



SAPIENZA
UNIVERSITÀ DI ROMA

Linear Programming

Exercises 3 — Duality and the dual simplex

Worked exercises on chapters 9–10: building the dual, graphical solution of primal and dual, complementary slackness, primal-dual outcomes, the dual simplex algorithm.

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Academic year 2026–2027

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Duality theory

9.1 Building the dual

Exercise 9.1 — Dual problem (*P71*)

Consider the following primal LP problem in canonical maximization form with $n = 4$ variables and $m = 3$ constraints:

$$\begin{array}{rcllclclclcl}
 \max & 4 & x_1 & + & & x_2 & + & 5 & x_3 & + & 3 & x_4 \\
 \text{subject to} & & x_1 & - & & x_2 & - & & x_3 & + & 3 & x_4 & \leq & 1 \\
 & & 5 & x_1 & + & & x_2 & + & 3 & x_3 & + & 8 & x_4 & \leq & 55 \\
 & & - & x_1 & + & 2 & x_2 & + & 3 & x_3 & - & 5 & x_4 & \leq & 3 \\
 & & & x_1, & & x_2, & & x_3, & & x_4 & & \geq & 0
 \end{array}$$

The optimal dictionary of LP problem in standard form is:

$$\left\{ \begin{array}{l}
 x_2 = 14 - 2x_1 - 4x_3 - 5x_5 - 3x_7 \\
 x_4 = 5 - x_1 - x_3 - 2x_5 - x_7 \\
 x_6 = 1 + 5x_1 + 9x_3 + 21x_5 + 11x_7 \\
 z = 29 - x_1 - 2x_3 - 11x_5 - 6x_7
 \end{array} \right.$$

Questions

1. Compute lower bounds and upper bounds on the optimal solution value $z(\text{LP})$ of the LP problem.
2. Compute the linear combination of the three constraints using the multipliers π_1 , π_2 and π_3 . Give the necessary conditions under which the RHS of the linear combination provides an upper bound for the optimal solution value $z(\text{LP})$ of the LP problem.

3. Write the dual problem of the LP problem. Then, using the optimal dictionary, find an optimal solution of the dual problem.

Solution

1. A lower bound on the optimal solution value is given, for example, by the following feasible solution:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 1, \quad \bar{x}_3 = 2, \quad \bar{x}_4 = 1 \quad \text{with value} \quad 14.$$

Hence, we have $z(\text{LP}) \geq 14$.

We can improve the lower bound using, for example, the following feasible solution:

$$\bar{x}_1 = 5, \quad \bar{x}_2 = 4, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{with value} \quad 24.$$

Hence, we have $z(\text{LP}) \geq 24$.

An upper bound on the optimal solution value can be obtained, for example, multiplying the second constraint by $\frac{5}{3}$:

$$\underbrace{\frac{25}{3} x_1 + \frac{5}{3} x_2 + 5x_3 + \frac{40}{3} x_4}_{\geq 4x_1 + x_2 + 5x_3 + 3x_4} \leq \frac{275}{3}$$

Hence, we have $z(\text{LP}) \leq \frac{275}{3}$.

We can improve the upper bound using, for example, summing up the second and third constraints:

$$\underbrace{4x_1 + 3x_2 + 6x_3 + 3x_4}_{\geq 4x_1 + x_2 + 5x_3 + 3x_4} \leq 58$$

Hence, we have $z(\text{LP}) \leq 58$.

2. Multiplying each constraint by its multiplier we get:

$$(x_1 - x_2 - x_3 + 3x_4) \pi_1 \leq 1 \pi_1$$

$$(5x_1 + x_2 + 3x_3 + 8x_4) \pi_2 \leq 55 \pi_2$$

$$(-x_1 + 2x_2 + 3x_3 - 5x_4) \pi_3 \leq 3 \pi_3$$

The linear combination of the three constraints generates the following inequality:

$$\begin{aligned} & (\pi_1 + 5\pi_2 - \pi_3) x_1 + (-\pi_1 + \pi_2 + 2\pi_3) x_2 + \\ & + (-\pi_1 + 3\pi_2 + 3\pi_3) x_3 + (3\pi_1 + 8\pi_2 - 5\pi_3) x_4 \leq \\ & \leq \pi_1 + 55\pi_2 + 3\pi_3 \end{aligned}$$

The RHS $\pi_1 + 55\pi_2 + 3\pi_3$ of this inequality is an upper bound for $z(\text{LP})$ if and only if:

$$\begin{aligned} \pi_1 + 5\pi_2 - \pi_3 &\geq 4 \\ -\pi_1 + \pi_2 + 2\pi_3 &\geq 1 \\ -\pi_1 + 3\pi_2 + 3\pi_3 &\geq 5 \\ 3\pi_1 + 8\pi_2 - 5\pi_3 &\geq 3 \\ \pi_1, \pi_2, \pi_3 &\geq 0 \end{aligned}$$

3. The dual problem of the LP problem with $m = 3$ non-negative variables and $n = 4$ greater-than-or-equal-to constraints reads as follows:

$$\begin{aligned} \min \quad & \pi_1 + 55\pi_2 + 3\pi_3 \\ \text{subject to} \quad & \pi_1 + 5\pi_2 - \pi_3 \geq 4 \\ & -\pi_1 + \pi_2 + 2\pi_3 \geq 1 \\ & -\pi_1 + 3\pi_2 + 3\pi_3 \geq 5 \\ & 3\pi_1 + 8\pi_2 - 5\pi_3 \geq 3 \\ & \pi_1, \pi_2, \pi_3 \geq 0 \end{aligned}$$

An optimal dual solution is given by

$$\tilde{\pi}_i = -\hat{c}_{n+i}, \quad \forall i \in \{1, 2, \dots, m\} \quad \text{where} \quad \hat{c}_j = \begin{cases} \bar{c}_j & \text{if } j \in \mathcal{N} \\ 0 & \text{otherwise} \end{cases}$$

The values $\bar{c}_j$ are the reduced costs of the non-basic variables.

In the optimal dictionary we have $\mathcal{N} = \{1, 3, 5, 7\}$, accordingly:

$$\hat{c}_5 = -11, \quad \hat{c}_6 = 0, \quad \hat{c}_7 = -6 \quad \text{and} \quad \tilde{\pi}_1 = 11, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 6$$

is an optimal solution of the dual problem.

9.2 Graphical solution of primal and dual

Exercise 9.2 — Primal and dual problems ($P2$)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & & x_1 & + & 3 & x_2 \\ \text{subject to} & -4 & x_1 & + & 10 & x_2 \leq 35 \\ & & 3 & x_1 & + & x_2 \leq 12 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$ constraints:

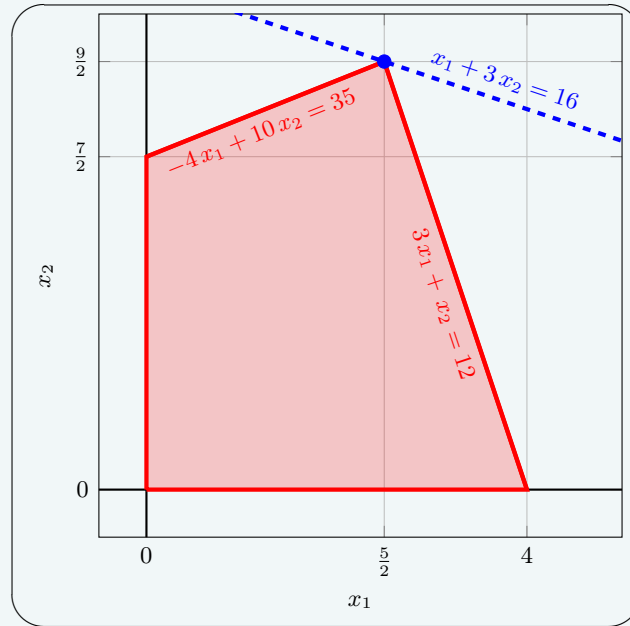
$$\begin{array}{llllll} \min & 35 & \pi_1 & + & 12 & \pi_2 \\ \text{subject to} & -4 & \pi_1 & + & 3 & \pi_2 \geq 1 \\ & 10 & \pi_1 & + & & \pi_2 \geq 3 \\ & & \pi_1, & & \pi_2 & \geq 0 \end{array}$$

Questions

1. Find graphically an optimal solution of the primal problem and an optimal solution of the dual problem. Compute the optimal values of the two problems.

Solution

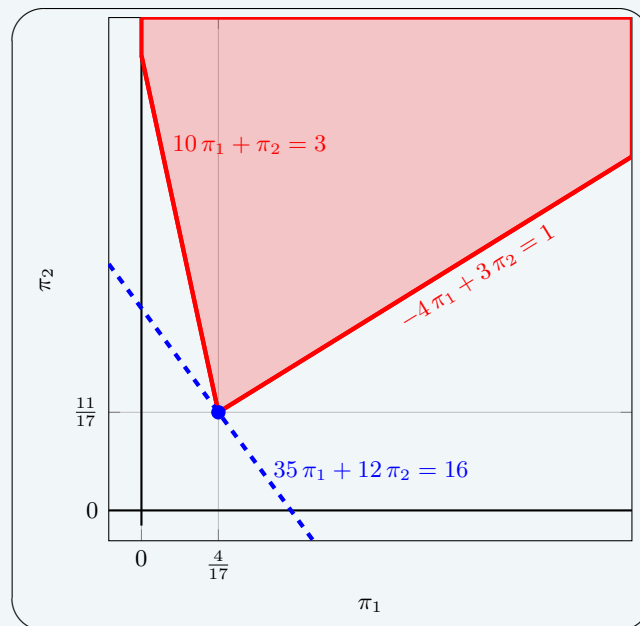
- For the primal LP problem, we have:



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = \frac{9}{2} \quad \text{and} \quad z(LP) = \tilde{x}_1 + 3\tilde{x}_2 = \frac{5}{2} + 3 \cdot \frac{9}{2} = 16$$

- For the dual LP problem, we have:



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{4}{17}, \quad \tilde{\pi}_2 = \frac{11}{17} \quad \text{and} \quad z(D(LP)) = 35\tilde{\pi}_1 + 12\tilde{\pi}_2 = 35 \cdot \frac{4}{17} + 12 \cdot \frac{11}{17} = 16$$

- The two optimal values coincide, $z(LP) = z(D(LP)) = 16$, as stated by the Strong Duality theorem.

Exercise 9.3 — Primal and dual problems (P3)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & 2 & x_1 & + & 4 & x_2 \\ \text{subject to} & - & x_1 & + & & x_2 & \leq & 2 \\ & 8 & x_1 & + & 2 & x_2 & \leq & 19 \\ & & x_1, & & & x_2 & \geq & 0 \end{array}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$ constraints:

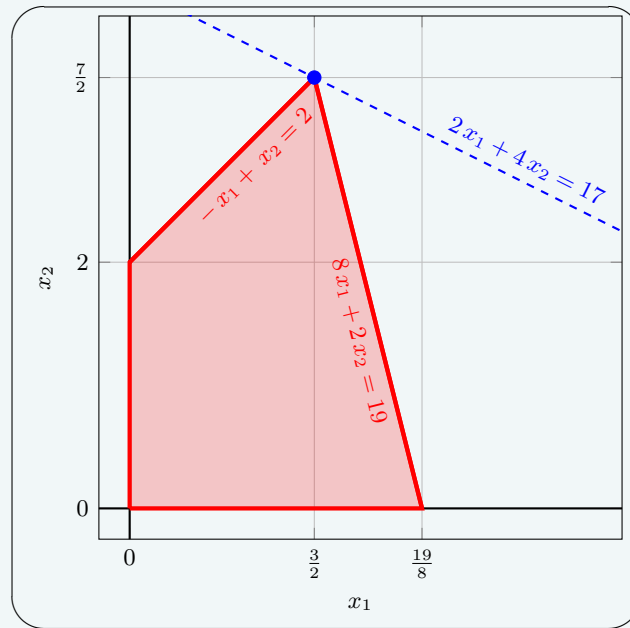
$$\begin{array}{llllll} \min & 2 & \pi_1 & + & 19 & \pi_2 \\ \text{subject to} & - & \pi_1 & + & 8 & \pi_2 & \geq & 2 \\ & & \pi_1 & + & 2 & \pi_2 & \geq & 4 \\ & & \pi_1, & & & \pi_2 & \geq & 0 \end{array}$$

Questions

1. Find graphically an optimal solution of the primal problem and an optimal solution of the dual problem. Compute the optimal values of the two problems.

Solution

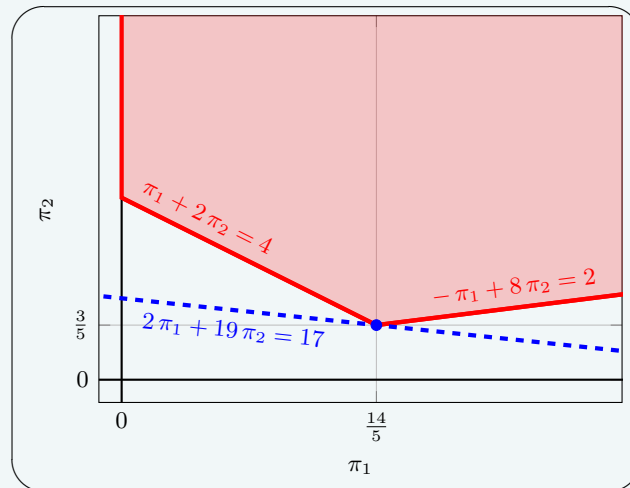
- For the primal LP problem, we have:



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{3}{2}, \quad \tilde{x}_2 = \frac{7}{2} \quad \text{and} \quad z(LP) = 2\tilde{x}_1 + 4\tilde{x}_2 = 2 \cdot \frac{3}{2} + 4 \cdot \frac{7}{2} = 17$$

- For the dual LP problem, we have:



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{14}{5}, \quad \tilde{\pi}_2 = \frac{3}{5} \quad \text{and} \quad z(D(LP)) = 2\tilde{\pi}_1 + 19\tilde{\pi}_2 = 2 \cdot \frac{14}{5} + 19 \cdot \frac{3}{5} = 17$$

- The two optimal values coincide, $z(LP) = z(D(LP)) = 17$, as stated by the Strong Duality theorem.

