde{\mu}_3 = 11, \tilde{\mu}_2 = 14, \tilde{\mu}_1 = 12, \tilde{\pi}_1 = -2$ and $\tilde{\pi}_2 = -2$. The remaining dual constraints, of s_2 and b_1 , are satisfied: $-14 + 11 = -3 \leq 2$ and $12 - 14 = -2 \leq 3$; moreover $\tilde{\pi}_1, \tilde{\pi}_2 \leq 0$. Hence the dual solution

$$\tilde{\mu}_1 = 12, \tilde{\mu}_2 = 14, \tilde{\mu}_3 = 11, \tilde{\pi}_1 = -2, \tilde{\pi}_2 = -2, \tilde{\pi}_3 = 0$$

is feasible, and its value is:

$$30 \cdot 12 + 50 \cdot 14 + 20 \cdot 11 + 40 \cdot (-2) + 30 \cdot (-2) + 40 \cdot 0 = 1140.$$

The primal and the dual solutions have the same value: by the strong duality theorem both are optimal, and

$$z(LP) = 1140.$$

Exercise 16.6 — Min-max objective (1)

A printing company has to print $r \in \mathbb{Z}_{\geq 1}$ jobs (for example, the copies of a book or of a catalogue) on $s \in \mathbb{Z}_{\geq 1}$ printing machines, which start working at the same time. Each job can be split among several machines: each machine prints a fraction of the copies of the job. For each machine $i \in \{1, 2, \dots, s\}$ and each job $j \in \{1, 2, \dots, r\}$, the value $t_{ij} \in \mathbb{Q}_{>0}$ is the number of hours that machine i would need to print the whole job j alone; printing a fraction of the job takes the same fraction of this time. The company aims to split the jobs among the machines so as to minimize the time at which the last machine finishes its work (the *makespan*).

Questions

1. Write a Linear Programming (LP) model for the problem. Identify the decisions that must be taken and the corresponding decision variables. Identify and comment the objective function of the model and the constraints. What algorithm can be used to find an optimal solution of the model?
2. Consider now an instance of the problem with $s = 2$ machines and $r = 3$ jobs, and the following processing times:

t_{ij}	$j = 1$	$j = 2$	$j = 3$
$i = 1$	2	4	4
$i = 2$	4	12	12

Write the LP model for this instance.

3. Find a feasible solution of the LP model of the instance and accordingly an upper bound on the optimal solution value of the LP model for the instance.
4. Write the dual model of the LP model and the dual model for the instance. Find a feasible solution of the dual model of the instance and accordingly a lower bound on the optimal solution value of the LP model for the instance.
5. Consider the solution in which machine 1 prints $\frac{3}{4}$ of job 2 and the whole job 3, while machine 2 prints the whole job 1 and $\frac{1}{4}$ of job 2. Prove that it is optimal by finding, with the complementary slackness conditions, a feasible dual solution with the same objective function value. What is the optimal solution value?

Solution

1. The input data of the model are the following ones:

Symbol	Type	Meaning
s	$\mathbb{Z}_{\geq 1}$	number of machines
r	$\mathbb{Z}_{\geq 1}$	number of jobs
t_{ij}	$\mathbb{Q}_{>0}$	hours needed by machine i to print the whole job j

We introduce the following $s \cdot r$ non-negative variables:

$$x_{ij} = \text{fraction of job } j \text{ printed by machine } i, \\ \forall i \in \{1, 2, \dots, s\}, \forall j \in \{1, 2, \dots, r\}.$$

The working time of machine i is $\sum_{j=1}^r t_{ij} x_{ij}$, and the makespan is the maximum of these s linear functions: the objective function

$$\min \max_{i \in \{1, 2, \dots, s\}} \left\{ \sum_{j=1}^r t_{ij} x_{ij} \right\}$$

is not linear. We linearize it by introducing one additional free variable:

$$w = \text{makespan (hours)}.$$

Using these variables, an LP model for the problem reads as follows:

$$\min w \quad (16.9a)$$

$$\text{subject to } \sum_{i=1}^s x_{ij} = 1, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.9b)$$

$$w - \sum_{j=1}^r t_{ij} x_{ij} \geq 0, \quad \forall i \in \{1, 2, \dots, s\}, \quad (16.9c)$$

$$x_{ij} \geq 0, \quad \forall i \in \{1, 2, \dots, s\}, \forall j \in \{1, 2, \dots, r\}, \quad (16.9d)$$

$$w \geq 0. \quad (16.9e)$$

Description of the objective function and constraints of the LP model:

