



SAPIENZA
UNIVERSITÀ DI ROMA

Linear Programming

Exercises 4 — Matrix methods, sensitivity and geometry

Worked exercises on chapters 11–15: revised primal and dual simplex, primal-dual method, sensitivity analysis, geometry of linear programming.

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Revised simplex algorithm

11.1 Problems with two variables

Exercise 11.1 — Revised simplex algorithm (P_0)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 1$ constraint:

$$\begin{array}{llll} \max & 2x_1 & + & 4x_2 \\ \text{subject to} & 2x_1 & + & 3x_2 \leq 4 \\ & x_1, & & x_2 \geq 0 \end{array}$$

By introducing the slack variable, we get the following LP problem in standard maximization form with $n + m = 3$ variables and $m = 1$ constraint:

$$\begin{array}{llllll} \max & 2x_1 & + & 4x_2 & & \\ \text{subject to} & 2x_1 & + & 3x_2 & + & x_3 = 4 \\ & x_1, & & x_2, & & x_3 \geq 0 \end{array}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \quad \mathbf{b} = (4) \quad \mathbf{c} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{3\}$ and $\mathcal{N} = \{1, 2\}$ is associated with the following basic feasible solution:

$$(x_3) = (4) \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{3\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = (x_3) \quad \mathbf{A}_{\mathcal{B}} = (1) \quad \mathbf{c}_{\mathcal{B}} = (0) \quad \bar{\mathbf{b}} = (4)$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 2 & 3 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{l} \pi_3 = 0 \\ \boldsymbol{\pi} = (0) \end{array} \right.$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0) (4) = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 2 & 4 \end{pmatrix} - (0) \begin{pmatrix} 2 & 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 4 \end{pmatrix}}_{\mathbf{c}'_1 \quad \mathbf{c}'_2} - \underbrace{\begin{pmatrix} 0 & 0 \end{pmatrix}}_{\boldsymbol{\pi} \mathbf{A}_1 \quad \boldsymbol{\pi} \mathbf{A}_2} = \underbrace{\begin{pmatrix} 2 & 4 \end{pmatrix}}_{\bar{\mathbf{c}}_1 \quad \bar{\mathbf{c}}_2}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \boldsymbol{\pi} \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = (2)$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_1$, second system of linear equations:

$$\left\{ \begin{array}{l} d_3 = 2 \\ \mathbf{d} = (2) \end{array} \right.$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\left\{ \underbrace{4 - t \cdot 2}_{x_3 \geq 0} \geq 0 \right\} \quad \left\{ t \leq 2 \right\} \quad \text{then} \quad t_m = 2$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = 2$. The value of x_3 becomes zero and x_3 is the pivot row.

- **Iteration 2**

With $\mathcal{B} = \{1\}$ and $\mathcal{N} = \{2, 3\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = (x_1) \quad \mathbf{A}_{\mathcal{B}} = (2) \quad \mathbf{c}_{\mathcal{B}} = (2) \quad \bar{\mathbf{b}} = (2)$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 3 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{l} 2 \pi_1 = 2 \\ \pi = (1) \end{array} \right.$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (1) (4) = 4$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 4 & 0 \end{pmatrix} - (1) \begin{pmatrix} 3 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & 0 \end{pmatrix}}_{\mathbf{c}_2 \quad \mathbf{c}_3} - \underbrace{\begin{pmatrix} 3 & 1 \end{pmatrix}}_{\boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_3} = \underbrace{\begin{pmatrix} 1 & -1 \end{pmatrix}}_{\bar{\mathbf{c}}_2 \quad \bar{\mathbf{c}}_3}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \boldsymbol{\pi} \mathbf{A}_2$, x_2 is the pivot column and we have $\mathbf{A}_2 = (3)$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_2$, second system of linear equations:

$$\left\{ \begin{array}{l} 2 d_1 = 3 \\ \mathbf{d} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \end{array} \right.$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\left\{ \underbrace{2 - t \cdot \frac{3}{2}}_{x_1 \geq 0} \geq 0 \right\} \quad \left\{ t \leq \frac{4}{3} \right\} \quad \text{then} \quad t_m = \frac{4}{3}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = \frac{4}{3}$. The value of x_1 becomes zero and x_1 is the pivot row.

• **Iteration 3**

With $\mathcal{B} = \{2\}$ and $\mathcal{N} = \{1, 3\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = (x_2) \quad \mathbf{A}_{\mathcal{B}} = (3) \quad \mathbf{c}_{\mathcal{B}} = (4) \quad \bar{\mathbf{b}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 2 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{lcl} 3 \pi_2 & = & 4 \end{array} \right. \quad \boldsymbol{\pi} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} (4) = \frac{16}{3}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 2 & 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 \end{pmatrix}}_{\mathbf{c}'_{\mathcal{N}}} - \underbrace{\begin{pmatrix} 8 & 4 \\ 3 & 3 \end{pmatrix}}_{\boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}} = \underbrace{\begin{pmatrix} -2 & -4 \\ -3 & -3 \end{pmatrix}}_{\bar{\mathbf{c}}'}$$

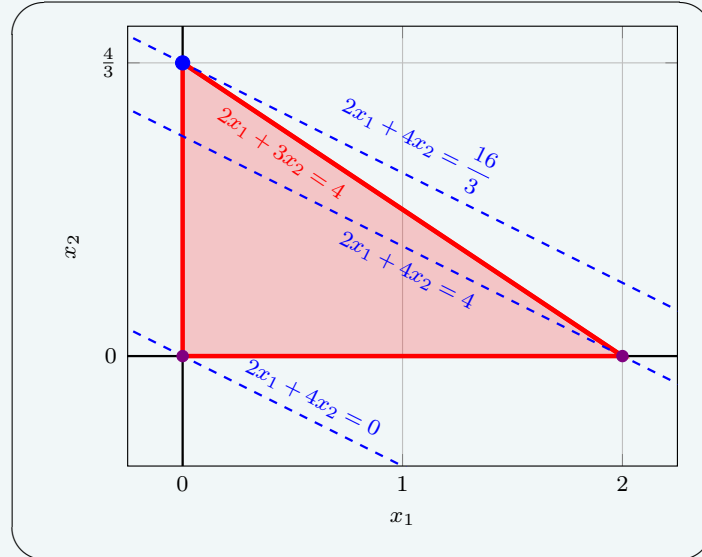
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_{\mathcal{B}} = (x_2) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = \frac{16}{3}$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = \frac{4}{3}$$

the optimal value $z(LP)$ is $\frac{16}{3}$.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 11.2 — Revised simplex algorithm (P1)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 1$ constraint:

$$\begin{aligned} \max \quad & x_1 + 8x_2 \\ \text{subject to} \quad & 3x_1 + 6x_2 \leq 16 \\ & x_1, \quad x_2 \geq 0 \end{aligned}$$

By introducing the slack variable, we get the following LP problem in standard maximiza-

tion form with $n + m = 3$ variables and $m = 1$ constraint:

$$\begin{aligned} \max \quad & x_1 + 8x_2 \\ \text{subject to} \quad & 3x_1 + 6x_2 + x_3 = 16 \\ & x_1, \quad x_2, \quad x_3 \geq 0 \end{aligned}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 3 & 6 & 1 \end{pmatrix} \quad \mathbf{b} = (16) \quad \mathbf{c} = \begin{pmatrix} 1 \\ 8 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{3\}$ and $\mathcal{N} = \{1, 2\}$ is associated with the following basic feasible solution:

$$(x_3) = (16) \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{3\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = (x_3) \quad \mathbf{A}_{\mathcal{B}} = (1) \quad \mathbf{c}_{\mathcal{B}} = (0) \quad \bar{\mathbf{b}} = (16)$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 3 & 6 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{l} \pi_3 = 0 \\ \boldsymbol{\pi} = (0) \end{array} \right.$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0) (16) = 0$

We compute the reduced costs $\bar{c}' = c'_{\mathcal{N}} - \pi A_{\mathcal{N}}$:

$$\bar{c}' = (1 \ 8) - (0) (3 \ 6) = \underbrace{(1 \ 8)}_{c_1 \ c_2} - \underbrace{(0 \ 0)}_{\pi A_1 \ \pi A_2} = \underbrace{(1 \ 8)}_{\bar{c}_1 \ \bar{c}_2}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \pi A_1$, x_1 is the pivot column and we have $A_1 = (3)$

We solve $A_B d = A_1$ to compute $d = A_B^{-1} A_1$, second system of linear equations:

$$\begin{cases} d_3 = 3 & d = (3) \end{cases}$$

We have to find the largest t such that $\bar{b} - t d \geq 0$:

$$\begin{cases} \underbrace{16 - t \cdot 3}_{x_3 \geq 0} \geq 0 & \begin{cases} t \leq \frac{16}{3} \end{cases} \quad \text{then} \quad t_m = \frac{16}{3} \end{cases}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = \frac{16}{3}$. The value of x_3 becomes zero and x_3 is the pivot row.

• Iteration 2

With $\mathcal{B} = \{1\}$ and $\mathcal{N} = \{2, 3\}$, we have:

$$x_B = (x_1) \quad A_B = (3) \quad c_B = (1) \quad \bar{b} = \left(\frac{16}{3}\right)$$

$$x_N = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad A_N = (6 \ 1) \quad c_N = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi A_B = c'_B$ to compute $\pi = c'_B A_B^{-1}$, first system of linear equations:

$$\begin{cases} 3 \pi_1 = 1 & \pi = \left(\frac{1}{3}\right) \end{cases}$$

We compute the basic feasible solution value $\bar{z} = \pi b = \left(\frac{1}{3}\right) (16) = \frac{16}{3}$

We compute the reduced costs $\bar{c}' = c'_{\mathcal{N}} - \pi A_{\mathcal{N}}$:

$$\bar{c}' = (8 \ 0) - \left(\frac{1}{3}\right) (6 \ 1) = \underbrace{(8 \ 0)}_{c_2 \ c_3} - \underbrace{\left(2 \ \frac{1}{3}\right)}_{\pi A_2 \ \pi A_3} = \underbrace{\left(6 \ -\frac{1}{3}\right)}_{\bar{c}_2 \ \bar{c}_3}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \pi A_2$, x_2 is the pivot column and we have $A_2 = (6)$

We solve $A_B d = A_2$ to compute $d = A_B^{-1} A_2$, second system of linear equations:

$$\begin{cases} 3d_1 = 6 & d = (2) \end{cases}$$

We have to find the largest t such that $\bar{b} - t d \geq 0$:

$$\underbrace{\begin{cases} \frac{16}{3} - t \cdot 2 \geq 0 \\ x_1 \geq 0 \end{cases}}_{x_1 \geq 0} \quad \begin{cases} t \leq \frac{8}{3} \end{cases} \quad \text{then } t_m = \frac{8}{3}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = \frac{8}{3}$. The value of x_1 becomes zero and x_1 is the pivot row.

- Iteration 3**

With $\mathcal{B} = \{2\}$ and $\mathcal{N} = \{1, 3\}$, we have:

$$x_B = (x_2) \quad A_B = (6) \quad c_B = (8) \quad \bar{b} = \left(\frac{8}{3}\right)$$

$$x_N = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad A_N = \begin{pmatrix} 3 & 1 \end{pmatrix} \quad c_N = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi A_B = c'_B$ to compute $\pi = c'_B A_B^{-1}$, first system of linear equations:

$$\begin{cases} 6\pi_2 = 8 & \pi = \left(\frac{4}{3}\right) \end{cases}$$

We compute the basic feasible solution value $\bar{z} = \pi b = \left(\frac{4}{3}\right) (16) = \frac{64}{3}$

We compute the reduced costs $\bar{c}' = c'_N - \pi A_N$:

$$\bar{c}' = (1 \ 0) - \left(\frac{4}{3}\right) \begin{pmatrix} 3 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\substack{c_1 \quad c_3}} - \underbrace{\begin{pmatrix} 4 & \frac{4}{3} \end{pmatrix}}_{\substack{\pi A_1 \quad \pi A_3}} = \underbrace{\begin{pmatrix} -3 & -\frac{4}{3} \end{pmatrix}}_{\substack{\bar{c}_1 \quad \bar{c}_3}}$$

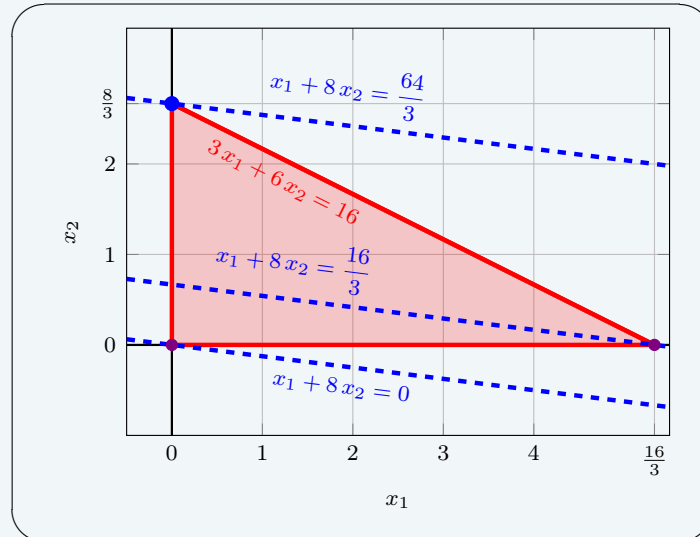
All reduced costs are non-positive, then the basic feasible solution

$$x_B = (x_2) = \left(\frac{8}{3}\right) \quad \text{and} \quad x_N = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = \frac{64}{3}$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = \frac{8}{3}$$

the optimal value $z(LP)$ is $\frac{64}{3}$.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 11.3 — Revised simplex algorithm ($P2$)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & x_1 + 3x_2 \\ \text{subject to} \quad & -4x_1 + 10x_2 \leq 35 \\ & 3x_1 + x_2 \leq 12 \\ & x_1, \quad x_2 \geq 0 \end{aligned}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 4$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & x_1 + 3x_2 \\ \text{subject to} \quad & -4x_1 + 10x_2 + x_3 = 35 \\ & 3x_1 + x_2 + x_4 = 12 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} -4 & 10 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 35 \\ 12 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 35 \\ 12 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 35 \\ 12 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -4 & 10 \\ 3 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_3 & = & 0 \\ \pi_4 & = & 0 \end{cases} \quad \boldsymbol{\pi} = (0 \ 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \ 0) \begin{pmatrix} 35 \\ 12 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (1 \ 3) - (0 \ 0) \begin{pmatrix} -4 & 10 \\ 3 & 1 \end{pmatrix} = \underbrace{(1 \ 3)}_{c_1 \ c_2} - \underbrace{(0 \ 0)}_{\boldsymbol{\pi} \mathbf{A}_1 \ \boldsymbol{\pi} \mathbf{A}_2} = \underbrace{(1 \ 3)}_{\bar{c}_1 \ \bar{c}_2}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \boldsymbol{\pi} \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_3 & = & -4 \\ d_4 & = & 3 \end{cases} \quad \mathbf{d} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{35 - t \cdot (-4)}_{x_3 \geq 0} \geq 0 \\ \underbrace{12 - t \cdot 3}_{x_4 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \geq -\frac{35}{4} \\ t \leq 4 \end{cases} \quad \text{then } t_m = 4$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = 4$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_3 = 35 - 4 \cdot (-4) = 51$$

• Iteration 2

With $\mathcal{B} = \{1, \ 3\}$ and $\mathcal{N} = \{2, \ 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -4 & 1 \\ 3 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 4 \\ 51 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 10 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi \mathbf{A}_B = \mathbf{c}'_B$ to compute $\pi = \mathbf{c}'_B \mathbf{A}_B^{-1}$, first system of linear equations:

$$\begin{cases} -4\pi_1 + 3\pi_3 = 1 \\ \pi_1 = 0 \end{cases} \quad \pi = \begin{pmatrix} 0 & \frac{1}{3} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \pi \mathbf{b} = \begin{pmatrix} 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 35 \\ 12 \end{pmatrix} = 4$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \pi \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 0 \end{pmatrix}}_{c_2 \quad c_4} - \underbrace{\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix}}_{\pi \mathbf{A}_2 \quad \pi \mathbf{A}_4} = \underbrace{\begin{pmatrix} \frac{8}{3} & -\frac{1}{3} \end{pmatrix}}_{\bar{c}_2 \quad \bar{c}_4}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > 0$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} -4d_1 + d_3 = 10 \\ 3d_1 = 1 \end{cases} \quad \mathbf{d} = \begin{pmatrix} \frac{1}{3} \\ \frac{34}{3} \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{4 - t \cdot \frac{1}{3} \geq 0}_{x_1 \geq 0} \\ \underbrace{51 - t \cdot \frac{34}{3} \geq 0}_{x_3 \geq 0} \end{cases} \quad \begin{cases} t \leq 12 \\ t \leq \frac{9}{2} \end{cases} \quad \text{then } t_m = \frac{9}{2}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = \frac{9}{2}$. The value of x_3 becomes zero and x_3 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = 4 - \frac{9}{2} \cdot \frac{1}{3} = \frac{5}{2}$$

• **Iteration 3**

With $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -4 & 10 \\ 3 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{5}{2} \\ \frac{9}{2} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} -4\pi_1 + 3\pi_2 = 1 \\ 10\pi_1 + \pi_2 = 3 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{4}{17} & \frac{11}{17} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} \frac{4}{17} & \frac{11}{17} \end{pmatrix} \begin{pmatrix} 35 \\ 12 \end{pmatrix} = 16$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{4}{17} & \frac{11}{17} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \end{pmatrix}}_{\substack{c_3 \quad c_4}} - \underbrace{\begin{pmatrix} \frac{4}{17} & \frac{11}{17} \end{pmatrix}}_{\substack{\boldsymbol{\pi} \mathbf{A}_3 \quad \boldsymbol{\pi} \mathbf{A}_4}} = \underbrace{\begin{pmatrix} -\frac{4}{17} & -\frac{11}{17} \end{pmatrix}}_{\substack{\bar{c}_3 \quad \bar{c}_4}}$$

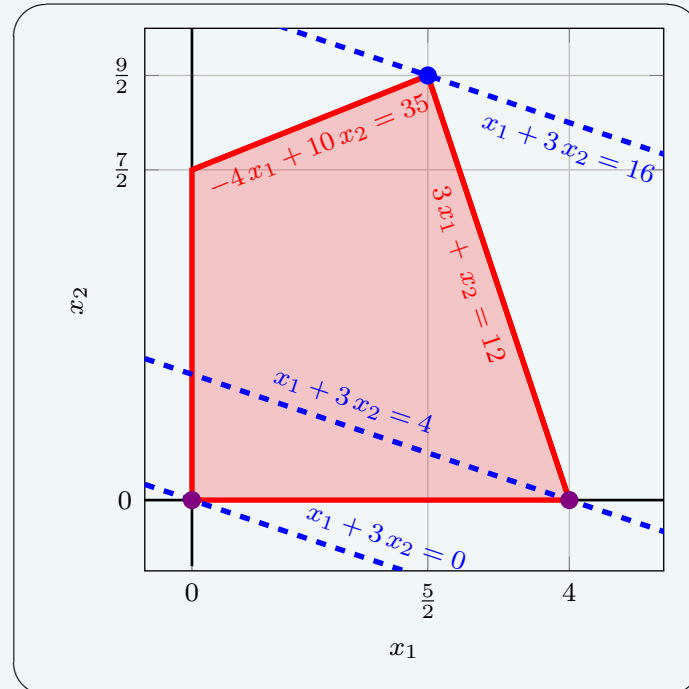
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{9}{2} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = 16$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = \frac{9}{2}$$

the optimal value $z(LP)$ is 16.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 11.4 — Revised simplex algorithm (P3)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & 2x_1 & + & 4x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & 2 \\ & 8x_1 & + & 2x_2 & \leq & 19 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 4$ variables and $m = 2$ constraints:

$$\begin{array}{llllllll} \max & 2x_1 & + & 4x_2 & & & & \\ \text{subject to} & -x_1 & + & x_2 & + & x_3 & & = & 2 \\ & 8x_1 & + & 2x_2 & & & + & x_4 & = & 19 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 8 & 2 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 19 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 19 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 19 \end{pmatrix}$$

$$\mathbf{x}_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_N = \begin{pmatrix} -1 & 1 \\ 8 & 2 \end{pmatrix} \quad \mathbf{c}_N = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_B = \mathbf{c}'_B$ to compute $\boldsymbol{\pi} = \mathbf{c}'_B \mathbf{A}_B^{-1}$, first system of linear equations:

$$\begin{cases} \pi_3 & = & 0 \\ \pi_4 & = & 0 \end{cases} \quad \boldsymbol{\pi} = (0 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \quad 0) \begin{pmatrix} 2 \\ 19 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_N - \boldsymbol{\pi} \mathbf{A}_N$:

$$\bar{\mathbf{c}}' = (2 \quad 4) - (0 \quad 0) \begin{pmatrix} -1 & 1 \\ 8 & 2 \end{pmatrix} = \underbrace{(2 \quad 4)}_{c_1 \quad c_2} - \underbrace{(0 \quad 0)}_{\boldsymbol{\pi} \mathbf{A}_1 \quad \boldsymbol{\pi} \mathbf{A}_2} = \underbrace{(2 \quad 4)}_{\bar{c}_1 \quad \bar{c}_2}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \boldsymbol{\pi} \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_3 & = & -1 \\ d_4 & = & 8 \end{cases} \quad \mathbf{d} = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{2 - t \cdot (-1) \geq 0}_{x_3 \geq 0} \\ \underbrace{19 - t \cdot 8 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} t \geq -2 \\ t \leq \frac{19}{8} \end{cases} \quad \text{then } t_m = \frac{19}{8}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = \frac{19}{8}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_3 = 2 - \frac{19}{8} \cdot (-1) = \frac{35}{8}$$

• **Iteration 2**

With $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 1 \\ 8 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{19}{8} \\ \frac{35}{8} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} -\pi_1 + 8\pi_3 = 2 \\ \pi_1 = 0 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} 0 & \frac{1}{4} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 2 \\ 19 \end{pmatrix} = \frac{19}{4}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 4 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & 0 \end{pmatrix}}_{\mathbf{c}'_{\mathcal{N}}} - \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \end{pmatrix}}_{\boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}} = \underbrace{\begin{pmatrix} \frac{7}{2} & -\frac{1}{4} \end{pmatrix}}_{\bar{\mathbf{c}}'}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \boldsymbol{\pi} \mathbf{A}_2$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} -d_1 + d_3 = 1 \\ 8d_1 = 2 \end{cases} \quad \mathbf{d} = \begin{pmatrix} \frac{1}{4} \\ \frac{5}{4} \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{\frac{19}{8} - t \cdot \frac{1}{4}}_{x_1 \geq 0} \geq 0 \\ \underbrace{\frac{35}{8} - t \cdot \frac{5}{4}}_{x_3 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \leq \frac{19}{2} \\ t \leq \frac{7}{2} \end{cases} \quad \text{then } t_m = \frac{7}{2}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = \frac{7}{2}$. The value of x_3 becomes zero and x_3 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = \frac{19}{8} - \frac{7}{2} \cdot \frac{1}{4} = \frac{3}{2}$$

• **Iteration 3**

With $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 1 \\ 8 & 2 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} -\pi_1 + 8\pi_2 = 2 \\ \pi_1 + 2\pi_2 = 4 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{14}{5} & \frac{3}{5} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} \frac{14}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 2 \\ 19 \end{pmatrix} = 17$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{14}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \end{pmatrix}}_{\substack{c_3 \quad c_4}} - \underbrace{\begin{pmatrix} \frac{14}{5} & \frac{3}{5} \end{pmatrix}}_{\substack{\boldsymbol{\pi} \mathbf{A}_3 \quad \boldsymbol{\pi} \mathbf{A}_4}} = \underbrace{\begin{pmatrix} -\frac{14}{5} & -\frac{3}{5} \end{pmatrix}}_{\substack{\bar{c}_3 \quad \bar{c}_4}}$$

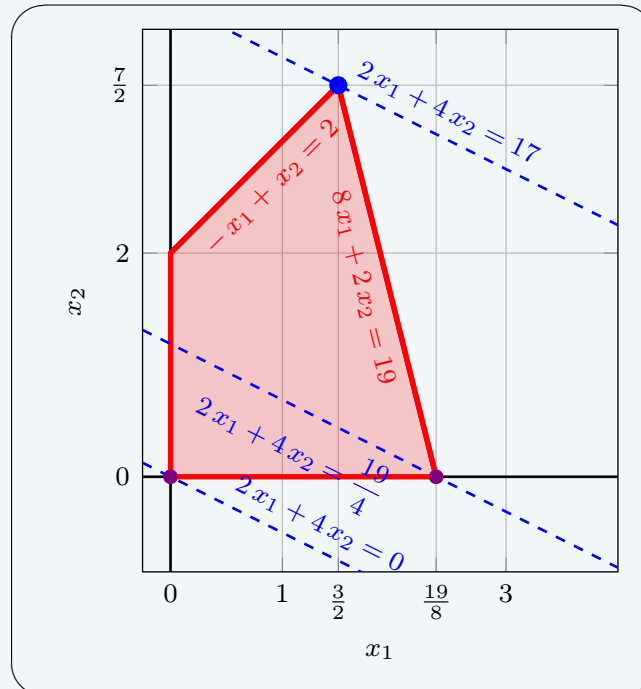
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{7}{2} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = 17$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{3}{2}, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value $z(LP)$ is 17.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 11.5 — Revised simplex algorithm (P_4)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 5x_1 + 3x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 14 \\ & x_1 + 8x_2 \leq 32 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 4$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 5x_1 + 3x_2 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 = 14 \\ & x_1 + 8x_2 + x_4 = 32 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 8 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 14 \\ 32 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 5 \\ 3 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 14 \\ 32 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 14 \\ 32 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_3 & = & 0 \\ \pi_4 & = & 0 \end{cases} \quad \boldsymbol{\pi} = (0 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \quad 0) \begin{pmatrix} 14 \\ 32 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (5 \quad 3) - (0 \quad 0) \begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix} = \underbrace{(5 \quad 3)}_{c_1 \quad c_2} - \underbrace{(0 \quad 0)}_{\boldsymbol{\pi} \mathbf{A}_1 \quad \boldsymbol{\pi} \mathbf{A}_2} = \underbrace{(5 \quad 3)}_{\bar{c}_1 \quad \bar{c}_2}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \boldsymbol{\pi} \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_3 & = & 2 \\ d_4 & = & 1 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{14 - t \cdot 2}_{x_3 \geq 0} \geq 0 \\ \underbrace{32 - t \cdot 1}_{x_4 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \leq 7 \\ t \leq 32 \end{cases} \quad \text{then } t_m = 7$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = 7$. The value of x_3 becomes zero and x_3 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_4 = 32 - 7 \cdot 1 = 25$$

• **Iteration 2**

With $\mathcal{B} = \{1, 4\}$ and $\mathcal{N} = \{2, 3\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 7 \\ 25 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 1 \\ 8 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} 2\pi_1 + \pi_4 = 5 \\ \pi_4 = 0 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{5}{2} & 0 \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} \frac{5}{2} & 0 \end{pmatrix} \begin{pmatrix} 14 \\ 32 \end{pmatrix} = 35$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 3 & 0 \end{pmatrix} - \begin{pmatrix} \frac{5}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 8 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 0 \end{pmatrix}}_{\mathbf{c}'_2 \quad \mathbf{c}'_3} - \underbrace{\begin{pmatrix} \frac{5}{2} & \frac{5}{2} \end{pmatrix}}_{\boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_3} = \underbrace{\begin{pmatrix} \frac{1}{2} & -\frac{5}{2} \end{pmatrix}}_{\bar{\mathbf{c}}_2 \quad \bar{\mathbf{c}}_3}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \boldsymbol{\pi} \mathbf{A}_2$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} 2d_1 = 1 \\ d_1 + d_4 = 8 \end{cases} \quad \mathbf{d} = \begin{pmatrix} \frac{1}{2} \\ \frac{15}{2} \\ 8 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{7 - t \cdot \frac{1}{2} \geq 0}_{x_1 \geq 0} \\ \underbrace{25 - t \cdot \frac{15}{2} \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} t \leq 14 \\ t \leq \frac{10}{3} \end{cases} \quad \text{then } t_m = \frac{10}{3}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = \frac{10}{3}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{b} - t_m \mathbf{d}$, then we have:

$$x_1 = 7 - \frac{10}{3} \cdot \frac{1}{2} = \frac{16}{3}$$

• **Iteration 3**

With $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{16}{3} \\ \frac{10}{3} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} 2\pi_1 + \pi_2 = 5 \\ \pi_1 + 8\pi_2 = 3 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{37}{15} & \frac{1}{15} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} \frac{37}{15} & \frac{1}{15} \end{pmatrix} \begin{pmatrix} 14 \\ 32 \end{pmatrix} = \frac{110}{3}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{37}{15} & \frac{1}{15} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \end{pmatrix}}_{\mathbf{c}'_{\mathcal{N}}} - \underbrace{\begin{pmatrix} \frac{37}{15} & \frac{1}{15} \end{pmatrix}}_{\boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}} = \underbrace{\begin{pmatrix} -\frac{37}{15} & -\frac{1}{15} \end{pmatrix}}_{\bar{\mathbf{c}}'}$$

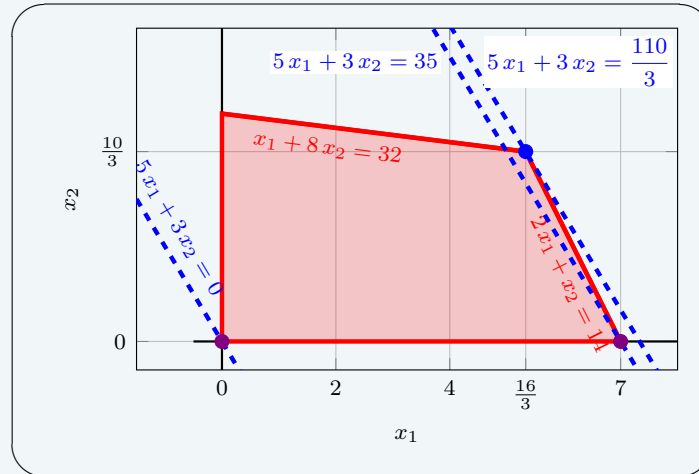
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{16}{3} \\ \frac{10}{3} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = \frac{110}{3}$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{16}{3}, \quad \tilde{x}_2 = \frac{10}{3}$$

the optimal value $z(LP)$ is $\frac{110}{3}$.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 11.6 — Revised simplex algorithm (P5)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{subject to} \quad & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 4$ variables and $m = 2$ constraints:

$$\begin{aligned}
 & \max \quad 2x_1 + 3x_2 \\
 & \text{subject to} \quad 6x_1 + 21x_2 + x_3 = 77 \\
 & \quad \quad \quad 8x_1 + 3x_2 + x_4 = 36 \\
 & \quad \quad \quad x_1, x_2, x_3, x_4 \geq 0
 \end{aligned}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 6 & 21 & 1 & 0 \\ 8 & 3 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 77 \\ 36 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 77 \\ 36 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 77 \\ 36 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 6 & 21 \\ 8 & 3 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_3 = 0 \\ \pi_4 = 0 \end{cases} \quad \boldsymbol{\pi} = (0 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \pi \mathbf{b} = (0 \ 0) \begin{pmatrix} 77 \\ 36 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \pi \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (2 \ 3) - (0 \ 0) \begin{pmatrix} 6 & 21 \\ 8 & 3 \end{pmatrix} = \underbrace{(2 \ 3)}_{c_1 \ c_2} - \underbrace{(0 \ 0)}_{\pi \mathbf{A}_1 \ \pi \mathbf{A}_2} = \underbrace{(2 \ 3)}_{\bar{c}_1 \ \bar{c}_2}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \pi \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_3 & = & 6 \\ & d_4 & = & 8 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{77 - t \cdot 6 \geq 0}_{x_3 \geq 0} \\ \underbrace{36 - t \cdot 8 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} t \leq \frac{77}{6} \\ t \leq \frac{9}{2} \end{cases} \quad \text{then } t_m = \frac{9}{2}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = \frac{9}{2}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_3 = 77 - \frac{9}{2} \cdot 6 = 50$$