Exercise 9.4 — Primal and dual problems (P4)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 5x_1 + 3x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 14 \\ & x_1 + 8x_2 \leq 32 \\ & x_1, x_2 \geq 0 \end{aligned}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$ constraints:

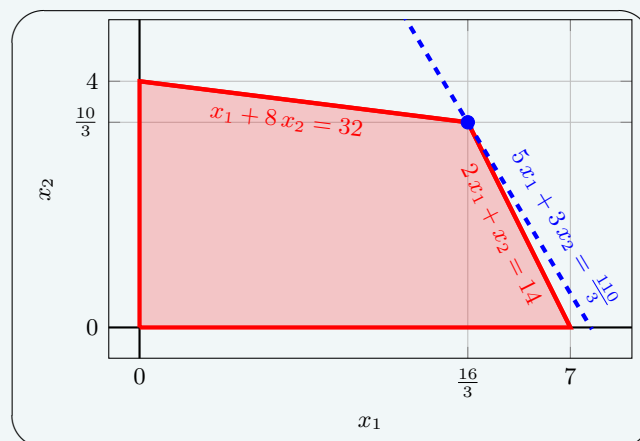
$$\begin{aligned} \min \quad & 14\pi_1 + 32\pi_2 \\ \text{subject to} \quad & 2\pi_1 + \pi_2 \geq 5 \\ & \pi_1 + 8\pi_2 \geq 3 \\ & \pi_1, \pi_2 \geq 0 \end{aligned}$$

Questions

1. Find graphically an optimal solution of the primal problem and an optimal solution of the dual problem. Compute the optimal values of the two problems.

Solution

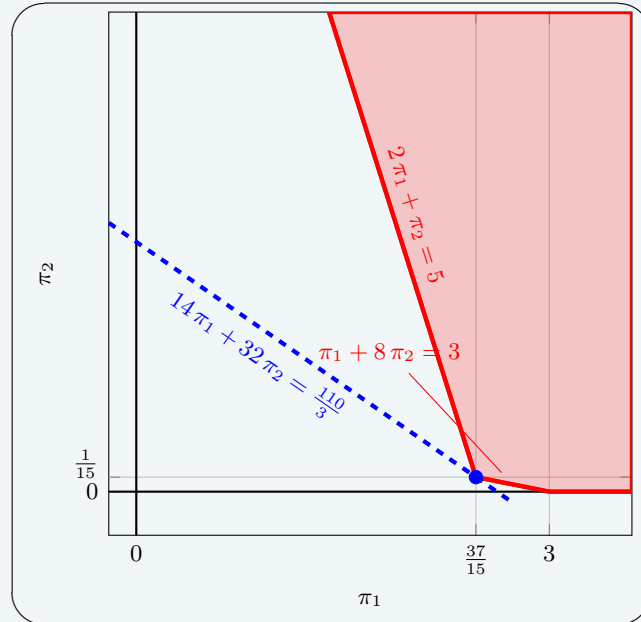
- For the primal LP problem, we have:



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{16}{3}, \quad \tilde{x}_2 = \frac{10}{3} \quad \text{and} \quad z(LP) = 5\tilde{x}_1 + 3\tilde{x}_2 = 5 \cdot \frac{16}{3} + 3 \cdot \frac{10}{3} = \frac{110}{3}$$

- For the dual LP problem, we have:



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{37}{15}, \quad \tilde{\pi}_2 = \frac{1}{15} \quad \text{and} \quad z(D(LP)) = 14\tilde{\pi}_1 + 32\tilde{\pi}_2 = 14 \cdot \frac{37}{15} + 32 \cdot \frac{1}{15} = \frac{110}{3}$$

- The two optimal values coincide, $z(LP) = z(D(LP)) = \frac{110}{3}$, as stated by the Strong Duality theorem.

Exercise 9.5 — Primal and dual problems (P5)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{subject to} \quad & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1, x_2 \geq 0 \end{aligned}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$ constraints:

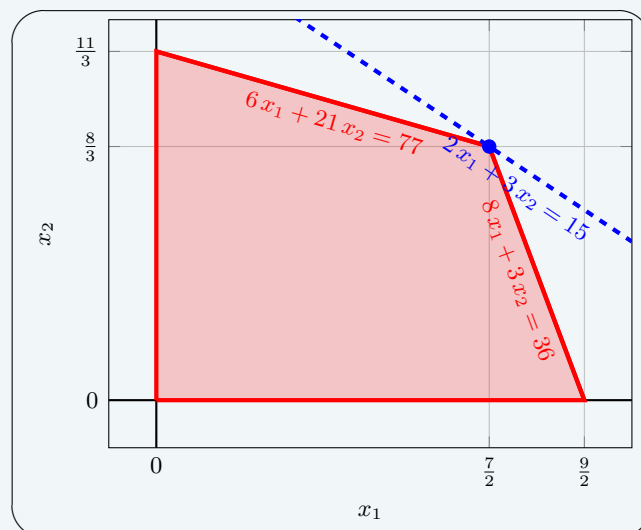
$$\begin{aligned}
 \min \quad & 77 \pi_1 + 36 \pi_2 \\
 \text{subject to} \quad & 6 \pi_1 + 8 \pi_2 \geq 2 \\
 & 21 \pi_1 + 3 \pi_2 \geq 3 \\
 & \pi_1, \pi_2 \geq 0
 \end{aligned}$$

Questions

1. Find graphically an optimal solution of the primal problem and an optimal solution of the dual problem. Compute the optimal values of the two problems.

Solution

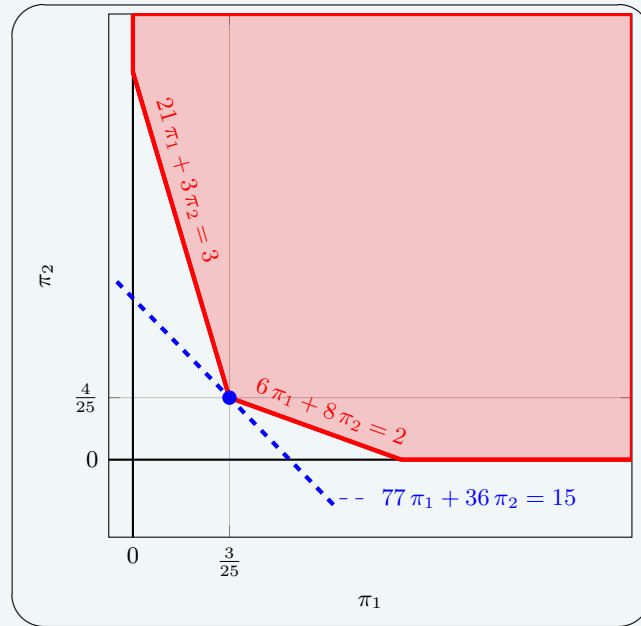
- For the primal LP problem, we have:



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{7}{2}, \quad \tilde{x}_2 = \frac{8}{3} \quad \text{and} \quad z(LP) = 2\tilde{x}_1 + 3\tilde{x}_2 = 2 \cdot \frac{7}{2} + 3 \cdot \frac{8}{3} = 15$$

- For the dual LP problem, we have:



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{3}{25}, \quad \tilde{\pi}_2 = \frac{4}{25} \quad \text{and} \quad z(D(LP)) = 77 \tilde{\pi}_1 + 36 \tilde{\pi}_2 = 77 \cdot \frac{3}{25} + 36 \cdot \frac{4}{25} = 15$$

- The two optimal values coincide, $z(LP) = z(D(LP)) = 15$, as stated by the Strong Duality theorem.

Exercise 9.6 — Primal and dual problems (P6)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{subject to} \quad & 4x_1 + 15x_2 \leq 70 \\ & 12x_1 + 3x_2 \leq 52 \\ & x_1, x_2 \geq 0 \end{aligned}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$ constraints:

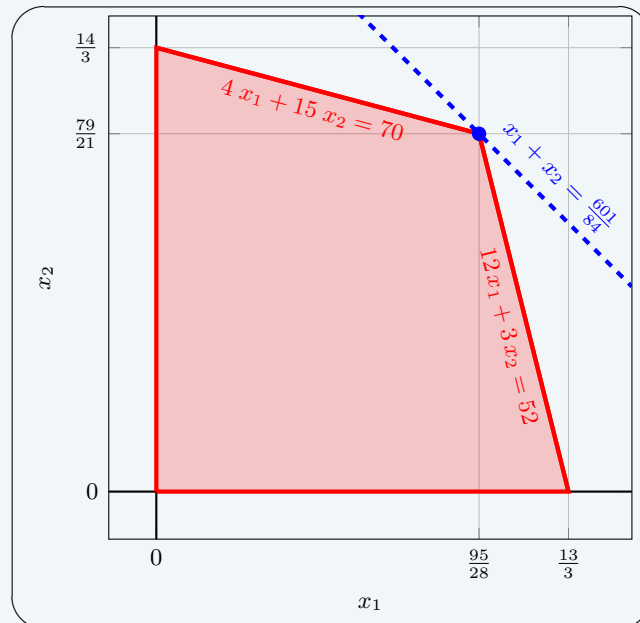
$$\begin{aligned} \min \quad & 70\pi_1 + 52\pi_2 \\ \text{subject to} \quad & 4\pi_1 + 12\pi_2 \geq 1 \\ & 15\pi_1 + 3\pi_2 \geq 1 \\ & \pi_1, \pi_2 \geq 0 \end{aligned}$$

Questions

1. Find graphically an optimal solution of the primal problem and an optimal solution of the dual problem. Compute the optimal values of the two problems.

Solution

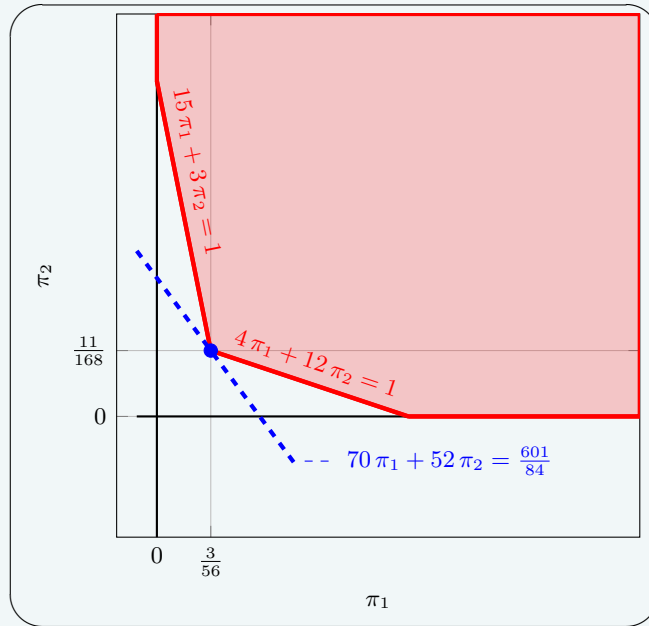
- For the primal LP problem, we have:



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{95}{28}, \quad \tilde{x}_2 = \frac{79}{21} \quad \text{and} \quad z(LP) = \tilde{x}_1 + \tilde{x}_2 = \frac{95}{28} + \frac{79}{21} = \frac{601}{84}$$

- For the dual LP problem, we have:



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{3}{56}, \quad \tilde{\pi}_2 = \frac{11}{168} \quad \text{and} \quad z(D(LP)) = 70 \tilde{\pi}_1 + 52 \tilde{\pi}_2 = 70 \cdot \frac{3}{56} + 52 \cdot \frac{11}{168} = \frac{601}{84}$$

- The two optimal values coincide, $z(LP) = z(D(LP)) = \frac{601}{84}$, as stated by the Strong Duality theorem.

Exercise 9.7 — Primal and dual problems (P7)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 3x_1 + x_2 \\ \text{subject to} \quad & x_1 + 4x_2 \leq 12 \\ & 2x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$ constraints:

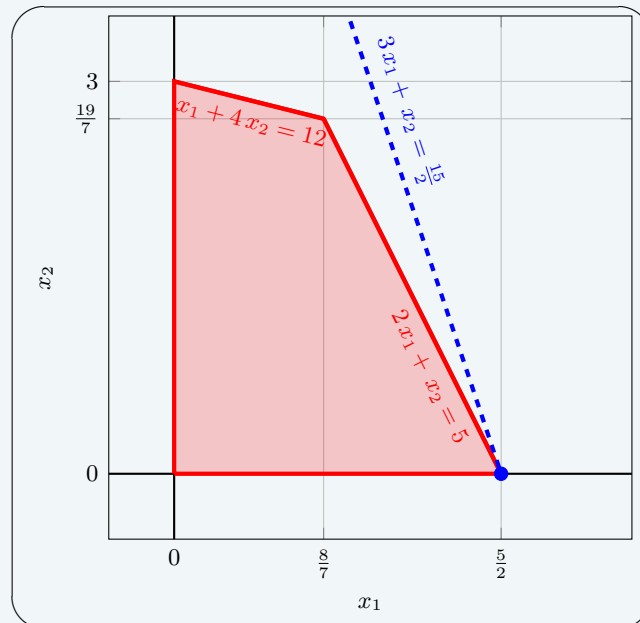
$$\begin{aligned} \min \quad & 12\pi_1 + 5\pi_2 \\ \text{subject to} \quad & \pi_1 + 2\pi_2 \geq 3 \\ & 4\pi_1 + \pi_2 \geq 1 \\ & \pi_1, \pi_2 \geq 0 \end{aligned}$$

Questions

1. Find graphically an optimal solution of the primal problem and an optimal solution of the dual problem. Compute the optimal values of the two problems.