- the linear objective function (16.9a) minimizes the makespan;
- the linear constraints (16.9b) ensure that each job is printed completely (r linear constraints);
- the linear constraints (16.9c) ensure that w is not smaller than the working time of each machine, that is, not smaller than their maximum (s linear constraints). Since w is minimized, at an optimal solution w is equal to the working time of the busiest machine;
- the linear constraints (16.9d) and (16.9e) define the variables of the LP model.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. The LP model (16.9) for the instance reads as follows:

$$\begin{array}{llllllll} \min & & & & & & & w \\ \text{subject to} & x_{11} & & + & x_{21} & & & = 1 \\ & & x_{12} & & & + & x_{22} & = 1 \\ & & & x_{13} & & + & x_{23} & = 1 \\ & -2x_{11} - 4x_{12} - 4x_{13} & & & & & + w & \geq 0 \\ & & & & -4x_{21} - 12x_{22} - 12x_{23} + w & & & \geq 0 \\ & x_{11}, & x_{12}, & x_{13}, & x_{21}, & x_{22}, & x_{23} & \geq 0 \\ & & & & & & & w \geq 0 \end{array}$$

3. For example, by assigning each job entirely to one machine, jobs 2 and 3 to machine 1 and job 1 to machine 2:

$$\bar{x}_{11} = 0, \quad \bar{x}_{12} = 1, \quad \bar{x}_{13} = 1, \quad \bar{x}_{21} = 1, \quad \bar{x}_{22} = 0, \quad \bar{x}_{23} = 0,$$

the working times of the two machines are $4 + 4 = 8$ and 4 hours, and by setting

$$\bar{w} = \max\{8, 4\} = 8$$

we have a feasible solution. Its objective function value is 8, accordingly 8 is an upper bound on the optimal solution value of the LP model (16.9). So, for this instance, we have:

$$z(LP) \leq 8$$

4. By using

- a free dual variable μ_j for each constraint (16.9b) ($j \in \{1, 2, \dots, r\}$)
- a non-negative dual variable λ_i for each constraint (16.9c) ($i \in \{1, 2, \dots, s\}$)

the dual of the LP model (16.9) reads as follows:

$$\max \sum_{j=1}^r \mu_j \quad (16.10a)$$

$$\text{subject to } \mu_j - t_{ij} \lambda_i \leq 0, \quad \forall i \in \{1, 2, \dots, s\}, \forall j \in \{1, 2, \dots, r\}, \quad (16.10b)$$

$$\sum_{i=1}^s \lambda_i = 1, \quad (16.10c)$$

$$\mu_j \geq 0, \quad \forall j \in \{1, 2, \dots, r\}, \quad (16.10d)$$

$$\lambda_i \geq 0, \quad \forall i \in \{1, 2, \dots, s\}. \quad (16.10e)$$

The constraint (16.10c) corresponds to the free variable w , and it is an equality. The dual variables λ_i are non-negative weights of the machines with sum 1, and the constraints (16.10b) impose $\mu_j \leq \min_i t_{ij} \lambda_i$.