• Iteration 2

With $\mathcal{B} = \{1, \ 3\}$ and $\mathcal{N} = \{2, \ 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 6 & 1 \\ 8 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{9}{2} \\ 50 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 21 & 0 \\ 3 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi \mathbf{A}_B = \mathbf{c}'_B$ to compute $\pi = \mathbf{c}'_B \mathbf{A}_B^{-1}$, first system of linear equations:

$$\begin{cases} 6\pi_1 + 8\pi_3 = 2 \\ \pi_1 = 0 \end{cases} \quad \pi = \begin{pmatrix} 0 & \frac{1}{4} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \pi \mathbf{b} = \begin{pmatrix} 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 77 \\ 36 \end{pmatrix} = 9$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_N - \pi \mathbf{A}_N$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 21 & 0 \\ 3 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 0 \end{pmatrix}}_{c_2 \quad c_4} - \underbrace{\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \end{pmatrix}}_{\pi \mathbf{A}_2 \quad \pi \mathbf{A}_4} = \underbrace{\begin{pmatrix} \frac{9}{4} & -\frac{1}{4} \end{pmatrix}}_{\bar{c}_2 \quad \bar{c}_4}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > 0$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} 21 \\ 3 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} 6d_1 + d_3 = 21 \\ 8d_1 = 3 \end{cases} \quad \mathbf{d} = \begin{pmatrix} \frac{3}{8} \\ \frac{75}{4} \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{\frac{9}{2} - t \cdot \frac{3}{8} \geq 0}_{x_1 \geq 0} \\ \underbrace{50 - t \cdot \frac{75}{4} \geq 0}_{x_3 \geq 0} \end{cases} \quad \begin{cases} t \leq 12 \\ t \leq \frac{8}{3} \end{cases} \quad \text{then } t_m = \frac{8}{3}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = \frac{8}{3}$. The value of x_3 becomes zero and x_3 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = \frac{9}{2} - \frac{8}{3} \cdot \frac{3}{8} = \frac{7}{2}$$

• Iteration 3

With $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 6 & 21 \\ 8 & 3 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{7}{2} \\ 8 \\ \frac{8}{3} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} 6\pi_1 + 8\pi_2 = 2 \\ 21\pi_1 + 3\pi_2 = 3 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{3}{25} & \frac{4}{25} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} \frac{3}{25} & \frac{4}{25} \end{pmatrix} \begin{pmatrix} 77 \\ 36 \end{pmatrix} = 15$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{3}{25} & \frac{4}{25} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \end{pmatrix}}_{\mathbf{c}'_{\mathcal{N}}} - \underbrace{\begin{pmatrix} \frac{3}{25} & \frac{4}{25} \end{pmatrix}}_{\boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}} = \underbrace{\begin{pmatrix} -\frac{3}{25} & -\frac{4}{25} \end{pmatrix}}_{\bar{\mathbf{c}}'}$$

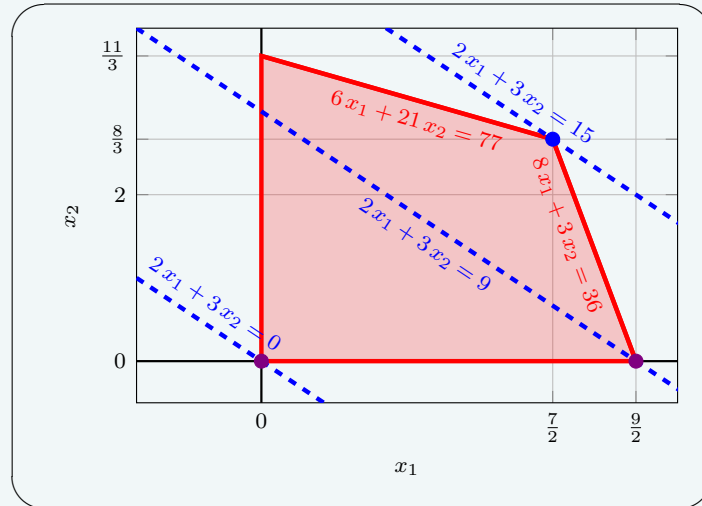
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ 8 \\ \frac{8}{3} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = 15$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{7}{2}, \quad \tilde{x}_2 = \frac{8}{3}$$

the optimal value $z(LP)$ is 15.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 11.7 — Revised simplex algorithm (P6)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & x_1 & + & x_2 & & \\ \text{subject to} & 4x_1 & + & 15x_2 & \leq & 70 \\ & 12x_1 & + & 3x_2 & \leq & 52 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

By introducing the slack variables, we get the following LP problem in standard maximiza-

tion form with $n + m = 4$ variables and $m = 2$ constraints:

$$\begin{array}{llllllll} \max & x_1 & + & x_2 & & & & \\ \text{subject to} & 4x_1 & + & 15x_2 & + & x_3 & & = 70 \\ & 12x_1 & + & 3x_2 & & & + & x_4 = 52 \\ & x_1, & & x_2, & & x_3, & & x_4 \geq 0 \end{array}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 4 & 15 & 1 & 0 \\ 12 & 3 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 70 \\ 52 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 70 \\ 52 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 70 \\ 52 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 4 & 15 \\ 12 & 3 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_3 & = & 0 \\ \pi_4 & = & 0 \end{cases} \quad \boldsymbol{\pi} = (0 \ 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \ 0) \begin{pmatrix} 70 \\ 52 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (1 \ 1) - (0 \ 0) \begin{pmatrix} 4 & 15 \\ 12 & 3 \end{pmatrix} = \underbrace{(1 \ 1)}_{c_1 \ c_2} - \underbrace{(0 \ 0)}_{\boldsymbol{\pi} \mathbf{A}_1 \ \boldsymbol{\pi} \mathbf{A}_2} = \underbrace{(1 \ 1)}_{\bar{c}_1 \ \bar{c}_2}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \boldsymbol{\pi} \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_3 & = & 4 \\ d_4 & = & 12 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 4 \\ 12 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{70 - t \cdot 4 \geq 0}_{x_3 \geq 0} \\ \underbrace{52 - t \cdot 12 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} t \leq \frac{35}{2} \\ t \leq \frac{13}{3} \end{cases} \quad \text{then } t_m = \frac{13}{3}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = \frac{13}{3}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_3 = 70 - \frac{13}{3} \cdot 4 = \frac{158}{3}$$

• Iteration 2

With $\mathcal{B} = \{1, \ 3\}$ and $\mathcal{N} = \{2, \ 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 4 & 1 \\ 12 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{13}{3} \\ \frac{158}{3} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 15 & 0 \\ 3 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} 4\pi_1 + 12\pi_3 = 1 \\ \pi_1 = 0 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} 0 & \frac{1}{12} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} 0 & \frac{1}{12} \end{pmatrix} \begin{pmatrix} 70 \\ 52 \end{pmatrix} = \frac{13}{3}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{12} \end{pmatrix} \begin{pmatrix} 15 & 0 \\ 3 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{c}'_2 \quad \mathbf{c}'_4} - \underbrace{\begin{pmatrix} \frac{1}{4} & \frac{1}{12} \end{pmatrix}}_{\boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_4} = \underbrace{\begin{pmatrix} \frac{3}{4} & -\frac{1}{12} \end{pmatrix}}_{\bar{\mathbf{c}}_2 \quad \bar{\mathbf{c}}_4}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \boldsymbol{\pi} \mathbf{A}_2$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} 15 \\ 3 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} 4d_1 + d_3 = 15 \\ 12d_1 = 3 \end{cases} \quad \mathbf{d} = \begin{pmatrix} \frac{1}{4} \\ 14 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{\frac{13}{3} - t \cdot \frac{1}{4} \geq 0}_{x_1 \geq 0} \\ \underbrace{\frac{158}{3} - t \cdot 14 \geq 0}_{x_3 \geq 0} \end{cases} \quad \begin{cases} t \leq \frac{52}{3} \\ t \leq \frac{79}{21} \end{cases} \quad \text{then } t_m = \frac{79}{21}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = \frac{79}{21}$. The value of x_3 becomes zero and x_3 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = \frac{13}{3} - \frac{79}{21} \cdot \frac{1}{4} = \frac{95}{28}$$

• **Iteration 3**

With $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 4 & 15 \\ 12 & 3 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{95}{28} \\ \frac{79}{21} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} 4\pi_1 + 12\pi_2 = 1 \\ 15\pi_1 + 3\pi_2 = 1 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{3}{56} & \frac{11}{168} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} \frac{3}{56} & \frac{11}{168} \end{pmatrix} \begin{pmatrix} 70 \\ 52 \end{pmatrix} = \frac{601}{84}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{3}{56} & \frac{11}{168} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \end{pmatrix}}_{\substack{c_3 \quad c_4}} - \underbrace{\begin{pmatrix} \frac{3}{56} & \frac{11}{168} \end{pmatrix}}_{\substack{\boldsymbol{\pi} \mathbf{A}_3 \quad \boldsymbol{\pi} \mathbf{A}_4}} = \underbrace{\begin{pmatrix} -\frac{3}{56} & -\frac{11}{168} \end{pmatrix}}_{\substack{\bar{c}_3 \quad \bar{c}_4}}$$

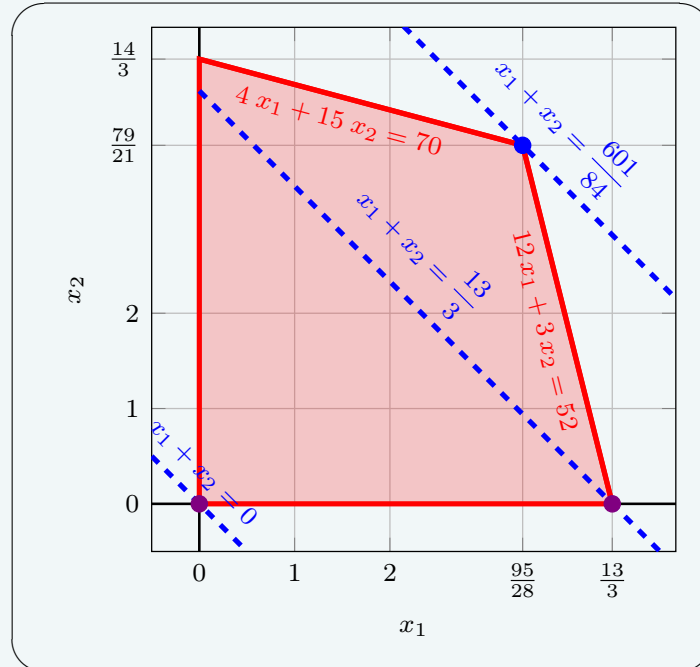
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{95}{28} \\ \frac{79}{21} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = \frac{601}{84}$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{x}(LP)$ is:

$$\tilde{x}_1 = \frac{95}{28}, \quad \tilde{x}_2 = \frac{79}{21}$$

the optimal value $z(LP)$ is $\frac{601}{84}$.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 11.8 — Revised simplex algorithm (P7)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & 3x_1 & + & x_2 & & \\ \text{subject to} & x_1 & + & 4x_2 & \leq & 12 \\ & 2x_1 & + & x_2 & \leq & 5 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 4$ variables and $m = 2$ constraints:

$$\begin{array}{llllllll} \max & 3x_1 & + & x_2 & & & & \\ \text{subject to} & x_1 & + & 4x_2 & + & x_3 & = & 12 \\ & 2x_1 & + & x_2 & & & + & x_4 = 5 \\ & x_1, & & x_2, & & x_3, & & x_4 \geq 0 \end{array}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 12 \\ 5 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_3 & = & 0 \\ \pi_4 & = & 0 \end{cases} \quad \boldsymbol{\pi} = (0 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \quad 0) \begin{pmatrix} 12 \\ 5 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (3 \quad 1) - (0 \quad 0) \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} = \underbrace{(3 \quad 1)}_{c_1 \quad c_2} - \underbrace{(0 \quad 0)}_{\boldsymbol{\pi} \mathbf{A}_1 \quad \boldsymbol{\pi} \mathbf{A}_2} = \underbrace{(3 \quad 1)}_{\bar{c}_1 \quad \bar{c}_2}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \boldsymbol{\pi} \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_3 & = & 1 \\ d_4 & = & 2 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{12 - t \cdot 1 \geq 0}_{x_3 \geq 0} \\ \underbrace{5 - t \cdot 2 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} t \leq 12 \\ t \leq \frac{5}{2} \end{cases} \quad \text{then } t_m = \frac{5}{2}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = \frac{5}{2}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_3 = 12 - \frac{5}{2} \cdot 1 = \frac{19}{2}$$

• **Iteration 2**

With $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{5}{2} \\ \frac{19}{2} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_1 + 2\pi_3 = 3 \\ \pi_1 = 0 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} 0 & \frac{3}{2} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 12 \\ 5 \end{pmatrix} = \frac{15}{2}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{c}'_{\mathcal{N}}} - \underbrace{\begin{pmatrix} \frac{3}{2} & \frac{3}{2} \end{pmatrix}}_{\boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}} = \underbrace{\begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}}_{\bar{\mathbf{c}}'}$$

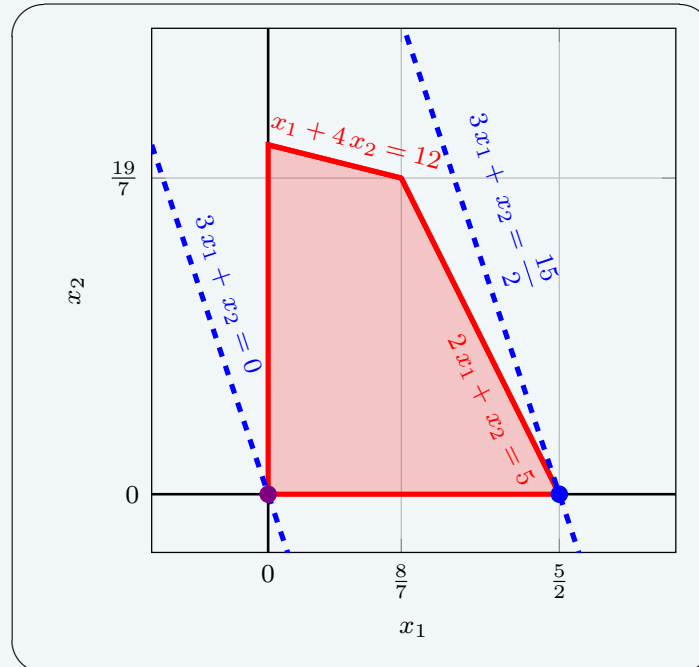
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{19}{2} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = \frac{15}{2}$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = 0$$

the optimal value $z(LP)$ is $\frac{15}{2}$.

In violet the first vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the two vertices.

Exercise 11.9 — Revised simplex algorithm (P8)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & x_1 & + & 2x_2 & & \\ \text{subject to} & -x_1 & + & 2x_2 & \leq & 2 \\ & x_1 & + & x_2 & \leq & 3 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 4$ variables and $m = 2$ constraints:

$$\begin{array}{llllllll} \max & x_1 & + & 2x_2 & & & & \\ \text{subject to} & -x_1 & + & 2x_2 & + & x_3 & = & 2 \\ & x_1 & + & x_2 & & & + & x_4 = 3 \\ & x_1, & & x_2, & & x_3, & & x_4 \geq 0 \end{array}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_3 & = & 0 \\ \pi_4 & = & 0 \end{cases} \quad \boldsymbol{\pi} = (0 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \quad 0) \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (1 \quad 2) - (0 \quad 0) \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} = \underbrace{(1 \quad 2)}_{c_1 \quad c_2} - \underbrace{(0 \quad 0)}_{\boldsymbol{\pi} \mathbf{A}_1 \quad \boldsymbol{\pi} \mathbf{A}_2} = \underbrace{(1 \quad 2)}_{\bar{c}_1 \quad \bar{c}_2}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \boldsymbol{\pi} \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_3 & = & -1 \\ d_4 & = & 1 \end{cases} \quad \mathbf{d} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{2 - t \cdot (-1) \geq 0}_{x_3 \geq 0} \\ \underbrace{3 - t \cdot 1 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} t \geq -2 \\ t \leq 3 \end{cases} \quad \text{then } t_m = 3$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = 3$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_3 = 2 - 3 \cdot (-1) = 5$$

• **Iteration 2**

With $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} -\pi_1 + \pi_3 = 1 \\ \pi_1 = 0 \end{cases} \quad \boldsymbol{\pi} = (0 \quad 1)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \quad 1) \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 3$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (2 \quad 0) - (0 \quad 1) \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \underbrace{(2 \quad 0)}_{c_2 \quad c_4} - \underbrace{(1 \quad 1)}_{\boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_4} = \underbrace{(1 \quad -1)}_{\bar{c}_2 \quad \bar{c}_4}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \boldsymbol{\pi} \mathbf{A}_2$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} -d_1 + d_3 = 2 \\ d_1 = 1 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{3 - t \cdot 1}_{x_1 \geq 0} \geq 0 \\ \underbrace{5 - t \cdot 3}_{x_3 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \leq 3 \\ t \leq \frac{5}{3} \end{cases} \quad \text{then} \quad t_m = \frac{5}{3}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = \frac{5}{3}$. The value of x_3 becomes zero and x_3 is the pivot row. The other basic variables take values $\bar{b} - t_m \mathbf{d}$, then we have:

$$x_1 = 3 - \frac{5}{3} \cdot 1 = \frac{4}{3}$$

• **Iteration 3**

With $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{4}{3} \\ \frac{5}{3} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} -\pi_1 + \pi_2 = 1 \\ 2\pi_1 + \pi_2 = 2 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{14}{3}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (0 \ 0) - \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{(0 \ 0)}_{\mathbf{c}'_{\mathcal{N}}} - \underbrace{\begin{pmatrix} \frac{1}{3} & \frac{4}{3} \end{pmatrix}}_{\boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}} = \underbrace{\begin{pmatrix} -\frac{1}{3} & -\frac{4}{3} \end{pmatrix}}_{\bar{\mathbf{c}}'}$$

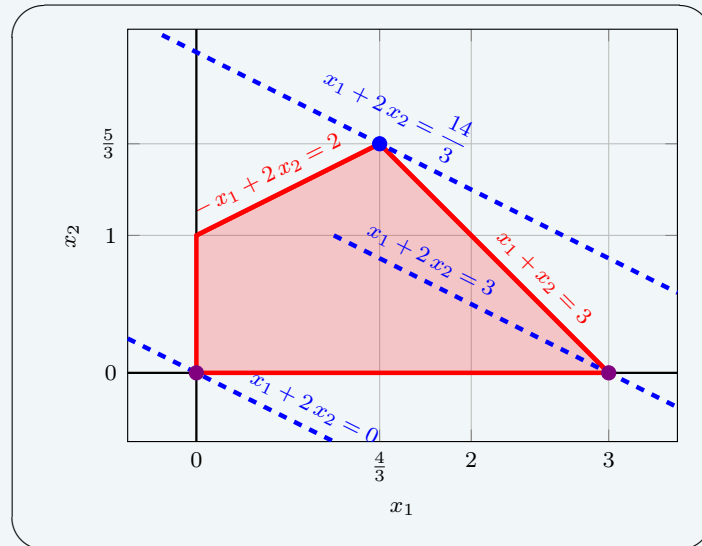
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{5}{3} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = \frac{14}{3}$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{4}{3}, \quad \tilde{x}_2 = \frac{5}{3}$$

the optimal value $z(LP)$ is $\frac{14}{3}$.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 11.10 — Revised simplex algorithm (P9)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & 6x_1 + 4x_2 \leq 39 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 4$ variables and $m = 2$ constraints:

$$\begin{array}{llllllll} \max & 2x_1 & + & x_2 & & & & \\ \text{subject to} & -x_1 & + & x_2 & + & x_3 & & = 1 \\ & 6x_1 & + & 4x_2 & & & + & x_4 = 39 \\ & x_1, & & x_2, & & x_3, & & x_4 \geq 0 \end{array}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 6 & 4 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 39 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 39 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 1 \\ 39 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & 1 \\ 6 & 4 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_3 & = & 0 \\ \pi_4 & = & 0 \end{cases} \quad \boldsymbol{\pi} = (0 \ 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \ 0) \begin{pmatrix} 1 \\ 39 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (2 \ 1) - (0 \ 0) \begin{pmatrix} -1 & 1 \\ 6 & 4 \end{pmatrix} = \underbrace{(2 \ 1)}_{c_1 \ c_2} - \underbrace{(0 \ 0)}_{\boldsymbol{\pi} \mathbf{A}_1 \ \boldsymbol{\pi} \mathbf{A}_2} = \underbrace{(2 \ 1)}_{\bar{c}_1 \ \bar{c}_2}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \boldsymbol{\pi} \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_3 & = & -1 \\ d_4 & = & 6 \end{cases} \quad \mathbf{d} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{1 - t \cdot (-1) \geq 0}_{x_3 \geq 0} \\ \underbrace{39 - t \cdot 6 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} t \geq -1 \\ t \leq \frac{13}{2} \end{cases} \quad \text{then } t_m = \frac{13}{2}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = \frac{13}{2}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_3 = 1 - \frac{13}{2} \cdot (-1) = \frac{15}{2}$$

• Iteration 2

With $\mathcal{B} = \{1, \ 3\}$ and $\mathcal{N} = \{2, \ 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 1 \\ 6 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{13}{2} \\ \frac{15}{2} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} -\pi_1 + 6\pi_3 = 2 \\ \pi_1 = 0 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} 0 & \frac{1}{3} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 39 \end{pmatrix} = 13$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{c}'_2 \quad \mathbf{c}'_4} - \underbrace{\begin{pmatrix} \frac{4}{3} & \frac{1}{3} \end{pmatrix}}_{\boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_4} = \underbrace{\begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}}_{\bar{\mathbf{c}}_2 \quad \bar{\mathbf{c}}_4}$$

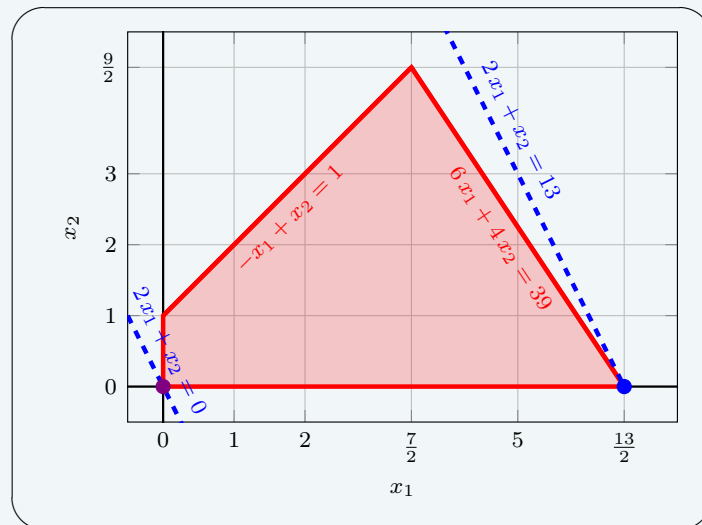
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{13}{2} \\ \frac{15}{2} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = 13$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{13}{2}, \quad \tilde{x}_2 = 0$$

the optimal value $z(LP)$ is 13.

In violet the first vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the two vertices.

Exercise 11.11 — Revised simplex algorithm (P10)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{array}{llllll} \max & 5x_1 & + & 2x_2 & & \\ \text{subject to} & -x_1 & + & 3x_2 & \leq & 12 \\ & x_1 & + & 2x_2 & \leq & 12 \\ & 3x_1 & - & 2x_2 & \leq & 8 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 5$ variables and $m = 3$ constraints:

$$\begin{array}{llllllllll} \max & 5x_1 & + & 2x_2 & & & & & & \\ \text{subject to} & -x_1 & + & 3x_2 & + & x_3 & & & & = & 12 \\ & x_1 & + & 2x_2 & & & + & x_4 & & = & 12 \\ & 3x_1 & - & 2x_2 & & & & & + & x_5 & = & 8 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0 \end{array}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} -1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 5 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & 3 \\ 1 & 2 \\ 3 & -2 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{rcl} \pi_3 & & = 0 \\ & \pi_4 & = 0 \\ & & \pi_5 = 0 \end{array} \right. \quad \boldsymbol{\pi} = (0 \quad 0 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \quad 0 \quad 0) \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned}\bar{c}' &= (5 \ 2) - (0 \ 0 \ 0) \begin{pmatrix} -1 & 3 \\ 1 & 2 \\ 3 & -2 \end{pmatrix} \\ &= \underbrace{(5 \ 2)}_{c_1 \ c_2} - \underbrace{(0 \ 0)}_{\pi \ A_1 \ \pi \ A_2} = \underbrace{(5 \ 2)}_{\bar{c}_1 \ \bar{c}_2}\end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \pi A_1$, x_1 is the pivot column and we have $A_1 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$

We solve $A_B d = A_1$ to compute $d = A_B^{-1} A_1$, second system of linear equations:

$$\begin{cases} d_3 & = & -1 \\ & d_4 & = & 1 \\ & & d_5 & = & 3 \end{cases} \quad d = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

We have to find the largest t such that $\bar{b} - t d \geq 0$:

$$\begin{cases} \underbrace{12 - t \cdot (-1) \geq 0}_{x_3 \geq 0} \\ \underbrace{12 - t \cdot 1 \geq 0}_{x_4 \geq 0} \\ \underbrace{8 - t \cdot 3 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} t \geq -12 \\ t \leq 12 \\ t \leq \frac{8}{3} \end{cases} \quad \text{then } t_m = \frac{8}{3}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = \frac{8}{3}$. The value of x_5 becomes zero and x_5 is the pivot row. The other basic variables take values $\bar{b} - t_m d$, then we have:

$$x_3 = 12 - \frac{8}{3} \cdot (-1) = \frac{44}{3}$$

$$x_4 = 12 - \frac{8}{3} \cdot 1 = \frac{28}{3}$$

• Iteration 2

With $B = \{1, 3, 4\}$ and $N = \{2, 5\}$, we have:

$$x_B = \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} \quad A_B = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \quad c_B = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \quad \bar{b} = \begin{pmatrix} \frac{8}{3} \\ \frac{44}{3} \\ \frac{28}{3} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 3 & 0 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} -\pi_1 + \pi_3 + 3\pi_4 = 5 \\ \pi_1 = 0 \\ \pi_3 = 0 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} 0 & 0 & \frac{5}{3} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} 0 & 0 & \frac{5}{3} \end{pmatrix} \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = \frac{40}{3}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= \begin{pmatrix} 2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \frac{5}{3} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 2 & 0 \end{pmatrix}}_{\substack{c_2 \quad c_5}} - \underbrace{\begin{pmatrix} -10 & 5 \\ 3 & 3 \end{pmatrix}}_{\substack{\boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_5}} = \underbrace{\begin{pmatrix} \frac{16}{3} & -\frac{5}{3} \end{pmatrix}}_{\substack{\bar{c}_2 \quad \bar{c}_5}} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \boldsymbol{\pi} \mathbf{A}_2$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} -d_1 + d_3 = 3 \\ d_1 + d_4 = 2 \\ 3d_1 = -2 \end{cases} \quad \mathbf{d} = \begin{pmatrix} -\frac{2}{3} \\ \frac{7}{3} \\ \frac{8}{3} \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{\frac{8}{3} - t \cdot \left(-\frac{2}{3}\right)}_{x_1 \geq 0} \geq 0 \\ \underbrace{\frac{44}{3} - t \cdot \frac{7}{3}}_{x_3 \geq 0} \geq 0 \\ \underbrace{\frac{28}{3} - t \cdot \frac{8}{3}}_{x_4 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \geq -4 \\ t \leq \frac{44}{7} \\ t \leq \frac{7}{2} \end{cases} \quad \text{then } t_m = \frac{7}{2}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = \frac{7}{2}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = \frac{8}{3} - \frac{7}{2} \cdot \left(-\frac{2}{3}\right) = 5$$

$$x_3 = \frac{44}{3} - \frac{7}{2} \cdot \frac{7}{3} = \frac{13}{2}$$

• **Iteration 3**

With $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 3 & 1 \\ 1 & 2 & 0 \\ 3 & -2 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 5 \\ \frac{7}{2} \\ \frac{13}{2} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} -\pi_1 + \pi_2 + 3\pi_3 = 5 \\ 3\pi_1 + 2\pi_2 - 2\pi_3 = 2 \\ \pi_1 = 0 \end{cases} \quad \boldsymbol{\pi} = (0 \quad 2 \quad 1)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \quad 2 \quad 1) \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = 32$

We compute the reduced costs $\bar{c}' = c'_{\mathcal{N}} - \pi A_{\mathcal{N}}$:

$$\begin{aligned}\bar{c}' &= \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 0 & 0 \end{pmatrix}}_{c_4 \quad c_5} - \underbrace{\begin{pmatrix} 2 & 1 \end{pmatrix}}_{\pi A_4 \quad \pi A_5} = \underbrace{\begin{pmatrix} -2 & -1 \end{pmatrix}}_{\bar{c}_4 \quad \bar{c}_5}\end{aligned}$$

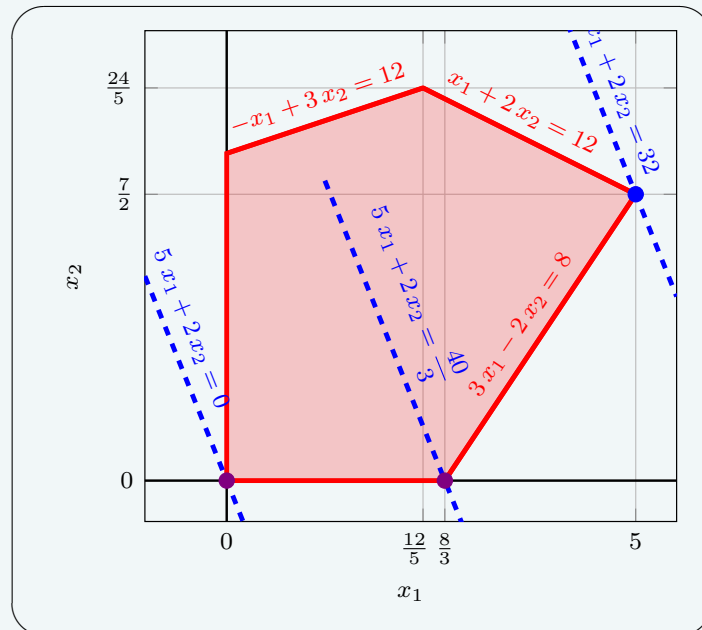
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ \frac{7}{2} \\ \frac{13}{2} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = 32$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value $z(LP)$ is 32.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 11.12 — First phase of the revised simplex algorithm (P21)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{array}{llllll} \max & x_1 & + & 2 & x_2 & \\ \text{subject to} & - & x_1 & - & x_2 & \leq -2 \\ & 4 & x_1 & - & 2 & x_2 \leq 7 \\ & & x_1 & + & 5 & x_2 \leq 21 \\ & x_1, & & & x_2 & \geq 0 \end{array}$$

Questions

1. Solve the first phase of the simplex algorithm using Bland's rule to select the pivot column and the pivot row.
2. Starting from the basic feasible solution obtained in the previous question, solve this LP problem using the revised simplex algorithm and Bland's rule to select the pivot column and the pivot row.