Solution

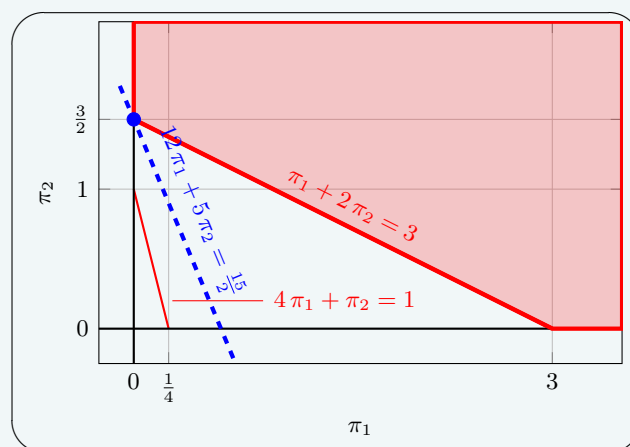
- For the primal LP problem, we have:



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = 0 \quad \text{and} \quad z(LP) = 3\tilde{x}_1 + \tilde{x}_2 = 3 \cdot \frac{5}{2} + 0 = \frac{15}{2}$$

- For the dual LP problem, we have:



An optimal dual LP solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = \frac{3}{2} \quad \text{and} \quad z(D(LP)) = 12 \tilde{\pi}_1 + 5 \tilde{\pi}_2 = 12 \cdot 0 + 5 \cdot \frac{3}{2} = \frac{15}{2}$$

- The two optimal values coincide, $z(LP) = z(D(LP)) = \frac{15}{2}$, as stated by the Strong Duality theorem.

Exercise 9.8 — Primal and dual problems (P8)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & & x_1 & + & 2 & x_2 \\ \text{subject to} & - & x_1 & + & 2 & x_2 \leq 2 \\ & & x_1 & + & & x_2 \leq 3 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$ constraints:

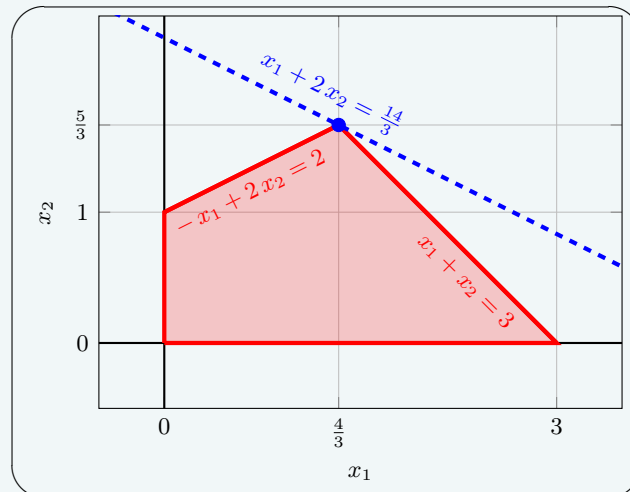
$$\begin{array}{llllll} \min & 2 & \pi_1 & + & 3 & \pi_2 \\ \text{subject to} & - & \pi_1 & + & & \pi_2 \geq 1 \\ & 2 & \pi_1 & + & & \pi_2 \geq 2 \\ & & \pi_1, & & \pi_2 & \geq 0 \end{array}$$

Questions

1. Find graphically an optimal solution of the primal problem and an optimal solution of the dual problem. Compute the optimal values of the two problems.

Solution

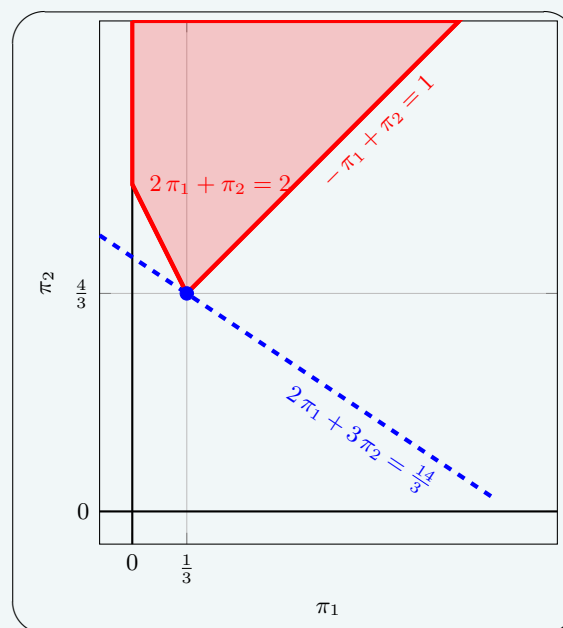
- For the primal LP problem, we have:



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{4}{3}, \quad \tilde{x}_2 = \frac{5}{3} \quad \text{and} \quad z(LP) = \tilde{x}_1 + 2\tilde{x}_2 = \frac{4}{3} + 2 \cdot \frac{5}{3} = \frac{14}{3}$$

- For the dual LP problem, we have:



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{1}{3}, \quad \tilde{\pi}_2 = \frac{4}{3} \quad \text{and} \quad z(D(LP)) = 2\tilde{\pi}_1 + 3\tilde{\pi}_2 = 2 \cdot \frac{1}{3} + 3 \cdot \frac{4}{3} = \frac{14}{3}$$

- The two optimal values coincide, $z(LP) = z(D(LP)) = \frac{14}{3}$, as stated by the Strong Duality theorem.

Exercise 9.9 — Primal and dual problems ($P9$)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & 2 & x_1 & + & & x_2 \\ \text{subject to} & - & x_1 & + & & x_2 & \leq & 1 \\ & 6 & x_1 & + & 4 & x_2 & \leq & 39 \\ & & x_1, & & & x_2 & \geq & 0 \end{array}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$ constraints:

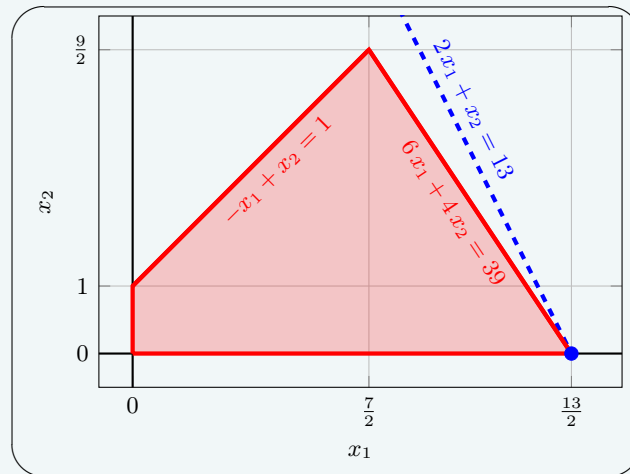
$$\begin{array}{llllll} \min & & \pi_1 & + & 39 & \pi_2 \\ \text{subject to} & - & \pi_1 & + & 6 & \pi_2 & \geq & 2 \\ & & \pi_1 & + & 4 & \pi_2 & \geq & 1 \\ & & \pi_1, & & & \pi_2 & \geq & 0 \end{array}$$

Questions

1. Find graphically an optimal solution of the primal problem and an optimal solution of the dual problem. Compute the optimal values of the two problems.

Solution

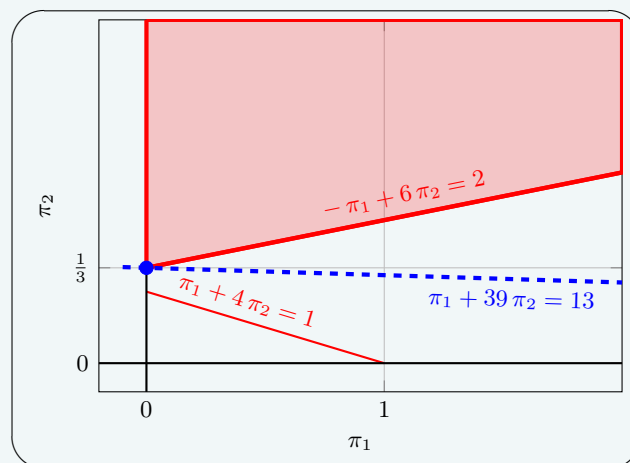
- For the primal LP problem, we have:



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{13}{2}, \quad \tilde{x}_2 = 0 \quad \text{and} \quad z(LP) = 2\tilde{x}_1 + \tilde{x}_2 = 2 \cdot \frac{13}{2} + 0 = 13$$

- For the dual LP problem, we have:



An optimal dual LP solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = \frac{1}{3} \quad \text{and} \quad z(D(LP)) = \tilde{\pi}_1 + 39\tilde{\pi}_2 = 0 + 39 \cdot \frac{1}{3} = 13$$

- The two optimal values coincide, $z(LP) = z(D(LP)) = 13$, as stated by the Strong Duality theorem.

Exercise 9.10 — Primal and dual problems (P10)

Consider the following primal problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{aligned}
 \max \quad & 5 x_1 + 2 x_2 \\
 \text{subject to} \quad & -x_1 + 3 x_2 \leq 12 \\
 & x_1 + 2 x_2 \leq 12 \\
 & 3 x_1 - 2 x_2 \leq 8 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

and its dual LP problem in canonical minimization form with $m = 3$ variables and $n = 2$ constraints:

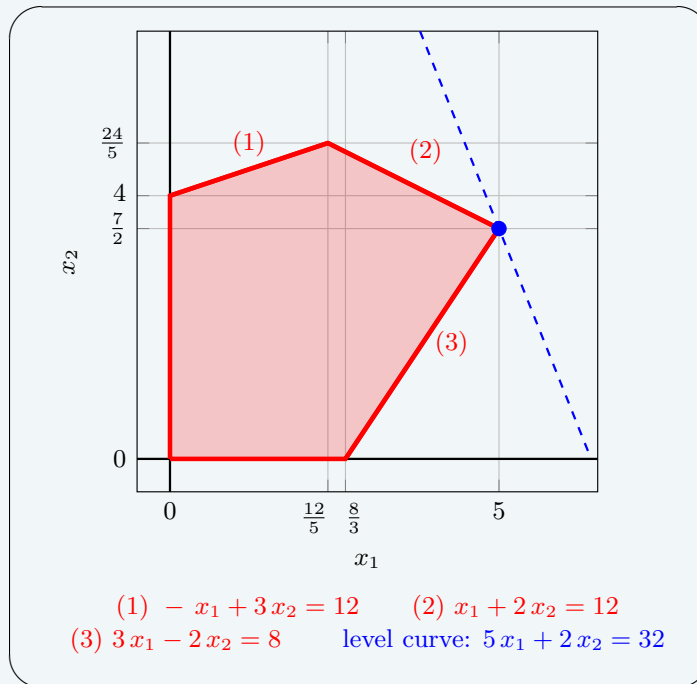
$$\begin{aligned}
 \min \quad & 12 \pi_1 + 12 \pi_2 + 8 \pi_3 \\
 \text{subject to} \quad & -\pi_1 + \pi_2 + 3 \pi_3 \geq 5 \\
 & 3 \pi_1 + 2 \pi_2 - 2 \pi_3 \geq 2 \\
 & \pi_1, \pi_2, \pi_3 \geq 0
 \end{aligned}$$

Questions

1. Find graphically an optimal solution of the primal problem and an optimal solution of the dual problem. Compute the optimal values of the two problems.