The dual model (16.10) for the instance reads as follows:

$$\begin{aligned} \max \quad & \mu_1 + \mu_2 + \mu_3 \\ \text{subject to} \quad & \mu_1 - 2\lambda_1 \leq 0 \\ & \mu_2 - 4\lambda_1 \leq 0 \\ & \mu_3 - 4\lambda_1 \leq 0 \\ & \mu_1 - 4\lambda_2 \leq 0 \\ & \mu_2 - 12\lambda_2 \leq 0 \\ & \mu_3 - 12\lambda_2 \leq 0 \\ & \lambda_1 + \lambda_2 = 1 \\ & \mu_1, \mu_2, \mu_3 \geq 0 \\ & \lambda_1, \lambda_2 \geq 0 \end{aligned}$$

For example, by giving the same weight to the two machines:

$$\bar{\lambda}_1 = \frac{1}{2}, \quad \bar{\lambda}_2 = \frac{1}{2},$$

and by setting:

$$\begin{aligned}\bar{\mu}_1 &= \min \left\{ 2 \cdot \frac{1}{2}, 4 \cdot \frac{1}{2} \right\} = 1, & \bar{\mu}_2 &= \min \left\{ 4 \cdot \frac{1}{2}, 12 \cdot \frac{1}{2} \right\} = 2, \\ \bar{\mu}_3 &= \min \left\{ 4 \cdot \frac{1}{2}, 12 \cdot \frac{1}{2} \right\} = 2\end{aligned}$$

we have a feasible dual solution. The objective function value of this feasible dual solution is:

$$\bar{\mu}_1 + \bar{\mu}_2 + \bar{\mu}_3 = 1 + 2 + 2 = 5$$

Accordingly, 5 is a lower bound on the optimal solution value of the LP model. So, for this instance, we have:

$$5 \leq z(LP) \leq 8$$

This lower bound has a simple meaning: the makespan is at least the average working time of the two machines, which is at least half of the sum, over the jobs, of the fastest processing time of each job, that is, $\frac{1}{2}(2 + 4 + 4) = 5$.

5. The solution of the question is

$$\tilde{x}_{11} = 0, \quad \tilde{x}_{12} = \frac{3}{4}, \quad \tilde{x}_{13} = 1, \quad \tilde{x}_{21} = 1, \quad \tilde{x}_{22} = \frac{1}{4}, \quad \tilde{x}_{23} = 0,$$

in which the working times of the two machines are $4 \cdot \frac{3}{4} + 4 \cdot 1 = 7$ and $4 \cdot 1 + 12 \cdot \frac{1}{4} = 7$ hours: with $\tilde{w} = 7$ it is feasible, with value 7.

The variables $\tilde{x}_{12}$, $\tilde{x}_{13}$, $\tilde{x}_{21}$, $\tilde{x}_{22}$ and $\tilde{w}$ are positive, hence, by the complementary slackness conditions, the corresponding dual constraints are satisfied at equality:

$$\begin{cases} \mu_2 = 4\lambda_1 \\ \mu_3 = 4\lambda_1 \\ \mu_1 = 4\lambda_2 \\ \mu_2 = 12\lambda_2 \\ \lambda_1 + \lambda_2 = 1 \end{cases}$$

From the first and the fourth equations, $\lambda_1 = 3\lambda_2$, hence $\tilde{\lambda}_2 = \frac{1}{4}$ and $\tilde{\lambda}_1 = \frac{3}{4}$; then $\tilde{\mu}_1 = 1$, $\tilde{\mu}_2 = 3$ and $\tilde{\mu}_3 = 3$. The remaining dual constraints, of x_{11} and x_{23} , are satisfied: $1 - 2 \cdot \frac{3}{4} = -\frac{1}{2} \leq 0$ and $3 - 12 \cdot \frac{1}{4} = 0 \leq 0$; moreover $\tilde{\lambda}_1, \tilde{\lambda}_2 \geq 0$. Hence the dual solution

$$\tilde{\mu}_1 = 1, \quad \tilde{\mu}_2 = 3, \quad \tilde{\mu}_3 = 3, \quad \tilde{\lambda}_1 = \frac{3}{4}, \quad \tilde{\lambda}_2 = \frac{1}{4}$$

is feasible, and its value is $1 + 3 + 3 = 7$. The primal and the dual solutions have the same value: by the strong duality theorem both are optimal, and

$$z(LP) = 7.$$

At the optimal solution the two machines finish at the same time: if one of them finished earlier, part of the work of the other one could be moved to it.