Solution

1. We write the auxiliary problem:

$$\begin{array}{llllllllll} \max & - & x_0 & & & & & & & \\ \text{subject to} & - & x_0 & - & x_1 & - & x_2 & + & x_3 & = -2 \\ & & - & x_0 & + & 4 & x_1 & - & 2 & x_2 & + & x_4 & = 7 \\ & & - & x_0 & + & x_1 & + & 5 & x_2 & & + & x_5 & = 21 \\ & & x_0, & & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq 0 \end{array}$$

Now there are $n + m + 1 = 6$ variables and $m = 3$ constraints. We have:

$$\mathbf{A} = \begin{pmatrix} -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 4 & -2 & 0 & 1 & 0 \\ -1 & 1 & 5 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 7 \\ 21 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

- **Initialization**

We consider $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{0, 1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} -2 \\ 7 \\ 21 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 4 & -2 \\ -1 & 1 & 5 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

First Pivoting operation

The variable x_0 enters the basis and x_3 leaves the basis, since x_3 is the slack variable associated with the most negative RHS (also the only negative one). We have $x_0 = -b_1 = 2$.

The other basic variables x_j with $j \in \mathcal{B}$ take values $x_j = b_j - b_1$, then we have:

$$x_4 = 7 - (-2) = 9$$

$$x_5 = 21 - (-2) = 23$$

The new basic variables are $\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_0 \\ x_4 \\ x_5 \end{pmatrix}$. The basic feasible solution is $\bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 9 \\ 23 \end{pmatrix}$

- **Iteration 1**

With $\mathcal{B} = \{0, 4, 5\}$ and $\mathcal{N} = \{1, 2, 3\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_0 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 9 \\ 23 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & -1 & 1 \\ 4 & -2 & 0 \\ 1 & 5 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi \mathbf{A}_B = \mathbf{c}'_B$ to compute $\pi = \mathbf{c}'_B \mathbf{A}_B^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{cccccc} - & \pi_0 & - & \pi_4 & - & \pi_5 & = & -1 \\ & & & \pi_4 & & & = & 0 \\ & & & & & \pi_5 & = & 0 \end{array} \right. \quad \pi = (1 \ 0 \ 0)$$

We compute the basic feasible solution value $\bar{z} = \pi \mathbf{b} = (1 \ 0 \ 0) \begin{pmatrix} -2 \\ 7 \\ 21 \end{pmatrix} = -2$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \pi \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (0 \ 0 \ 0) - (1 \ 0 \ 0) \begin{pmatrix} -1 & -1 & 1 \\ 4 & -2 & 0 \\ 1 & 5 & 0 \end{pmatrix} \\ &= \underbrace{(0 \ 0 \ 0)}_{c_1 \ c_2 \ c_3} - \underbrace{(-1 \ -1 \ 1)}_{\pi \mathbf{A}_1 \ \pi \mathbf{A}_2 \ \pi \mathbf{A}_3} = \underbrace{(1 \ 1 \ -1)}_{\bar{c}_1 \ \bar{c}_2 \ \bar{c}_3} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \pi \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_1$, second system of linear equations:

$$\left\{ \begin{array}{cccccc} - & d_0 & & & & = & -1 \\ - & d_0 & + & d_4 & & = & 4 \\ - & d_0 & & & + & d_5 & = & 1 \end{array} \right. \quad \mathbf{d} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\left\{ \begin{array}{l} \underbrace{2 - t \cdot (1)}_{x_0 \geq 0} \geq 0 \\ \underbrace{9 - t \cdot (5)}_{x_4 \geq 0} \geq 0 \\ \underbrace{23 - t \cdot (2)}_{x_5 \geq 0} \geq 0 \end{array} \right. \quad \left\{ \begin{array}{l} t \leq 2 \\ t \leq \frac{9}{5} \\ t \leq \frac{23}{2} \end{array} \right. \quad \text{then } t_m = \frac{9}{5}$$

Accordingly, the value of the pivot column becomes t_m , i.e., $x_1 = \frac{9}{5}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{b} - t_m \mathbf{d}$, then we have:

$$\begin{aligned} x_0 &= 2 - \frac{9}{5} \cdot (1) = \frac{1}{5} \\ x_5 &= 23 - \frac{9}{5} \cdot (2) = \frac{97}{5} \end{aligned}$$

• **Iteration 2**

With $\mathcal{B} = \{0, 1, 5\}$ and $\mathcal{N} = \{2, 3, 4\}$, we have:

$$\begin{aligned} \mathbf{x}_{\mathcal{B}} &= \begin{pmatrix} x_0 \\ x_1 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 4 & 0 \\ -1 & 1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{1}{5} \\ \frac{9}{5} \\ \frac{97}{5} \end{pmatrix} \\ \mathbf{x}_{\mathcal{N}} &= \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} -\pi_0 - \pi_1 - \pi_5 = -1 \\ -\pi_0 + 4\pi_1 = 0 \\ \pi_5 = 0 \end{cases} \quad \boldsymbol{\pi} = \left(\frac{4}{5} \quad \frac{1}{5} \quad 0 \right)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \left(\frac{4}{5} \quad \frac{1}{5} \quad 0 \right) \begin{pmatrix} -2 \\ 7 \\ 21 \end{pmatrix} = -\frac{1}{5}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (0 \quad 0 \quad 0) - \left(\frac{4}{5} \quad \frac{1}{5} \quad 0 \right) \begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 5 & 0 & 0 \end{pmatrix} \\ &= \underbrace{(0 \quad 0 \quad 0)}_{c_2 \quad c_3 \quad c_4} - \underbrace{\left(-\frac{6}{5} \quad \frac{4}{5} \quad \frac{1}{5} \right)}_{\boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_3 \quad \boldsymbol{\pi} \mathbf{A}_4} = \underbrace{\left(\frac{6}{5} \quad -\frac{4}{5} \quad -\frac{1}{5} \right)}_{\bar{c}_2 \quad \bar{c}_3 \quad \bar{c}_4} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \pi \mathbf{A}_2$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} -d_0 - d_1 = -1 \\ -d_0 + 4d_1 = -2 \\ -d_0 + d_1 + d_5 = 5 \end{cases} \quad \mathbf{d} = \begin{pmatrix} \frac{6}{5} \\ -\frac{1}{5} \\ \frac{32}{5} \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{\frac{1}{5} - t \cdot \left(\frac{6}{5}\right)}_{x_0 \geq 0} \geq 0 \\ \underbrace{\frac{9}{5} - t \cdot \left(-\frac{1}{5}\right)}_{x_1 \geq 0} \geq 0 \\ \underbrace{\frac{97}{5} - t \cdot \left(\frac{32}{5}\right)}_{x_5 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \leq \frac{1}{6} \\ t \geq -9 \\ t \leq \frac{97}{32} \end{cases} \quad \text{then } t_m = \frac{1}{6}$$

Accordingly, the value of the pivot column becomes t_m , i.e., $x_2 = \frac{1}{6}$. The value of x_0 becomes zero and x_0 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = \frac{9}{5} - \frac{1}{6} \left(-\frac{1}{5}\right) = \frac{11}{6}$$

$$x_5 = \frac{97}{5} - \frac{1}{6} \left(\frac{32}{5}\right) = \frac{55}{3}$$

- Iteration 3**

With $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{0, 3, 4\}$, we have:

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_2 \\ x_5 \end{pmatrix} \quad \mathbf{A}_B = \begin{pmatrix} -1 & -1 & 0 \\ 4 & -2 & 0 \\ 1 & 5 & 1 \end{pmatrix} \quad \mathbf{c}_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{11}{6} \\ \frac{1}{6} \\ \frac{55}{3} \end{pmatrix}$$

$$\mathbf{x}_N = \begin{pmatrix} x_0 \\ x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_N = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad \mathbf{c}_N = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi \mathbf{A}_B = \mathbf{c}'_B$ to compute $\pi = \mathbf{c}'_B \mathbf{A}_B^{-1}$, first system of linear equations:

$$\begin{cases} -\pi_1 + 4\pi_2 + \pi_5 = 0 \\ -\pi_1 - 2\pi_2 + 5\pi_5 = 0 \\ \pi_5 = 0 \end{cases} \quad \pi = (0 \ 0 \ 0)$$

We compute the basic feasible solution value $\bar{z} = \pi \mathbf{b} = (0 \ 0 \ 0) \begin{pmatrix} -2 \\ 7 \\ 21 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \pi \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (-1 \ 0 \ 0) - (0 \ 0 \ 0) \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \\ &= \underbrace{(-1 \ 0 \ 0)}_{c_0 \ c_3 \ c_4} - \underbrace{(0 \ 0 \ 0)}_{\pi \mathbf{A}_0 \ \pi \mathbf{A}_3 \ \pi \mathbf{A}_4} = \underbrace{(-1 \ 0 \ 0)}_{\bar{c}_0 \ \bar{c}_3 \ \bar{c}_4} \end{aligned}$$

All reduced costs are non-positive, then the basic feasible solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{11}{6} \\ \frac{1}{6} \\ \frac{55}{3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_0 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the auxiliary LP problem and $\bar{z} = 0$ is the optimal solution value.

Warning

An optimal basic solution of the auxiliary problem of value zero (where the auxiliary variable is a non-basic variable) corresponds to a basic feasible solution of the original LP problem.

By removing the auxiliary variable x_0 from the non-basic variables, we obtain $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$ and the basic feasible solution of the original LP problem:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{11}{6} \\ \frac{1}{6} \\ \frac{55}{3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2. We execute the revised simplex algorithm for the original LP problem. We have:

$$\mathbf{A} = \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ 4 & -2 & 0 & 1 & 0 \\ 1 & 5 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 7 \\ 21 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

• **Iteration 1**

With $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & -1 & 0 \\ 4 & -2 & 0 \\ 1 & 5 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{11}{6} \\ \frac{1}{6} \\ \frac{55}{3} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} -\pi_1 + 4\pi_2 + \pi_5 = 1 \\ -\pi_1 - 2\pi_2 + 5\pi_5 = 2 \\ \pi_5 = 0 \end{cases} \quad \boldsymbol{\pi} = \left(-\frac{5}{3} \quad -\frac{1}{6} \quad 0 \right)$$

We compute the basic feasible solution value

$$\bar{z} = \boldsymbol{\pi} \mathbf{b} = \left(-\frac{5}{3} \quad -\frac{1}{6} \quad 0 \right) \begin{pmatrix} -2 \\ 7 \\ 21 \end{pmatrix} = \frac{13}{6}$$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (0 \quad 0) - \left(-\frac{5}{3} \quad -\frac{1}{6} \quad 0 \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \underbrace{(0 \quad 0)}_{\mathbf{c}'_{\mathcal{N}}} - \underbrace{\left(-\frac{5}{3} \quad -\frac{1}{6} \right)}_{\boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}} = \underbrace{\left(\frac{5}{3} \quad \frac{1}{6} \right)}_{\bar{\mathbf{c}}'} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_3 > \pi \mathbf{A}_3$, x_3 is the pivot column and we have $\mathbf{A}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_3$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_3$, second system of linear equations:

$$\begin{cases} -d_1 - d_2 = 1 \\ 4d_1 - 2d_2 = 0 \\ d_1 + 5d_2 + d_5 = 0 \end{cases} \quad \mathbf{d} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{11}{3} \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{\frac{11}{6} - t \cdot \left(-\frac{1}{3}\right)}_{x_1 \geq 0} \geq 0 \\ \underbrace{\frac{1}{6} - t \cdot \left(-\frac{2}{3}\right)}_{x_2 \geq 0} \geq 0 \\ \underbrace{\frac{55}{3} - t \cdot \left(\frac{11}{3}\right)}_{x_5 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \geq -\frac{11}{2} \\ t \geq -\frac{1}{4} \\ t \leq 5 \end{cases} \quad \text{then } t_m = 5$$

Accordingly, the value of the pivot column becomes t_m , i.e., $x_3 = 5$. The value of x_5 becomes zero and x_5 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = \frac{11}{6} - 5 \cdot \left(-\frac{1}{3}\right) = \frac{7}{2}$$

$$x_2 = \frac{1}{6} - 5 \cdot \left(-\frac{2}{3}\right) = \frac{7}{2}$$

• Iteration 2

With $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$, we have:

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_B = \begin{pmatrix} -1 & -1 & 1 \\ 4 & -2 & 0 \\ 1 & 5 & 0 \end{pmatrix} \quad \mathbf{c}_B = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{7}{2} \\ \frac{7}{2} \\ 5 \end{pmatrix}$$

$$\mathbf{x}_N = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi \mathbf{A}_B = \mathbf{c}'_B$ to compute $\pi = \mathbf{c}'_B \mathbf{A}_B^{-1}$, first system of linear equations:

$$\begin{cases} -\pi_1 + 4\pi_2 + \pi_3 = 1 \\ -\pi_1 - 2\pi_2 + 5\pi_3 = 2 \\ \pi_1 = 0 \end{cases} \quad \pi = \begin{pmatrix} 0 & \frac{3}{22} & \frac{5}{11} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \pi \mathbf{b} = \begin{pmatrix} 0 & \frac{3}{22} & \frac{5}{11} \end{pmatrix} \begin{pmatrix} -2 \\ 7 \\ 21 \end{pmatrix} = \frac{21}{2}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \pi \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \frac{3}{22} & \frac{5}{11} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 0 & 0 \end{pmatrix}}_{\mathbf{c}_4 \quad \mathbf{c}_5} - \underbrace{\begin{pmatrix} \frac{3}{22} & \frac{5}{11} \end{pmatrix}}_{\pi \mathbf{A}_4 \quad \pi \mathbf{A}_5} = \underbrace{\begin{pmatrix} -\frac{3}{22} & -\frac{5}{11} \end{pmatrix}}_{\bar{\mathbf{c}}_4 \quad \bar{\mathbf{c}}_5} \end{aligned}$$

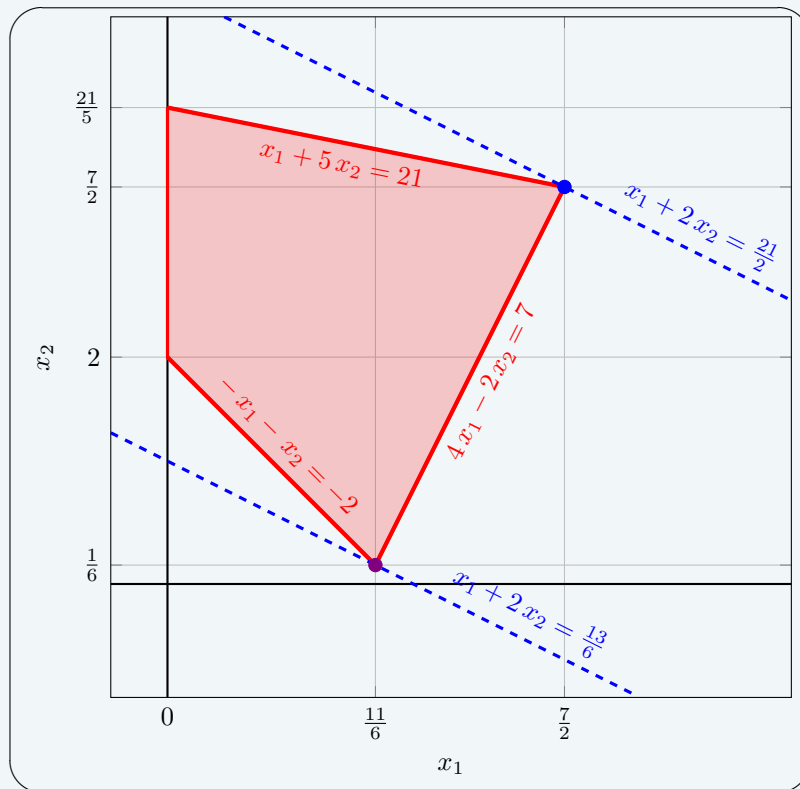
All reduced costs are non-positive, then the basic feasible solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ \frac{7}{2} \\ 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = \frac{21}{2}$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red. In violet the basic feasible solution obtained by the first phase, $(\frac{11}{6}, \frac{1}{6})$, in blue the optimal one, $(\frac{7}{2}, \frac{7}{2})$. In dashed blue, the contour lines of the objective function associated with the values $\frac{13}{6}$ and $\frac{21}{2}$ taken in the two vertices.



Exercise 11.13 — Revised simplex algorithm (B1)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned}
 \max \quad & 25x_1 + 16x_2 \\
 \text{subject to} \quad & 50x_1 + 31x_2 \leq 250 \\
 & -3x_1 + 2x_2 \leq 4 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 4$ variables and $m = 2$ constraints:

$$\begin{aligned}
 & \max \quad 25x_1 + 16x_2 \\
 & \text{subject to} \quad 50x_1 + 31x_2 + x_3 = 250 \\
 & \quad \quad \quad -3x_1 + 2x_2 + x_4 = 4 \\
 & \quad \quad \quad x_1, x_2, x_3, x_4 \geq 0
 \end{aligned}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 50 & 31 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 250 \\ 4 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 25 \\ 16 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 250 \\ 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 250 \\ 4 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 50 & 31 \\ -3 & 2 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 25 \\ 16 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_3 & = & 0 \\ & \pi_4 & = & 0 \end{cases} \quad \boldsymbol{\pi} = (0 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \pi \mathbf{b} = (0 \ 0) \begin{pmatrix} 250 \\ 4 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \pi \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = (25 \ 16) - (0 \ 0) \begin{pmatrix} 50 & 31 \\ -3 & 2 \end{pmatrix} = \underbrace{(25 \ 16)}_{c_1 \ c_2} - \underbrace{(0 \ 0)}_{\pi \mathbf{A}_1 \ \pi \mathbf{A}_2} = \underbrace{(25 \ 16)}_{\bar{c}_1 \ \bar{c}_2}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \pi \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} 50 \\ -3 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_3 = 50 \\ d_4 = -3 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 50 \\ -3 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{250 - t \cdot 50}_{x_3 \geq 0} \geq 0 \\ \underbrace{4 - t \cdot (-3)}_{x_4 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \leq 5 \\ t \geq -\frac{4}{3} \end{cases} \quad \text{then } t_m = 5$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = 5$. The value of x_3 becomes zero and x_3 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_4 = 4 - 5 \cdot (-3) = 19$$

• Iteration 2

With $\mathcal{B} = \{1, 4\}$ and $\mathcal{N} = \{2, 3\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 50 & 0 \\ -3 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 25 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 5 \\ 19 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 31 & 1 \\ 2 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 16 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\pi = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} 50\pi_1 - 3\pi_4 = 25 \\ \pi_4 = 0 \end{cases} \quad \pi = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \pi \mathbf{b} = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 250 \\ 4 \end{pmatrix} = 125$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \pi \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 16 & 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 31 & 1 \\ 2 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 16 & 0 \end{pmatrix}}_{\mathbf{c}_2 \quad \mathbf{c}_3} - \underbrace{\begin{pmatrix} \frac{31}{2} & \frac{1}{2} \end{pmatrix}}_{\pi \mathbf{A}_2 \quad \pi \mathbf{A}_3} = \underbrace{\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}_{\bar{\mathbf{c}}_2 \quad \bar{\mathbf{c}}_3}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \pi \mathbf{A}_2$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} 31 \\ 2 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} 50d_1 = 31 \\ -3d_1 + d_4 = 2 \end{cases} \quad \mathbf{d} = \begin{pmatrix} \frac{31}{50} \\ \frac{193}{50} \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{5 - t \cdot \frac{31}{50} \geq 0}_{x_1 \geq 0} \\ \underbrace{19 - t \cdot \frac{193}{50} \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} t \leq \frac{250}{31} \\ t \leq \frac{950}{193} \end{cases} \quad \text{then } t_m = \frac{950}{193}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = \frac{950}{193}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = 5 - \frac{950}{193} \cdot \frac{31}{50} = \frac{376}{193}$$

• Iteration 3

With $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 50 & 31 \\ -3 & 2 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 25 \\ 16 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{376}{193} \\ \frac{950}{193} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} 50 \pi_1 - 3 \pi_2 = 25 \\ 31 \pi_1 + 2 \pi_2 = 16 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{98}{193} & \frac{25}{193} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} \frac{98}{193} & \frac{25}{193} \end{pmatrix} \begin{pmatrix} 250 \\ 4 \end{pmatrix} = \frac{24600}{193}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\bar{\mathbf{c}}' = \begin{pmatrix} 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{98}{193} & \frac{25}{193} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \end{pmatrix}}_{\substack{c_3 \quad c_4}} - \underbrace{\begin{pmatrix} \frac{98}{193} & \frac{25}{193} \end{pmatrix}}_{\substack{\boldsymbol{\pi} \mathbf{A}_3 \quad \boldsymbol{\pi} \mathbf{A}_4}} = \underbrace{\begin{pmatrix} -\frac{98}{193} & -\frac{25}{193} \end{pmatrix}}_{\substack{\bar{c}_3 \quad \bar{c}_4}}$$

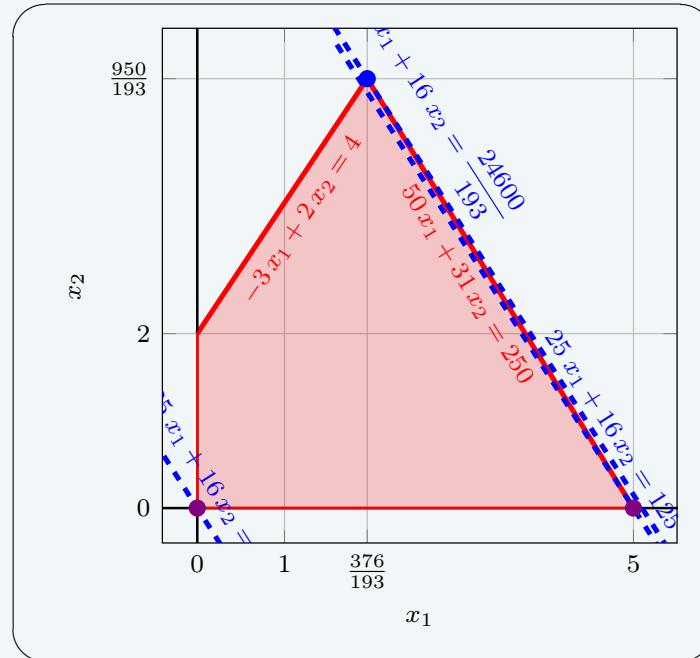
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{376}{193} \\ \frac{950}{193} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = \frac{24600}{193}$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{376}{193}, \quad \tilde{x}_2 = \frac{950}{193}$$

the optimal value $z(LP)$ is $\frac{24600}{193}$.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

11.2 Problems with three variables

Exercise 11.14 — Revised simplex algorithm (*P41*)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 4$ constraints:

$$\begin{array}{llllll}
 \max & 2x_1 & + & x_2 & + & 3x_3 \\
 \text{subject to} & x_1 & + & x_2 & + & x_3 & \leq & 4 \\
 & & & x_2 & & & \leq & 2 \\
 & & & & & x_3 & \leq & 3 \\
 & 3x_1 & & & + & x_3 & \leq & 6 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
 \end{array}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 7$ variables and $m = 4$ constraints:

$$\begin{array}{llllllll}
 \max & 2x_1 & + & x_2 & + & 3x_3 & & \\
 \text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 4 \\
 & & & x_2 & & & + & x_5 & = & 2 \\
 & & & & & x_3 & & + & x_6 & = & 3 \\
 & 3x_1 & & & + & x_3 & & & + & x_7 & = & 6 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7 & \geq & 0
 \end{array}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix} \quad c = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $b \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{4, 5, 6, 7\}$ and $\mathcal{N} = \{1, 2, 3\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Iteration 1**

With $\mathcal{B} = \{4, 5, 6, 7\}$ and $\mathcal{N} = \{1, 2, 3\}$, we have:

$$x_{\mathcal{B}} = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} \quad A_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad c_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \bar{b} = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix}$$

$$x_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad A_{\mathcal{N}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \quad c_{\mathcal{N}} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

Optimality check

We solve $\pi A_{\mathcal{B}} = c'_{\mathcal{B}}$ to compute $\pi = c'_{\mathcal{B}} A_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{rcl} \pi_4 & & = 0 \\ & \pi_5 & = 0 \\ & & \pi_6 & = 0 \\ & & & \pi_7 = 0 \end{array} \right. \quad \pi = (0 \quad 0 \quad 0 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \ 0 \ 0 \ 0) \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (2 \ 1 \ 3) - (0 \ 0 \ 0 \ 0) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix} \\ &= \underbrace{(2 \ 1 \ 3)}_{c_1 \ c_2 \ c_3} - \underbrace{(0 \ 0 \ 0)}_{\pi \mathbf{A}_1 \ \pi \mathbf{A}_2 \ \pi \mathbf{A}_3} = \underbrace{(2 \ 1 \ 3)}_{\bar{c}_1 \ \bar{c}_2 \ \bar{c}_3} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \boldsymbol{\pi} \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_1$, second system of linear equations:

$$\left\{ \begin{array}{rcl} d_4 & = & 1 \\ & d_5 & = 0 \\ & & d_6 = 0 \\ & & & d_7 = 3 \end{array} \right. \quad \mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\left\{ \begin{array}{l} \underbrace{4 - t \cdot 1 \geq 0}_{x_4 \geq 0} \\ \underbrace{6 - t \cdot 3 \geq 0}_{x_7 \geq 0} \end{array} \right. \quad \left\{ \begin{array}{l} t \leq 4 \\ t \leq 2 \end{array} \right. \quad \text{then } t_m = 2$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = 2$. The value of x_7 becomes zero and x_7 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_4 = 4 - 2 \cdot 1 = 2$$

$$x_5 = 2 - 2 \cdot 0 = 2$$

$$x_6 = 3 - 2 \cdot 0 = 3$$

• **Iteration 2**

With $\mathcal{B} = \{1, 4, 5, 6\}$ and $\mathcal{N} = \{2, 3, 7\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 3 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_3 \\ x_7 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{rcl} \pi_1 & + & 3\pi_6 = 2 \\ \pi_1 & & = 0 \\ & \pi_4 & = 0 \\ & \pi_5 & = 0 \end{array} \right. \quad \boldsymbol{\pi} = \begin{pmatrix} 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} 0 & 0 & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix} = 4$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (1 \quad 3 \quad 0) - \begin{pmatrix} 0 & 0 & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \underbrace{(1 \quad 3 \quad 0)}_{c_2 \quad c_3 \quad c_7} - \underbrace{\begin{pmatrix} 0 & \frac{2}{3} & \frac{2}{3} \end{pmatrix}}_{\boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_3 \quad \boldsymbol{\pi} \mathbf{A}_7} = \underbrace{\begin{pmatrix} 1 & \frac{7}{3} & -\frac{2}{3} \end{pmatrix}}_{\bar{c}_2 \quad \bar{c}_3 \quad \bar{c}_7} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \boldsymbol{\pi} \mathbf{A}_2$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

We solve $A_B d = A_2$ to compute $d = A_B^{-1} A_2$, second system of linear equations:

$$\left\{ \begin{array}{rcl} d_1 + d_4 & = & 1 \\ & d_5 & = 1 \\ & d_6 & = 0 \\ 3d_1 & = & 0 \end{array} \right. \quad d = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

We have to find the largest t such that $\bar{b} - t d \geq 0$:

$$\left\{ \begin{array}{l} \underbrace{2 - t \cdot 1 \geq 0}_{x_4 \geq 0} \\ \underbrace{2 - t \cdot 1 \geq 0}_{x_5 \geq 0} \end{array} \right\} \quad \left\{ \begin{array}{l} t \leq 2 \\ t \leq 2 \end{array} \right. \quad \text{then } t_m = 2$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = 2$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{b} - t_m d$, then we have:

$$x_1 = 2 - 2 \cdot 0 = 2$$

$$x_5 = 2 - 2 \cdot 1 = 0$$

$$x_6 = 3 - 2 \cdot 0 = 3$$

• Iteration 3

With $B = \{1, 2, 5, 6\}$ and $N = \{3, 4, 7\}$, we have:

$$x_B = \begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} \quad A_B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{pmatrix} \quad c_B = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \bar{b} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 3 \end{pmatrix}$$

$$x_N = \begin{pmatrix} x_3 \\ x_4 \\ x_7 \end{pmatrix} \quad A_N = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad c_N = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi A_B = c'_B$ to compute $\pi = c'_B A_B^{-1}$, first system of linear equations:

$$\begin{cases} \pi_1 & + & 3\pi_6 & = & 2 \\ \pi_1 & + & \pi_2 & & = & 1 \\ & & \pi_2 & & = & 0 \\ & & & \pi_5 & = & 0 \end{cases} \quad \pi = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \pi \mathbf{b} = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix} = 6$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \pi \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (3 \quad 0 \quad 0) - \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \underbrace{(3 \quad 0 \quad 0)}_{c_3 \quad c_4 \quad c_7} - \underbrace{\begin{pmatrix} \frac{4}{3} & 1 & \frac{1}{3} \end{pmatrix}}_{\pi \mathbf{A}_3 \quad \pi \mathbf{A}_4 \quad \pi \mathbf{A}_7} = \underbrace{\begin{pmatrix} \frac{5}{3} & -1 & -\frac{1}{3} \end{pmatrix}}_{\bar{c}_3 \quad \bar{c}_4 \quad \bar{c}_7} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_3 > \pi \mathbf{A}_3$, x_3 is the pivot column and we have $\mathbf{A}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_3$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_3$, second system of linear equations:

$$\begin{cases} d_1 + d_2 & = & 1 \\ & d_2 + d_5 & = & 0 \\ & & d_6 & = & 1 \\ 3d_1 & = & 1 \end{cases} \quad \mathbf{d} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\left\{ \begin{array}{l} \underbrace{2 - t \cdot \frac{1}{3}}_{x_1 \geq 0} \geq 0 \\ \underbrace{2 - t \cdot \frac{2}{3}}_{x_2 \geq 0} \geq 0 \\ \underbrace{0 - t \cdot \left(-\frac{2}{3}\right)}_{x_5 \geq 0} \geq 0 \\ \underbrace{3 - t \cdot 1}_{x_6 \geq 0} \geq 0 \end{array} \right. \quad \left\{ \begin{array}{l} t \leq 6 \\ t \leq 3 \\ t \geq 0 \\ t \leq 3 \end{array} \right. \quad \text{then } t_m = 3$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_3 = 3$. The value of x_2 becomes zero and x_2 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = 2 - 3 \cdot \frac{1}{3} = 1$$

$$x_5 = 0 - 3 \cdot \left(-\frac{2}{3}\right) = 2$$

$$x_6 = 3 - 3 \cdot 1 = 0$$

• Iteration 4

With $\mathcal{B} = \{1, 3, 5, 6\}$ and $\mathcal{N} = \{2, 4, 7\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \\ x_5 \\ x_6 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \\ x_7 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{rcl} \pi_1 & + & 3\pi_6 = 2 \\ \pi_1 & + & \pi_5 + \pi_6 = 3 \\ & \pi_3 & = 0 \\ & \pi_5 & = 0 \end{array} \right. \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{7}{2} & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix} = 11$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 7 \\ 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}}_{c_2 \quad c_4 \quad c_7} - \underbrace{\begin{pmatrix} 7 & 7 & -\frac{1}{2} \end{pmatrix}}_{\pi \mathbf{A}_2 \quad \pi \mathbf{A}_4 \quad \pi \mathbf{A}_7} = \underbrace{\begin{pmatrix} -\frac{5}{2} & -\frac{7}{2} & \frac{1}{2} \end{pmatrix}}_{\bar{c}_2 \quad \bar{c}_4 \quad \bar{c}_7} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_7 > \pi \mathbf{A}_7$, x_7 is the pivot column and we have $\mathbf{A}_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_7$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_7$, second system of linear equations:

$$\left\{ \begin{array}{rcl} d_1 & + & d_3 & = & 0 \\ & & d_5 & = & 0 \\ & & d_3 & + & d_6 & = & 0 \\ 3d_1 & + & d_3 & = & 1 \end{array} \right. \quad \mathbf{d} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\left\{ \begin{array}{l} \underbrace{1 - t \cdot \frac{1}{2} \geq 0}_{x_1 \geq 0} \\ \underbrace{3 - t \cdot \left(-\frac{1}{2}\right) \geq 0}_{x_3 \geq 0} \\ \underbrace{0 - t \cdot \frac{1}{2} \geq 0}_{x_6 \geq 0} \end{array} \right. \quad \left\{ \begin{array}{l} t \leq 2 \\ t \geq -6 \\ t \leq 0 \end{array} \right. \quad \text{then } t_m = 0$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_7 = 0$. The value of x_6 remains zero and x_6 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = 1 - 0 \cdot \frac{1}{2} = 1$$

$$x_3 = 3 - 0 \cdot \left(-\frac{1}{2}\right) = 3$$

$$x_5 = 2 - 0 \cdot 0 = 2$$

• **Iteration 5**

With $\mathcal{B} = \{1, 3, 5, 7\}$ and $\mathcal{N} = \{2, 4, 6\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \\ x_6 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{rclcl} \pi_1 & & + & 3\pi_7 & = & 2 \\ \pi_1 & & & \pi_5 & + & \pi_7 & = & 3 \\ & \pi_3 & & & & & = & 0 \\ & & & & \pi_7 & = & 0 \end{array} \right. \quad \boldsymbol{\pi} = (2 \quad 0 \quad 1 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (2 \quad 0 \quad 1 \quad 0) \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix} = 11$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (1 \quad 0 \quad 0) - (2 \quad 0 \quad 1 \quad 0) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \underbrace{(1 \quad 0 \quad 0)}_{c_2 \quad c_4 \quad c_6} - \underbrace{(2 \quad 2 \quad 1)}_{\boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_4 \quad \boldsymbol{\pi} \mathbf{A}_6} = \underbrace{(-1 \quad -2 \quad -1)}_{\bar{c}_2 \quad \bar{c}_4 \quad \bar{c}_6} \end{aligned}$$