Solution

- For the primal LP problem we have:



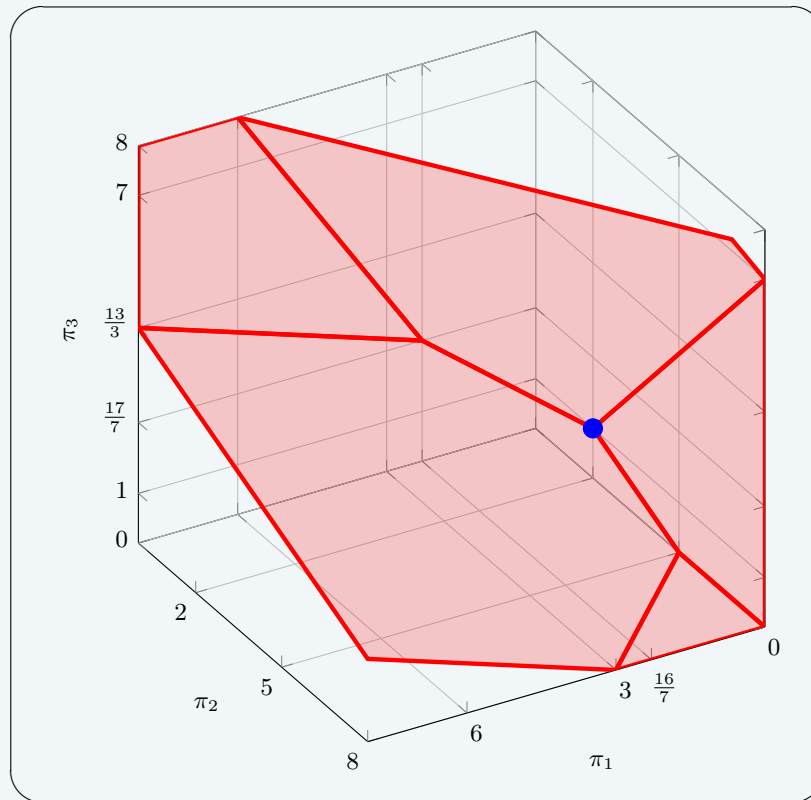
An optimal primal solution is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}$$

and the optimal value of the primal problem is:

$$z(\text{LP}) = 5\tilde{x}_1 + 2\tilde{x}_2 = 5 \cdot 5 + 2 \cdot \frac{7}{2} = 32$$

- For the dual LP problem we have:



The feasible region of the dual problem is unbounded, and the optimal solution is the vertex marked in blue:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 2, \quad \tilde{\pi}_3 = 1$$

and the optimal value of the dual problem is:

$$z(\text{D(LP)}) = 12 \tilde{\pi}_1 + 12 \tilde{\pi}_2 + 8 \tilde{\pi}_3 = 12 \cdot 0 + 12 \cdot 2 + 8 \cdot 1 = 32$$

- The two optimal values coincide, $z(\text{LP}) = z(\text{D(LP)}) = 32$, as stated by the Strong Duality theorem.

Exercise 9.11 — Primal and dual problems (P11)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & -2x_1 - 4x_2 \\ \text{subject to} \quad & -x_1 - x_2 \leq -2 \\ & x_1 - x_2 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$

constraints:

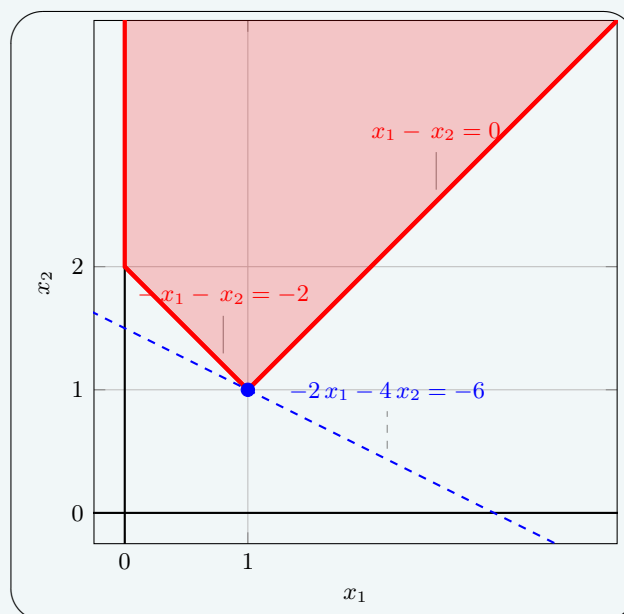
$$\begin{aligned}
 \min \quad & -2 \pi_1 \\
 \text{subject to} \quad & -\pi_1 + \pi_2 \geq -2 \\
 & -\pi_1 - \pi_2 \geq -4 \\
 & \pi_1, \pi_2 \geq 0
 \end{aligned}$$

Questions

1. Find graphically an optimal solution of the primal problem and an optimal solution of the dual problem. Compute the optimal values of the two problems.

Solution

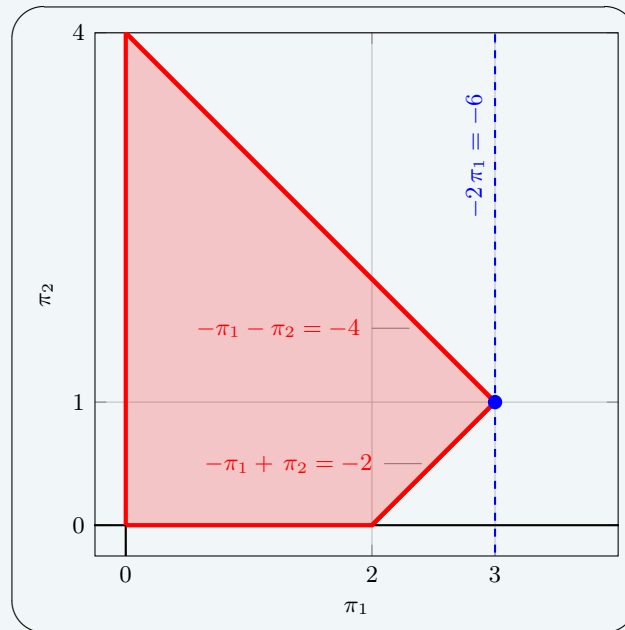
- For the primal LP problem, we have:



An optimal primal LP solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1 \quad \text{and} \quad z(\text{LP}) = -2\tilde{x}_1 - 4\tilde{x}_2 = -2 \cdot 1 - 4 \cdot 1 = -6$$

- For the dual LP problem, we have:



An optimal dual LP solution is:

$$\tilde{\pi}_1 = 3, \quad \tilde{\pi}_2 = 1 \quad \text{and} \quad z(\text{D(LP)}) = -2 \tilde{\pi}_1 + 0 \tilde{\pi}_2 = -2 \cdot 3 + 0 \cdot 1 = -6$$

- The two optimal values coincide, $z(\text{LP}) = z(\text{D(LP)}) = -6$, as stated by the Strong Duality theorem.

9.3 Complementary slackness

Exercise 9.12 — Complementary slackness ($P3$)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 2x_1 + 4x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 2 \\ & 8x_1 + 2x_2 \leq 19 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Questions

1. Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 2$$

is an optimal solution of the LP problem.

2. Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = \frac{3}{2}, \quad \bar{x}_2 = \frac{7}{2}$$

is an optimal solution of the LP problem.

Solution

- We check if the feasible solution

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 2$$

of the LP problem is optimal.

1. Since

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following linear equation with two variables:

$$\bar{\pi}_1 + 2\bar{\pi}_2 = 4 \quad (\bar{x}_2 = 2 > 0)$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij}\bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$-1 \cdot 0 + 1 \cdot 2 = 2 \not\leq 2$$

$$8 \cdot 0 + 2 \cdot 2 = 4 < 19 \quad \text{so} \quad \bar{\pi}_2 = 0$$

We then solve the previous equation with one variable and we get:

$$\bar{\pi}_1 = 4, \quad \bar{\pi}_2 = 0$$

3. We need to check

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$-1 \cdot 4 + 8 \cdot 0 = -4 \not\geq 2$$

$$1 \cdot 4 + 2 \cdot 0 = 4 \geq 4$$

so we stop the check and conclude that the solution is not optimal.

- We check if the feasible solution

$$\bar{x}_1 = \frac{3}{2}, \quad \bar{x}_2 = \frac{7}{2}$$

of the LP problem is optimal.

1. Since

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if } \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of two linear equations with two variables:

$$- \bar{\pi}_1 + 8 \bar{\pi}_2 = 2 \quad (\bar{x}_1 = \frac{3}{2} > 0)$$

$$\bar{\pi}_1 + 2 \bar{\pi}_2 = 4 \quad (\bar{x}_2 = \frac{7}{2} > 0)$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij} \bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$-1 \cdot \frac{3}{2} + 1 \cdot \frac{7}{2} = 2 \not\leq 2$$

$$8 \cdot \frac{3}{2} + 2 \cdot \frac{7}{2} = 19 \not\leq 19$$

We then solve the previous system of two equations with two variables and we get:

$$\bar{\pi}_1 = \frac{14}{5}, \quad \bar{\pi}_2 = \frac{3}{5}$$

3. We need to check

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$-1 \cdot \frac{14}{5} + 8 \cdot \frac{3}{5} = 2 \geq 2$$

$$1 \cdot \frac{14}{5} + 2 \cdot \frac{3}{5} = 4 \geq 4$$

4. We need to check

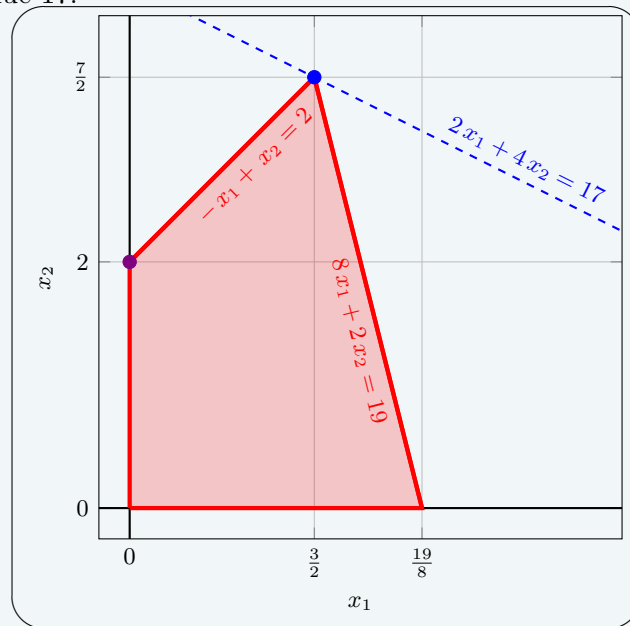
$$\bar{\pi}_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}$$

so since

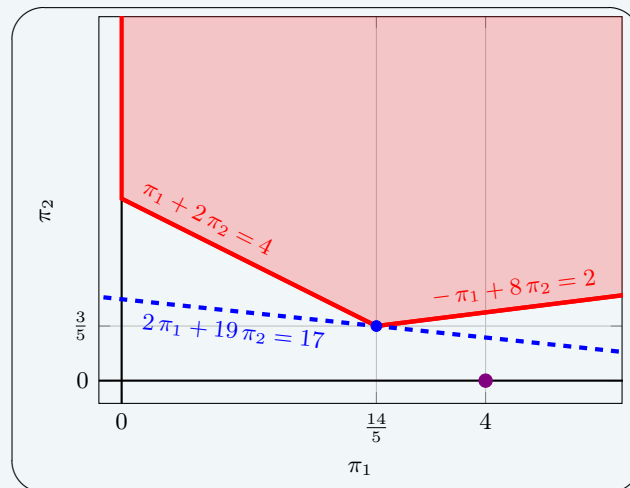
$$\bar{\pi}_1 = \frac{14}{5} \geq 0, \quad \bar{\pi}_2 = \frac{3}{5} \geq 0$$

the solution is optimal.

Graphical solution: the feasible region of the LP problem is the polyhedron shaded in red; the solution $\bar{x}_1 = 0, \bar{x}_2 = 2$, which is not optimal, is marked in violet and the optimal solution $\tilde{x}_1 = \frac{3}{2}, \tilde{x}_2 = \frac{7}{2}$, of value 17, is marked in blue, together with the level curve of value 17.



The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. The dual solution $\bar{\pi}_1 = 4, \bar{\pi}_2 = 0$ obtained from the first solution (in violet) violates the first dual constraint, while the dual solution $\bar{\pi}_1 = \frac{14}{5}, \bar{\pi}_2 = \frac{3}{5}$ obtained from the second solution (in blue) is feasible and has value 17, equal to the value of the primal solution.



Exercise 9.13 — Complementary slackness (P10)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{array}{ll}
 \max & 5 x_1 + 2 x_2 \\
 \text{subject to} & - x_1 + 3 x_2 \leq 12 \\
 & x_1 + 2 x_2 \leq 12 \\
 & 3 x_1 - 2 x_2 \leq 8 \\
 & x_1, x_2 \geq 0
 \end{array}$$

Questions

1. Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = \frac{8}{3}, \quad \bar{x}_2 = 0$$

is an optimal solution of the LP problem.

2. Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = 3, \quad \bar{x}_2 = 4$$

is an optimal solution of the LP problem.

Solution

- We check if the feasible solution

$$\bar{x}_1 = \frac{8}{3}, \quad \bar{x}_2 = 0$$

of the LP problem is optimal.

1. Since

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of one linear equation with three variables:

$$- \bar{\pi}_1 + \bar{\pi}_2 + 3 \bar{\pi}_3 = 5 \quad (\bar{x}_1 = \frac{8}{3} > 0)$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij} \bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$\begin{aligned} -1 \cdot \frac{8}{3} + 3 \cdot 0 &= -\frac{8}{3} < 12 & \text{so} & \bar{\pi}_1 = 0 \\ 1 \cdot \frac{8}{3} + 2 \cdot 0 &= \frac{8}{3} < 12 & \text{so} & \bar{\pi}_2 = 0 \\ 3 \cdot \frac{8}{3} - 2 \cdot 0 &= 8 \not< 8 \end{aligned}$$

We then solve the previous equation with one variable and we get:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = \frac{5}{3}$$

3. We need to check

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$\begin{aligned} -1 \cdot 0 + 1 \cdot 0 + 3 \cdot \frac{5}{3} &= 5 \geq 5 \\ 3 \cdot 0 + 2 \cdot 0 - 2 \cdot \frac{5}{3} &= -\frac{10}{3} \not\geq 2 \end{aligned}$$

Since the second dual constraint is violated, we stop the check and conclude that the solution is not optimal.

- We check if the feasible solution

$$\bar{x}_1 = 3, \quad \bar{x}_2 = 4$$

of the LP problem is optimal.