Exercise 16.7 — Absolute values (1)

A shop recorded the sales of a product in four consecutive weeks: 3, 4, 8 and 9 hundreds of units in weeks 1, 2, 3 and 4, respectively. To forecast the sales of the next weeks, the shop wants to describe the sales with a *trend line*, that is, with a linear function $a + bt$ of the week t , where the intercept a and the slope b can be any real numbers. In week t , the deviation of the line from the recorded sales y_t is $|y_t - a - bt|$. The goal of the shop is to determine the values of a and b that minimize the sum of the deviations in the four weeks.

Questions

1. Write a Linear Programming (LP) model for the problem. Identify the decisions that must be made and define the corresponding decision variables. Describe the objective function of the model and its constraints. What algorithm can be used to find an optimal solution to the model?
2. Find a feasible solution of the LP model and accordingly an upper bound on its optimal solution value.
3. Write the dual model of the LP model. Find a feasible solution of the dual model and accordingly a lower bound on the optimal solution value of the LP model.
4. Prove that the trend line $1+2t$ is optimal, by finding with the complementary slackness conditions a feasible dual solution with the same objective function value. What is the optimal solution value?

Solution

1. The decisions are the intercept and the slope of the line. We introduce the following two free decision variables:

$$\begin{cases} a = \text{intercept of the trend line} \\ b = \text{slope of the trend line} \end{cases}$$

The objective function

$$\min \quad |3 - a - b| + |4 - a - 2b| + |8 - a - 3b| + |9 - a - 4b|$$

is not linear. Each absolute value has a positive coefficient in a minimized objective function, hence it can be linearized: we introduce the following four non-negative decision variables:

$$e_t = \text{deviation of the line in week } t, \quad \forall t \in \{1, 2, 3, 4\},$$

and we impose $e_t \geq y_t - a - bt$ and $e_t \geq -(y_t - a - bt)$, that is, $e_t + a + bt \geq y_t$ and $e_t - a - bt \geq -y_t$. Using these variables, a Linear Programming (LP) model for the problem reads as follows:

$$\begin{array}{llllll}
 \min & e_1 + e_2 + e_3 + e_4 & & & & \\
 \text{subject to} & e_1 & & + a + b & \geq & 3 \\
 & & e_2 & & + a + 2b & \geq 4 \\
 & & & e_3 & + a + 3b & \geq 8 \\
 & & & & e_4 + a + 4b & \geq 9 \\
 & e_1 & & - a - b & \geq & -3 \\
 & & e_2 & & - a - 2b & \geq -4 \\
 & & & e_3 & - a - 3b & \geq -8 \\
 & & & & e_4 - a - 4b & \geq -9 \\
 & e_1, & e_2, & e_3, & e_4 & \geq 0 \\
 & & & & a, & b \geq 0
 \end{array}$$

Description of the objective function and constraints of the LP model:

- the linear objective function minimizes the sum of the deviations;
- the first four linear constraints impose that each deviation e_t is not smaller than $y_t - a - bt$;
- the next four linear constraints impose that each deviation e_t is not smaller than $-(y_t - a - bt)$; together with the previous ones, they impose $e_t \geq |y_t - a - bt|$, and since the deviations are minimized, at an optimal solution each e_t is exactly equal to the absolute value;
- the final constraints define the variables of the model: the deviations are non-negative, while the intercept and the slope are free.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. Every choice of a and b gives a feasible solution, by setting each e_t equal to the corresponding absolute value. For example, the line $2t$ ($\bar{a} = 0, \bar{b} = 2$) takes the values 2, 4, 6 and 8 in the four weeks, hence

$$\bar{a} = 0, \quad \bar{b} = 2, \quad \bar{e}_1 = |3 - 2| = 1, \quad \bar{e}_2 = |4 - 4| = 0, \quad \bar{e}_3 = |8 - 6| = 2, \quad \bar{e}_4 = |9 - 8| = 1$$

is a feasible solution. The objective function value of this feasible solution is:

$$1 + 0 + 2 + 1 = 4$$

accordingly 4 is an upper bound on the optimal solution value of the LP model. So, we have:

$$z(LP) \leq 4$$

3. By using a non-negative dual variable α_t for each of the first four constraints and a non-negative dual variable β_t for each of the next four constraints ($t \in \{1, 2, 3, 4\}$), the dual of the LP model reads as follows:

$$\begin{aligned}
 \max \quad & 3\alpha_1 + 4\alpha_2 + 8\alpha_3 + 9\alpha_4 - 3\beta_1 - 4\beta_2 - 8\beta_3 - 9\beta_4 \\
 \text{subject to} \quad & \alpha_1 + \beta_1 \leq 1 \\
 & \alpha_2 + \beta_2 \leq 1 \\
 & \alpha_3 + \beta_3 \leq 1 \\
 & \alpha_4 + \beta_4 \leq 1 \\
 & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \beta_1 - \beta_2 - \beta_3 - \beta_4 = 0 \\
 & \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 - \beta_1 - 2\beta_2 - 3\beta_3 - 4\beta_4 = 0 \\
 & \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0
 \end{aligned}$$

The first four constraints correspond to the non-negative variables e_t , while the last two constraints are equalities, since they correspond to the free variables a and b .

For example, by setting:

$$\bar{\alpha}_1 = \frac{1}{2}, \bar{\alpha}_2 = 0, \bar{\alpha}_3 = \frac{1}{2}, \bar{\alpha}_4 = 0, \bar{\beta}_1 = 0, \bar{\beta}_2 = 1, \bar{\beta}_3 = 0, \bar{\beta}_4 = 0$$

we have a feasible dual solution: the first four constraints read $\frac{1}{2} \leq 1$, $1 \leq 1$, $\frac{1}{2} \leq 1$ and $0 \leq 1$, and the two equalities read $\frac{1}{2} + \frac{1}{2} - 1 = 0$ and $\frac{1}{2} + \frac{3}{2} - 2 = 0$. The objective function value of this feasible dual solution is:

$$3 \cdot \frac{1}{2} + 8 \cdot \frac{1}{2} - 4 \cdot 1 = \frac{3}{2}$$

Accordingly, $\frac{3}{2}$ is a lower bound on the optimal solution value of the LP model. So, we have:

$$\frac{3}{2} \leq z(LP) \leq 4$$

4. The line $1 + 2t$ takes the values 3, 5, 7 and 9 in the four weeks, hence it corresponds to the feasible solution

$$\tilde{a} = 1, \tilde{b} = 2, \tilde{e}_1 = 0, \tilde{e}_2 = 1, \tilde{e}_3 = 1, \tilde{e}_4 = 0,$$

whose value is $0 + 1 + 1 + 0 = 2$.

The complementary slackness conditions give:

- the constraint $e_2 + a + 2b \geq 4$ is not tight ($1 + 1 + 4 = 6 > 4$), hence $\tilde{\alpha}_2 = 0$; the constraint $e_3 - a - 3b \geq -8$ is not tight ($1 - 1 - 6 = -6 > -8$), hence $\tilde{\beta}_3 = 0$;
- the variables $\tilde{e}_2$ and $\tilde{e}_3$ are positive, hence the corresponding dual constraints are satisfied at equality: $\alpha_2 + \beta_2 = 1$ and $\alpha_3 + \beta_3 = 1$, that is, $\tilde{\beta}_2 = 1$ and $\tilde{\alpha}_3 = 1$;

- the two equalities of the dual model become:

$$\begin{cases} \alpha_1 - \beta_1 + \alpha_4 - \beta_4 = 0 \\ \alpha_1 - \beta_1 + 4(\alpha_4 - \beta_4) = -1 \end{cases}$$

whose solution is $\alpha_1 - \beta_1 = \frac{1}{3}$ and $\alpha_4 - \beta_4 = -\frac{1}{3}$.