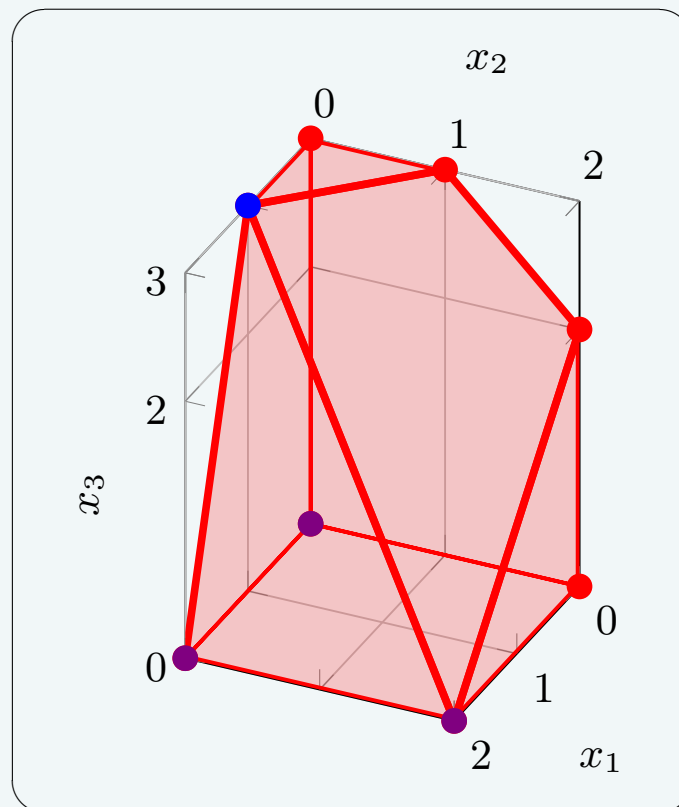
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_3 \\ x_5 \\ x_7 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_N = \begin{pmatrix} x_2 \\ x_4 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = 11$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 3$$

the optimal value $z(LP)$ is 11.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

Exercise 11.15 — Revised simplex algorithm (P42)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 3$ constraints:

$$\begin{array}{llllll} \max & 3x_1 & + & 2x_2 & + & 4x_3 \\ \text{subject to} & x_1 & + & x_2 & + & 2x_3 \leq 4 \\ & 2x_1 & & & + & 3x_3 \leq 5 \\ & 2x_1 & + & x_2 & + & 3x_3 \leq 7 \\ & x_1, & & x_2, & & x_3 \geq 0 \end{array}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 6$ variables and $m = 3$ constraints:

$$\begin{array}{llllllll} \max & 3x_1 & + & 2x_2 & + & 4x_3 & & \\ \text{subject to} & x_1 & + & x_2 & + & 2x_3 & + & x_4 = 4 \\ & 2x_1 & & & + & 3x_3 & + & x_5 = 5 \\ & 2x_1 & + & x_2 & + & 3x_3 & & + x_6 = 7 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 \geq 0 \end{array}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

- **Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- **Iteration 1**

With $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{rcl} \pi_4 & = & 0 \\ \pi_5 & = & 0 \\ \pi_6 & = & 0 \end{array} \right. \quad \boldsymbol{\pi} = (0 \quad 0 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \quad 0 \quad 0) \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (3 \quad 2 \quad 4) - (0 \quad 0 \quad 0) \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\ &= \underbrace{(3 \quad 2 \quad 4)}_{c_1 \quad c_2 \quad c_3} - \underbrace{(0 \quad 0 \quad 0)}_{\boldsymbol{\pi} \mathbf{A}_1 \quad \boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_3} = \underbrace{(3 \quad 2 \quad 4)}_{\bar{c}_1 \quad \bar{c}_2 \quad \bar{c}_3} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \pi$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_4 & = & 1 \\ & d_5 & = & 2 \\ & & d_6 & = & 2 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{4 - t \cdot 1}_{x_4 \geq 0} \geq 0 \\ \underbrace{5 - t \cdot 2}_{x_5 \geq 0} \geq 0 \\ \underbrace{7 - t \cdot 2}_{x_6 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \leq 4 \\ t \leq \frac{5}{2} \\ t \leq \frac{7}{2} \end{cases} \quad \text{then } t_m = \frac{5}{2}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = \frac{5}{2}$. The value of x_5 becomes zero and x_5 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_4 = 4 - \frac{5}{2} \cdot 1 = \frac{3}{2}$$

$$x_6 = 7 - \frac{5}{2} \cdot 2 = 2$$

• Iteration 2

With $\mathcal{B} = \{1, 4, 6\}$ and $\mathcal{N} = \{2, 3, 5\}$, we have:

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_4 \\ x_6 \end{pmatrix} \quad \mathbf{A}_B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_B = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{5}{2} \\ \frac{3}{2} \\ 2 \end{pmatrix}$$

$$\mathbf{x}_N = \begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix} \quad \mathbf{A}_N = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 3 & 0 \end{pmatrix} \quad \mathbf{c}_N = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi \mathbf{A}_B = \mathbf{c}'_B$ to compute $\pi = \mathbf{c}'_B \mathbf{A}_B^{-1}$, first system of linear equations:

$$\begin{cases} \pi_1 + 2\pi_4 + 2\pi_6 = 3 \\ \pi_1 = 0 \\ \pi_6 = 0 \end{cases} \quad \pi = \begin{pmatrix} 0 & \frac{3}{2} & 0 \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \pi \mathbf{b} = \begin{pmatrix} 0 & \frac{3}{2} & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = \frac{15}{2}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \pi \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= \begin{pmatrix} 2 & 4 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \frac{3}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 3 & 0 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 2 & 4 & 0 \end{pmatrix}}_{\substack{c_2 \quad c_3 \quad c_5}} - \underbrace{\begin{pmatrix} 0 & \frac{9}{2} & \frac{3}{2} \end{pmatrix}}_{\substack{\pi \mathbf{A}_2 \quad \pi \mathbf{A}_3 \quad \pi \mathbf{A}_5}} = \underbrace{\begin{pmatrix} 2 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}}_{\substack{\bar{c}_2 \quad \bar{c}_3 \quad \bar{c}_5}} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \pi \mathbf{A}_2$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} d_1 + d_4 = 1 \\ 2d_1 = 0 \\ 2d_1 + d_6 = 1 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{\frac{3}{2} - t \cdot 1}_{x_4 \geq 0} \geq 0 \\ \underbrace{2 - t \cdot 1}_{x_6 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \leq \frac{3}{2} \\ t \leq 2 \end{cases} \quad \text{then } t_m = \frac{3}{2}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = \frac{3}{2}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{b} - t_m \mathbf{d}$, then we have:

$$x_1 = \frac{5}{2} - \frac{3}{2} \cdot 0 = \frac{5}{2}$$

$$x_6 = 2 - \frac{3}{2} \cdot 1 = \frac{1}{2}$$

• **Iteration 3**

With $\mathcal{B} = \{1, 2, 6\}$ and $\mathcal{N} = \{3, 4, 5\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_1 + 2\pi_2 + 2\pi_6 = 3 \\ \pi_1 + \pi_6 = 2 \\ \pi_6 = 0 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} 2 & \frac{1}{2} & 0 \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} 2 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix} = \frac{21}{2}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= \begin{pmatrix} 4 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 2 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 4 & 0 & 0 \end{pmatrix}}_{\substack{c_3 \quad c_4 \quad c_5}} - \underbrace{\begin{pmatrix} \frac{11}{2} & 2 & \frac{1}{2} \end{pmatrix}}_{\substack{\boldsymbol{\pi} \mathbf{A}_3 \quad \boldsymbol{\pi} \mathbf{A}_4 \quad \boldsymbol{\pi} \mathbf{A}_5}} = \underbrace{\begin{pmatrix} -\frac{3}{2} & -2 & -\frac{1}{2} \end{pmatrix}}_{\substack{\bar{c}_3 \quad \bar{c}_4 \quad \bar{c}_5}} \end{aligned}$$

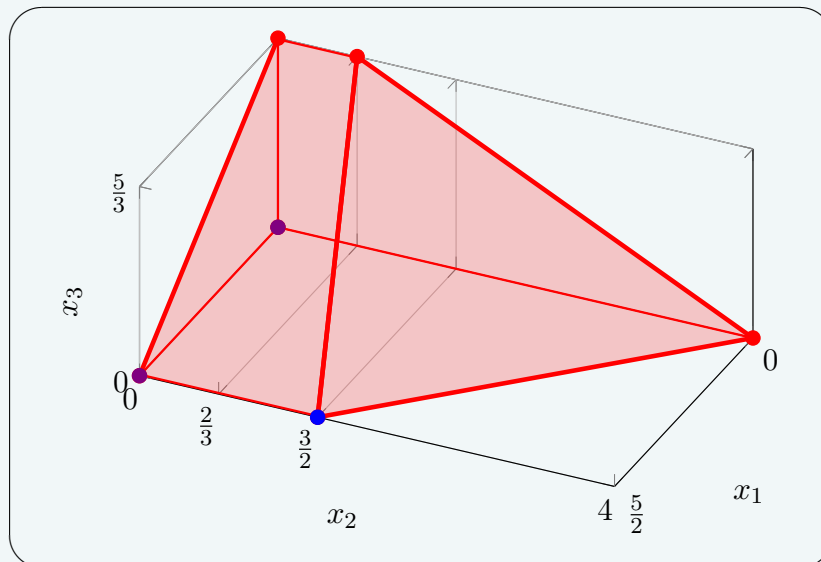
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_N = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = \frac{21}{2}$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = \frac{3}{2}, \quad \tilde{x}_3 = 0$$

the optimal value $z(LP)$ is $\frac{21}{2}$.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

Exercise 11.16 — Revised simplex algorithm (P43)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & x_1 & + & 2x_2 & + & 7x_3 \\ \text{subject to} & x_1 & & & + & 2x_3 \leq 2 \\ & & & x_2 & + & 2x_3 \leq 2 \\ & x_1, & & x_2, & & x_3 \geq 0 \end{array}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 5$ variables and $m = 2$ constraints:

$$\begin{array}{llllllll} \max & x_1 & + & 2x_2 & + & 7x_3 & & \\ \text{subject to} & x_1 & & & + & 2x_3 & + & x_4 = 2 \\ & & & x_2 & + & 2x_3 & & + x_5 = 2 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5 \geq 0 \end{array}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 7 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{4, 5\}$ and $\mathcal{N} =$

$\{1, 2, 3\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

• **Iteration 1**

With $\mathcal{B} = \{4, 5\}$ and $\mathcal{N} = \{1, 2, 3\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_4 & = & 0 \\ & \pi_5 & = & 0 \end{cases} \quad \boldsymbol{\pi} = (0 \ 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \ 0) \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (1 \ 2 \ 7) - (0 \ 0) \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \\ &= \underbrace{(1 \ 2 \ 7)}_{\substack{c_1 \quad c_2 \quad c_3}} - \underbrace{(0 \ 0 \ 0)}_{\substack{\pi \mathbf{A}_1 \quad \pi \mathbf{A}_2 \quad \pi \mathbf{A}_3}} = \underbrace{(1 \ 2 \ 7)}_{\substack{\bar{c}_1 \quad \bar{c}_2 \quad \bar{c}_3}} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > 0$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_4 & = & 1 \\ & d_5 & = & 0 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\left\{ \underbrace{2 - t \cdot 1}_{x_4 \geq 0} \geq 0 \right\} \quad \left\{ t \leq 2 \right\} \quad \text{then} \quad t_m = 2$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = 2$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_5 = 2 - 2 \cdot 0 = 2$$

• **Iteration 2**

With $\mathcal{B} = \{1, 5\}$ and $\mathcal{N} = \{2, 3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 2 \\ 7 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{rcl} \pi_1 & = & 1 \\ & \pi_5 & = 0 \end{array} \right. \quad \boldsymbol{\pi} = (1 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (1 \quad 0) \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (2 \quad 7 \quad 0) - (1 \quad 0) \begin{pmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix} \\ &= \underbrace{(2 \quad 7 \quad 0)}_{\substack{c_2 \quad c_3 \quad c_4}} - \underbrace{(0 \quad 2 \quad 1)}_{\substack{\pi \mathbf{A}_2 \quad \pi \mathbf{A}_3 \quad \pi \mathbf{A}_4}} = \underbrace{(2 \quad 5 \quad -1)}_{\substack{\bar{c}_2 \quad \bar{c}_3 \quad \bar{c}_4}} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \boldsymbol{\pi} \mathbf{A}_2$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} d_1 & = & 0 \\ & d_5 & = & 1 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\underbrace{\begin{cases} 2 - t \cdot 1 \geq 0 \\ x_5 \geq 0 \end{cases}}_{x_5 \geq 0} \quad \begin{cases} t \leq 2 \end{cases} \quad \text{then} \quad t_m = 2$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = 2$. The value of x_5 becomes zero and x_5 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = 2 - 2 \cdot 0 = 2$$

• Iteration 3

With $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4, 5\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_1 & = & 1 \\ & \pi_2 & = & 2 \end{cases} \quad \boldsymbol{\pi} = (1 \quad 2)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (1 \quad 2) \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 6$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (7 \quad 0 \quad 0) - (1 \quad 2) \begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \\ &= \underbrace{(7 \quad 0 \quad 0)}_{\substack{c_3 \quad c_4 \quad c_5}} - \underbrace{(6 \quad 1 \quad 2)}_{\substack{\pi \mathbf{A}_3 \quad \pi \mathbf{A}_4 \quad \pi \mathbf{A}_5}} = \underbrace{(1 \quad -1 \quad -2)}_{\substack{\bar{c}_3 \quad \bar{c}_4 \quad \bar{c}_5}} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_3 > \pi A_3$, x_3 is the pivot column and we have $A_3 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

We solve $A_B d = A_3$ to compute $d = A_B^{-1} A_3$, second system of linear equations:

$$\begin{cases} d_1 & = & 2 \\ & d_2 & = & 2 \end{cases} \quad d = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

We have to find the largest t such that $\bar{b} - t d \geq 0$:

$$\begin{cases} \underbrace{2 - t \cdot 2 \geq 0}_{x_1 \geq 0} \\ \underbrace{2 - t \cdot 2 \geq 0}_{x_2 \geq 0} \end{cases} \quad \begin{cases} t \leq 1 \\ t \leq 1 \end{cases} \quad \text{then } t_m = 1$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_3 = 1$. The value of x_1 becomes zero and x_1 is the pivot row. The other basic variables take values $\bar{b} - t_m d$, then we have:

$$x_2 = 2 - 1 \cdot 2 = 0$$

• Iteration 4

With $\mathcal{B} = \{2, 3\}$ and $\mathcal{N} = \{1, 4, 5\}$, we have:

$$x_B = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad A_B = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \quad c_B = \begin{pmatrix} 2 \\ 7 \end{pmatrix} \quad \bar{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_N = \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} \quad A_N = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad c_N = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi A_B = c'_B$ to compute $\pi = c'_B A_B^{-1}$, first system of linear equations:

$$\begin{cases} \pi_3 & = & 2 \\ 2\pi_2 + 2\pi_3 & = & 7 \end{cases} \quad \pi = \begin{pmatrix} \frac{3}{2} & 2 \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \pi b = \begin{pmatrix} \frac{3}{2} & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 7$

We compute the reduced costs $\bar{c}' = c'_N - \pi A_N$:

$$\begin{aligned}\bar{c}' &= (1 \ 0 \ 0) - \begin{pmatrix} \frac{3}{2} & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \underbrace{(1 \ 0 \ 0)}_{c_1 \ c_4 \ c_5} - \underbrace{\begin{pmatrix} \frac{3}{2} & \frac{3}{2} & 2 \end{pmatrix}}_{\pi A_1 \ \pi A_4 \ \pi A_5} = \underbrace{\begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & -2 \end{pmatrix}}_{\bar{c}_1 \ \bar{c}_4 \ \bar{c}_5}\end{aligned}$$

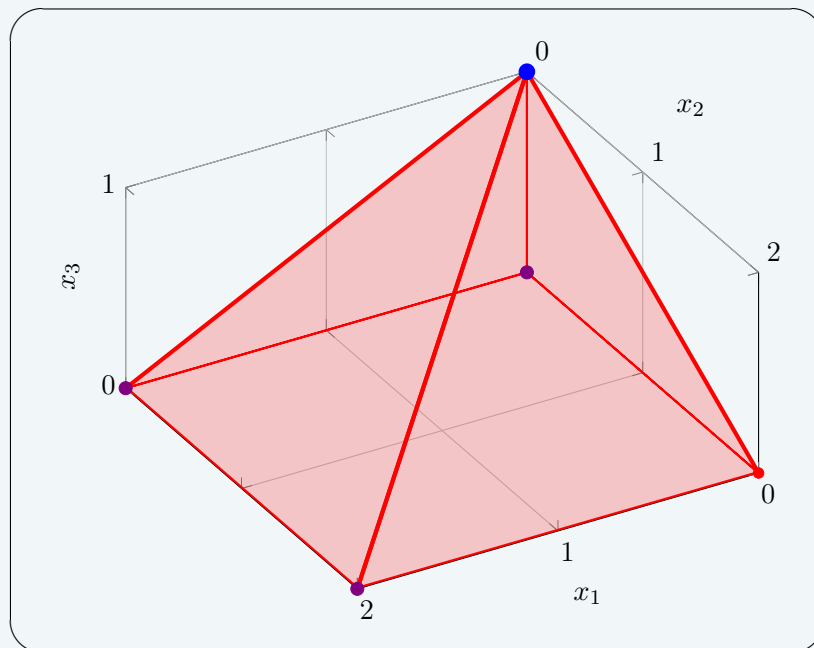
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_B = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_N = \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = 7$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 1$$

the optimal value $z(LP)$ is 7.

In violet the first, the second and the third vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

Exercise 11.17 — Revised simplex algorithm (P44)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 3$ constraints:

$$\begin{array}{llllll}
 \max & x_1 & + & 2x_2 & + & 3x_3 \\
 \text{subject to} & x_1 & + & 2x_2 & - & 2x_3 & \leq & 1 \\
 & 2x_1 & - & x_2 & - & 3x_3 & \leq & 4 \\
 & x_1 & + & 2x_2 & + & 5x_3 & \leq & 2 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
 \end{array}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 6$ variables and $m = 3$ constraints:

$$\begin{array}{llllllll}
 \max & x_1 & + & 2x_2 & + & 3x_3 & & \\
 \text{subject to} & x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & = & 1 \\
 & 2x_1 & - & x_2 & - & 3x_3 & & + & x_5 & = & 4 \\
 & x_1 & + & 2x_2 & + & 5x_3 & & & + & x_6 & = & 2 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
 \end{array}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -2 & 1 & 0 & 0 \\ 2 & -1 & -3 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

• **Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

• **Iteration 1**

With $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -1 & -3 \\ 1 & 2 & 5 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{rcl} \pi_4 & = & 0 \\ \pi_5 & = & 0 \\ \pi_6 & = & 0 \end{array} \right. \quad \boldsymbol{\pi} = (0 \quad 0 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \quad 0 \quad 0) \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (1 \quad 2 \quad 3) - (0 \quad 0 \quad 0) \begin{pmatrix} 1 & 2 & -2 \\ 2 & -1 & -3 \\ 1 & 2 & 5 \end{pmatrix} \\ &= \underbrace{(1 \quad 2 \quad 3)}_{c_1 \quad c_2 \quad c_3} - \underbrace{(0 \quad 0 \quad 0)}_{\boldsymbol{\pi} \mathbf{A}_1 \quad \boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_3} = \underbrace{(1 \quad 2 \quad 3)}_{\bar{c}_1 \quad \bar{c}_2 \quad \bar{c}_3} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \pi \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_4 & = & 1 \\ & d_5 & = & 2 \\ & & d_6 & = & 1 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{1 - t \cdot 1}_{x_4 \geq 0} \geq 0 \\ \underbrace{4 - t \cdot 2}_{x_5 \geq 0} \geq 0 \\ \underbrace{2 - t \cdot 1}_{x_6 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \leq 1 \\ t \leq 2 \\ t \leq 2 \end{cases} \quad \text{then } t_m = 1$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = 1$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_5 = 4 - 1 \cdot 2 = 2$$

$$x_6 = 2 - 1 \cdot 1 = 1$$

- **Iteration 2**

With $\mathcal{B} = \{1, 5, 6\}$ and $\mathcal{N} = \{2, 3, 4\}$, we have:

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_5 \\ x_6 \end{pmatrix} \quad \mathbf{A}_B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\mathbf{x}_N = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_N = \begin{pmatrix} 2 & -2 & 1 \\ -1 & -3 & 0 \\ 2 & 5 & 0 \end{pmatrix} \quad \mathbf{c}_N = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi \mathbf{A}_B = \mathbf{c}'_B$ to compute $\pi = \mathbf{c}'_B \mathbf{A}_B^{-1}$, first system of linear equations:

$$\begin{cases} \pi_1 + 2\pi_5 + \pi_6 = 1 \\ \pi_5 = 0 \\ \pi_6 = 0 \end{cases} \quad \boldsymbol{\pi} = (1 \ 0 \ 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (1 \ 0 \ 0) \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 1$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (2 \ 3 \ 0) - (1 \ 0 \ 0) \begin{pmatrix} 2 & -2 & 1 \\ -1 & -3 & 0 \\ 2 & 5 & 0 \end{pmatrix} \\ &= \underbrace{(2 \ 3 \ 0)}_{c_2 \ c_3 \ c_4} - \underbrace{(2 \ -2 \ 1)}_{\pi \mathbf{A}_2 \ \pi \mathbf{A}_3 \ \pi \mathbf{A}_4} = \underbrace{(0 \ 5 \ -1)}_{\bar{c}_2 \ \bar{c}_3 \ \bar{c}_4} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_3 > \pi \mathbf{A}_3$, x_3 is the pivot column and we have $\mathbf{A}_3 = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_3$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_3$, second system of linear equations:

$$\begin{cases} d_1 = -2 \\ 2d_1 + d_5 = -3 \\ d_1 + d_6 = 5 \end{cases} \quad \mathbf{d} = \begin{pmatrix} -2 \\ 1 \\ 7 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{1 - t \cdot (-2)}_{x_1 \geq 0} \geq 0 \\ \underbrace{2 - t \cdot 1}_{x_5 \geq 0} \geq 0 \\ \underbrace{1 - t \cdot 7}_{x_6 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \geq -\frac{1}{2} \\ t \leq 2 \\ t \leq \frac{1}{7} \end{cases} \quad \text{then } t_m = \frac{1}{7}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_3 = \frac{1}{7}$. The value of x_6 becomes zero and x_6 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$,

then we have:

$$x_1 = 1 - \frac{1}{7} \cdot (-2) = \frac{9}{7}$$

$$x_5 = 2 - \frac{1}{7} \cdot 1 = \frac{13}{7}$$

• **Iteration 3**

With $\mathcal{B} = \{1, 3, 5\}$ and $\mathcal{N} = \{2, 4, 6\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -3 & 1 \\ 1 & 5 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{9}{7} \\ \frac{1}{7} \\ \frac{13}{7} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \\ x_6 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_1 + 2\pi_3 + \pi_5 = 1 \\ -2\pi_1 - 3\pi_3 + 5\pi_5 = 3 \\ \pi_3 = 0 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{2}{7} & 0 & \frac{5}{7} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} \frac{2}{7} & 0 & \frac{5}{7} \end{pmatrix} \begin{pmatrix} \frac{9}{7} \\ \frac{1}{7} \\ \frac{13}{7} \end{pmatrix} = \frac{12}{7}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= \begin{pmatrix} 2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \frac{2}{7} & 0 & \frac{5}{7} \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 2 & 0 & 0 \end{pmatrix}}_{\substack{c_2 \quad c_4 \quad c_6}} - \underbrace{\begin{pmatrix} 2 & \frac{2}{7} & \frac{5}{7} \end{pmatrix}}_{\substack{\pi \mathbf{A}_2 \quad \pi \mathbf{A}_4 \quad \pi \mathbf{A}_6}} = \underbrace{\begin{pmatrix} 0 & -\frac{2}{7} & -\frac{5}{7} \end{pmatrix}}_{\substack{\bar{c}_2 \quad \bar{c}_4 \quad \bar{c}_6}} \end{aligned}$$

All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{9}{7} \\ \frac{1}{7} \\ \frac{13}{7} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

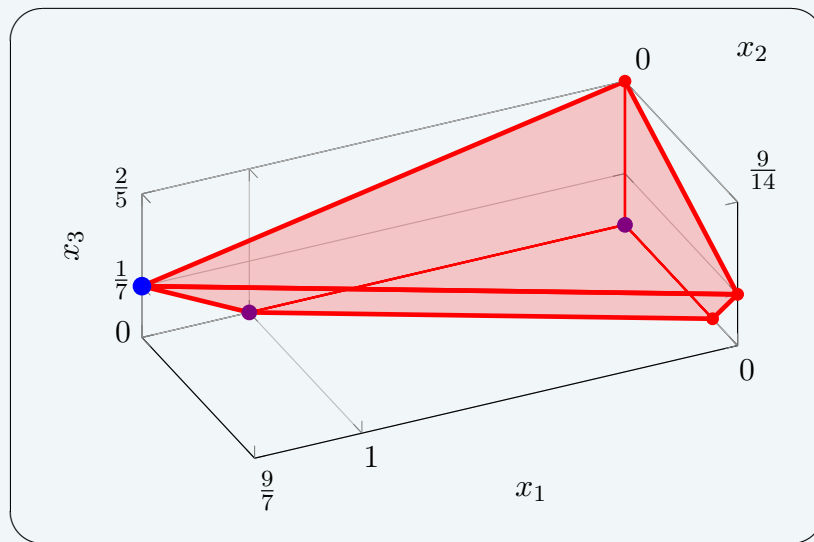
is an optimal solution of the LP problem and $\bar{z} = \frac{12}{7}$ is the optimal solution value.