1. Since

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of two linear equations with three variables:

$$\begin{aligned} -\bar{\pi}_1 + \bar{\pi}_2 + 3\bar{\pi}_3 &= 5 & (\bar{x}_1 = 3 > 0) \\ 3\bar{\pi}_1 + 2\bar{\pi}_2 - 2\bar{\pi}_3 &= 2 & (\bar{x}_2 = 4 > 0) \end{aligned}$$

2. Since

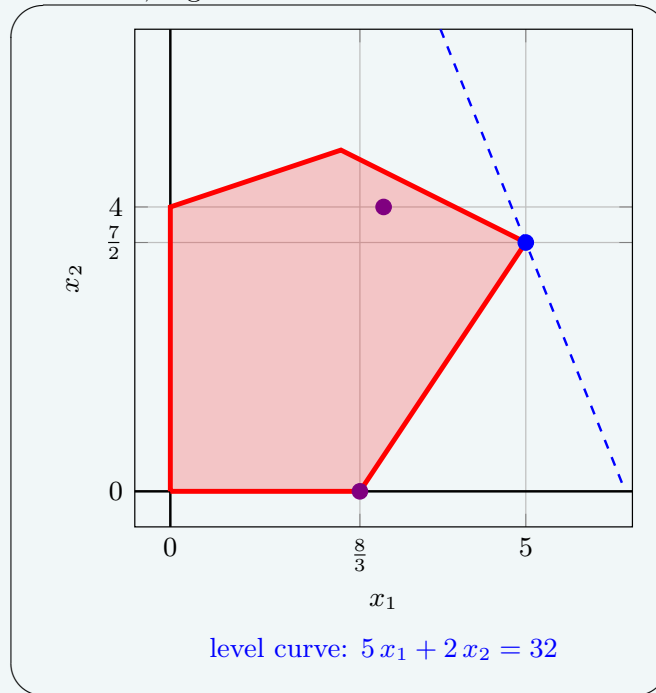
$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij} \bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$\begin{aligned} -1 \cdot 3 + 3 \cdot 4 &= 9 < 12 & \text{so} & \bar{\pi}_1 = 0 \\ 1 \cdot 3 + 2 \cdot 4 &= 11 < 12 & \text{so} & \bar{\pi}_2 = 0 \\ 3 \cdot 3 - 2 \cdot 4 &= 1 < 8 & \text{so} & \bar{\pi}_3 = 0 \end{aligned}$$

With $\bar{\pi}_1 = \bar{\pi}_2 = \bar{\pi}_3 = 0$ the two equations of the previous system read $0 = 5$ and $0 = 2$. Since the system cannot be solved, we stop the check and conclude that the solution is not optimal.

Graphical solution: the feasible region is the polygon shaded in red; the two solutions checked above are marked in violet and the optimal solution $\tilde{x}_1 = 5$, $\tilde{x}_2 = \frac{7}{2}$, of value 32, is marked in blue, together with the level curve of value 32.



Exercise 9.14 — Complementary slackness ($P11$)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned}
 \max \quad & -2x_1 - 4x_2 \\
 \text{subject to} \quad & -x_1 - x_2 \leq -2 \\
 & x_1 - x_2 \leq 0 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

Questions

- Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = 1, \quad \bar{x}_2 = 1$$

is an optimal solution of the LP problem.

- Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = 1, \quad \bar{x}_2 = 2$$

is an optimal solution of the LP problem.

Solution

- We check if the feasible solution

$$\bar{x}_1 = 1, \quad \bar{x}_2 = 1$$

of the LP problem is optimal.

1. Since

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of two linear equations with two variables:

$$- \bar{\pi}_1 + \bar{\pi}_2 = -2 \quad (\bar{x}_1 = 1 > 0)$$

$$- \bar{\pi}_1 - \bar{\pi}_2 = -4 \quad (\bar{x}_2 = 1 > 0)$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij} \bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$-1 \cdot 1 - 1 \cdot 1 = -2 \not\geq -2$$

$$1 \cdot 1 - 1 \cdot 1 = 0 \not\geq 0$$

We then solve the previous system of two equations with two variables and we get:

$$\bar{\pi}_1 = 3, \quad \bar{\pi}_2 = 1$$

3. We need to check

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$-1 \cdot 3 + 1 \cdot 1 = -2 \geq -2$$

$$-1 \cdot 3 - 1 \cdot 1 = -4 \geq -4$$

4. We need to check

$$\bar{\pi}_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}$$

so since

$$\bar{\pi}_1 = 3 \geq 0, \quad \bar{\pi}_2 = 1 \geq 0$$

the solution is optimal.

- We check if the feasible solution

$$\bar{x}_1 = 1, \quad \bar{x}_2 = 2$$

of the LP problem is optimal.

1. Since

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i = c_j \quad \text{if } \bar{x}_j > 0, \forall j \in \{1, 2, \dots, n\}$$

we build the following system of two linear equations with two variables:

$$- \bar{\pi}_1 + \bar{\pi}_2 = -2 \quad (\bar{x}_1 = 1 > 0)$$

$$- \bar{\pi}_1 - \bar{\pi}_2 = -4 \quad (\bar{x}_2 = 2 > 0)$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij}\bar{x}_j < b_i, \forall i \in \{1, 2, \dots, m\}$$

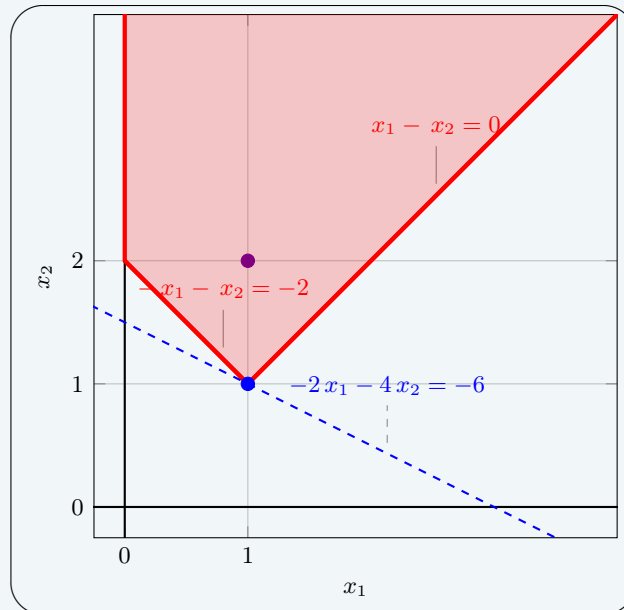
we compute:

$$-1 \cdot 1 - 1 \cdot 2 = -3 < -2 \quad \text{so} \quad \bar{\pi}_1 = 0$$

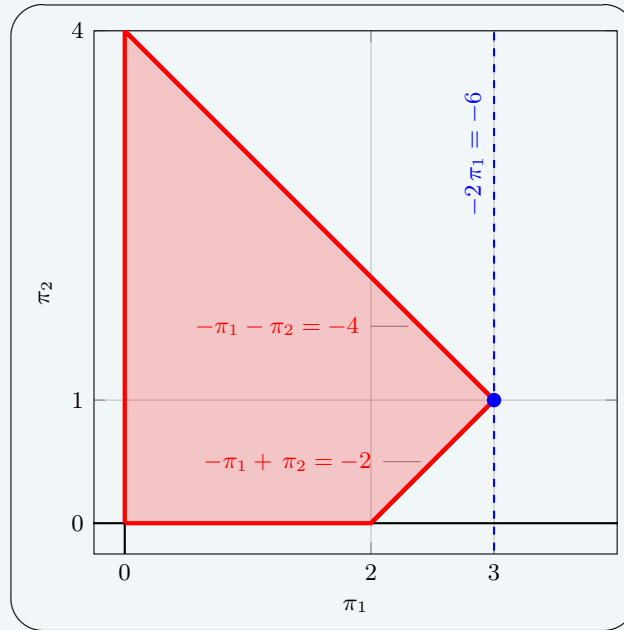
$$1 \cdot 1 - 1 \cdot 2 = -1 < 0 \quad \text{so} \quad \bar{\pi}_2 = 0$$

Since the previous system of equations cannot be solved, we stop the check and conclude that the solution is not optimal.

Graphical solution: the feasible region of the LP problem is the polyhedron shaded in red (it is unbounded); the solution $\bar{x}_1 = 1, \bar{x}_2 = 2$, which is not optimal, is marked in violet and the optimal solution $\tilde{x}_1 = 1, \tilde{x}_2 = 1$, of value -6 , is marked in blue, together with the level curve of value -6 .



The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. The dual solution $\bar{\pi}_1 = 3, \bar{\pi}_2 = 1$ obtained from the first solution is marked in blue: it is feasible and has value -6 , equal to the value of the primal solution.



Exercise 9.15 — Complementary slackness (P44)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 3$ constraints:

$$\begin{array}{ll}
 \max & x_1 + 2x_2 + 3x_3 \\
 \text{subject to} & x_1 + 2x_2 - 2x_3 \leq 1 \\
 & 2x_1 - x_2 - 3x_3 \leq 4 \\
 & x_1 + 2x_2 + 5x_3 \leq 2 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

Questions

- Using the Complementary Slackness theorem, check whether the following feasible solutions are optimal solutions of the LP problem:

$$\bar{x}_1 = \frac{9}{7}, \bar{x}_2 = 0, \bar{x}_3 = \frac{1}{7} \quad \text{and} \quad \bar{x}_1 = 0, \bar{x}_2 = \frac{9}{14}, \bar{x}_3 = \frac{1}{7}$$

- Calculate the objective function values of the two solutions.
- Replace the objective function with the following one:

$$\max \quad 7x_1 + 14x_2 + 30x_3$$

and check whether the following feasible solutions are optimal solutions of the LP problem:

$$\bar{x}_1 = \frac{9}{7}, \bar{x}_2 = 0, \bar{x}_3 = \frac{1}{7} \quad \text{and} \quad \bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = \frac{2}{5}$$

Solution

1. We check if the feasible solution

$$\bar{x}_1 = \frac{9}{7}, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = \frac{1}{7}$$

of the LP problem is optimal.

(a) Since

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of two linear equations with three variables:

$$\begin{aligned} 1 \bar{\pi}_1 + 2 \bar{\pi}_2 + 1 \bar{\pi}_3 &= 1 & (\bar{x}_1 = \frac{9}{7} > 0) \\ -2 \bar{\pi}_1 - 3 \bar{\pi}_2 + 5 \bar{\pi}_3 &= 3 & (\bar{x}_3 = \frac{1}{7} > 0) \end{aligned}$$

(b) Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij} \bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$\begin{aligned} 1 \cdot \frac{9}{7} + 2 \cdot 0 - 2 \cdot \frac{1}{7} &= 1 \not< 1 \\ 2 \cdot \frac{9}{7} - 1 \cdot 0 - 3 \cdot \frac{1}{7} &= \frac{15}{7} < 4 & \text{so} \quad \bar{\pi}_2 = 0 \\ 1 \cdot \frac{9}{7} + 2 \cdot 0 + 5 \cdot \frac{1}{7} &= 2 \not< 2 \end{aligned}$$

We then solve the previous system of two equations with two variables and we get:

$$\bar{\pi}_1 = \frac{2}{7}, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = \frac{5}{7}$$

(c) We need to check

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$\begin{aligned} 1 \cdot \frac{2}{7} + 2 \cdot 0 + 1 \cdot \frac{5}{7} &= 1 \geq 1 \\ 2 \cdot \frac{2}{7} - 1 \cdot 0 + 2 \cdot \frac{5}{7} &= 2 \geq 2 \\ -2 \cdot \frac{2}{7} - 3 \cdot 0 + 5 \cdot \frac{5}{7} &= 3 \geq 3 \end{aligned}$$

(d) We need to check

$$\bar{\pi}_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}$$

so since

$$\bar{\pi}_1 = \frac{2}{7} \geq 0, \quad \bar{\pi}_2 = 0 \geq 0, \quad \bar{\pi}_3 = \frac{5}{7} \geq 0$$

the solution is optimal.

We check if the feasible solution

$$\bar{x}_1 = 0, \quad \bar{x}_2 = \frac{9}{14}, \quad \bar{x}_3 = \frac{1}{7}$$

of the LP problem is optimal.

(a) Since

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of two linear equations with three variables:

$$\begin{aligned} 2 \bar{\pi}_1 - 1 \bar{\pi}_2 + 2 \bar{\pi}_3 &= 2 & (\bar{x}_2 = \frac{9}{14} > 0) \\ -2 \bar{\pi}_1 - 3 \bar{\pi}_2 + 5 \bar{\pi}_3 &= 3 & (\bar{x}_3 = \frac{1}{7} > 0) \end{aligned}$$

(b) Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij} \bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$\begin{aligned} 1 \cdot 0 + 2 \cdot \frac{9}{14} - 2 \cdot \frac{1}{7} &= 1 \not\leq 1 \\ 2 \cdot 0 - 1 \cdot \frac{9}{14} - 3 \cdot \frac{1}{7} &= -\frac{15}{14} < 4 & \text{so} \quad \bar{\pi}_2 = 0 \\ 1 \cdot 0 + 2 \cdot \frac{9}{14} + 5 \cdot \frac{1}{7} &= 2 \not\leq 2 \end{aligned}$$

We then solve the previous system of two equations with two variables and we get:

$$\bar{\pi}_1 = \frac{2}{7}, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = \frac{5}{7}$$

(c) We need to check

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$\begin{aligned} 1 \cdot \frac{2}{7} + 2 \cdot 0 + 1 \cdot \frac{5}{7} &= 1 \geq 1 \\ 2 \cdot \frac{2}{7} - 1 \cdot 0 + 2 \cdot \frac{5}{7} &= 2 \geq 2 \\ -2 \cdot \frac{2}{7} - 3 \cdot 0 + 5 \cdot \frac{5}{7} &= 3 \geq 3 \end{aligned}$$

(d) We need to check

$$\bar{\pi}_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}$$

so since

$$\bar{\pi}_1 = \frac{2}{7} \geq 0, \quad \bar{\pi}_2 = 0 \geq 0, \quad \bar{\pi}_3 = \frac{5}{7} \geq 0$$

the solution is optimal.