We choose, for example, $\tilde{\alpha}_1 = \frac{1}{3}$, $\tilde{\beta}_1 = 0$, $\tilde{\alpha}_4 = 0$ and $\tilde{\beta}_4 = \frac{1}{3}$, which satisfy $\alpha_t + \beta_t \leq 1$ and the sign conditions. The dual solution

$$\tilde{\alpha}_1 = \frac{1}{3}, \quad \tilde{\alpha}_2 = 0, \quad \tilde{\alpha}_3 = 1, \quad \tilde{\alpha}_4 = 0, \quad \tilde{\beta}_1 = 0, \quad \tilde{\beta}_2 = 1, \quad \tilde{\beta}_3 = 0, \quad \tilde{\beta}_4 = \frac{1}{3}$$

is then feasible, and its value is:

$$3 \cdot \frac{1}{3} + 8 \cdot 1 - 4 \cdot 1 - 9 \cdot \frac{1}{3} = 2.$$

The primal and the dual solutions have the same value: by the strong duality theorem both are optimal, and

$$z(LP) = 2.$$

Exercise 16.8 — Covering (1)

The emergency department of a hospital divides the day into six periods of four hours: period 1 from 0 to 4, period 2 from 4 to 8, period 3 from 8 to 12, period 4 from 12 to 16, period 5 from 16 to 20 and period 6 from 20 to 24. The minimum number of nurses needed in the six periods is 4, 8, 10, 7, 12 and 4, respectively. Each nurse works a shift of eight consecutive hours, which starts at the beginning of one of the six periods: the shift that starts at the beginning of period 6 continues in period 1 of the following day. The cost of a nurse is 400 euros per shift, except for the night shifts that start at the beginning of periods 1 and 6, which cost 500 euros. The goal of the hospital is to determine how many nurses start their shift at the beginning of each period, so as to cover the requirements of all the periods at minimum cost.

Questions

1. Write a Linear Programming (LP) model for the problem, measuring the costs in hundreds of euros. Identify the decisions that must be made and define the corresponding decision variables. Describe the objective function of the model and its constraints. What algorithm can be used to find an optimal solution to the model?
2. Find a feasible solution of the LP model and accordingly an upper bound on its optimal solution value.
3. Write the dual model of the LP model. Find a feasible solution of the dual model and accordingly a lower bound on the optimal solution value of the LP model.
4. Prove that the plan in which 4, 10, 0, 8, 4 and 0 nurses start their shift at the beginning of periods 1, 2, ..., 6, respectively, is optimal, by finding with the complementary slackness conditions a feasible dual solution with the same objective function value. What is the optimal solution value?

Solution

Warning

We assume that the number of nurses of each shift can be fractional, that is, it is measured in full-time equivalents (for example, two nurses who work every other day count as one nurse). Without this assumption the variables would have to be integer, and the model would not be an LP model.

1. We introduce the following six non-negative decision variables:

$$x_k = \text{nurses whose shift starts in period } k, \quad \forall k \in \{1, 2, \dots, 6\}.$$

In period t the nurses on duty are those who started their shift at the beginning of period t and those who started it at the beginning of the previous period, which for period 1 is period 6. Using these variables, a Linear Programming (LP) model for the problem reads as follows:

$$\begin{array}{ll} \min & 5x_1 + 4x_2 + 4x_3 + 4x_4 + 4x_5 + 5x_6 \\ \text{subject to} & x_1 + x_6 \geq 4 \\ & x_1 + x_2 \geq 8 \\ & x_2 + x_3 \geq 10 \\ & x_3 + x_4 \geq 7 \\ & x_4 + x_5 \geq 12 \\ & x_5 + x_6 \geq 4 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

Description of the objective function and constraints of the LP model:

- the linear objective function minimizes the total cost of the nurses, in hundreds of euros;
- the six linear constraints are *covering constraints*: for each period, the nurses on duty are at least the minimum number required;
- the final constraint defines the variables of the model.

The LP model can be solved to optimality, for example, by the simplex algorithm.