- **Multiple optimal solutions**

The optimal solution is not unique: in the final basis the non-basic variable x_2 has reduced cost $\bar{c}_2 = 0$, and every point of the edge joining $\left(\frac{9}{7}, 0, \frac{1}{7}\right)$ and $\left(0, \frac{9}{14}, \frac{1}{7}\right)$ is optimal, with value $\frac{12}{7}$.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{x}(LP)$ is:

$$\tilde{x}_1 = \frac{9}{7}, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = \frac{1}{7}$$

the optimal value $z(LP)$ is $\frac{12}{7}$.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

Exercise 11.18 — Revised simplex algorithm (P45)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 2$ constraints:

$$\begin{array}{ll}
 \max & 4x_1 - 13x_2 + 7x_3 \\
 \text{subject to} & 3x_1 + 2x_2 + 5x_3 \leq 5 \\
 & x_1 - 3x_2 + 5x_3 \leq 3 \\
 & x_1, \quad x_2, \quad x_3 \geq 0
 \end{array}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 5$ variables and $m = 2$ constraints:

$$\begin{array}{ll}
 \max & 4x_1 - 13x_2 + 7x_3 \\
 \text{subject to} & 3x_1 + 2x_2 + 5x_3 + x_4 = 5 \\
 & x_1 - 3x_2 + 5x_3 + x_5 = 3 \\
 & x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5 \geq 0
 \end{array}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 5 & 1 & 0 \\ 1 & -3 & 5 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 4 \\ -13 \\ 7 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

- **Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{4, 5\}$ and $\mathcal{N} = \{1, 2, 3\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

• **Iteration 1**

With $\mathcal{B} = \{4, 5\}$ and $\mathcal{N} = \{1, 2, 3\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 3 & 2 & 5 \\ 1 & -3 & 5 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 4 \\ -13 \\ 7 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} \pi_4 & = & 0 \\ \pi_5 & = & 0 \end{cases} \quad \boldsymbol{\pi} = (0 \ 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \ 0) \begin{pmatrix} 5 \\ 3 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (4 \ -13 \ 7) - (0 \ 0) \begin{pmatrix} 3 & 2 & 5 \\ 1 & -3 & 5 \end{pmatrix} \\ &= \underbrace{(4 \ -13 \ 7)}_{c_1 \ c_2 \ c_3} - \underbrace{(0 \ 0 \ 0)}_{\pi \mathbf{A}_1 \ \pi \mathbf{A}_2 \ \pi \mathbf{A}_3} = \underbrace{(4 \ -13 \ 7)}_{\bar{c}_1 \ \bar{c}_2 \ \bar{c}_3} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > 0$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_4 & = & 3 \\ d_5 & = & 1 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{5 - t \cdot 3}_{x_4 \geq 0} \geq 0 \\ \underbrace{3 - t \cdot 1}_{x_5 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \leq \frac{5}{3} \\ t \leq 3 \end{cases} \quad \text{then } t_m = \frac{5}{3}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = \frac{5}{3}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_5 = 3 - \frac{5}{3} \cdot 1 = \frac{4}{3}$$

• **Iteration 2**

With $\mathcal{B} = \{1, 5\}$ and $\mathcal{N} = \{2, 3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{5}{3} \\ 4 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 2 & 5 & 1 \\ -3 & 5 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} -13 \\ 7 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} 3\pi_1 + \pi_5 = 4 \\ \pi_5 = 0 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{4}{3} & 0 \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} \frac{4}{3} & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{3} \\ 4 \end{pmatrix} = \frac{20}{3}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= \begin{pmatrix} -13 & 7 & 0 \end{pmatrix} - \begin{pmatrix} \frac{4}{3} & 0 \end{pmatrix} \begin{pmatrix} 2 & 5 & 1 \\ -3 & 5 & 0 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} -13 & 7 & 0 \end{pmatrix}}_{\substack{c_2 \quad c_3 \quad c_4}} - \underbrace{\begin{pmatrix} \frac{8}{3} & \frac{20}{3} & \frac{4}{3} \end{pmatrix}}_{\substack{\boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_3 \quad \boldsymbol{\pi} \mathbf{A}_4}} = \underbrace{\begin{pmatrix} -\frac{47}{3} & \frac{1}{3} & -\frac{4}{3} \end{pmatrix}}_{\substack{\bar{c}_2 \quad \bar{c}_3 \quad \bar{c}_4}} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_3 > \boldsymbol{\pi} \mathbf{A}_3$, x_3 is the pivot column and we have $\mathbf{A}_3 = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

We solve $\mathbf{A}_{\mathcal{B}} \mathbf{d} = \mathbf{A}_3$ to compute $\mathbf{d} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_3$, second system of linear equations:

$$\begin{cases} 3d_1 & = & 5 \\ d_1 + d_5 & = & 5 \end{cases} \quad \mathbf{d} = \begin{pmatrix} \frac{5}{3} \\ \frac{10}{3} \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{\frac{5}{3} - t \cdot \frac{5}{3} \geq 0}_{x_1 \geq 0} \\ \underbrace{\frac{4}{3} - t \cdot \frac{10}{3} \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} t \leq 1 \\ t \leq \frac{2}{5} \end{cases} \quad \text{then } t_m = \frac{2}{5}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_3 = \frac{2}{5}$. The value of x_5 becomes zero and x_5 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_1 = \frac{5}{3} - \frac{2}{5} \cdot \frac{5}{3} = 1$$

• Iteration 3

With $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4, 5\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 3 & 5 \\ 1 & 5 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 1 \\ \frac{2}{5} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} -13 \\ 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} 3\pi_1 + \pi_3 = 4 \\ 5\pi_1 + 5\pi_3 = 7 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{13}{10} & \frac{1}{10} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} \frac{13}{10} & \frac{1}{10} \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \frac{34}{5}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned}\bar{c}' &= (-13 \quad 0 \quad 0) - \left(\frac{13}{10} \quad \frac{1}{10} \right) \begin{pmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} -13 & 0 & 0 \end{pmatrix}}_{\substack{c_2 \quad c_4 \quad c_5}} - \underbrace{\begin{pmatrix} \frac{23}{10} & \frac{13}{10} & \frac{1}{10} \end{pmatrix}}_{\substack{\pi A_2 \quad \pi A_4 \quad \pi A_5}} = \underbrace{\begin{pmatrix} -\frac{153}{10} & -\frac{13}{10} & -\frac{1}{10} \end{pmatrix}}_{\substack{\bar{c}_2 \quad \bar{c}_4 \quad \bar{c}_5}}\end{aligned}$$

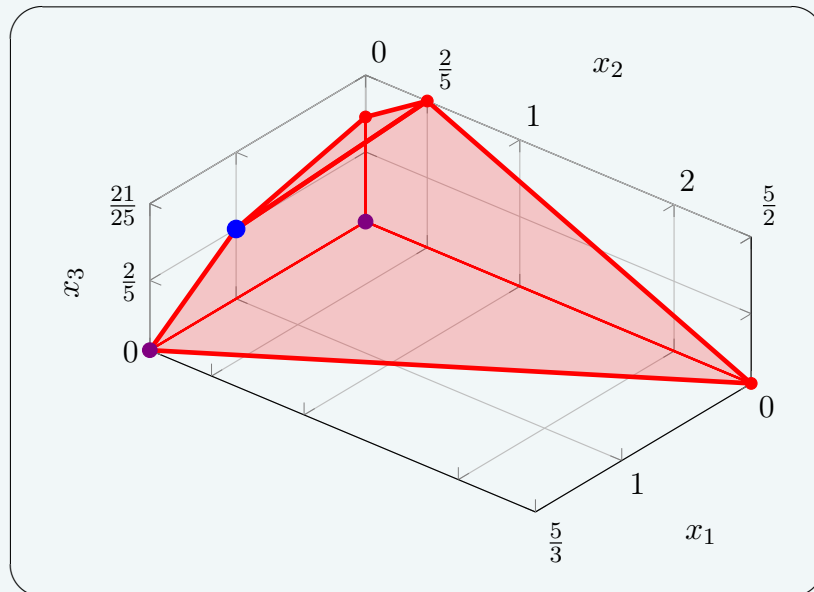
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2}{5} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_N = \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = \frac{34}{5}$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = \frac{2}{5}$$

the optimal value $z(LP)$ is $\frac{34}{5}$.

In violet the first and the second vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

Exercise 11.19 — Revised simplex algorithm (P46)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 3$ constraints:

$$\begin{aligned} \max \quad & 5x_1 + 6x_2 + 10x_3 \\ \text{subject to} \quad & 9x_1 + 2x_2 + 2x_3 \leq 11 \\ & 2x_1 + 6x_2 + 6x_3 \leq 16 \\ & 4x_1 + 2x_2 + 7x_3 \leq 10 \\ & x_1, \quad x_2, \quad x_3 \geq 0 \end{aligned}$$

By introducing the slack variables, we get the following LP problem in standard maximization form with $n + m = 6$ variables and $m = 3$ constraints:

$$\begin{aligned} \max \quad & 5x_1 + 6x_2 + 10x_3 \\ \text{subject to} \quad & 9x_1 + 2x_2 + 2x_3 + x_4 = 11 \\ & 2x_1 + 6x_2 + 6x_3 + x_5 = 16 \\ & 4x_1 + 2x_2 + 7x_3 + x_6 = 10 \\ & x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_6 \geq 0 \end{aligned}$$

Questions

Execute the revised simplex algorithm to find an optimal solution of this LP problem using Bland's rule to select the pivot column and the pivot row.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 9 & 2 & 2 & 1 & 0 & 0 \\ 2 & 6 & 6 & 0 & 1 & 0 \\ 4 & 2 & 7 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 11 \\ 16 \\ 10 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 5 \\ 6 \\ 10 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

- **Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{b} \geq 0$ by adding the slack variables, the partition $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$ is associated with the following basic feasible solution:

$$\begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 11 \\ 16 \\ 10 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- **Iteration 1**

With $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} 11 \\ 16 \\ 10 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 9 & 2 & 2 \\ 2 & 6 & 6 \\ 4 & 2 & 7 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 5 \\ 6 \\ 10 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\left\{ \begin{array}{rcl} \pi_4 & & = 0 \\ & \pi_5 & = 0 \\ & & \pi_6 = 0 \end{array} \right. \quad \boldsymbol{\pi} = (0 \quad 0 \quad 0)$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = (0 \quad 0 \quad 0) \begin{pmatrix} 11 \\ 16 \\ 10 \end{pmatrix} = 0$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (5 \quad 6 \quad 10) - (0 \quad 0 \quad 0) \begin{pmatrix} 9 & 2 & 2 \\ 2 & 6 & 6 \\ 4 & 2 & 7 \end{pmatrix} \\ &= \underbrace{(5 \quad 6 \quad 10)}_{c_1 \quad c_2 \quad c_3} - \underbrace{(0 \quad 0 \quad 0)}_{\boldsymbol{\pi} \mathbf{A}_1 \quad \boldsymbol{\pi} \mathbf{A}_2 \quad \boldsymbol{\pi} \mathbf{A}_3} = \underbrace{(5 \quad 6 \quad 10)}_{\bar{c}_1 \quad \bar{c}_2 \quad \bar{c}_3} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_1 > \pi \mathbf{A}_1$, x_1 is the pivot column and we have $\mathbf{A}_1 = \begin{pmatrix} 9 \\ 2 \\ 4 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_1$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_1$, second system of linear equations:

$$\begin{cases} d_4 & = & 9 \\ & d_5 & = & 2 \\ & & d_6 & = & 4 \end{cases} \quad \mathbf{d} = \begin{pmatrix} 9 \\ 2 \\ 4 \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{11 - t \cdot 9 \geq 0}_{x_4 \geq 0} \\ \underbrace{16 - t \cdot 2 \geq 0}_{x_5 \geq 0} \\ \underbrace{10 - t \cdot 4 \geq 0}_{x_6 \geq 0} \end{cases} \quad \begin{cases} t \leq \frac{11}{9} \\ t \leq 8 \\ t \leq \frac{5}{2} \end{cases} \quad \text{then } t_m = \frac{11}{9}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_1 = \frac{11}{9}$. The value of x_4 becomes zero and x_4 is the pivot row. The other basic variables take values $\bar{\mathbf{b}} - t_m \mathbf{d}$, then we have:

$$x_5 = 16 - \frac{11}{9} \cdot 2 = \frac{122}{9}$$

$$x_6 = 10 - \frac{11}{9} \cdot 4 = \frac{46}{9}$$

• Iteration 2

With $\mathcal{B} = \{1, 5, 6\}$ and $\mathcal{N} = \{2, 3, 4\}$, we have:

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_5 \\ x_6 \end{pmatrix} \quad \mathbf{A}_B = \begin{pmatrix} 9 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_B = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{11}{9} \\ \frac{122}{9} \\ \frac{46}{9} \end{pmatrix}$$

$$\mathbf{x}_N = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_N = \begin{pmatrix} 2 & 2 & 1 \\ 6 & 6 & 0 \\ 2 & 7 & 0 \end{pmatrix} \quad \mathbf{c}_N = \begin{pmatrix} 6 \\ 10 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi \mathbf{A}_B = \mathbf{c}'_B$ to compute $\pi = \mathbf{c}'_B \mathbf{A}_B^{-1}$, first system of linear equations:

$$\begin{cases} 9\pi_1 + 2\pi_5 + 4\pi_6 = 5 \\ \pi_5 = 0 \\ \pi_6 = 0 \end{cases} \quad \pi = \begin{pmatrix} \frac{5}{9} & 0 & 0 \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \pi \mathbf{b} = \begin{pmatrix} \frac{5}{9} & 0 & 0 \end{pmatrix} \begin{pmatrix} 11 \\ 16 \\ 10 \end{pmatrix} = \frac{55}{9}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_N - \pi \mathbf{A}_N$:

$$\begin{aligned} \bar{\mathbf{c}}' &= \begin{pmatrix} 6 & 10 & 0 \end{pmatrix} - \begin{pmatrix} \frac{5}{9} & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ 6 & 6 & 0 \\ 2 & 7 & 0 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 6 & 10 & 0 \end{pmatrix}}_{\substack{c_2 \quad c_3 \quad c_4}} - \underbrace{\begin{pmatrix} \frac{10}{9} & \frac{10}{9} & \frac{5}{9} \end{pmatrix}}_{\substack{\pi \mathbf{A}_2 \quad \pi \mathbf{A}_3 \quad \pi \mathbf{A}_4}} = \underbrace{\begin{pmatrix} \frac{44}{9} & \frac{80}{9} & -\frac{5}{9} \end{pmatrix}}_{\substack{\bar{c}_2 \quad \bar{c}_3 \quad \bar{c}_4}} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_2 > \pi \mathbf{A}_2$, x_2 is the pivot column and we have $\mathbf{A}_2 = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}$

We solve $\mathbf{A}_B \mathbf{d} = \mathbf{A}_2$ to compute $\mathbf{d} = \mathbf{A}_B^{-1} \mathbf{A}_2$, second system of linear equations:

$$\begin{cases} 9d_1 = 2 \\ 2d_1 + d_5 = 6 \\ 4d_1 + d_6 = 2 \end{cases} \quad \mathbf{d} = \begin{pmatrix} \frac{2}{9} \\ \frac{50}{9} \\ \frac{10}{9} \end{pmatrix}$$

We have to find the largest t such that $\bar{\mathbf{b}} - t \mathbf{d} \geq 0$:

$$\begin{cases} \underbrace{\frac{11}{9} - t \cdot \frac{2}{9}}_{x_1 \geq 0} \geq 0 \\ \underbrace{\frac{122}{9} - t \cdot \frac{50}{9}}_{x_5 \geq 0} \geq 0 \\ \underbrace{\frac{46}{9} - t \cdot \frac{10}{9}}_{x_6 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \leq \frac{11}{2} \\ t \leq \frac{61}{25} \\ t \leq \frac{23}{5} \end{cases} \quad \text{then } t_m = \frac{61}{25}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_2 = \frac{61}{25}$. The value of x_5 becomes zero and x_5 is the pivot row. The other basic variables take values $\bar{b} - t_m \mathbf{d}$, then we have:

$$x_1 = \frac{11}{9} - \frac{61}{25} \cdot \frac{2}{9} = \frac{17}{25}$$

$$x_6 = \frac{46}{9} - \frac{61}{25} \cdot \frac{10}{9} = \frac{12}{5}$$

• **Iteration 3**

With $\mathcal{B} = \{1, 2, 6\}$ and $\mathcal{N} = \{3, 4, 5\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 9 & 2 & 0 \\ 2 & 6 & 0 \\ 4 & 2 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{17}{25} \\ \frac{61}{25} \\ \frac{12}{5} \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 2 & 1 & 0 \\ 6 & 0 & 1 \\ 7 & 0 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\boldsymbol{\pi} \mathbf{A}_{\mathcal{B}} = \mathbf{c}'_{\mathcal{B}}$ to compute $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1}$, first system of linear equations:

$$\begin{cases} 9\pi_1 + 2\pi_2 + 4\pi_6 = 5 \\ 2\pi_1 + 6\pi_2 + 2\pi_6 = 6 \\ \pi_6 = 0 \end{cases} \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{9}{25} & \frac{22}{25} & 0 \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \boldsymbol{\pi} \mathbf{b} = \begin{pmatrix} \frac{9}{25} & \frac{22}{25} & 0 \end{pmatrix} \begin{pmatrix} \frac{11}{25} \\ \frac{16}{25} \\ 10 \end{pmatrix} = \frac{451}{25}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \boldsymbol{\pi} \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (10 \quad 0 \quad 0) - \begin{pmatrix} \frac{9}{25} & \frac{22}{25} & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 6 & 0 & 1 \\ 7 & 0 & 0 \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 10 & 0 & 0 \end{pmatrix}}_{\substack{c_3 \quad c_4 \quad c_5}} - \underbrace{\begin{pmatrix} 6 & \frac{9}{25} & \frac{22}{25} \end{pmatrix}}_{\substack{\boldsymbol{\pi} \mathbf{A}_3 \quad \boldsymbol{\pi} \mathbf{A}_4 \quad \boldsymbol{\pi} \mathbf{A}_5}} = \underbrace{\begin{pmatrix} 4 & -\frac{9}{25} & -\frac{22}{25} \end{pmatrix}}_{\substack{\bar{c}_3 \quad \bar{c}_4 \quad \bar{c}_5}} \end{aligned}$$

Not all reduced costs are non-positive then we perform a pivoting operation.

Pivoting operation

Since $c_3 > \pi$, x_3 is the pivot column and we have $A_3 = \begin{pmatrix} 2 \\ 6 \\ 7 \end{pmatrix}$

We solve $A_B d = A_3$ to compute $d = A_B^{-1} A_3$, second system of linear equations:

$$\begin{cases} 9d_1 + 2d_2 = 2 \\ 2d_1 + 6d_2 = 6 \\ 4d_1 + 2d_2 + d_6 = 7 \end{cases} \quad d = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$$

We have to find the largest t such that $\bar{b} - t d \geq 0$:

$$\begin{cases} \underbrace{\frac{61}{25} - t \cdot 1}_{x_2 \geq 0} \geq 0 \\ \underbrace{\frac{12}{5} - t \cdot 5}_{x_6 \geq 0} \geq 0 \end{cases} \quad \begin{cases} t \leq \frac{61}{25} \\ t \leq \frac{12}{25} \end{cases} \quad \text{then } t_m = \frac{12}{25}$$

Accordingly the value of the pivot column becomes t_m , i.e., $x_3 = \frac{12}{25}$. The value of x_6 becomes zero and x_6 is the pivot row. The other basic variables take values $\bar{b} - t_m d$, then we have:

$$x_1 = \frac{17}{25} - \frac{12}{25} \cdot 0 = \frac{17}{25}$$

$$x_2 = \frac{61}{25} - \frac{12}{25} \cdot 1 = \frac{49}{25}$$

- Iteration 4**

With $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5, 6\}$, we have:

$$x_B = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad A_B = \begin{pmatrix} 9 & 2 & 2 \\ 2 & 6 & 6 \\ 4 & 2 & 7 \end{pmatrix} \quad c_B = \begin{pmatrix} 5 \\ 6 \\ 10 \end{pmatrix} \quad \bar{b} = \begin{pmatrix} \frac{17}{25} \\ \frac{49}{25} \\ \frac{12}{25} \end{pmatrix}$$

$$x_N = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \quad A_N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad c_N = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Optimality check

We solve $\pi \mathbf{A}_B = \mathbf{c}'_B$ to compute $\pi = \mathbf{c}'_B \mathbf{A}_B^{-1}$, first system of linear equations:

$$\begin{cases} 9\pi_1 + 2\pi_2 + 4\pi_3 = 5 \\ 2\pi_1 + 6\pi_2 + 2\pi_3 = 6 \\ 2\pi_1 + 6\pi_2 + 7\pi_3 = 10 \end{cases} \quad \pi = \begin{pmatrix} \frac{1}{25} & \frac{18}{25} & \frac{4}{5} \end{pmatrix}$$

We compute the basic feasible solution value $\bar{z} = \pi \mathbf{b} = \begin{pmatrix} \frac{1}{25} & \frac{18}{25} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 11 \\ 16 \\ 10 \end{pmatrix} = \frac{499}{25}$

We compute the reduced costs $\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \pi \mathbf{A}_{\mathcal{N}}$:

$$\begin{aligned} \bar{\mathbf{c}}' &= (0 \quad 0 \quad 0) - \begin{pmatrix} \frac{1}{25} & \frac{18}{25} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \underbrace{(0 \quad 0 \quad 0)}_{\substack{c_4 \quad c_5 \quad c_6}} - \underbrace{\begin{pmatrix} \frac{1}{25} & \frac{18}{25} & \frac{4}{5} \end{pmatrix}}_{\substack{\pi \mathbf{A}_4 \quad \pi \mathbf{A}_5 \quad \pi \mathbf{A}_6}} = \underbrace{\begin{pmatrix} -\frac{1}{25} & -\frac{18}{25} & -\frac{4}{5} \end{pmatrix}}_{\substack{\bar{c}_4 \quad \bar{c}_5 \quad \bar{c}_6}} \end{aligned}$$

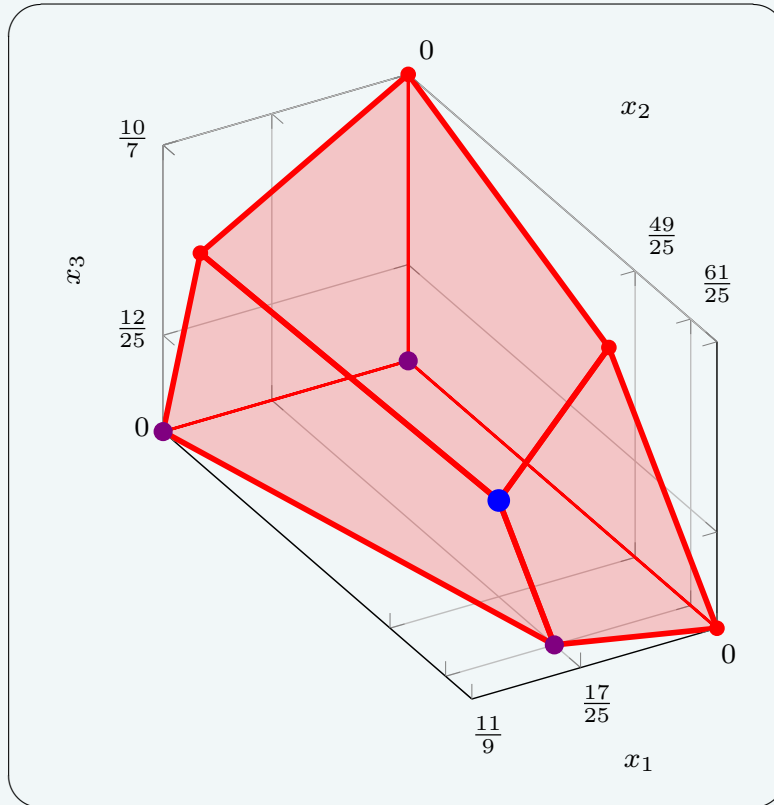
All reduced costs are non-positive, then the basic feasible solution

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{17}{25} \\ \frac{49}{25} \\ \frac{12}{25} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is an optimal solution of the LP problem and $\bar{z} = \frac{499}{25}$ is the optimal solution value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{17}{25}, \quad \tilde{x}_2 = \frac{49}{25}, \quad \tilde{x}_3 = \frac{12}{25}$$

the optimal value $z(LP)$ is $\frac{499}{25}$.

In violet the first, the second and the third vertex explored by the previous execution of the revised simplex algorithm (before reaching the optimal one depicted in blue).

Revised dual simplex algorithm

Exercise 12.1 — Revised dual simplex algorithm (P11)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & -2 & x_1 & - & 4 & x_2 \\ \text{subject to} & - & x_1 & - & & x_2 \leq -2 \\ & & x_1 & - & & x_2 \leq 0 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Execute the revised dual simplex algorithm to find an optimal solution of this LP problem, solving at each iteration the two systems of linear equations. As pivot row choose the basic variable with the smallest index among those with $\bar{b}_r < 0$, and break ties in the ratio test for the pivot column by choosing the smallest index.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} -1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -2 \\ -4 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- **Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{c} \leq \mathbf{0}$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is **dual feasible**, since $\mathbf{A}_{\mathcal{B}} = \mathbf{I}$ and $\mathbf{c}_{\mathcal{B}} = \mathbf{0}$ give

$$\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} = (-2 \quad -4) \leq \mathbf{0}.$$

The associated basic solution is not feasible, and then the dual simplex algorithm is used.

- Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-2 \quad -4)$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} \bar{b}_3 & = & -2 \\ \bar{b}_4 & = & 0 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_3 = -2 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_3 < 0$, x_3 is the pivot row and we have $\mathbf{e}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_{\mathcal{B}} = \mathbf{e}_3'$ to compute $\mathbf{u} = \mathbf{e}_3' \mathbf{A}_{\mathcal{B}}^{-1}$, second system of linear equations:

$$\begin{cases} u_3 & = & 1 \\ u_4 & = & 0 \end{cases} \quad \mathbf{u} = (1 \quad 0)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_3 , $\bar{a}_{3j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-1}_{\bar{a}_{31}} \quad \underbrace{-1}_{\bar{a}_{32}}$$

The candidates are the non-basic variables with $\bar{a}_{3j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{\frac{-2}{-1}}_{=2}, \underbrace{\frac{-4}{-1}}_{=4} \right\} = 2,$$

then x_1 is the pivot column.

We update the reduced costs with the row $\bar{a}_{3j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - 2 \bar{a}_{3j}$:

$$\underbrace{-2}_{\bar{c}_2} \quad \underbrace{-2}_{\bar{c}_3}$$

The new basis is $\mathcal{B} = \{1, 4\}$.

- **Iteration 2**

With $\mathcal{B} = \{1, 4\}$ and $\mathcal{N} = \{2, 3\}$, we have:

$$\begin{aligned} \mathbf{x}_{\mathcal{B}} &= \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \\ \mathbf{x}_{\mathcal{N}} &= \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-2 \quad -2) \end{aligned}$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -\bar{b}_1 & = & -2 \\ \bar{b}_1 + \bar{b}_4 & = & 0 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_4 = -2 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_4 < 0$, x_4 is the pivot row and we have $\mathbf{e}_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_{\mathcal{B}} = \mathbf{e}_4'$ to compute $\mathbf{u} = \mathbf{e}_4' \mathbf{A}_{\mathcal{B}}^{-1}$, second system of linear equations:

$$\begin{cases} -u_1 + u_4 & = & 0 \\ u_4 & = & 1 \end{cases} \quad \mathbf{u} = (1 \quad 1)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_4 , $\bar{a}_{4j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-2}_{\bar{a}_{42}} \quad \underbrace{1}_{\bar{a}_{43}}$$

The candidates are the non-basic variables with $\bar{a}_{4j} < 0$, and the ratio test gives:

$$\min \left\{ \begin{array}{c} -2 \\ \underbrace{-2}_{=1} \end{array} \right\} = 1,$$

then x_2 is the pivot column.

We update the reduced costs with the row $\bar{a}_{4j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - 1 \bar{a}_{4j}$:

$$\underbrace{-3}_{\bar{c}_3} \quad \underbrace{-1}_{\bar{c}_4}$$

The new basis is $\mathcal{B} = \{1, 2\}$.

- **Iteration 3**

With $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-3 \quad -1)$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -\bar{b}_1 & -\bar{b}_2 & = & -2 \\ \bar{b}_1 & -\bar{b}_2 & = & 0 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

All the basic variables are non-negative, then the basis is primal feasible and dual feasible: it is **optimal**.

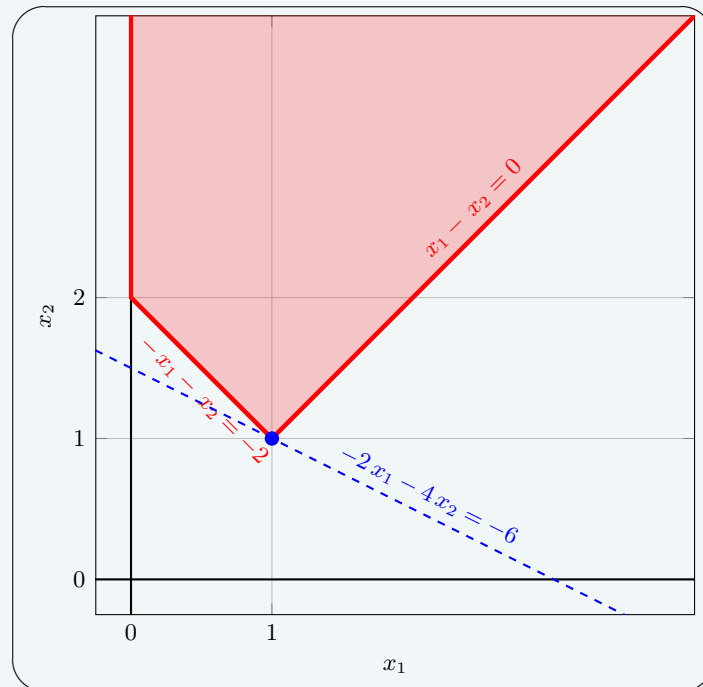
We compute the optimal value $\bar{z} = \mathbf{c}'_{\mathcal{B}} \bar{\mathbf{b}} = (-2 \quad -4) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -6$

An optimal solution of this LP problem is

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1 \quad \text{and} \quad z(LP) = -6.$$

Graphical solution:

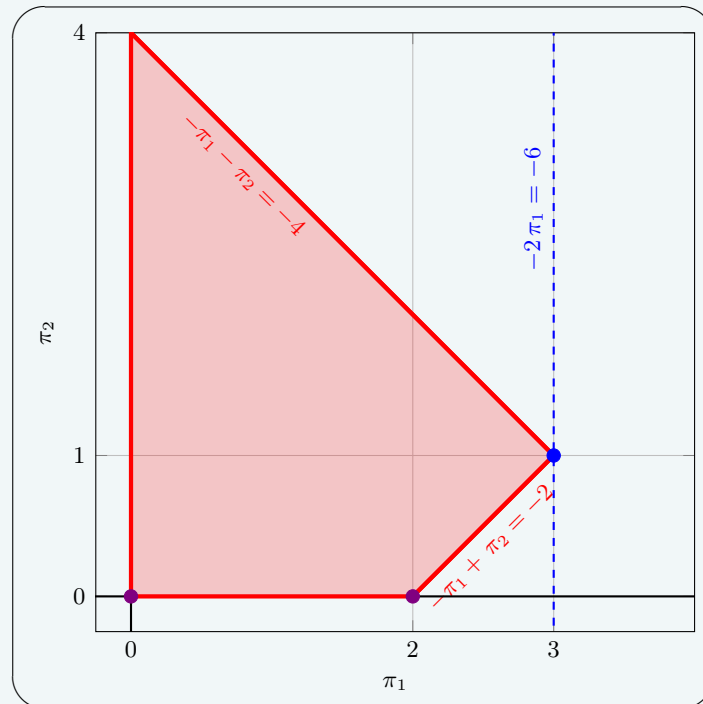
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1 \quad \text{and} \quad z(LP) = -6$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the two original dual variables: π_1, π_2 .



An optimal dual LP solution is:

$$\tilde{\pi}_1 = 3, \quad \tilde{\pi}_2 = 1 \quad \text{and} \quad z(D(LP)) = -6.$$

In violet, we show the first and the second vertex explored by the previous execution of the revised dual simplex algorithm, before reaching the optimal one depicted in blue.

Exercise 12.2 — Revised dual simplex algorithm (P12)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & -x_1 - x_2 \leq -3 \\ & x_1 - 2x_2 \leq -1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Execute the revised dual simplex algorithm to find an optimal solution of this LP problem, solving at each iteration the two systems of linear equations. As pivot row choose the basic variable with the smallest index among those with $\bar{b}_r < 0$, and break ties in the ratio test for the pivot column by choosing the smallest index.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} -1 & -1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- **Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{c} \leq \mathbf{0}$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is **dual feasible**, since $\mathbf{A}_{\mathcal{B}} = \mathbf{I}$ and $\mathbf{c}_{\mathcal{B}} = \mathbf{0}$ give

$$\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} = (-1 \quad -2) \leq \mathbf{0}.$$

The associated basic solution is not feasible, and then the dual simplex algorithm is used.

- **Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-1 \quad -2)$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} \bar{b}_3 & = & -3 \\ \bar{b}_4 & = & -1 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_3 = -3 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_3 < 0$, x_3 is the pivot row and we have $\mathbf{e}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_B = \mathbf{e}_3'$ to compute $\mathbf{u} = \mathbf{e}_3' \mathbf{A}_B^{-1}$, second system of linear equations:

$$\begin{cases} u_3 & = & 1 \\ & u_4 & = & 0 \end{cases} \quad \mathbf{u} = (1 \quad 0)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_3 , $\bar{a}_{3j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-1}_{\bar{a}_{31}} \quad \underbrace{-1}_{\bar{a}_{32}}$$

The candidates are the non-basic variables with $\bar{a}_{3j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{\frac{-1}{-1}}_{=1}, \underbrace{\frac{-2}{-1}}_{=2} \right\} = 1,$$

then x_1 is the pivot column.

We update the reduced costs with the row $\bar{a}_{3j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - 1 \bar{a}_{3j}$:

$$\underbrace{-1}_{\bar{c}_2} \quad \underbrace{-1}_{\bar{c}_3}$$

The new basis is $\mathcal{B} = \{1, 4\}$.

- Iteration 2**

With $\mathcal{B} = \{1, 4\}$ and $\mathcal{N} = \{2, 3\}$, we have:

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \quad \mathbf{A}_B = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathbf{c}_B = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_N = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_N = \begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-1 \quad -1)$$

Feasibility check

We solve $\mathbf{A}_B \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -\bar{b}_1 & = & -3 \\ \bar{b}_1 + \bar{b}_4 & = & -1 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_4 = -4 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_4 < 0$, x_4 is the pivot row and we have $\mathbf{e}_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_B = \mathbf{e}_4'$ to compute $\mathbf{u} = \mathbf{e}_4' \mathbf{A}_B^{-1}$, second system of linear equations:

$$\begin{cases} -u_1 + u_4 = 0 \\ u_4 = 1 \end{cases} \quad \mathbf{u} = (1 \quad 1)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_4 , $\bar{a}_{4j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-3}_{\bar{a}_{42}} \quad \underbrace{1}_{\bar{a}_{43}}$$

The candidates are the non-basic variables with $\bar{a}_{4j} < 0$, and the ratio test gives:

$$\min \left\{ \begin{array}{c} -1 \\ -3 \\ = \frac{1}{3} \end{array} \right\} = \frac{1}{3},$$

then x_2 is the pivot column.