2. The objective function value of the first solution is:

$$\bar{x}_1 + 2\bar{x}_2 + 3\bar{x}_3 = \frac{9}{7} + 0 + 3 \cdot \frac{1}{7} = \frac{12}{7}$$

and the objective function value of the second is:

$$\bar{x}_1 + 2\bar{x}_2 + 3\bar{x}_3 = 0 + 2 \cdot \frac{9}{14} + 3 \cdot \frac{1}{7} = \frac{12}{7}$$

Since the solutions are both optimal, the two values must coincide. Moreover, both solutions satisfy the first and the third constraints at equality, and every point of the segment joining them,

$$\bar{\mathbf{x}}(\lambda) = \lambda \begin{pmatrix} \frac{9}{7} \\ 0 \\ \frac{1}{7} \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 0 \\ \frac{9}{14} \\ \frac{1}{7} \end{pmatrix}, \quad \lambda \in [0, 1],$$

is a feasible solution of value $\frac{12}{7}$. Accordingly, the LP problem has infinitely many optimal solutions: the whole edge of the polyhedron between the two solutions is optimal.

3. We check if the feasible solution

$$\bar{x}_1 = \frac{9}{7}, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = \frac{1}{7}$$

of the LP problem is optimal when considering the new objective function:

$$\max \quad 7x_1 + 14x_2 + 30x_3$$

(a) Since

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i = c_j \quad \text{if } \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of two linear equations with three variables:

$$\begin{aligned} 1 \bar{\pi}_1 + 2 \bar{\pi}_2 + 1 \bar{\pi}_3 &= 7 & (\bar{x}_1 = \frac{9}{7} > 0) \\ -2 \bar{\pi}_1 - 3 \bar{\pi}_2 + 5 \bar{\pi}_3 &= 30 & (\bar{x}_3 = \frac{1}{7} > 0) \end{aligned}$$

(b) From point 1(b) we have that $\bar{\pi}_2 = 0$. We then solve the previous system of two equations with two variables and we get:

$$\bar{\pi}_1 = \frac{5}{7}, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = \frac{44}{7}$$

(c) We need to check

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$\begin{aligned} 1 \cdot \frac{5}{7} + 2 \cdot 0 + 1 \cdot \frac{44}{7} &= 7 \geq 7 \\ 2 \cdot \frac{5}{7} - 1 \cdot 0 + 2 \cdot \frac{44}{7} &= 14 \geq 14 \\ -2 \cdot \frac{5}{7} - 3 \cdot 0 + 5 \cdot \frac{44}{7} &= 30 \geq 30 \end{aligned}$$

(d) We need to check

$$\bar{\pi}_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}$$

so since

$$\bar{\pi}_1 = \frac{5}{7} \geq 0, \quad \bar{\pi}_2 = 0 \geq 0, \quad \bar{\pi}_3 = \frac{44}{7} \geq 0$$

the solution is optimal.

We then check if the feasible solution

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = \frac{2}{5}$$

of the LP problem is optimal when considering the new objective function:

$$\max \quad 7x_1 + 14x_2 + 30x_3$$

(a) Since

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following linear equation with three variables:

$$-2\bar{\pi}_1 - 3\bar{\pi}_2 + 5\bar{\pi}_3 = 30 \quad (\bar{x}_3 = \frac{2}{5} > 0)$$

(b) Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij}\bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$1 \cdot 0 + 2 \cdot 0 - 2 \cdot \frac{2}{5} = -\frac{4}{5} < 1 \quad \text{so} \quad \bar{\pi}_1 = 0$$

$$2 \cdot 0 - 1 \cdot 0 - 3 \cdot \frac{2}{5} = -\frac{6}{5} < 4 \quad \text{so} \quad \bar{\pi}_2 = 0$$

$$1 \cdot 0 + 2 \cdot 0 + 5 \cdot \frac{2}{5} = 2 \not< 2$$

We then solve the previous equation with one variable and we get:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 6$$

(c) We need to check

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$1 \cdot 0 + 2 \cdot 0 + 1 \cdot 6 = 6 \not\geq 7$$

we stop the check and conclude that the solution is not optimal.

Exercise 9.16 — Complementary slackness (P71)

Consider the following LP problem in canonical maximization form with $n = 4$ variables and $m = 3$ constraints:

$$\begin{array}{llllllllll}
 \max & 4 & x_1 & + & & x_2 & + & 5 & x_3 & + & 3 & x_4 \\
 \text{subject to} & & x_1 & - & & x_2 & - & & x_3 & + & 3 & x_4 & \leq & 1 \\
 & & 5 & x_1 & + & & x_2 & + & 3 & x_3 & + & 8 & x_4 & \leq & 55 \\
 & & - & x_1 & + & 2 & x_2 & + & 3 & x_3 & - & 5 & x_4 & \leq & 3 \\
 & & & x_1, & & & x_2, & & & x_3, & & & x_4 & \geq & 0
 \end{array}$$

Questions

- Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 14, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 5$$

is an optimal solution of the LP problem.

- Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 1, \quad \bar{x}_3 = 2, \quad \bar{x}_4 = 1$$

is an optimal solution of the LP problem.

- Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = 5, \quad \bar{x}_2 = 4, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0$$

is an optimal solution of the LP problem.

Solution

- We check if the feasible solution

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 14, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 5$$

of the LP problem is optimal.

- Since

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of two linear equations with three variables:

$$\begin{aligned} -\bar{\pi}_1 + \bar{\pi}_2 + 2\bar{\pi}_3 &= 1 & (\bar{x}_2 = 14 > 0) \\ 3\bar{\pi}_1 + 8\bar{\pi}_2 - 5\bar{\pi}_3 &= 3 & (\bar{x}_4 = 5 > 0) \end{aligned}$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij}\bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$\begin{aligned} 1 \cdot 0 - 1 \cdot 14 - 1 \cdot 0 + 3 \cdot 5 &= -1 < 1 \\ 5 \cdot 0 + 1 \cdot 14 + 3 \cdot 0 + 8 \cdot 5 &= 54 < 55 \quad \text{so} \quad \bar{\pi}_2 = 0 \\ -1 \cdot 0 + 2 \cdot 14 + 3 \cdot 0 - 5 \cdot 5 &= -3 < 3 \end{aligned}$$

We then solve the previous system of two equations with two variables and we get:

$$\bar{\pi}_1 = 11, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 6$$

3. We need to check

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$\begin{aligned} 1 \cdot 11 + 5 \cdot 0 - 1 \cdot 6 &= 5 \geq 4 \\ -1 \cdot 11 + 1 \cdot 0 + 2 \cdot 6 &= 1 \geq 1 \\ -1 \cdot 11 + 3 \cdot 0 + 3 \cdot 6 &= 7 \geq 5 \\ 3 \cdot 11 + 8 \cdot 0 - 5 \cdot 6 &= 3 \geq 3 \end{aligned}$$

4. We need to check

$$\bar{\pi}_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}$$

so since

$$\bar{\pi}_1 = 11 \geq 0, \quad \bar{\pi}_2 = 0 \geq 0, \quad \bar{\pi}_3 = 6 \geq 0$$

the solution is optimal.

- We check if the feasible solution

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 1, \quad \bar{x}_3 = 2, \quad \bar{x}_4 = 1$$

of the LP problem is optimal.

1. Since

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of three linear equations with three variables:

$$\begin{aligned} -\bar{\pi}_1 + \bar{\pi}_2 + 2\bar{\pi}_3 &= 1 & (\bar{x}_2 = 1 > 0) \\ -\bar{\pi}_1 + 3\bar{\pi}_2 + 3\bar{\pi}_3 &= 5 & (\bar{x}_3 = 2 > 0) \\ 3\bar{\pi}_1 + 8\bar{\pi}_2 - 5\bar{\pi}_3 &= 3 & (\bar{x}_4 = 1 > 0) \end{aligned}$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij}\bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$\begin{aligned} 1 \cdot 0 - 1 \cdot 1 - 1 \cdot 2 + 3 \cdot 1 &= 0 < 1 & \text{so} \quad \bar{\pi}_1 = 0 \\ 5 \cdot 0 + 1 \cdot 1 + 3 \cdot 2 + 8 \cdot 1 &= 15 < 55 & \text{so} \quad \bar{\pi}_2 = 0 \\ -1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 - 5 \cdot 1 &= 3 \not< 3 \end{aligned}$$

Since the previous system of equations cannot be solved, we stop the check and conclude that the solution is not optimal.

- We check if the feasible solution

$$\bar{x}_1 = 5, \quad \bar{x}_2 = 4, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0$$

of the LP problem is optimal.

1. Since

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of two linear equations with three variables:

$$\begin{aligned} \bar{\pi}_1 + 5\bar{\pi}_2 - \bar{\pi}_3 &= 4 & (\bar{x}_1 = 5 > 0) \\ -\bar{\pi}_1 + \bar{\pi}_2 + 2\bar{\pi}_3 &= 1 & (\bar{x}_2 = 4 > 0) \end{aligned}$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij}\bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$\begin{aligned} 1 \cdot 5 - 1 \cdot 4 - 1 \cdot 0 + 3 \cdot 0 &= 1 \not< 1 \\ 5 \cdot 5 + 1 \cdot 4 + 3 \cdot 0 + 8 \cdot 0 &= 29 < 55 & \text{so} \quad \bar{\pi}_2 = 0 \\ -1 \cdot 5 + 2 \cdot 4 + 3 \cdot 0 - 5 \cdot 0 &= 3 \not< 3 \end{aligned}$$

We then solve the previous system of two equations with two variables and we get:

$$\bar{\pi}_1 = 9, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_3 = 5$$

3. We need to check

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$\begin{aligned} 1 \cdot 9 + 5 \cdot 0 - 1 \cdot 5 &= 4 \geq 4 \\ -1 \cdot 9 + 1 \cdot 0 + 2 \cdot 5 &= 1 \geq 1 \\ -1 \cdot 9 + 3 \cdot 0 + 3 \cdot 5 &= 6 \geq 5 \\ 3 \cdot 9 + 8 \cdot 0 - 5 \cdot 5 &= 2 \not\geq 3 \end{aligned}$$

so we stop the check and conclude that the solution is not optimal.

Exercise 9.17 — Complementary slackness (P72)

Consider the following LP problem in canonical maximization form with $n = 5$ variables and $m = 4$ constraints:

$$\begin{aligned} \max \quad & 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5 \\ \text{subject to} \quad & x_1 + 3x_2 + 5x_3 - 2x_4 + 2x_5 \leq 4 \\ & 4x_1 + 2x_2 - 2x_3 + x_4 + x_5 \leq 3 \\ & 2x_1 + 4x_2 + 4x_3 - 2x_4 + 5x_5 \leq 5 \\ & 3x_1 + x_2 + 2x_3 - x_4 - 2x_5 \leq 1 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Questions

1. Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = 0, \quad \bar{x}_2 = \frac{4}{3}, \quad \bar{x}_3 = \frac{2}{3}, \quad \bar{x}_4 = \frac{5}{3}, \quad \bar{x}_5 = 0$$

is an optimal solution of the LP problem.

2. Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = \frac{39}{59}, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = \frac{91}{59}, \quad \bar{x}_4 = \frac{166}{59}, \quad \bar{x}_5 = \frac{37}{59}$$

is an optimal solution of the LP problem.

Solution

- We check if the feasible solution

$$\bar{x}_1 = 0, \quad \bar{x}_2 = \frac{4}{3}, \quad \bar{x}_3 = \frac{2}{3}, \quad \bar{x}_4 = \frac{5}{3}, \quad \bar{x}_5 = 0$$

of the LP problem is optimal.

1. Since

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of three linear equations with four variables:

$$\begin{aligned} 3 \bar{\pi}_1 + 2 \bar{\pi}_2 + 4 \bar{\pi}_3 + \bar{\pi}_4 &= 6 & (\bar{x}_2 = \frac{4}{3} > 0) \\ 5 \bar{\pi}_1 - 2 \bar{\pi}_2 + 4 \bar{\pi}_3 + 2 \bar{\pi}_4 &= 5 & (\bar{x}_3 = \frac{2}{3} > 0) \\ -2 \bar{\pi}_1 + \bar{\pi}_2 - 2 \bar{\pi}_3 - \bar{\pi}_4 &= -2 & (\bar{x}_4 = \frac{5}{3} > 0) \end{aligned}$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij} \bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$\begin{aligned} 1 \cdot 0 + 3 \cdot \frac{4}{3} + 5 \cdot \frac{2}{3} - 2 \cdot \frac{5}{3} + 2 \cdot 0 &= 4 \not\leq 4 \\ 4 \cdot 0 + 2 \cdot \frac{4}{3} - 2 \cdot \frac{2}{3} + 1 \cdot \frac{5}{3} + 1 \cdot 0 &= 3 \not\leq 3 \\ 2 \cdot 0 + 4 \cdot \frac{4}{3} + 4 \cdot \frac{2}{3} - 2 \cdot \frac{5}{3} + 5 \cdot 0 &= \frac{14}{3} < 5 \quad \text{so} \quad \bar{\pi}_3 = 0 \\ 3 \cdot 0 + 1 \cdot \frac{4}{3} + 2 \cdot \frac{2}{3} - 1 \cdot \frac{5}{3} - 2 \cdot 0 &= 1 \not\leq 1 \end{aligned}$$

We then solve the previous system of three equations with three variables and we get:

$$\bar{\pi}_1 = 1, \quad \bar{\pi}_2 = 1, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 1$$

3. We need to check

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$\begin{aligned} 1 \cdot 1 + 4 \cdot 1 + 2 \cdot 0 + 3 \cdot 1 &= 8 \geq 7 \\ 3 \cdot 1 + 2 \cdot 1 + 4 \cdot 0 + 1 \cdot 1 &= 6 \geq 6 \\ 5 \cdot 1 - 2 \cdot 1 + 4 \cdot 0 + 2 \cdot 1 &= 5 \geq 5 \\ -2 \cdot 1 + 1 \cdot 1 - 2 \cdot 0 - 1 \cdot 1 &= -2 \geq -2 \\ 2 \cdot 1 + 1 \cdot 1 + 5 \cdot 0 - 2 \cdot 1 &= 1 \not\geq 3 \end{aligned}$$

so we stop the check and conclude that the solution is not optimal.