2. For example, we can use only the shifts that start at the beginning of the odd periods, each one with the largest requirement of the two periods it covers:

$$\bar{x}_1 = \max\{4, 8\} = 8, \quad \bar{x}_3 = \max\{10, 7\} = 10, \quad \bar{x}_5 = \max\{12, 4\} = 12, \\ \bar{x}_2 = \bar{x}_4 = \bar{x}_6 = 0.$$

This is a feasible solution: the nurses on duty in the six periods are 8, 8, 10, 10, 12 and 12. The objective function value of this feasible solution is:

$$5 \cdot 8 + 4 \cdot 10 + 4 \cdot 12 = 128$$

accordingly 128 is an upper bound on the optimal solution value of the LP model. So, we have:

$$z(LP) \leq 128$$

3. By using a non-negative dual variable y_t for the covering constraint of period t ($t \in \{1, 2, \dots, 6\}$), the dual of the LP model reads as follows:

$$\begin{array}{llllll} \max & 4y_1 + 8y_2 + 10y_3 + 7y_4 + 12y_5 + 4y_6 \\ \text{subject to} & y_1 + y_2 & & & & \leq 5 \\ & & y_2 + y_3 & & & \leq 4 \\ & & & y_3 + y_4 & & \leq 4 \\ & & & & y_4 + y_5 & \leq 4 \\ & & & & & y_5 + y_6 \leq 4 \\ & & y_1 & & & + y_6 \leq 5 \\ & y_1, & y_2, & y_3, & y_4, & y_5, & y_6 \geq 0 \end{array}$$

Each dual constraint corresponds to a shift: the sum of the dual variables of the two periods covered by the shift cannot exceed its cost. The dual variable y_t can be read as the price of one nurse in period t .

For example, by splitting the cheapest cost of a shift equally between the two periods it covers:

$$\bar{y}_1 = \bar{y}_2 = \bar{y}_3 = \bar{y}_4 = \bar{y}_5 = \bar{y}_6 = 2$$

we have a feasible dual solution, since every dual constraint reads $4 \leq 4$ or $4 \leq 5$. The objective function value of this feasible dual solution is:

$$(4 + 8 + 10 + 7 + 12 + 4) \cdot 2 = 90$$

Accordingly, 90 is a lower bound on the optimal solution value of the LP model. So, we have:

$$90 \leq z(LP) \leq 128$$

4. The plan of the question is the solution

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 10, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 8, \quad \tilde{x}_5 = 4, \quad \tilde{x}_6 = 0,$$

in which the nurses on duty in the six periods are 4, 14, 10, 8, 12 and 4: it is feasible, and its value is $5 \cdot 4 + 4 \cdot 10 + 4 \cdot 8 + 4 \cdot 4 = 108$.

The complementary slackness conditions give:

- the covering constraints of periods 2 and 4 are not tight ($14 > 8$ and $8 > 7$), hence $\tilde{y}_2 = 0$ and $\tilde{y}_4 = 0$;
- the variables $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_4$ and $\tilde{x}_5$ are positive, hence the corresponding dual constraints are satisfied at equality:

$$\begin{cases} y_1 + y_2 = 5 \\ y_2 + y_3 = 4 \\ y_4 + y_5 = 4 \\ y_5 + y_6 = 4 \end{cases}$$

We obtain $\tilde{y}_1 = 5$, $\tilde{y}_3 = 4$, $\tilde{y}_5 = 4$ and $\tilde{y}_6 = 0$. The remaining dual constraints, of x_3 and x_6 , are satisfied: $4 + 0 = 4 \leq 4$ and $5 + 0 = 5 \leq 5$. Hence the dual solution

$$\tilde{y}_1 = 5, \quad \tilde{y}_2 = 0, \quad \tilde{y}_3 = 4, \quad \tilde{y}_4 = 0, \quad \tilde{y}_5 = 4, \quad \tilde{y}_6 = 0$$

is feasible, and its value is:

$$4 \cdot 5 + 10 \cdot 4 + 12 \cdot 4 = 108.$$

The primal and the dual solutions have the same value: by the strong duality theorem both are optimal, and

$$z(LP) = 108,$$

that is, the minimum cost is 10800 euros per day. In this instance the optimal solution is integer, hence it is also an optimal plan with integer numbers of nurses.