We update the reduced costs with the row $\bar{a}_{4j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - \frac{1}{3} \bar{a}_{4j}$:

$$\underbrace{-\frac{4}{3}}_{\bar{c}_3} \quad \underbrace{-\frac{1}{3}}_{\bar{c}_4}$$

The new basis is $\mathcal{B} = \{1, 2\}$.

• Iteration 3

With $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_B = \begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix} \quad \mathbf{c}_B = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$\mathbf{x}_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \bar{\mathbf{c}}' = \begin{pmatrix} -\frac{4}{3} & -\frac{1}{3} \end{pmatrix}$$

Feasibility check

We solve $A_B \bar{b} = b$ to compute $\bar{b} = A_B^{-1} b$, first system of linear equations:

$$\begin{cases} -\bar{b}_1 & -\bar{b}_2 & = & -3 \\ \bar{b}_1 & -2\bar{b}_2 & = & -1 \end{cases} \quad \bar{b} = \begin{pmatrix} \frac{5}{3} \\ \frac{4}{3} \end{pmatrix}$$

All the basic variables are non-negative, then the basis is primal feasible and dual feasible: it is **optimal**.

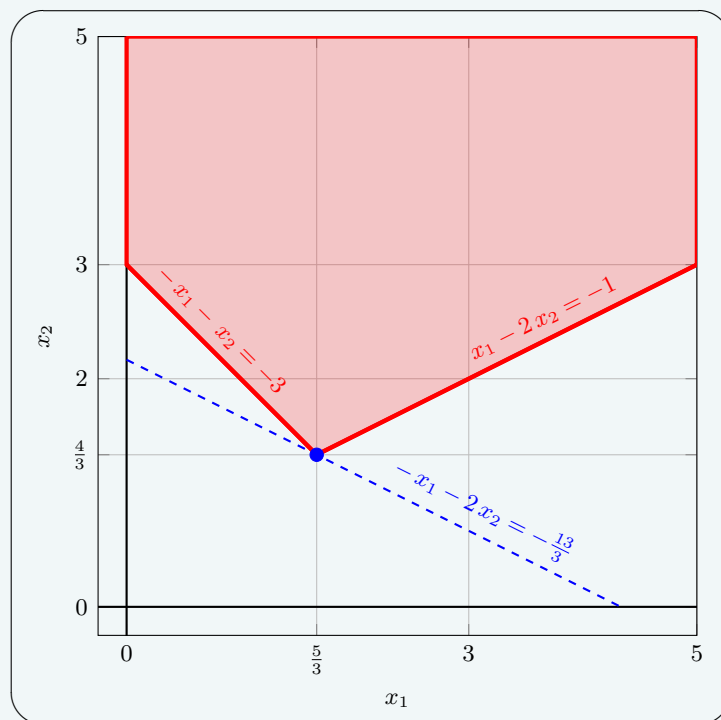
We compute the optimal value $\bar{z} = c'_B \bar{b} = (-1 \ -2) \begin{pmatrix} \frac{5}{3} \\ \frac{4}{3} \end{pmatrix} = -\frac{13}{3}$

An optimal solution of this LP problem is

$$\tilde{x}_1 = \frac{5}{3}, \quad \tilde{x}_2 = \frac{4}{3} \quad \text{and} \quad z(LP) = -\frac{13}{3}.$$

Graphical solution:

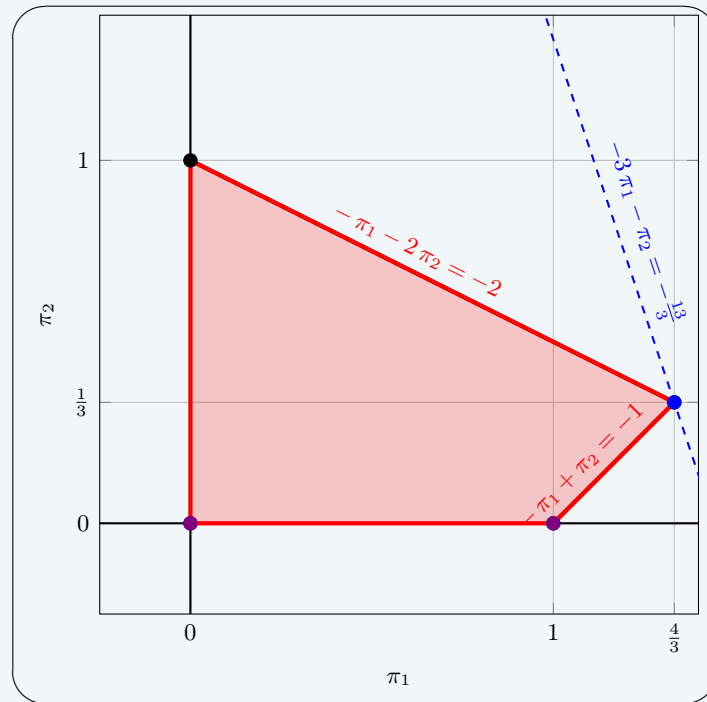
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{5}{3}, \quad \tilde{x}_2 = \frac{4}{3} \quad \text{and} \quad z(LP) = -\frac{13}{3}.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the two original dual variables: π_1, π_2 .



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{4}{3}, \quad \tilde{\pi}_2 = \frac{1}{3}, \quad \text{and} \quad z(D(LP)) = -\frac{13}{3}.$$

In violet, we show the first and the second vertex explored by the previous execution of the revised dual simplex algorithm, before reaching the optimal one depicted in blue.

Exercise 12.3 — Revised dual simplex algorithm (P13)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{aligned} \max \quad & -x_1 - 3x_2 \\ \text{subject to} \quad & -x_1 - 2x_2 \leq -6 \\ & x_1 - 4x_2 \leq -6 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Execute the revised dual simplex algorithm to find an optimal solution of this LP problem, solving at each iteration the two systems of linear equations. As pivot row choose the basic variable with the smallest index among those with $\bar{b}_r < 0$, and break ties in the ratio test for the pivot column by choosing the smallest index.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} -1 & -2 & 1 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 \\ 2 & -2 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -6 \\ -6 \\ 3 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -1 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

- **Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{c} \leq \mathbf{0}$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$ is **dual feasible**, since $\mathbf{A}_{\mathcal{B}} = \mathbf{I}$ and $\mathbf{c}_{\mathcal{B}} = \mathbf{0}$ give

$$\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} = (-1 \quad -3) \leq \mathbf{0}.$$

The associated basic solution is not feasible, and then the dual simplex algorithm is used.

- **Iteration 1**

With $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & -2 \\ 1 & -4 \\ 2 & -2 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-1 \quad -3)$$

Feasibility check

We solve $\mathbf{A}_B \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} \bar{b}_3 & = & -6 \\ \bar{b}_4 & = & -6 \\ \bar{b}_5 & = & 3 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} -6 \\ -6 \\ 3 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_3 = -6 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_3 < 0$, x_3 is the pivot row and we have $\mathbf{e}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_B = \mathbf{e}_3'$ to compute $\mathbf{u} = \mathbf{e}_3' \mathbf{A}_B^{-1}$, second system of linear equations:

$$\begin{cases} u_3 & = & 1 \\ u_4 & = & 0 \\ u_5 & = & 0 \end{cases} \quad \mathbf{u} = (1 \quad 0 \quad 0)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_3 , $\bar{a}_{3j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-1}_{\bar{a}_{31}} \quad \underbrace{-2}_{\bar{a}_{32}}$$

The candidates are the non-basic variables with $\bar{a}_{3j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{\frac{-1}{-1}}_{=1}, \underbrace{\frac{-3}{-2}}_{=\frac{3}{2}} \right\} = 1,$$

then x_1 is the pivot column.

We update the reduced costs with the row $\bar{a}_{3j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - 1 \bar{a}_{3j}$:

$$\underbrace{-1}_{\bar{c}_2} \quad \underbrace{-1}_{\bar{c}_3}$$

The new basis is $\mathcal{B} = \{1, 4, 5\}$.

• **Iteration 2**

With $\mathcal{B} = \{1, 4, 5\}$ and $\mathcal{N} = \{2, 3\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -2 & 1 \\ -4 & 0 \\ -2 & 0 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-1 \quad -1)$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -\bar{b}_1 & = & -6 \\ \bar{b}_1 + \bar{b}_4 & = & -6 \\ 2\bar{b}_1 + \bar{b}_5 & = & 3 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} 6 \\ -12 \\ -9 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_4 = -12 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_4 < 0$, x_4 is the pivot row and we have $\mathbf{e}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_{\mathcal{B}} = \mathbf{e}_4'$ to compute $\mathbf{u} = \mathbf{e}_4' \mathbf{A}_{\mathcal{B}}^{-1}$, second system of linear equations:

$$\begin{cases} -u_1 + u_4 + 2u_5 & = & 0 \\ u_4 & = & 1 \\ u_5 & = & 0 \end{cases} \quad \mathbf{u} = (1 \quad 1 \quad 0)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_4 , $\bar{a}_{4j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-6}_{\bar{a}_{42}} \quad \underbrace{1}_{\bar{a}_{43}}$$

The candidates are the non-basic variables with $\bar{a}_{4j} < 0$, and the ratio test gives:

$$\min \left\{ \begin{array}{c} -1 \\ -6 \\ \underbrace{\quad}_{=\frac{1}{6}} \end{array} \right\} = \frac{1}{6},$$

then x_2 is the pivot column.

We update the reduced costs with the row $\bar{a}_{4j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - \frac{1}{6} \bar{a}_{4j}$:

$$\underbrace{-\frac{7}{6}}_{\bar{c}_3} \quad \underbrace{-\frac{1}{6}}_{\bar{c}_4}$$

The new basis is $\mathcal{B} = \{1, 2, 5\}$.

- **Iteration 3**

With $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & -2 & 0 \\ 1 & -4 & 0 \\ 2 & -2 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \bar{\mathbf{c}}' = \begin{pmatrix} -\frac{7}{6} & -\frac{1}{6} \end{pmatrix}$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\left\{ \begin{array}{lcl} -\bar{b}_1 & -2\bar{b}_2 & = -6 \\ \bar{b}_1 & -4\bar{b}_2 & = -6 \\ 2\bar{b}_1 & -2\bar{b}_2 & +\bar{b}_5 = 3 \end{array} \right. \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

All the basic variables are non-negative, then the basis is primal feasible and dual feasible: it is **optimal**.

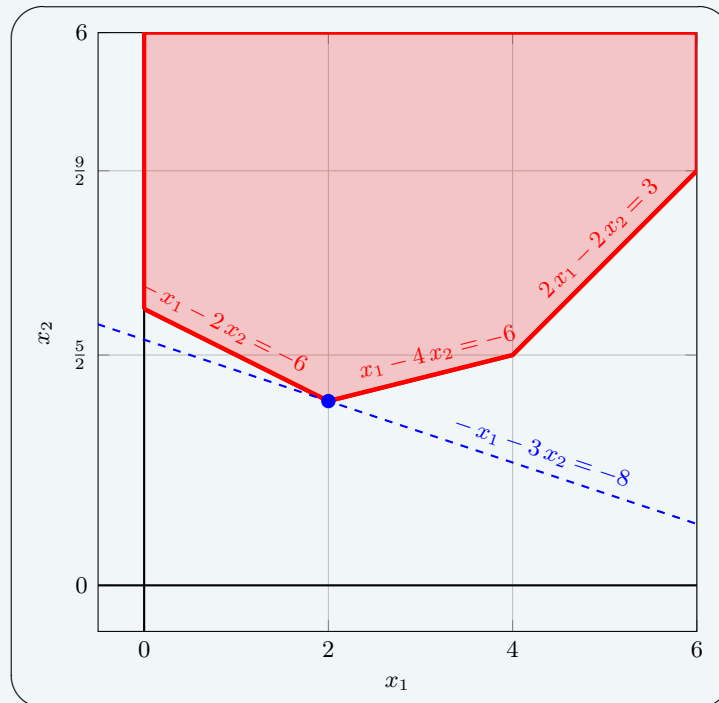
$$\text{We compute the optimal value } \bar{z} = \mathbf{c}'_{\mathcal{B}} \bar{\mathbf{b}} = \begin{pmatrix} -1 & -3 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = -8$$

An optimal solution of this LP problem is

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 2 \quad \text{and} \quad z(LP) = -8.$$

Graphical solution:

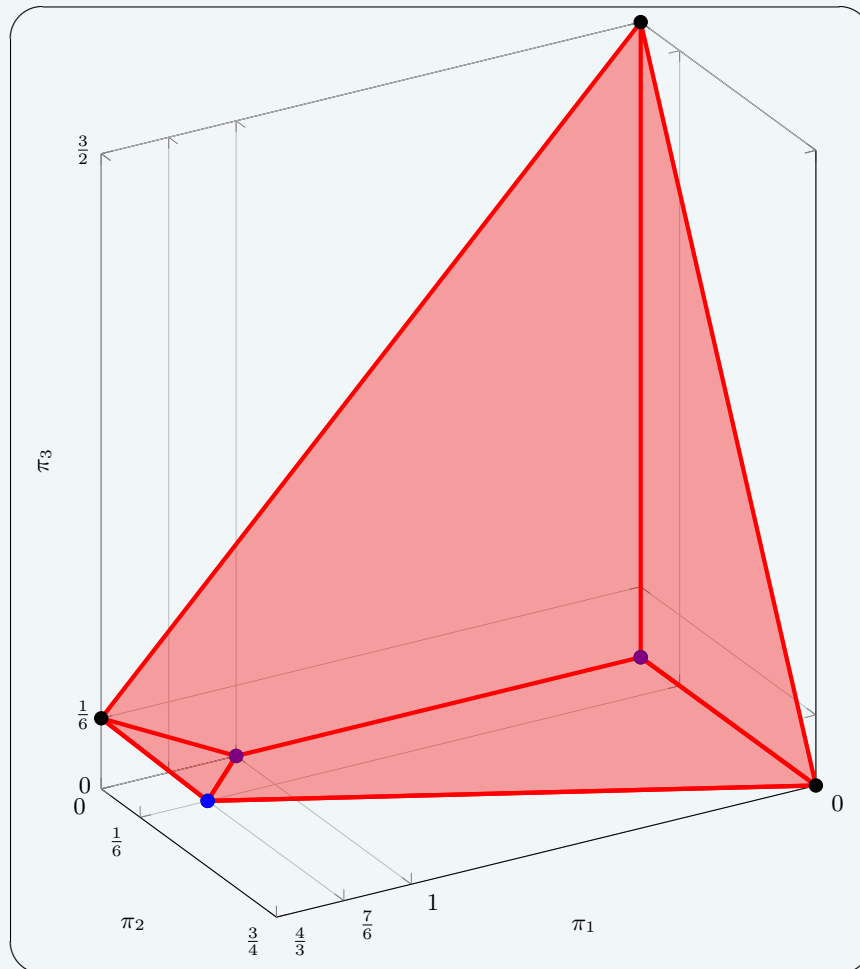
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 2 \quad \text{and} \quad z(LP) = -8.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the three original dual variables: π_1, π_2, π_3 .



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{7}{6}, \quad \tilde{\pi}_2 = \frac{1}{6}, \quad \tilde{\pi}_3 = 0, \quad \text{and} \quad z(D(LP)) = -8.$$

In violet, we show the first and the second vertex explored by the previous execution of the revised dual simplex algorithm, before reaching the optimal one depicted in blue.

Exercise 12.4 — Revised dual simplex algorithm (P14)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{array}{ll}
 \max & -x_1 - 2x_2 \\
 \text{subject to} & -x_1 + x_2 \leq 2 \\
 & x_1 - 2x_2 \leq 2 \\
 & -x_1 - x_2 \leq -3 \\
 & x_1, x_2 \geq 0
 \end{array}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Execute the revised dual simplex algorithm to find an optimal solution of this LP problem, solving at each iteration the two systems of linear equations. As pivot row choose the basic variable with the smallest index among those with $\bar{b}_r < 0$, and break ties in the ratio test for the pivot column by choosing the smallest index.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

- Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{c} \leq \mathbf{0}$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$ is **dual feasible**, since $\mathbf{A}_{\mathcal{B}} = \mathbf{I}$ and $\mathbf{c}_{\mathcal{B}} = \mathbf{0}$ give

$$\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} = (-1 \quad -2) \leq \mathbf{0}.$$

The associated basic solution is not feasible, and then the dual simplex algorithm is used.

- Iteration 1**

With $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & 1 \\ 1 & -2 \\ -1 & -1 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-1 \quad -2)$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} \bar{b}_3 & = & 2 \\ \bar{b}_4 & = & 2 \\ \bar{b}_5 & = & -3 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_5 = -3 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_5 < 0$, x_5 is the pivot row and we have $\mathbf{e}_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_{\mathcal{B}} = \mathbf{e}_5'$ to compute $\mathbf{u} = \mathbf{e}_5' \mathbf{A}_{\mathcal{B}}^{-1}$, second system of linear equations:

$$\begin{cases} u_3 & = & 0 \\ u_4 & = & 0 \\ u_5 & = & 1 \end{cases} \quad \mathbf{u} = (0 \quad 0 \quad 1)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_5 , $\bar{a}_{5j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-1}_{\bar{a}_{51}} \quad \underbrace{-1}_{\bar{a}_{52}}$$

The candidates are the non-basic variables with $\bar{a}_{5j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{\frac{-1}{-1}}_{=1}, \underbrace{\frac{-2}{-1}}_{=2} \right\} = 1,$$

then x_1 is the pivot column.

We update the reduced costs with the row $\bar{a}_{5j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - 1 \bar{a}_{5j}$:

$$\underbrace{-1}_{\bar{c}_2} \quad \underbrace{-1}_{\bar{c}_5}$$

The new basis is $\mathcal{B} = \{1, 3, 4\}$.

• **Iteration 2**

With $\mathcal{B} = \{1, 3, 4\}$ and $\mathcal{N} = \{2, 5\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ -2 & 0 \\ -1 & 1 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-1 \quad -1)$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -\bar{b}_1 + \bar{b}_3 & = & 2 \\ \bar{b}_1 + \bar{b}_4 & = & 2 \\ -\bar{b}_1 & = & -3 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_4 = -1 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_4 < 0$, x_4 is the pivot row and we have $\mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_{\mathcal{B}} = \mathbf{e}_4'$ to compute $\mathbf{u} = \mathbf{e}_4' \mathbf{A}_{\mathcal{B}}^{-1}$, second system of linear equations:

$$\begin{cases} -u_1 + u_3 - u_4 & = & 0 \\ u_1 & = & 0 \\ u_3 & = & 1 \end{cases} \quad \mathbf{u} = (0 \quad 1 \quad 1)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_4 , $\bar{a}_{4j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-3}_{\bar{a}_{42}} \quad \underbrace{1}_{\bar{a}_{45}}$$

The candidates are the non-basic variables with $\bar{a}_{4j} < 0$, and the ratio test gives:

$$\min \left\{ \begin{array}{c} -1 \\ -3 \\ \underbrace{\quad}_{=\frac{1}{3}} \end{array} \right\} = \frac{1}{3},$$

then x_2 is the pivot column.

We update the reduced costs with the row $\bar{a}_{4j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - \frac{1}{3} \bar{a}_{4j}$:

$$\underbrace{-\frac{1}{3}}_{\bar{c}_4} \quad \underbrace{-\frac{4}{3}}_{\bar{c}_5}$$

The new basis is $\mathcal{B} = \{1, 2, 3\}$.

- **Iteration 3**

With $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ -1 & -1 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \bar{\mathbf{c}}' = \begin{pmatrix} -\frac{1}{3} & -\frac{4}{3} \end{pmatrix}$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\left\{ \begin{array}{lll} -\bar{b}_1 & +\bar{b}_2 & +\bar{b}_3 = 2 \\ \bar{b}_1 & -2\bar{b}_2 & = 2 \\ -\bar{b}_1 & -\bar{b}_2 & = -3 \end{array} \right. \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{8}{3} \\ \frac{1}{3} \\ \frac{13}{3} \end{pmatrix}$$

All the basic variables are non-negative, then the basis is primal feasible and dual feasible: it is **optimal**.

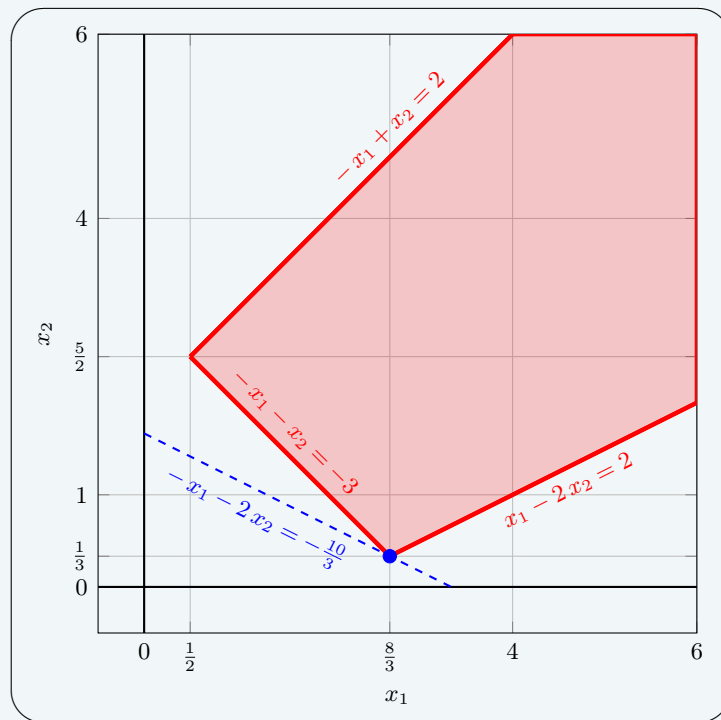
We compute the optimal value $\bar{z} = \mathbf{c}'_{\mathcal{B}} \bar{\mathbf{b}} = (-1 \quad -2 \quad 0) \begin{pmatrix} \frac{8}{3} \\ \frac{1}{3} \\ \frac{13}{3} \end{pmatrix} = -\frac{10}{3}$

An optimal solution of this LP problem is

$$\tilde{x}_1 = \frac{8}{3}, \quad \tilde{x}_2 = \frac{1}{3} \quad \text{and} \quad z(LP) = -\frac{10}{3}.$$

Graphical solution:

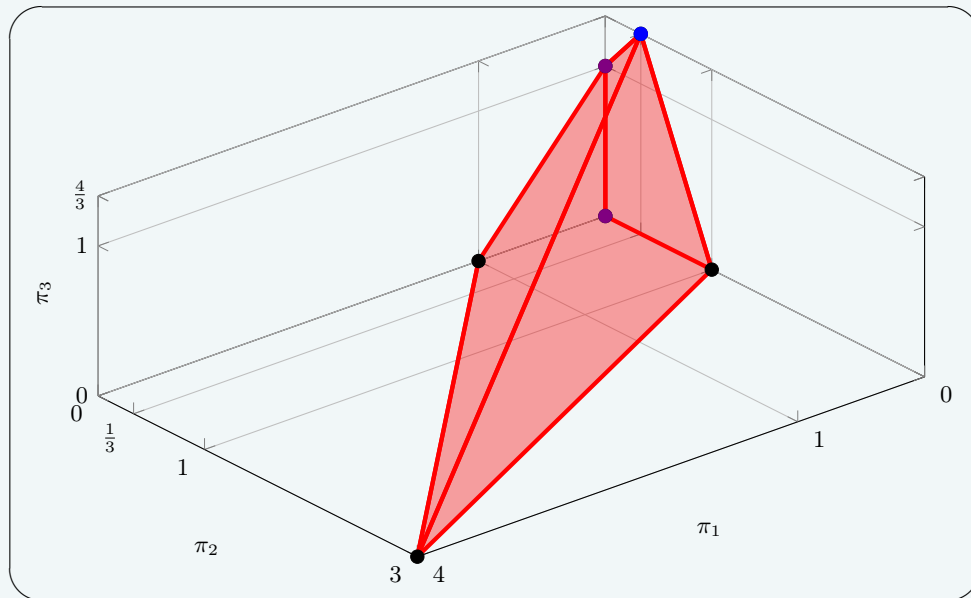
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{8}{3}, \quad \tilde{x}_2 = \frac{1}{3} \quad \text{and} \quad z(LP) = -\frac{10}{3}.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the three original dual variables: π_1, π_2, π_3 .



An optimal dual LP solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = \frac{1}{3}, \quad \tilde{\pi}_3 = \frac{4}{3}, \quad \text{and} \quad z(D(LP)) = -\frac{10}{3}.$$

In violet, we show the first and the second vertex explored by the previous execution of the revised dual simplex algorithm, before reaching the optimal one depicted in blue.

Exercise 12.5 — Revised dual simplex algorithm (P15)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & -3x_1 - 2x_2 \\ \text{subject to} \quad & -x_1 - x_2 \leq -4 \\ & -x_1 - 3x_2 \leq -6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Execute the revised dual simplex algorithm to find an optimal solution of this LP problem, solving at each iteration the two systems of linear equations. As pivot row choose the basic variable with the smallest index among those with $\bar{b}_r < 0$, and break ties in the ratio test for the pivot column by choosing the smallest index.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} -1 & -1 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -4 \\ -6 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -3 \\ -2 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- **Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{c} \leq \mathbf{0}$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is **dual feasible**, since $\mathbf{A}_{\mathcal{B}} = \mathbf{I}$ and $\mathbf{c}_{\mathcal{B}} = \mathbf{0}$ give

$$\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} = \begin{pmatrix} -3 & -2 \end{pmatrix} \leq \mathbf{0}.$$

The associated basic solution is not feasible, and then the dual simplex algorithm is used.

- **Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & -1 \\ -1 & -3 \end{pmatrix} \quad \bar{\mathbf{c}}' = \begin{pmatrix} -3 & -2 \end{pmatrix}$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} \bar{b}_3 & = & -4 \\ \bar{b}_4 & = & -6 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_3 = -4 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_3 < 0$, x_3 is the pivot row and we have $\mathbf{e}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_B = \mathbf{e}'_3$ to compute $\mathbf{u} = \mathbf{e}'_3 \mathbf{A}_B^{-1}$, second system of linear equations:

$$\begin{cases} u_3 & = & 1 \\ & u_4 & = & 0 \end{cases} \quad \mathbf{u} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_3 , $\bar{a}_{3j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-1}_{\bar{a}_{31}} \quad \underbrace{-1}_{\bar{a}_{32}}$$

The candidates are the non-basic variables with $\bar{a}_{3j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{\frac{-3}{-1}}_{=3}, \underbrace{\frac{-2}{-1}}_{=2} \right\} = 2,$$

then x_2 is the pivot column.

We update the reduced costs with the row $\bar{a}_{3j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - 2 \bar{a}_{3j}$:

$$\underbrace{-1}_{\bar{c}_1} \quad \underbrace{-2}_{\bar{c}_3}$$

The new basis is $\mathcal{B} = \{2, 4\}$.

- **Iteration 2**

With $\mathcal{B} = \{2, 4\}$ and $\mathcal{N} = \{1, 3\}$, we have:

$$\mathbf{x}_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_B = \begin{pmatrix} -1 & 0 \\ -3 & 1 \end{pmatrix} \quad \mathbf{c}_B = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_N = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad \mathbf{A}_N = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \quad \bar{\mathbf{c}}' = \begin{pmatrix} -1 & -2 \end{pmatrix}$$

Feasibility check

We solve $\mathbf{A}_B \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -\bar{b}_2 & = & -4 \\ -3\bar{b}_2 + \bar{b}_4 & = & -6 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

All the basic variables are non-negative, then the basis is primal feasible and dual feasible: it is **optimal**.

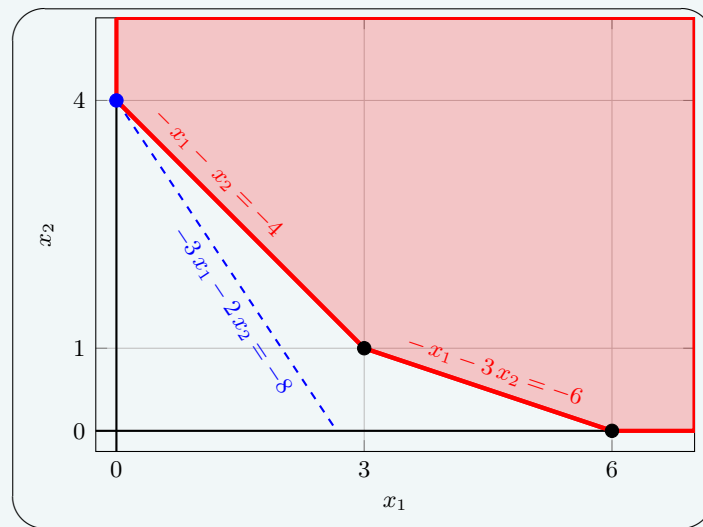
We compute the optimal value $\bar{z} = \mathbf{c}'_{\mathcal{B}} \bar{\mathbf{b}} = (-2 \ 0) \begin{pmatrix} 4 \\ 6 \end{pmatrix} = -8$

An optimal solution of this LP problem is

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 4 \quad \text{and} \quad z(LP) = -8.$$

Graphical solution:

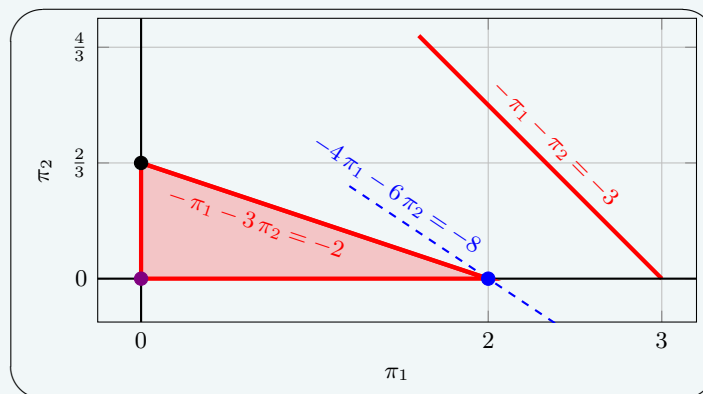
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 4 \quad \text{and} \quad z(LP) = -8.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the two original dual variables: π_1, π_2 . The constraint $-\pi_1 - \pi_2 \geq -3$ is redundant.



An optimal dual LP solution is:

$$\tilde{\pi}_1 = 2, \quad \tilde{\pi}_2 = 0, \quad \text{and} \quad z(D(LP)) = -8.$$

In violet, we show the first vertex explored by the previous execution of the revised dual simplex algorithm, before reaching the optimal one depicted in blue.

Exercise 12.6 — Revised dual simplex algorithm (P16)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 - x_2 \leq -4 \\ & -x_1 - 3x_2 \leq -6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Execute the revised dual simplex algorithm to find an optimal solution of this LP problem, solving at each iteration the two systems of linear equations. As pivot row choose the basic variable with the smallest index among those with $\bar{b}_r < 0$, and break ties in the ratio test for the pivot column by choosing the smallest index.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} -2 & -1 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -4 \\ -6 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

- **Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{c} \leq \mathbf{0}$ by adding the slack variables, the partition $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$ is **dual feasible**, since $\mathbf{A}_{\mathcal{B}} = \mathbf{I}$ and $\mathbf{c}_{\mathcal{B}} = \mathbf{0}$ give

$$\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} = (-1 \quad -2) \leq \mathbf{0}.$$

The associated basic solution is not feasible, and then the dual simplex algorithm is used.