- We check if the feasible solution

$$\bar{x}_1 = \frac{39}{59}, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = \frac{91}{59}, \quad \bar{x}_4 = \frac{166}{59}, \quad \bar{x}_5 = \frac{37}{59}$$

of the LP problem is optimal.

1. Since

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if} \quad \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of four linear equations with four variables:

$$\begin{aligned} \bar{\pi}_1 + 4 \bar{\pi}_2 + 2 \bar{\pi}_3 + 3 \bar{\pi}_4 &= 7 & (\bar{x}_1 = \frac{39}{59} > 0) \\ 5 \bar{\pi}_1 - 2 \bar{\pi}_2 + 4 \bar{\pi}_3 + 2 \bar{\pi}_4 &= 5 & (\bar{x}_3 = \frac{91}{59} > 0) \\ -2 \bar{\pi}_1 + \bar{\pi}_2 - 2 \bar{\pi}_3 - \bar{\pi}_4 &= -2 & (\bar{x}_4 = \frac{166}{59} > 0) \\ 2 \bar{\pi}_1 + \bar{\pi}_2 + 5 \bar{\pi}_3 - 2 \bar{\pi}_4 &= 3 & (\bar{x}_5 = \frac{37}{59} > 0) \end{aligned}$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij} \bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$\begin{aligned} 1 \cdot \frac{39}{59} + 3 \cdot 0 + 5 \cdot \frac{91}{59} - 2 \cdot \frac{166}{59} + 2 \cdot \frac{37}{59} &= 4 \not\leq 4 \\ 4 \cdot \frac{39}{59} + 2 \cdot 0 - 2 \cdot \frac{91}{59} + 1 \cdot \frac{166}{59} + 1 \cdot \frac{37}{59} &= 3 \not\leq 3 \\ 2 \cdot \frac{39}{59} + 4 \cdot 0 + 4 \cdot \frac{91}{59} - 2 \cdot \frac{166}{59} + 5 \cdot \frac{37}{59} &= 5 \not\leq 5 \\ 3 \cdot \frac{39}{59} + 1 \cdot 0 + 2 \cdot \frac{91}{59} - 1 \cdot \frac{166}{59} - 2 \cdot \frac{37}{59} &= 1 \not\leq 1 \end{aligned}$$

We then solve the previous system of four equations with four variables and we get:

$$\bar{\pi}_1 = 1, \quad \bar{\pi}_2 = \frac{58}{59}, \quad \bar{\pi}_3 = \frac{13}{59}, \quad \bar{\pi}_4 = \frac{32}{59}$$

3. We need to check

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$\begin{aligned} 1 \cdot 1 + 4 \cdot \frac{58}{59} + 2 \cdot \frac{13}{59} + 3 \cdot \frac{32}{59} &= 7 \geq 7 \\ 3 \cdot 1 + 2 \cdot \frac{58}{59} + 4 \cdot \frac{13}{59} + 1 \cdot \frac{32}{59} &= \frac{377}{59} \geq 6 \\ 5 \cdot 1 - 2 \cdot \frac{58}{59} + 4 \cdot \frac{13}{59} + 2 \cdot \frac{32}{59} &= 5 \geq 5 \\ -2 \cdot 1 + 1 \cdot \frac{58}{59} - 2 \cdot \frac{13}{59} - 1 \cdot \frac{32}{59} &= -2 \geq -2 \\ 2 \cdot 1 + 1 \cdot \frac{58}{59} + 5 \cdot \frac{13}{59} - 2 \cdot \frac{32}{59} &= 3 \geq 3 \end{aligned}$$

4. We need to check

$$\bar{\pi}_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}$$

so since

$$\bar{\pi}_1 = 1 \geq 0, \quad \bar{\pi}_2 = \frac{58}{59} \geq 0, \quad \bar{\pi}_3 = \frac{13}{59} \geq 0, \quad \bar{\pi}_4 = \frac{32}{59} \geq 0$$

the solution is optimal.

Exercise 9.18 — Complementary slackness (B1)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{array}{llllll} \max & 3 & x_1 & + & 3 & x_2 \\ \text{subject to} & & x_1 & + & 2 & x_2 & \leq & 6 \\ & & 2 & x_1 & + & & x_2 & \leq & 6 \\ & & & x_1 & + & & x_2 & \leq & 4 \\ & & & & & x_1, & & x_2 & \geq & 0 \end{array}$$

Questions

- Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = 2, \quad \bar{x}_2 = 2$$

is an optimal solution of the LP problem.

- Using the Complementary Slackness theorem, check whether the feasible solution

$$\bar{x}_1 = 3, \quad \bar{x}_2 = 0$$

is an optimal solution of the LP problem.

Solution

- We check if the feasible solution

$$\bar{x}_1 = 2, \quad \bar{x}_2 = 2$$

of the LP problem is optimal.

1. Since

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i = c_j \quad \text{if } \bar{x}_j > 0, \quad \forall j \in \{1, 2, \dots, n\}$$

we build the following system of two linear equations with three variables:

$$1 \cdot \bar{\pi}_1 + 2 \cdot \bar{\pi}_2 + 1 \cdot \bar{\pi}_3 = 3 \quad (\bar{x}_1 = 2 > 0)$$

$$2 \cdot \bar{\pi}_1 + 1 \cdot \bar{\pi}_2 + 1 \cdot \bar{\pi}_3 = 3 \quad (\bar{x}_2 = 2 > 0)$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij}\bar{x}_j < b_i, \quad \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$1 \cdot 2 + 2 \cdot 2 = 6 \not\leq 6$$

$$2 \cdot 2 + 1 \cdot 2 = 6 \not\leq 6$$

$$1 \cdot 2 + 1 \cdot 2 = 4 \not\leq 4$$

All three constraints are active, so no dual variable is forced to zero. The system of step 1 has two equations and three unknowns and therefore has **infinitely many solutions**. Subtracting the second equation from the first we get $\bar{\pi}_2 - \bar{\pi}_1 = 0$, i.e., $\bar{\pi}_1 = \bar{\pi}_2$. Setting $\bar{\pi}_1 = \bar{\pi}_2 = s$ and substituting into the first equation we get $\bar{\pi}_3 = 3 - 3s$. We choose $s = 1$ and we get:

$$\bar{\pi}_1 = 1, \quad \bar{\pi}_2 = 1, \quad \bar{\pi}_3 = 0$$

3. We need to check

$$\sum_{i=1}^m a_{ij}\bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

so we compute:

$$1 \cdot 1 + 2 \cdot 1 + 1 \cdot 0 = 3 \geq 3$$

$$2 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = 3 \geq 3$$

4. We need to check

$$\bar{\pi}_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}$$

so since

$$\bar{\pi}_1 = 1 \geq 0, \quad \bar{\pi}_2 = 1 \geq 0, \quad \bar{\pi}_3 = 0 \geq 0$$

the solution is optimal.

- We check if the feasible solution

$$\bar{x}_1 = 3, \quad \bar{x}_2 = 0$$

of the LP problem is optimal.

1. Since

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i = c_j \quad \text{if } \bar{x}_j > 0, \forall j \in \{1, 2, \dots, n\}$$

we build the following linear equation with three variables:

$$1 \cdot \bar{\pi}_1 + 2 \cdot \bar{\pi}_2 + 1 \cdot \bar{\pi}_3 = 3 \quad (\bar{x}_1 = 3 > 0)$$

2. Since

$$\bar{\pi}_i = 0 \quad \text{if} \quad \sum_{j=1}^n a_{ij} \bar{x}_j < b_i, \forall i \in \{1, 2, \dots, m\}$$

we compute:

$$1 \cdot 3 + 2 \cdot 0 = 3 < 6 \quad \text{so} \quad \bar{\pi}_1 = 0$$

$$2 \cdot 3 + 1 \cdot 0 = 6 \not< 6$$

$$1 \cdot 3 + 1 \cdot 0 = 3 < 4 \quad \text{so} \quad \bar{\pi}_3 = 0$$

We then solve the previous equation with one variable and we get:

$$\bar{\pi}_1 = 0, \quad \bar{\pi}_2 = \frac{3}{2}, \quad \bar{\pi}_3 = 0$$

3. We need to check

$$\sum_{i=1}^m a_{ij} \bar{\pi}_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}$$

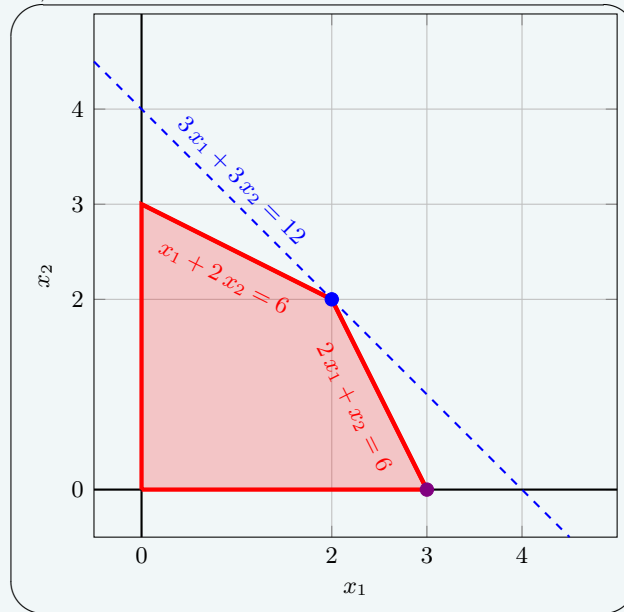
so we compute:

$$1 \cdot 0 + 2 \cdot \frac{3}{2} + 1 \cdot 0 = 3 \geq 3$$

$$2 \cdot 0 + 1 \cdot \frac{3}{2} + 1 \cdot 0 = \frac{3}{2} \not\geq 3$$

so we stop the check and conclude that the solution is not optimal.

Graphical solution: the feasible region of the LP problem is the polyhedron shaded in red; the solution $\bar{x}_1 = 3, \bar{x}_2 = 0$, which is not optimal, is marked in violet and the optimal solution $\tilde{x}_1 = 2, \tilde{x}_2 = 2$, of value 12, is marked in blue, together with the level curve of value 12. The level curve coincides with the line $x_1 + x_2 = 4$ of the third constraint, which touches the feasible region only at the optimal vertex: the optimal vertex is degenerate, since the three constraints are all active at it.



9.4 Primal-dual outcomes

Exercise 9.19 — Primal and dual outcomes (P31)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & & x_1 & + & x_2 & \\ \text{subject to} & - & x_1 & + & x_2 & \leq 1 \\ & & x_1 & - & x_2 & \leq 1 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

Questions

1. Write the dual problem.
2. Determine the outcome of the primal problem (optimal solution, unbounded or infeasible).
3. Determine the outcome of the dual problem and check that the pair is one of the four cases of the Theorem on the possible outcomes of a pair of primal and dual problems.

Solution

- The dual problem has $m = 2$ non-negative variables and $n = 2$ greater-than-or-equal-to linear constraints, obtained by transposing the constraint matrix:

$$\begin{array}{llllll} \min & & \pi_1 & + & \pi_2 & \\ \text{subject to} & - & \pi_1 & + & \pi_2 & \geq 1 \\ & & \pi_1 & - & \pi_2 & \geq 1 \\ & & \pi_1, & & \pi_2 & \geq 0 \end{array}$$

- The primal problem is **unbounded**. Indeed, for every value $t \geq 0$ the point $\bar{x}_1 = \bar{x}_2 = t$ is a feasible solution, since

$$-t + t = 0 \leq 1, \quad t - t = 0 \leq 1, \quad t \geq 0,$$

and its objective value is $2t$, which is larger than any fixed value M for t large enough.