• **Iteration 1**

With $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -2 & -1 \\ -1 & -3 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-1 \quad -2)$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} \bar{b}_3 & = & -4 \\ \bar{b}_4 & = & -6 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_3 = -4 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_3 < 0$, x_3 is the pivot row and we have $\mathbf{e}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_{\mathcal{B}} = \mathbf{e}_3'$ to compute $\mathbf{u} = \mathbf{e}_3' \mathbf{A}_{\mathcal{B}}^{-1}$, second system of linear equations:

$$\begin{cases} u_3 & = & 1 \\ u_4 & = & 0 \end{cases} \quad \mathbf{u} = (1 \quad 0)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_3 , $\bar{a}_{3j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-2}_{\bar{a}_{31}} \quad \underbrace{-1}_{\bar{a}_{32}}$$

The candidates are the non-basic variables with $\bar{a}_{3j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{\frac{-1}{-2}}_{=\frac{1}{2}}, \underbrace{\frac{-2}{-1}}_{=2} \right\} = \frac{1}{2},$$

then x_1 is the pivot column.

We update the reduced costs with the row $\bar{a}_{3j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - \frac{1}{2} \bar{a}_{3j}$:

$$\underbrace{-\frac{3}{2}}_{\bar{c}_2} \quad \underbrace{-\frac{1}{2}}_{\bar{c}_3}$$

The new basis is $\mathcal{B} = \{1, 4\}$.

- **Iteration 2**

With $\mathcal{B} = \{1, 4\}$ and $\mathcal{N} = \{2, 3\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & 1 \\ -3 & 0 \end{pmatrix} \quad \bar{\mathbf{c}}' = \left(-\frac{3}{2} \quad -\frac{1}{2}\right)$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -2\bar{b}_1 & = & -4 \\ -\bar{b}_1 + \bar{b}_4 & = & -6 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_4 = -4 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_4 < 0$, x_4 is the pivot row and we have $\mathbf{e}_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_{\mathcal{B}} = \mathbf{e}_4'$ to compute $\mathbf{u} = \mathbf{e}_4' \mathbf{A}_{\mathcal{B}}^{-1}$, second system of linear equations:

$$\begin{cases} -2u_1 - u_4 & = & 0 \\ u_4 & = & 1 \end{cases} \quad \mathbf{u} = \left(-\frac{1}{2} \quad 1\right)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_4 , $\bar{a}_{4j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-\frac{5}{2}}_{\bar{a}_{42}} \quad \underbrace{-\frac{1}{2}}_{\bar{a}_{43}}$$

The candidates are the non-basic variables with $\bar{a}_{4j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{-\frac{3}{2}}_{=\frac{3}{5}}, \underbrace{-\frac{1}{2}}_{=1} \right\} = \frac{3}{5},$$

then x_2 is the pivot column.

We update the reduced costs with the row $\bar{a}_{4j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - \frac{3}{5} \bar{a}_{4j}$:

$$\underbrace{-\frac{1}{5}}_{\bar{c}_3} \quad \underbrace{-\frac{3}{5}}_{\bar{c}_4}$$

The new basis is $\mathcal{B} = \{1, 2\}$.

- **Iteration 3**

With $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -2 & -1 \\ -1 & -3 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \bar{\mathbf{c}}' = \begin{pmatrix} -\frac{1}{5} & -\frac{3}{5} \end{pmatrix}$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -2\bar{b}_1 & -\bar{b}_2 & = & -4 \\ -\bar{b}_1 & -3\bar{b}_2 & = & -6 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{6}{5} \\ \frac{8}{5} \end{pmatrix}$$

All the basic variables are non-negative, then the basis is primal feasible and dual feasible: it is **optimal**.

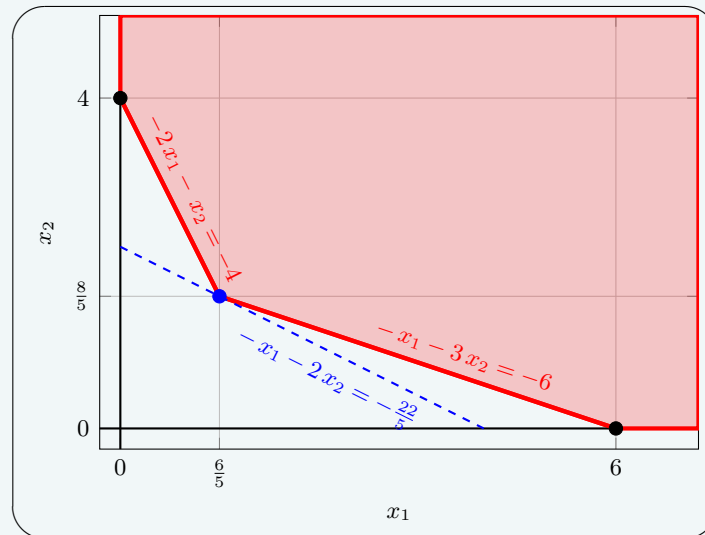
We compute the optimal value $\bar{z} = \mathbf{c}'_{\mathcal{B}} \bar{\mathbf{b}} = (-1 \quad -2) \begin{pmatrix} \frac{6}{5} \\ \frac{8}{5} \end{pmatrix} = -\frac{22}{5}$

An optimal solution of this LP problem is

$$\tilde{x}_1 = \frac{6}{5}, \quad \tilde{x}_2 = \frac{8}{5} \quad \text{and} \quad z(LP) = -\frac{22}{5}.$$

Graphical solution:

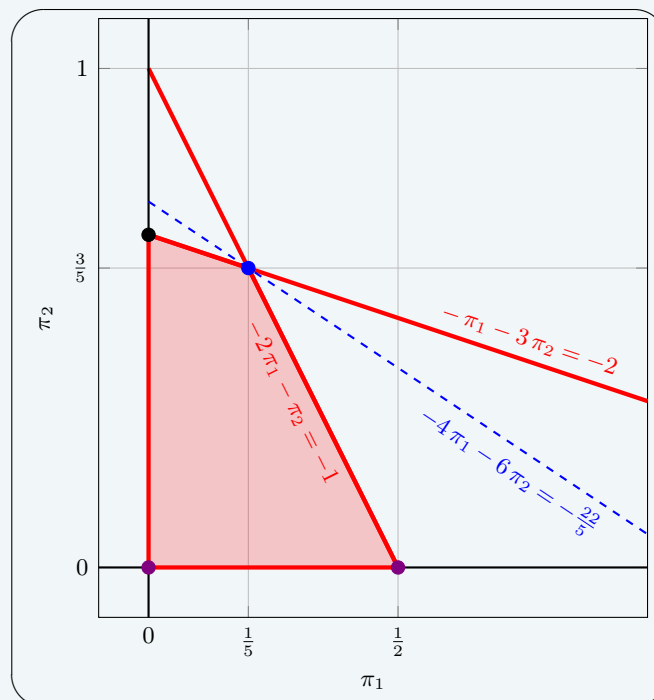
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{6}{5}, \quad \tilde{x}_2 = \frac{8}{5} \quad \text{and} \quad z(LP) = -\frac{22}{5}.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the two original dual variables: π_1, π_2 .



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{1}{5}, \quad \tilde{\pi}_2 = \frac{3}{5}, \quad \text{and} \quad z(D(LP)) = -\frac{22}{5}.$$

In violet, we show the first and the second vertex explored by the previous execution of the revised dual simplex algorithm, before reaching the optimal one depicted in blue.

Exercise 12.7 — Revised dual simplex algorithm (P51)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 2$ constraints:

$$\begin{array}{llllllll} \max & -2 & x_1 & - & x_2 & & & \\ \text{subject to} & & x_1 & - & x_2 & - & 3 & x_3 \leq -5 \\ & -3 & x_1 & - & 2 & x_2 & + & 2 & x_3 \leq -2 \\ & & x_1, & & x_2, & & & x_3 \geq 0 \end{array}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Execute the revised dual simplex algorithm to find an optimal solution of this LP problem, solving at each iteration the two systems of linear equations. As pivot row choose the basic variable with the smallest index among those with $\bar{b}_r < 0$, and break ties in the ratio test for the pivot column by choosing the smallest index.

Solution

We have:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -3 & 1 & 0 \\ -3 & -2 & 2 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

- **Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $\mathbf{c} \leq \mathbf{0}$ by adding the slack variables, the partition $\mathcal{B} = \{4, 5\}$ and $\mathcal{N} =$

$\{1, 2, 3\}$ is **dual feasible**, since $\mathbf{A}_B = \mathbf{I}$ and $\mathbf{c}_B = \mathbf{0}$ give

$$\bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} = \begin{pmatrix} -2 & -1 & 0 \end{pmatrix} \leq \mathbf{0}.$$

The associated basic solution is not feasible, and then the dual simplex algorithm is used.

- **Iteration 1**

With $B = \{4, 5\}$ and $\mathcal{N} = \{1, 2, 3\}$, we have:

$$\mathbf{x}_B = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & -1 & -3 \\ -3 & -2 & 2 \end{pmatrix} \quad \bar{\mathbf{c}}' = \begin{pmatrix} -2 & -1 & 0 \end{pmatrix}$$

Feasibility check

We solve $\mathbf{A}_B \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} \bar{b}_4 & = & -5 \\ \bar{b}_5 & = & -2 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_4 = -5 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_4 < 0$, x_4 is the pivot row and we have $\mathbf{e}_4 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_B = \mathbf{e}'_4$ to compute $\mathbf{u} = \mathbf{e}'_4 \mathbf{A}_B^{-1}$, second system of linear equations:

$$\begin{cases} u_4 & = & 1 \\ u_5 & = & 0 \end{cases} \quad \mathbf{u} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_4 , $\bar{a}_{4j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{1}_{\bar{a}_{41}} \quad \underbrace{-1}_{\bar{a}_{42}} \quad \underbrace{-3}_{\bar{a}_{43}}$$

The candidates are the non-basic variables with $\bar{a}_{4j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{\frac{-1}{-1}}_{=1}, \underbrace{\frac{0}{-3}}_{=0} \right\} = 0,$$

then x_3 is the pivot column.

We update the reduced costs with the row $\bar{a}_{4j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - 0 \bar{a}_{4j}$:

$$\underbrace{-2}_{\bar{c}_1} \quad \underbrace{-1}_{\bar{c}_2} \quad \underbrace{0}_{\bar{c}_4}$$

The new basis is $\mathcal{B} = \{3, 5\}$.

- Iteration 2**

With $\mathcal{B} = \{3, 5\}$ and $\mathcal{N} = \{1, 2, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -3 & 0 \\ 2 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & -1 & 1 \\ -3 & -2 & 0 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-2 \quad -1 \quad 0)$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -3\bar{b}_3 & = & -5 \\ 2\bar{b}_3 + \bar{b}_5 & = & -2 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} \frac{5}{3} \\ -\frac{16}{3} \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_5 = -\frac{16}{3} < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_5 < 0$, x_5 is the pivot row and we have $\mathbf{e}_5 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_{\mathcal{B}} = \mathbf{e}_5'$ to compute $\mathbf{u} = \mathbf{e}_5' \mathbf{A}_{\mathcal{B}}^{-1}$, second system of linear equations:

$$\begin{cases} -3u_3 + 2u_5 & = & 0 \\ & u_5 & = & 1 \end{cases} \quad \mathbf{u} = \begin{pmatrix} \frac{2}{3} & 1 \end{pmatrix}$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_5 , $\bar{a}_{5j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-\frac{7}{3}}_{\bar{a}_{51}} \quad \underbrace{-\frac{8}{3}}_{\bar{a}_{52}} \quad \underbrace{\frac{2}{3}}_{\bar{a}_{54}}$$

The candidates are the non-basic variables with $\bar{a}_{5j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{-\frac{2}{-\frac{7}{3}}}_{=\frac{6}{7}}, \underbrace{-\frac{1}{-\frac{8}{3}}}_{=\frac{3}{8}} \right\} = \frac{3}{8},$$

then x_2 is the pivot column.

We update the reduced costs with the row $\bar{a}_{5j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - \frac{3}{8} \bar{a}_{5j}$:

$$\underbrace{-\frac{9}{8}}_{\bar{c}_1} \quad \underbrace{-\frac{1}{4}}_{\bar{c}_4} \quad \underbrace{-\frac{3}{8}}_{\bar{c}_5}$$

The new basis is $\mathcal{B} = \{2, 3\}$.

- **Iteration 3**

With $\mathcal{B} = \{2, 3\}$ and $\mathcal{N} = \{1, 4, 5\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & -3 \\ -2 & 2 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \quad \bar{\mathbf{c}}' = \begin{pmatrix} -\frac{9}{8} & -\frac{1}{4} & -\frac{3}{8} \end{pmatrix}$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -\bar{b}_2 - 3\bar{b}_3 = -5 \\ -2\bar{b}_2 + 2\bar{b}_3 = -2 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

All the basic variables are non-negative, then the basis is primal feasible and dual feasible: it is **optimal**.

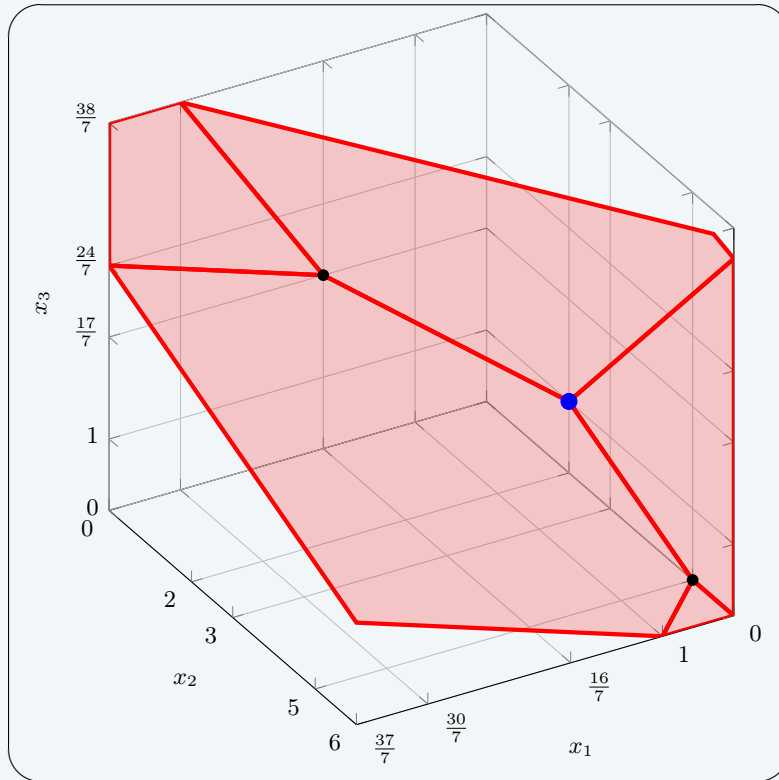
We compute the optimal value $\bar{z} = \mathbf{c}'_{\mathcal{B}} \bar{\mathbf{b}} = (-1 \ 0) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -2$

An optimal solution of this LP problem is

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 2, \quad \tilde{x}_3 = 1 \quad \text{and} \quad z(LP) = -2.$$

Graphical solution:

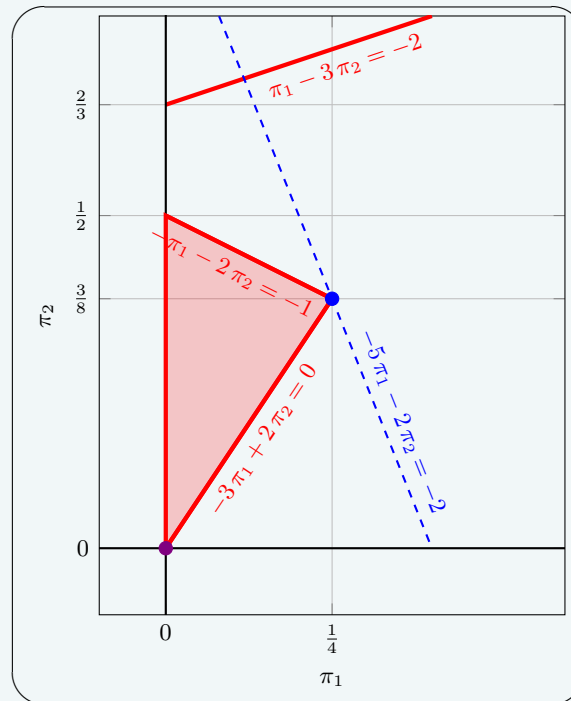
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 2, \quad \tilde{x}_3 = 1 \quad \text{and} \quad z(LP) = -2.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the two original dual variables: π_1, π_2 .



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{1}{4}, \quad \tilde{\pi}_2 = \frac{3}{8} \quad \text{and} \quad z(D(LP)) = -2.$$

In violet, we show the first vertex explored by the previous execution of the revised dual simplex algorithm, before reaching the optimal one depicted in blue.

Exercise 12.8 — Revised dual simplex algorithm (P53)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 3$ constraints:

$$\begin{array}{llllll} \max & - & x_1 & - & 2 & x_2 & - & 4 & x_3 \\ \text{subject to} & - & x_1 & - & & x_2 & & & \leq & -2 \\ & & & & - & x_2 & - & & x_3 & \leq & -3 \\ & - & x_1 & & & & - & & x_3 & \leq & -2 \\ & & x_1, & & x_2, & & & & x_3 & \geq & 0 \end{array}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Execute the revised dual simplex algorithm to find an optimal solution of this LP problem, solving at each iteration the two systems of linear equations. As pivot row choose the basic variable with the smallest index among those with $\bar{b}_r < 0$, and break ties in the ratio test for the pivot column by choosing the smallest index.

Solution

We have:

$$A = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} -2 \\ -3 \\ -2 \end{pmatrix} \quad c = \begin{pmatrix} -1 \\ -2 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

- **Initialization**

Since it is an LP problem in standard form obtained from an LP problem in canonical form with $c \leq 0$ by adding the slack variables, the partition $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$ is **dual feasible**, since $A_{\mathcal{B}} = I$ and $c_{\mathcal{B}} = 0$ give

$$\bar{c}' = c'_{\mathcal{N}} = (-1 \quad -2 \quad -4) \leq 0.$$

The associated basic solution is not feasible, and then the dual simplex algorithm is used.

- **Iteration 1**

With $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$, we have:

$$x_{\mathcal{B}} = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \quad A_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad c_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad A_{\mathcal{N}} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix} \quad \bar{c}' = (-1 \quad -2 \quad -4)$$

Feasibility check

We solve $\mathbf{A}_B \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} \bar{b}_4 & = & -2 \\ \bar{b}_5 & = & -3 \\ \bar{b}_6 & = & -2 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} -2 \\ -3 \\ -2 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_4 = -2 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_4 < 0$, x_4 is the pivot row and we have $\mathbf{e}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_B = \mathbf{e}_4'$ to compute $\mathbf{u} = \mathbf{e}_4' \mathbf{A}_B^{-1}$, second system of linear equations:

$$\begin{cases} u_4 & = & 1 \\ u_5 & = & 0 \\ u_6 & = & 0 \end{cases} \quad \mathbf{u} = (1 \quad 0 \quad 0)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_4 , $\bar{a}_{4j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-1}_{\bar{a}_{41}} \quad \underbrace{-1}_{\bar{a}_{42}} \quad \underbrace{0}_{\bar{a}_{43}}$$

The candidates are the non-basic variables with $\bar{a}_{4j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{\frac{-1}{-1}}_{=1}, \underbrace{\frac{-2}{-1}}_{=2} \right\} = 1,$$

then x_1 is the pivot column.

We update the reduced costs with the row $\bar{a}_{4j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - 1 \bar{a}_{4j}$:

$$\underbrace{-1}_{\bar{c}_2} \quad \underbrace{-4}_{\bar{c}_3} \quad \underbrace{-1}_{\bar{c}_4}$$

The new basis is $\mathcal{B} = \{1, \quad 5, \quad 6\}$.

• **Iteration 2**

With $\mathcal{B} = \{1, 5, 6\}$ and $\mathcal{N} = \{2, 3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_5 \\ x_6 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad \mathbf{c}' = (-1 \quad -4 \quad -1)$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -\bar{b}_1 & = & -2 \\ & \bar{b}_5 & = & -3 \\ -\bar{b}_1 & + \bar{b}_6 & = & -2 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_5 = -3 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_5 < 0$, x_5 is the pivot row and we have $\mathbf{e}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_{\mathcal{B}} = \mathbf{e}'_5$ to compute $\mathbf{u} = \mathbf{e}'_5 \mathbf{A}_{\mathcal{B}}^{-1}$, second system of linear equations:

$$\begin{cases} -u_1 & -u_6 & = & 0 \\ & u_5 & = & 1 \\ & u_6 & = & 0 \end{cases} \quad \mathbf{u} = (0 \quad 1 \quad 0)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_5 , $\bar{a}_{5j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-1}_{\bar{a}_{52}} \quad \underbrace{-1}_{\bar{a}_{53}} \quad \underbrace{0}_{\bar{a}_{54}}$$

The candidates are the non-basic variables with $\bar{a}_{5j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{\frac{-1}{-1}}_{=1}, \underbrace{\frac{-4}{-1}}_{=4} \right\} = 1,$$

then x_2 is the pivot column.

We update the reduced costs with the row $\bar{a}_{5j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - 1 \bar{a}_{5j}$:

$$\underbrace{-3}_{\bar{c}_3} \quad \underbrace{-1}_{\bar{c}_4} \quad \underbrace{-1}_{\bar{c}_5}$$

The new basis is $\mathcal{B} = \{1, 2, 6\}$.

- Iteration 3**

With $\mathcal{B} = \{1, 2, 6\}$ and $\mathcal{N} = \{3, 4, 5\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-3 \quad -1 \quad -1)$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -\bar{b}_1 - \bar{b}_2 & = & -2 \\ & -\bar{b}_2 & = & -3 \\ -\bar{b}_1 & + \bar{b}_6 & = & -2 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_1 = -1 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_1 < 0$, x_1 is the pivot row and we have $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_B = \mathbf{e}'_1$ to compute $\mathbf{u} = \mathbf{e}'_1 \mathbf{A}_B^{-1}$, second system of linear equations:

$$\begin{cases} -u_1 & -u_6 & = & 1 \\ -u_1 & -u_2 & & = & 0 \\ & & u_6 & = & 0 \end{cases} \quad \mathbf{u} = (-1 \quad 1 \quad 0)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_1 , $\bar{a}_{1j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-1}_{\bar{a}_{13}} \quad \underbrace{-1}_{\bar{a}_{14}} \quad \underbrace{1}_{\bar{a}_{15}}$$

The candidates are the non-basic variables with $\bar{a}_{1j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{\frac{-3}{-1}}_{=3}, \underbrace{\frac{-1}{-1}}_{=1} \right\} = 1,$$

then x_4 is the pivot column.

We update the reduced costs with the row $\bar{a}_{1j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - 1 \bar{a}_{1j}$:

$$\underbrace{-1}_{\bar{c}_1} \quad \underbrace{-2}_{\bar{c}_3} \quad \underbrace{-2}_{\bar{c}_5}$$

The new basis is $\mathcal{B} = \{2, 4, 6\}$.

- **Iteration 4**

With $\mathcal{B} = \{2, 4, 6\}$ and $\mathcal{N} = \{1, 3, 5\}$, we have:

$$\mathbf{x}_B = \begin{pmatrix} x_2 \\ x_4 \\ x_6 \end{pmatrix} \quad \mathbf{A}_B = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_B = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_N = \begin{pmatrix} x_1 \\ x_3 \\ x_5 \end{pmatrix} \quad \mathbf{A}_N = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & -1 & 0 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-1 \quad -2 \quad -2)$$

Feasibility check

We solve $\mathbf{A}_B \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -\bar{b}_2 + \bar{b}_4 & = & -2 \\ -\bar{b}_2 & = & -3 \\ & \bar{b}_6 & = & -2 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

Not all the basic variables are non-negative, since $\bar{b}_6 = -2 < 0$, then we perform a dual pivoting operation.

Dual pivoting operation

Since $\bar{b}_6 < 0$, x_6 is the pivot row and we have $\mathbf{e}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

We solve $\mathbf{u} \mathbf{A}_B = \mathbf{e}_6'$ to compute $\mathbf{u} = \mathbf{e}_6' \mathbf{A}_B^{-1}$, second system of linear equations:

$$\begin{cases} -u_2 - u_4 & = & 0 \\ u_2 & = & 0 \\ & u_6 & = & 1 \end{cases} \quad \mathbf{u} = (0 \quad 0 \quad 1)$$

We compute the row of $\bar{\mathbf{A}}$ associated with x_6 , $\bar{a}_{6j} = \mathbf{u} \mathbf{A}_j$:

$$\underbrace{-1}_{\bar{a}_{61}} \quad \underbrace{-1}_{\bar{a}_{63}} \quad \underbrace{0}_{\bar{a}_{65}}$$

The candidates are the non-basic variables with $\bar{a}_{6j} < 0$, and the ratio test gives:

$$\min \left\{ \underbrace{\frac{-1}{-1}}_{=1}, \underbrace{\frac{-2}{-1}}_{=2} \right\} = 1,$$

then x_1 is the pivot column.

We update the reduced costs with the row $\bar{a}_{6j}$ already computed, $\bar{c}_j \leftarrow \bar{c}_j - 1 \bar{a}_{6j}$:

$$\underbrace{-1}_{\bar{c}_3} \quad \underbrace{-2}_{\bar{c}_5} \quad \underbrace{-1}_{\bar{c}_6}$$

The new basis is $\mathcal{B} = \{1, 2, 4\}$.

• Iteration 5

With $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5, 6\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_5 \\ x_6 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \bar{\mathbf{c}}' = (-1 \quad -2 \quad -1)$$

Feasibility check

We solve $\mathbf{A}_{\mathcal{B}} \bar{\mathbf{b}} = \mathbf{b}$ to compute $\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b}$, first system of linear equations:

$$\begin{cases} -\bar{b}_1 & -\bar{b}_2 & +\bar{b}_4 & = & -2 \\ & -\bar{b}_2 & & = & -3 \\ -\bar{b}_1 & & & = & -2 \end{cases} \quad \bar{\mathbf{b}} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

All the basic variables are non-negative, then the basis is primal feasible and dual feasible: it is **optimal**.

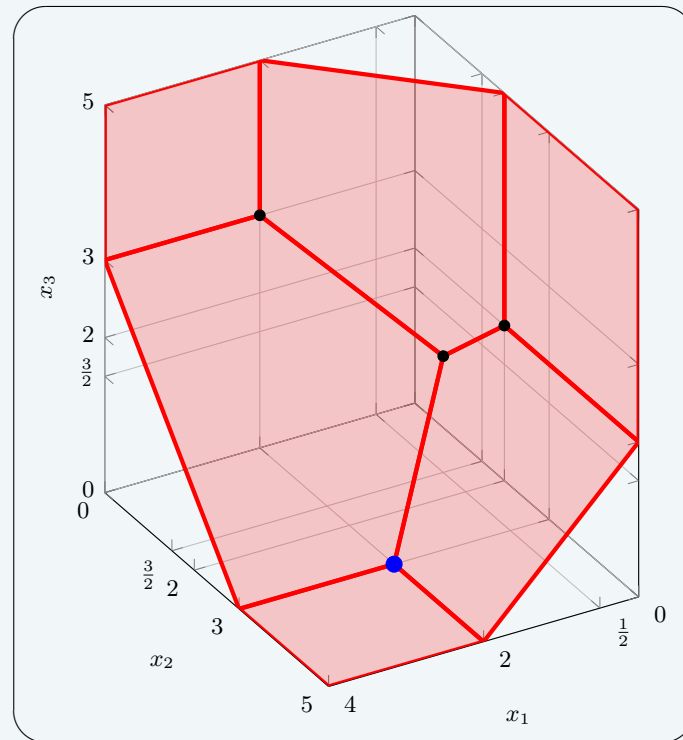
$$\text{We compute the optimal value } \bar{z} = \mathbf{c}'_{\mathcal{B}} \bar{\mathbf{b}} = (-1 \quad -2 \quad 0) \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = -8$$

An optimal solution of this LP problem is

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 0 \quad \text{and} \quad z(LP) = -8.$$

Graphical solution:

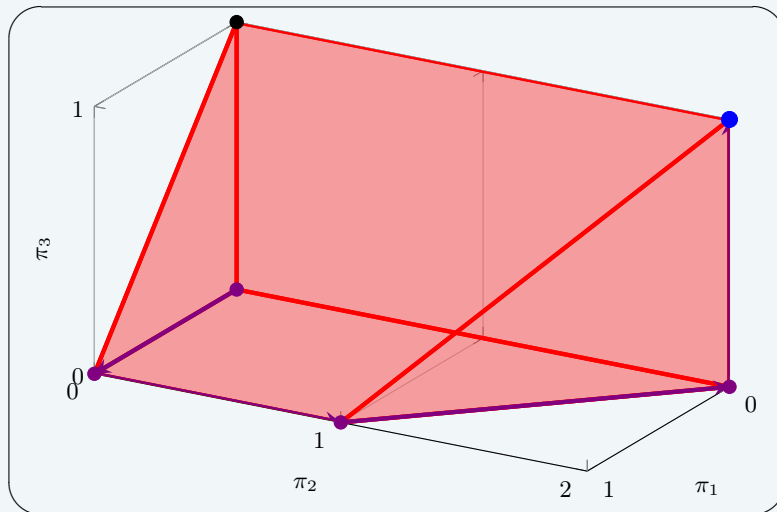
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded: we draw only its faces, cut at $x_1 = 4$, $x_2 = 5$ and $x_3 = 5$.



An optimal primal LP solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 0 \quad \text{and} \quad z(LP) = -8.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the three original dual variables: π_1, π_2, π_3 . The constraint $-\pi_2 - \pi_3 \geq -4$ is redundant.



An optimal dual LP solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 2, \quad \tilde{\pi}_3 = 1, \quad \text{and} \quad z(D(LP)) = -8.$$

In violet, we show the four vertices $(0,0,0) \rightarrow (1,0,0) \rightarrow (1,1,0) \rightarrow (0,2,0)$ explored by the previous execution of the revised dual simplex algorithm, before reaching the optimal one $(0,2,1)$ depicted in blue. The dual objective values are $0 \rightarrow -2 \rightarrow -5 \rightarrow -6 \rightarrow -8$.

The primal-dual method

Exercise 13.1 — The primal-dual method (PD1)

Consider the following LP problem in standard minimization form with $n = 3$ non-negative variables and $m = 2$ equality linear constraints:

$$\begin{array}{llllll} \min & 2 & x_1 & + & 2 & x_2 & + & 2 & x_3 \\ \text{subject to} & & x_1 & + & & x_2 & + & 2 & x_3 & = & 5 \\ & & x_1 & + & 2 & x_2 & + & & x_3 & = & 4 \\ & & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

Its right-hand side is non-negative and its objective-function vector is non-negative, and then $\pi = (0 \ 0)'$ is a feasible solution of its dual problem.

Questions

Execute the primal-dual method starting from $\pi = (0 \ 0)'$, showing at each iteration the admissible set, the optimal value of the restricted primal problem and the dual step.

Solution

- **Iteration 1**

With $\pi = (0 \ 0)'$ the dual slacks $s_j = c_j - \mathbf{A}'_j \pi$ are

$$s_1 = 2, \quad s_2 = 2, \quad s_3 = 2,$$

and then the admissible set is $\mathcal{J} = \emptyset$.

The restricted primal problem $RP(\mathcal{J})$ has only the m artificial variables, since no

original variable is admissible. Its optimal value is

$$\xi(\mathcal{J}) = 9 > 0.$$

The optimal solution of $DRP(\mathcal{J})$ is $\mathbf{y} = (1 \ 1)'$, and for the non-admissible columns we have

$$\mathbf{A}'_1 \mathbf{y} = 2, \quad \mathbf{A}'_2 \mathbf{y} = 3, \quad \mathbf{A}'_3 \mathbf{y} = 3.$$

Dual step. The candidates are the non-admissible columns with $\mathbf{A}'_j \mathbf{y} > 0$:

$$\theta = \min \left\{ \underbrace{\frac{2}{2}}_{=1}, \underbrace{\frac{2}{3}}_{=2/3}, \underbrace{\frac{2}{3}}_{=2/3} \right\} = \frac{2}{3},$$

attained by $j^* = 2, 3$. The new dual solution is

$$\boldsymbol{\pi} \leftarrow (0 \ 0)' + \frac{2}{3} \cdot (1 \ 1)' = \left(\frac{2}{3} \ \frac{2}{3}\right)',$$

whose dual objective value is $\mathbf{b}'\boldsymbol{\pi} = 6$.

• **Iteration 2**

With $\boldsymbol{\pi} = (\frac{2}{3} \ \frac{2}{3})'$ the dual slacks $s_j = c_j - \mathbf{A}'_j \boldsymbol{\pi}$ are

$$s_1 = \frac{2}{3}, \quad s_2 = 0, \quad s_3 = 0,$$

and then the admissible set is $\mathcal{J} = \{2, 3\}$.

The restricted primal problem $RP(\mathcal{J})$ has the variables x_2, x_3 and the m artificial variables. Its optimal value is

$$\xi(\mathcal{J}) = 0,$$

and then the method **stops**. An optimal solution of this LP problem is

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = 2$$

with optimal value $\mathbf{c}'\tilde{\mathbf{x}} = 6$. An optimal solution of the dual problem is $\tilde{\boldsymbol{\pi}} = (\frac{2}{3} \ \frac{2}{3})'$, whose objective value is $\mathbf{b}'\tilde{\boldsymbol{\pi}} = 6$: the two values coincide, in accordance with the Strong Duality theorem.

Exercise 13.2 — The primal-dual method (PD2)

Consider the following LP problem in standard minimization form with $n = 3$ non-negative variables and $m = 2$ equality linear constraints:

$$\begin{array}{llllll}
 \min & 2 & x_1 & + & 2 & x_2 & + & 2 & x_3 \\
 \text{subject to} & & x_1 & + & & x_2 & + & & x_3 & = & 4 \\
 & & x_1 & + & 2 & x_2 & + & & x_3 & = & 5 \\
 & & x_1, & & x_2, & & x_3 & \geq & 0
 \end{array}$$

Its right-hand side is non-negative and its objective-function vector is non-negative, and then $\pi = (0 \ 0)'$ is a feasible solution of its dual problem.

Questions

Execute the primal-dual method starting from $\pi = (0 \ 0)'$, showing at each iteration the admissible set, the optimal value of the restricted primal problem and the dual step.

Solution

• Iteration 1

With $\pi = (0 \ 0)'$ the dual slacks $s_j = c_j - \mathbf{A}'_j \pi$ are

$$s_1 = 2, \quad s_2 = 2, \quad s_3 = 2,$$

and then the admissible set is $\mathcal{J} = \emptyset$.

The restricted primal problem $RP(\mathcal{J})$ has only the m artificial variables, since no original variable is admissible. Its optimal value is

$$\xi(\mathcal{J}) = 9 > 0.$$

The optimal solution of $DRP(\mathcal{J})$ is $\mathbf{y} = (1 \ 1)'$, and for the non-admissible columns we have

$$\mathbf{A}'_1 \mathbf{y} = 2, \quad \mathbf{A}'_2 \mathbf{y} = 3, \quad \mathbf{A}'_3 \mathbf{y} = 2.$$

Dual step. The candidates are the non-admissible columns with $\mathbf{A}'_j \mathbf{y} > 0$:

$$\theta = \min \left\{ \underbrace{\frac{2}{2}}_{=1}, \underbrace{\frac{2}{3}}_{=2/3}, \underbrace{\frac{2}{2}}_{=1} \right\} = \frac{2}{3},$$

attained by $j^* = 2$. The new dual solution is

$$\pi \leftarrow (0 \ 0)' + \frac{2}{3} \cdot (1 \ 1)' = \left(\frac{2}{3} \ \frac{2}{3}\right)',$$

whose dual objective value is $\mathbf{b}'\pi = 6$.

• **Iteration 2**

With $\pi = (\frac{2}{3} \ \frac{2}{3})'$ the dual slacks $s_j = c_j - \mathbf{A}'_j \pi$ are

$$s_1 = \frac{2}{3}, \quad s_2 = 0, \quad s_3 = \frac{2}{3},$$

and then the admissible set is $\mathcal{J} = \{2\}$.

The restricted primal problem $RP(\mathcal{J})$ has the variables x_2 and the m artificial variables. Its optimal value is

$$\xi(\mathcal{J}) = \frac{3}{2} > 0.$$

The optimal solution of $DRP(\mathcal{J})$ is $\mathbf{y} = (1 \ -\frac{1}{2})'$, and for the non-admissible columns we have

$$\mathbf{A}'_1 \mathbf{y} = \frac{1}{2}, \quad \mathbf{A}'_3 \mathbf{y} = \frac{1}{2}.$$

Dual step. The candidates are the non-admissible columns with $\mathbf{A}'_j \mathbf{y} > 0$:

$$\theta = \min \left\{ \underbrace{\frac{2/3}{1/2}}_{=4/3}, \underbrace{\frac{2/3}{1/2}}_{=4/3} \right\} = \frac{4}{3},$$

attained by $j^* = 1, 3$. The new dual solution is

$$\pi \leftarrow (\frac{2}{3} \ \frac{2}{3})' + \frac{4}{3} \cdot (1 \ -\frac{1}{2})' = (2 \ 0)',$$

whose dual objective value is $\mathbf{b}'\pi = 8$.

• **Iteration 3**

With $\pi = (2 \ 0)'$ the dual slacks $s_j = c_j - \mathbf{A}'_j \pi$ are

$$s_1 = 0, \quad s_2 = 0, \quad s_3 = 0,$$

and then the admissible set is $\mathcal{J} = \{1, 2, 3\}$.

The restricted primal problem $RP(\mathcal{J})$ has the variables x_1, x_2, x_3 and the m artificial variables. Its optimal value is

$$\xi(\mathcal{J}) = 0,$$

and then the method **stops**. An optimal solution of this LP problem is

$$\tilde{x}_1 = 3, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = 0$$

with optimal value $\mathbf{c}'\tilde{\mathbf{x}} = 8$. An optimal solution of the dual problem is $\tilde{\pi} = (2 \ 0)'$, whose objective value is $\mathbf{b}'\tilde{\pi} = 8$: the two values coincide, in accordance with the Strong Duality theorem.

Exercise 13.3 — The primal-dual method (PD3)

Consider the following LP problem in standard minimization form with $n = 3$ non-negative variables and $m = 2$ equality linear constraints:

$$\begin{array}{llllll} \min & 2 & x_1 & + & 3 & x_2 & + & 4 & x_3 \\ \text{subject to} & & x_1 & + & 2 & x_2 & + & 2 & x_3 & = & 6 \\ & & 2 & x_1 & + & 2 & x_2 & + & & x_3 & = & 2 \\ & & & & & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

Its right-hand side is non-negative and its objective-function vector is non-negative, and then $\pi = (0 \ 0)'$ is a feasible solution of its dual problem.

Questions

Execute the primal-dual method starting from $\pi = (0 \ 0)'$, showing at each iteration the admissible set, the optimal value of the restricted primal problem and the dual step.

Solution
• Iteration 1

With $\pi = (0 \ 0)'$ the dual slacks $s_j = c_j - \mathbf{A}'_j \pi$ are

$$s_1 = 2, \quad s_2 = 3, \quad s_3 = 4,$$

and then the admissible set is $\mathcal{J} = \emptyset$.

The restricted primal problem $RP(\mathcal{J})$ has only the m artificial variables, since no original variable is admissible. Its optimal value is

$$\xi(\mathcal{J}) = 8 > 0.$$

The optimal solution of $DRP(\mathcal{J})$ is $\mathbf{y} = (1 \ 1)'$, and for the non-admissible columns we have

$$\mathbf{A}'_1 \mathbf{y} = 3, \quad \mathbf{A}'_2 \mathbf{y} = 4, \quad \mathbf{A}'_3 \mathbf{y} = 3.$$

Dual step. The candidates are the non-admissible columns with $\mathbf{A}'_j \mathbf{y} > 0$:

$$\theta = \min \left\{ \underbrace{\frac{2}{3}}_{=2/3}, \underbrace{\frac{3}{4}}_{=3/4}, \underbrace{\frac{4}{3}}_{=4/3} \right\} = \frac{2}{3},$$

attained by $j^* = 1$. The new dual solution is

$$\pi \leftarrow (0 \ 0)' + \frac{2}{3} \cdot (1 \ 1)' = \left(\frac{2}{3} \ \frac{2}{3}\right)',$$

whose dual objective value is $\mathbf{b}'\pi = \frac{16}{3}$.

• **Iteration 2**

With $\pi = (\frac{2}{3} \ \frac{2}{3})'$ the dual slacks $s_j = c_j - \mathbf{A}'_j\pi$ are

$$s_1 = 0, \quad s_2 = \frac{1}{3}, \quad s_3 = 2,$$

and then the admissible set is $\mathcal{J} = \{1\}$.

The restricted primal problem $RP(\mathcal{J})$ has the variables x_1 and the m artificial variables. Its optimal value is

$$\xi(\mathcal{J}) = 5 > 0.$$

The optimal solution of $DRP(\mathcal{J})$ is $\mathbf{y} = (1 \ -\frac{1}{2})'$, and for the non-admissible columns we have

$$\mathbf{A}'_2\mathbf{y} = 1, \quad \mathbf{A}'_3\mathbf{y} = \frac{3}{2}.$$

Dual step. The candidates are the non-admissible columns with $\mathbf{A}'_j\mathbf{y} > 0$:

$$\theta = \min \left\{ \underbrace{\frac{1/3}{1}}_{=1/3}, \underbrace{\frac{2}{3/2}}_{=4/3} \right\} = \frac{1}{3},$$

attained by $j^* = 2$. The new dual solution is

$$\pi \leftarrow \left(\frac{2}{3} \ \frac{2}{3}\right)' + \frac{1}{3} \cdot (1 \ -\frac{1}{2})' = (1 \ \frac{1}{2})',$$

whose dual objective value is $\mathbf{b}'\pi = 7$.

• **Iteration 3**

With $\pi = (1 \ \frac{1}{2})'$ the dual slacks $s_j = c_j - \mathbf{A}'_j\pi$ are

$$s_1 = 0, \quad s_2 = 0, \quad s_3 = \frac{3}{2},$$

and then the admissible set is $\mathcal{J} = \{1, 2\}$.

The restricted primal problem $RP(\mathcal{J})$ has the variables x_1, x_2 and the m artificial variables. Its optimal value is

$$\xi(\mathcal{J}) = 4 > 0.$$

The optimal solution of $DRP(\mathcal{J})$ is $\mathbf{y} = (1 \ -1)'$, and for the non-admissible columns we have

$$\mathbf{A}'_3\mathbf{y} = 1.$$

Dual step. The candidates are the non-admissible columns with $\mathbf{A}'_j \mathbf{y} > 0$:

$$\theta = \min \left\{ \frac{3/2}{\underbrace{1}_{=3/2}} \right\} = \frac{3}{2},$$

attained by $j^* = 3$. The new dual solution is

$$\boldsymbol{\pi} \leftarrow (1 \quad \frac{1}{2})' + \frac{3}{2} \cdot (1 \quad -1)' = (\frac{5}{2} \quad -1)',$$

whose dual objective value is $\mathbf{b}'\boldsymbol{\pi} = 13$.

• **Iteration 4**

With $\boldsymbol{\pi} = (\frac{5}{2} \quad -1)'$ the dual slacks $s_j = c_j - \mathbf{A}'_j \boldsymbol{\pi}$ are

$$s_1 = \frac{3}{2}, \quad s_2 = 0, \quad s_3 = 0,$$

and then the admissible set is $\mathcal{J} = \{2, 3\}$.

The restricted primal problem $RP(\mathcal{J})$ has the variables x_2, x_3 and the m artificial variables. Its optimal value is

$$\xi(\mathcal{J}) = 2 > 0.$$

The optimal solution of $DRP(\mathcal{J})$ is $\mathbf{y} = (1 \quad -2)'$, and for the non-admissible columns we have

$$\mathbf{A}'_1 \mathbf{y} = -3,$$

which is non-positive, and there is no other non-admissible column. Accordingly θ can grow without bound: the dual problem is **unbounded** and this LP problem is **infeasible**.

Sensitivity analysis

Exercise 14.1 — Sensitivity analysis (*S1*)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 4x_1 + 3x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 10 \\ & x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By adding $m = 2$ slack variables, the simplex algorithm determines the optimal basis $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, with the associated dictionary:

$$\begin{cases} x_1 = 2 - x_3 + x_4 \\ x_2 = 6 + x_3 - 2x_4 \\ z = 26 - x_3 - 2x_4 \end{cases}$$

and we have

$$\mathbf{A}_{\mathcal{B}}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\pi} = (1 \ 2).$$

Questions

1. Determine the interval of values of c_1 for which the basis $\mathcal{B} = \{1, 2\}$ remains optimal, and state how the optimal solution and the optimal value change inside that interval.
2. Determine the same interval for c_2 .
3. Determine the new optimal value when c_1 is increased from 4 to 5.

Solution

- The variable x_1 is **basic**, and then a change $c_1 \rightarrow 4 + \delta$ modifies all the reduced costs. The row of x_1 in the matrix $\bar{A}$ is $(1 \ -1)$, with one positive entry, which gives a lower bound on δ , and one negative entry, which gives an upper bound:

$$\underbrace{\frac{\bar{c}_3}{\bar{a}_{13}}}_{=\frac{-1}{1}=-1} \leq \delta \leq \underbrace{\frac{\bar{c}_4}{\bar{a}_{14}}}_{=\frac{-2}{-1}=2},$$

that is, $-1 \leq \delta \leq 2$ and then

$$c_1 \in [3, 6].$$

For every value of c_1 in this interval the optimal solution remains $\tilde{x}(LP) = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, while the optimal value becomes $26 + \delta \cdot \tilde{x}_1 = 26 + 2\delta$.

- The variable x_2 is basic as well. The row of x_2 in $\bar{A}$ is $(-1 \ 2)$, and then

$$\underbrace{\frac{\bar{c}_4}{\bar{a}_{24}}}_{=\frac{-2}{2}=-1} \leq \delta \leq \underbrace{\frac{\bar{c}_3}{\bar{a}_{23}}}_{=\frac{-1}{-1}=1},$$

that is, $-1 \leq \delta \leq 1$ and then

$$c_2 \in [2, 4].$$

For every value of c_2 in this interval the optimal solution does not change, while the optimal value becomes $26 + \delta \cdot \tilde{x}_2 = 26 + 6\delta$.

- With $c_1 = 5$ we have $\delta = 1$, which belongs to the interval $[-1, 2]$ determined at the first question. Accordingly, the optimal solution does not change and the new optimal value is

$$z(LP) = 26 + 1 \cdot 2 = 28,$$

which is confirmed by $5 \cdot 2 + 3 \cdot 6 = 28$.

Exercise 14.2 — Sensitivity analysis (S2)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & 4 & x_1 & + & 3 & x_2 \\ \text{subject to} & 2 & x_1 & + & & x_2 & \leq & 10 \\ & & x_1 & + & & x_2 & \leq & 8 \\ & & x_1, & & & x_2 & \geq & 0 \end{array}$$

By adding $m = 2$ slack variables, the simplex algorithm determines the optimal basis

$\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, with the associated dictionary:

$$\begin{cases} x_1 = 2 - x_3 + x_4 \\ x_2 = 6 + x_3 - 2x_4 \\ z = 26 - x_3 - 2x_4 \end{cases}$$

and we have

$$\mathbf{A}_{\mathcal{B}}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\pi} = (1 \ 2).$$

Questions

1. Write the shadow prices of the two linear constraints and explain their meaning.
2. Determine the interval of values of b_1 for which the basis $\mathcal{B} = \{1, 2\}$ remains optimal.
3. Determine the same interval for b_2 , and compute the new optimal value when b_2 is increased from 8 to 9.

Solution

- The shadow prices are the entries of the row vector $\boldsymbol{\pi} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} = (1 \ 2)$, that is,

$$\pi_1 = 1 \quad \text{and} \quad \pi_2 = 2.$$

The shadow price π_i is the increase of the optimal value per unit increase of b_i , as long as the optimal basis remains optimal. Both constraints are binding at the optimal solution, since both slack variables are non-basic with value 0. Both shadow prices are strictly positive, and then increasing any of the two right-hand sides increases the optimal value. A binding constraint may also have a zero shadow price, under dual degeneracy: here this does not happen, since $\pi_1 = 1$ and $\pi_2 = 2$.

- A change $b_1 \rightarrow 10 + \delta$ modifies the values of the basic variables by δ times the first column of $\mathbf{A}_{\mathcal{B}}^{-1}$, which is $(1 \ -1)'$:

$$\begin{cases} 2 + \delta \geq 0 \\ 6 - \delta \geq 0 \end{cases} \quad \Rightarrow \quad -2 \leq \delta \leq 6 \quad \Rightarrow \quad b_1 \in [8, 16].$$

- The second column of $\mathbf{A}_{\mathcal{B}}^{-1}$ is $(-1 \ 2)'$, and then

$$\begin{cases} 2 - \delta \geq 0 \\ 6 + 2\delta \geq 0 \end{cases} \quad \Rightarrow \quad -3 \leq \delta \leq 2 \quad \Rightarrow \quad b_2 \in [5, 10].$$

With $b_2 = 9$ we have $\delta = 1$, which belongs to the interval $[-3, 2]$, and then the new optimal value is

$$z(LP) = 26 + \delta \pi_2 = 26 + 1 \cdot 2 = 28.$$

The new optimal solution is $\tilde{x}_1 = 2 - 1 = 1$ and $\tilde{x}_2 = 6 + 2 = 8$, and indeed $4 \cdot 1 + 3 \cdot 8 = 28$.

Exercise 14.3 — Sensitivity analysis (S3)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & 4 & x_1 & + & 3 & x_2 \\ \text{subject to} & 2 & x_1 & + & & x_2 \leq 10 \\ & & x_1 & + & & x_2 \leq 8 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

By adding $m = 2$ slack variables, the simplex algorithm determines the optimal basis $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, with the associated dictionary:

$$\begin{cases} x_1 = 2 - x_3 + x_4 \\ x_2 = 6 + x_3 - 2x_4 \\ z = 26 - x_3 - 2x_4 \end{cases}$$

and we have

$$\mathbf{A}_{\mathcal{B}}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\pi} = (1 \ 2).$$

Questions

1. A new non-negative variable x_* is added to the LP problem, with objective-function coefficient $c_* = 7$ and column $\mathbf{A}_* = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Compute its reduced cost with respect to the optimal basis $\mathcal{B} = \{1, 2\}$ and state whether the basis is still optimal.
2. If it is not, restart the simplex algorithm from that basis and determine the new optimal solution and the new optimal value.

Solution

- The reduced cost of the new variable is

$$\bar{c}_\star = c_\star - \boldsymbol{\pi} \mathbf{A}_\star = 7 - (1 \ 2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 7 - 5 = 2 > 0.$$

Accordingly, the basis $\mathcal{B} = \{1, 2\}$ is no longer dual feasible and it is not optimal for the new LP problem: it is convenient to produce the new variable.

- We restart the simplex algorithm from the basis $\mathcal{B} = \{1, 2\}$, which is still primal feasible, with $x_\star$ as pivot column. Its column in the dictionary is

$$\bar{\mathbf{A}}_\star = \mathbf{A}_\mathcal{B}^{-1} \mathbf{A}_\star = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

The only positive entry is the one of the basic variable x_2 , and then

$$t_m = \frac{6}{3} = 2$$

and the pivot row is x_2 , which leaves the basis. After this single pivoting operation all the reduced costs become non-positive, and the new optimal solution is

$$\tilde{x}_1 = 4, \quad \tilde{x}_2 = 0, \quad \tilde{x}_\star = 2,$$

with optimal value

$$z(LP) = 4 \cdot 4 + 3 \cdot 0 + 7 \cdot 2 = 30.$$

Exercise 14.4 — Sensitivity analysis (S4)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & 4 & x_1 & + & 3 & x_2 \\ \text{subject to} & 2 & x_1 & + & & x_2 & \leq & 10 \\ & & x_1 & + & & x_2 & \leq & 8 \\ & & x_1, & & x_2 & \geq & 0 \end{array}$$

By adding $m = 2$ slack variables, the simplex algorithm determines the optimal basis $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, with the associated dictionary:

$$\begin{cases} x_1 = 2 - x_3 + x_4 \\ x_2 = 6 + x_3 - 2x_4 \\ z = 26 - x_3 - 2x_4 \end{cases}$$

and we have

$$\mathbf{A}_{\mathcal{B}}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\pi} = (1 \ 2).$$

Questions

1. A new linear constraint $x_1 + 2x_2 \leq 10$ is added to the LP problem. Check whether the optimal solution $\tilde{\mathbf{x}}(LP) = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ satisfies it.
2. Write the equation of the slack variable x_5 of the new constraint in the dictionary of the basis $\mathcal{B} = \{1, 2, 5\}$, and state which algorithm has to be used to restore optimality.
3. Execute one pivoting operation of that algorithm and determine the new optimal solution and the new optimal value.

Solution

- The optimal solution of the original LP problem violates the new constraint, since

$$\tilde{x}_1 + 2\tilde{x}_2 = 2 + 12 = 14 > 10.$$

- The slack variable of the new constraint is $x_5 = 10 - x_1 - 2x_2$. By substituting the equations of x_1 and x_2 of the dictionary, we obtain:

$$x_5 = 10 - \underbrace{(2 - x_3 + x_4)}_{=x_1} - 2 \underbrace{(6 + x_3 - 2x_4)}_{=x_2} = -4 - x_3 + 3x_4,$$

that is, $\bar{b}_5 = -4$, $\bar{a}_{53} = 1$ and $\bar{a}_{54} = -3$.

The extended basis $\mathcal{B} = \{1, 2, 5\}$ is still **dual feasible**, since the reduced costs $\bar{c}_3 = -1$ and $\bar{c}_4 = -2$ are unchanged, but it is not primal feasible, since $\bar{b}_5 = -4 < 0$. Accordingly, we use the **dual simplex algorithm**.

- The pivot row is x_5 . The only negative entry of its row is $\bar{a}_{54} = -3$, and then the ratio test gives

$$\frac{\bar{c}_4}{\bar{a}_{54}} = \frac{-2}{-3} = \frac{2}{3}$$

and the pivot column is x_4 . After this single dual pivoting operation the basis becomes primal feasible, and the new optimal solution is

$$\tilde{\mathbf{x}}(LP) = \begin{pmatrix} \frac{10}{3} \\ \frac{10}{3} \end{pmatrix} \quad \text{with} \quad z(LP) = 4 \cdot \frac{10}{3} + 3 \cdot \frac{10}{3} = \frac{70}{3}.$$

Adding a constraint cannot enlarge the feasible region, and then the optimal value of a maximization problem cannot increase: indeed $\frac{70}{3} < 26$.

The geometry of Linear Programming

Exercise 15.1 — The geometry of Linear Programming (*P2*)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & & x_1 & + & 3 & x_2 \\ \text{subject to} & -4 & x_1 & + & 10 & x_2 \leq 35 \\ & & 3 & x_1 & + & x_2 \leq 12 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

Questions

1. Determine the dimension of the feasible region F and say whether it is full-dimensional.
2. Determine the facets and the vertices of F .
3. Write its standard maximization form, enumerate all the bases and say which of them are associated with a basic feasible solution. Check the correspondence with the vertices found at the previous question.

Solution

- The feasible region is

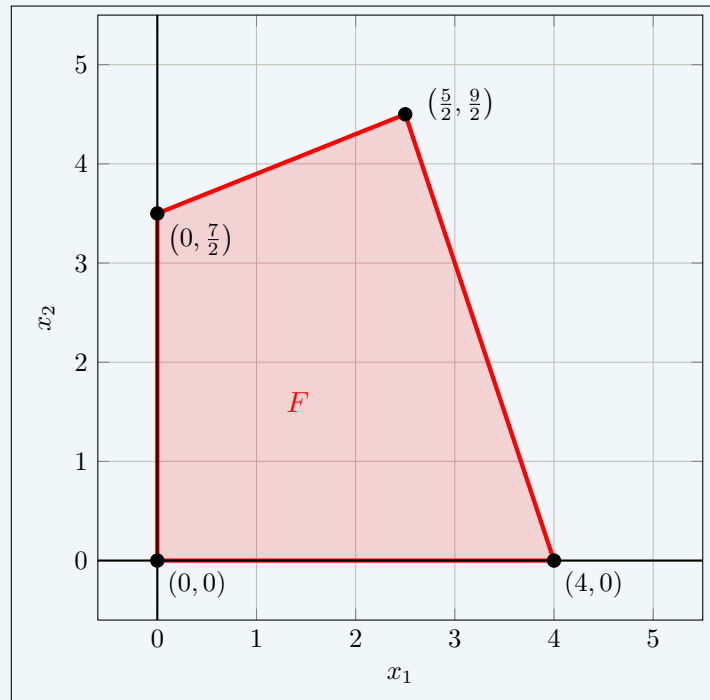
$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -4x_1 + 10x_2 \leq 35, \quad 3x_1 + x_2 \leq 12, \quad x_1 \geq 0, \quad x_2 \geq 0 \right\}.$$

The point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ satisfies all four constraints strictly, and then a small disc around it

is contained in F : we have $\dim(F) = 2$ and F is **full-dimensional**.

- Since $\dim(F) = 2$, its facets are the faces of dimension 1, that is, its edges, and its vertices are the faces of dimension 0. Each of the four constraints defines one facet, and the vertices are

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{5}{2} \\ \frac{9}{2} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \frac{7}{2} \end{pmatrix}.$$



- By adding $m = 2$ slack variables we obtain the standard maximization form with $n + m = 4$ non-negative variables and $m = 2$ equality linear constraints. A basis is a subset of 2 indices out of 4, and there are $\binom{4}{2} = 6$ such subsets, all of them giving an invertible matrix:

$\mathcal{B}$	$\mathbf{x}_{\mathcal{B}}$	feasible	vertex (x_1, x_2)
$\{1, 2\}$	$(\frac{5}{2}, \frac{9}{2})$	✓	$(\frac{5}{2}, \frac{9}{2})$
$\{1, 3\}$	$(4, 51)$	✓	$(4, 0)$
$\{1, 4\}$	$(-\frac{35}{4}, \frac{153}{4})$		
$\{2, 3\}$	$(12, -85)$		
$\{2, 4\}$	$(\frac{7}{2}, \frac{17}{2})$	✓	$(0, \frac{7}{2})$
$\{3, 4\}$	$(35, 12)$	✓	$(0, 0)$

Exactly 4 of the 6 bases are feasible, and they correspond to the 4 vertices found at the previous question. The correspondence is **one-to-one**, since no basic feasible solution is degenerate.

Exercise 15.2 — The geometry of Linear Programming (G2)

Consider the following LP problem in standard maximization form with $n = 3$ non-negative variables and $m = 1$ equality linear constraint:

$$\begin{array}{llllll} \max & 2 & x_1 & + & 2 & x_2 & + & x_3 \\ \text{subject to} & 2 & x_1 & + & 3 & x_2 & + & x_3 & = & 4 \\ & & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

Questions

1. Determine the dimension of the feasible region F and say whether it is full-dimensional.
2. Determine the vertices of F by enumerating all the bases.
3. Determine the set of the optimal solutions and say whether it is a face of F .

Solution

- All the points of F satisfy the equality constraint $2x_1 + 3x_2 + x_3 = 4$, and then $\text{aff}(F)$ is contained in that plane:

$$\dim(F) \leq n - \text{rank}(\mathbf{A}) = 3 - 1 = 2.$$

The three points $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \frac{4}{3} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ belong to F and are not aligned, and then $\dim(F) = 2$. The feasible region is therefore not full-dimensional, as it always happens for the standard maximization form.

- A basis is a subset of $m = 1$ index out of $n = 3$, and there are 3 such subsets, all of them giving an invertible matrix:

$\mathcal{B}$	$\mathbf{x}_{\mathcal{B}}$	feasible	vertex
$\{1\}$	(2)	✓	$(2, 0, 0)$
$\{2\}$	$(\frac{4}{3})$	✓	$(0, \frac{4}{3}, 0)$
$\{3\}$	(4)	✓	$(0, 0, 4)$

All three bases are feasible, and then F has exactly three vertices: it is a triangle.

- The objective values at the three vertices are

$$2 \cdot 2 = 4, \quad 2 \cdot \frac{4}{3} = \frac{8}{3}, \quad 1 \cdot 4 = 4,$$

and then the optimal value is $z(LP) = 4$, attained at two vertices. By the observation on multiple optimal solutions, every point of the segment joining them is optimal:

$$\left\{ \lambda \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} : \lambda \in [0, 1] \right\}.$$

This set is the **face** of F defined by the valid inequality $2x_1 + 2x_2 + x_3 \leq 4$, and it has dimension 1: it is an edge of the triangle.

Exercise 15.3 — The geometry of Linear Programming (G3)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 4$ constraints:

$$\begin{array}{llllll} \max & & x_1 & + & 2 & x_2 \\ \text{subject to} & & x_1 & + & & x_2 \leq 2 \\ & & - & x_1 & - & x_2 \leq -2 \\ & & & x_1 & - & x_2 \leq 0 \\ & & - & x_1 & + & x_2 \leq 0 \\ & & & x_1, & & x_2 \geq 0 \end{array}$$

Questions

1. Determine the feasible region F and its dimension.
2. Say how many bases of the standard maximization form are associated with the only point of F , and explain why the correspondence between bases and vertices is not one-to-one.

Solution

- The first two constraints impose $x_1 + x_2 \leq 2$ and $x_1 + x_2 \geq 2$, that is, $x_1 + x_2 = 2$. The last two impose $x_1 \leq x_2$ and $x_1 \geq x_2$, that is, $x_1 = x_2$. The two equalities give

$x_1 = x_2 = 1$, and then

$$F = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \dim(F) = 0.$$

The single point of F is therefore a vertex (here the vertex is the face $G = F$ itself, of dimension 0), and it is the optimal solution, with optimal value $z(LP) = 1 + 2 = 3$.

- In standard maximization form the problem has $n + m = 6$ non-negative variables and $m = 4$ equality linear constraints, and a basis is a subset of 4 indices out of 6. At the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ all four constraints are binding, and then all four slack variables take the value 0. Since $x_1 = x_2 = 1 > 0$, every basis associated with that point contains the indices 1 and 2 of the two structural variables, together with two of the four slack variables. Out of the $\binom{4}{2} = 6$ choices, $\{1, 2, 3, 4\}$ and $\{1, 2, 5, 6\}$ give a singular matrix $\mathbf{A}_B$ (the columns of x_1 and x_2 restricted to the rows of the other two slack variables are linearly dependent), while the remaining four are bases:

$$\{1, 2, 3, 5\}, \quad \{1, 2, 3, 6\}, \quad \{1, 2, 4, 5\}, \quad \{1, 2, 4, 6\},$$

that is, **four** different bases give the same vertex. In each of them two basic variables take the value 0: the basic feasible solution is **degenerate**.

Every basic feasible solution is a vertex, but a degenerate vertex may be associated with several bases, as in this case: the correspondence between bases and vertices is onto, not one-to-one.