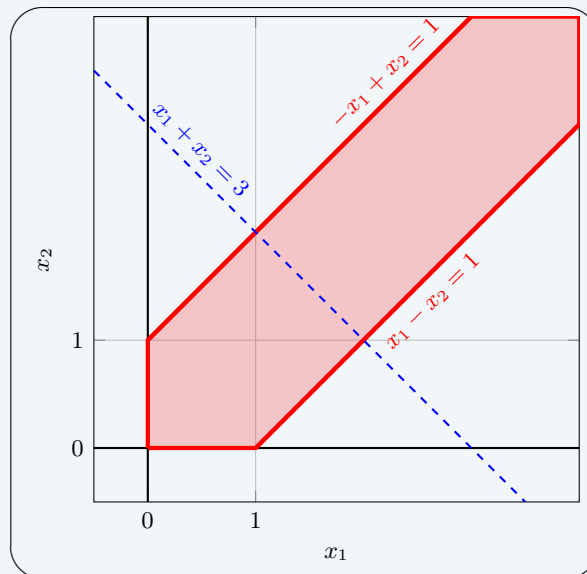
- The dual problem is **infeasible**. Indeed, by summing its two constraints we obtain

$$\underbrace{(-\pi_1 + \pi_2)}_{\geq 1} + \underbrace{(\pi_1 - \pi_2)}_{\geq 1} = 0 \geq 2,$$

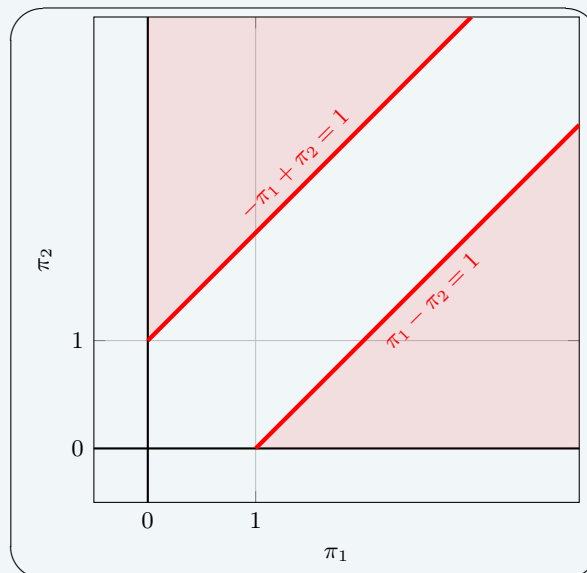
which is never satisfied.

The pair is then in the case “*primal unbounded, dual infeasible*”, as predicted by the Corollary: if the primal problem is unbounded, then the dual problem is infeasible.

Graphical solution: the feasible region of the primal problem is the unbounded polyhedron shaded in red; the level curve $x_1 + x_2 = 3$ is drawn in blue, and the level curves $x_1 + x_2 = k$ intersect the feasible region for every value of $k \geq 0$, so the objective function can increase without limit.



The two half-planes defined by the dual constraints, lightly shaded in red, have no point in common in the non-negative orthant: the dual feasible region is empty.



Exercise 9.20 — Primal and dual outcomes (P33)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{rclll} \max & x_1 & + & x_2 & \\ \text{subject to} & x_1 & + & x_2 & \leq 2 \\ & -x_1 & - & x_2 & \leq -4 \\ & x_1, & & x_2 & \geq 0 \end{array}$$

Questions

1. Write the dual problem.
2. Determine the outcome of the primal problem (optimal solution, unbounded or infeasible).
3. Determine the outcome of the dual problem and check that the pair is one of the four cases of the Theorem on the possible outcomes of a pair of primal and dual problems.

Solution

- The dual problem has $m = 2$ non-negative variables and $n = 2$ greater-than-or-equal-to linear constraints, obtained by transposing the constraint matrix:

$$\begin{array}{rclll} \min & 2 & \pi_1 & - & 4 & \pi_2 \\ \text{subject to} & \pi_1 & - & \pi_2 & \geq 1 \\ & \pi_1 & - & \pi_2 & \geq 1 \\ & \pi_1, & & \pi_2 & \geq 0 \end{array}$$

(the two dual constraints coincide, since the two columns of the primal constraint matrix are equal).

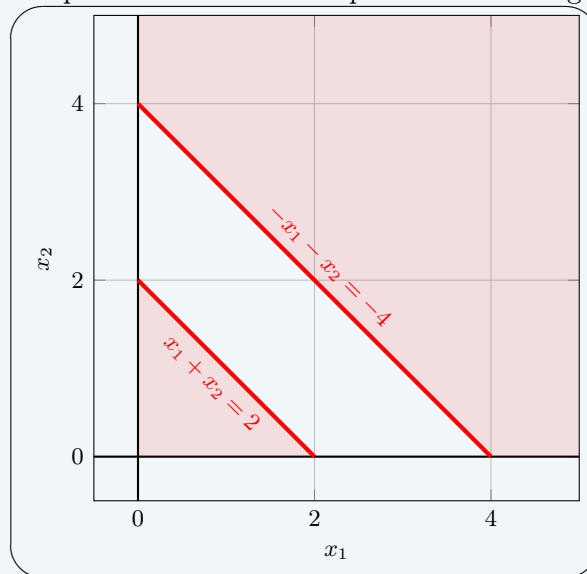
- The primal problem is **infeasible**: the first constraint imposes $x_1 + x_2 \leq 2$ and the second one, multiplied by -1 , imposes $x_1 + x_2 \geq 4$.
- The dual problem is **unbounded**. Indeed, for every value $t \geq 0$ the point $\bar{\pi}_1 = 1 + t$ and $\bar{\pi}_2 = t$ is a feasible solution, since $\bar{\pi}_1 - \bar{\pi}_2 = 1 \geq 1$, and its objective value is

$$2(1 + t) - 4t = 2 - 2t,$$

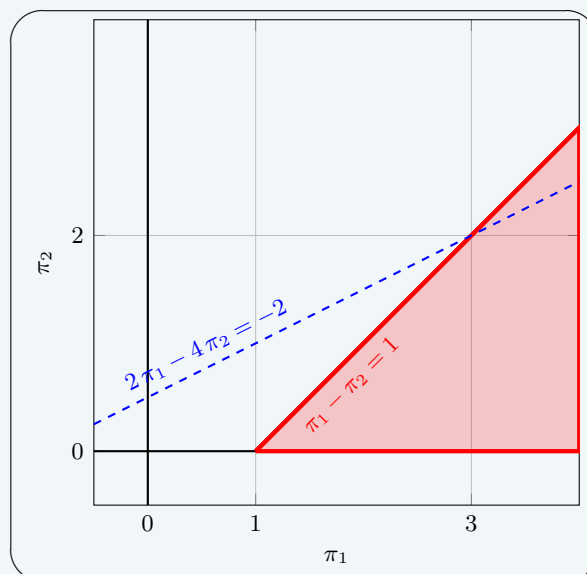
which tends to $-\infty$ as t grows. Since the dual problem is a minimization problem, it is unbounded.

The pair is then in the case “*primal infeasible, dual unbounded*”.

Graphical solution: the two half-planes defined by the primal constraints, lightly shaded in red, have no point in common: the primal feasible region is empty.



The dual feasible region is the unbounded polyhedron shaded in red (the two dual constraints coincide). The level curve $2\pi_1 - 4\pi_2 = -2$ is drawn in blue: moving along the direction $(1, 1)$ the objective function decreases without limit.



Exercise 9.21 — Primal and dual outcomes (P35)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll}
 \max & x_1 & + & x_2 & & \\
 \text{subject to} & x_1 & - & x_2 & \leq & 1 \\
 & - & x_1 & + & x_2 & \leq -2 \\
 & x_1, & & x_2 & \geq & 0
 \end{array}$$

Questions

1. Write the dual problem.
2. Determine the outcome of the primal problem (optimal solution, unbounded or infeasible).
3. Determine the outcome of the dual problem and check that the pair is one of the four cases of the Theorem on the possible outcomes of a pair of primal and dual problems.

Solution

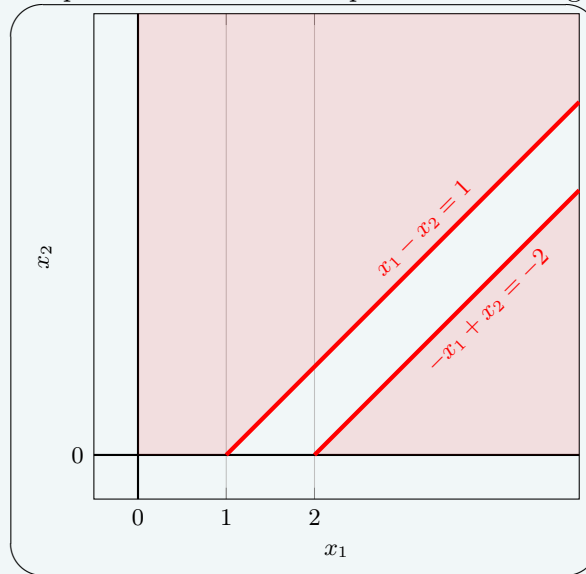
- The dual problem has $m = 2$ non-negative variables and $n = 2$ greater-than-or-equal-to linear constraints, obtained by transposing the constraint matrix:

$$\begin{array}{llllll}
 \min & \pi_1 & - & 2 & \pi_2 & \\
 \text{subject to} & \pi_1 & - & & \pi_2 & \geq 1 \\
 & - & \pi_1 & + & \pi_2 & \geq 1 \\
 & \pi_1, & & \pi_2 & \geq & 0
 \end{array}$$

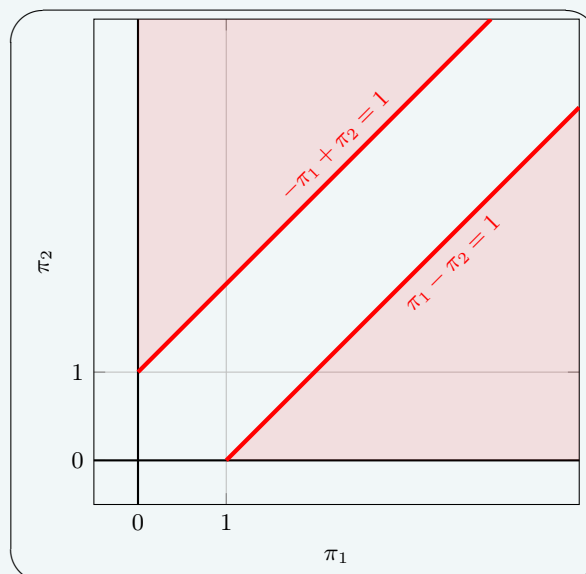
- The primal problem is **infeasible**: the two constraints impose $x_1 - x_2 \leq 1$ and $x_1 - x_2 \geq 2$.
- The dual problem is **infeasible** as well: by summing its two constraints we obtain $0 \geq 2$, which is never satisfied.

The pair is then in the case “*both problems infeasible*”. This example shows that the infeasibility of the primal problem alone does not determine the outcome of the dual problem: in the exercise *P33* the primal problem is infeasible and the dual problem is unbounded, while here both are infeasible.

Graphical solution: the two half-planes defined by the primal constraints, lightly shaded in red, have no point in common: the primal feasible region is empty.



Also the two half-planes defined by the dual constraints have no point in common: the dual feasible region is empty as well.



Exercise 9.22 — Primal and dual outcomes (P36)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{rcll}
 \max & - & x_1 & - & x_2 \\
 \text{subject to} & - & x_1 & - & x_2 \leq -2 \\
 & & x_1 & - & x_2 \leq 2 \\
 & & x_1, & & x_2 \geq 0
 \end{array}$$

Questions

1. Write the dual problem.
2. Determine the outcome of the primal problem (optimal solution, unbounded or infeasible).
3. Determine the outcome of the dual problem and check that the pair is one of the four cases of the Theorem on the possible outcomes of a pair of primal and dual problems.

Solution

- The dual problem has $m = 2$ non-negative variables and $n = 2$ greater-than-or-equal-to linear constraints, obtained by transposing the constraint matrix:

$$\begin{array}{rcll}
 \min & -2 & \pi_1 & + & 2 & \pi_2 \\
 \text{subject to} & - & \pi_1 & + & & \pi_2 \geq -1 \\
 & & - & \pi_1 & - & \pi_2 \geq -1 \\
 & & & \pi_1, & & \pi_2 \geq 0
 \end{array}$$

- The primal problem admits an **optimal solution**. Its feasible region is unbounded, but the first constraint imposes $x_1 + x_2 \geq 2$ for every feasible solution, and then

$$-x_1 - x_2 \leq -2, \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in F.$$

The two feasible solutions $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ attain this value, and then the optimal value is $z(LP) = -2$.

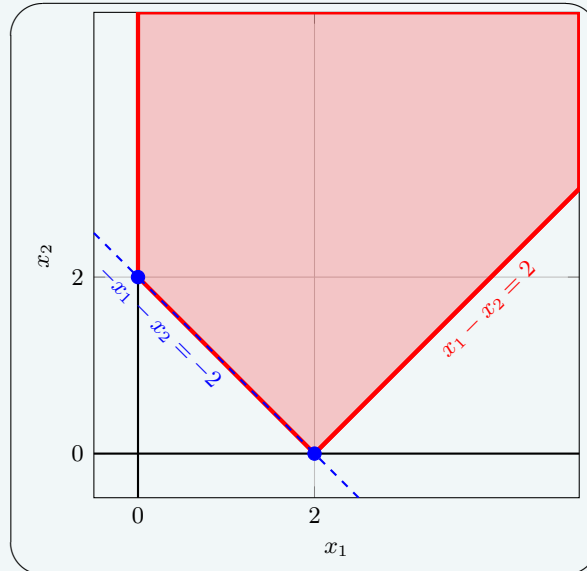
- The dual problem admits an **optimal solution** as well, and by Strong Duality its optimal value is $z(D(LP)) = -2$. An optimal dual solution is

$$\tilde{\pi}_1 = 1 \quad \text{and} \quad \tilde{\pi}_2 = 0,$$

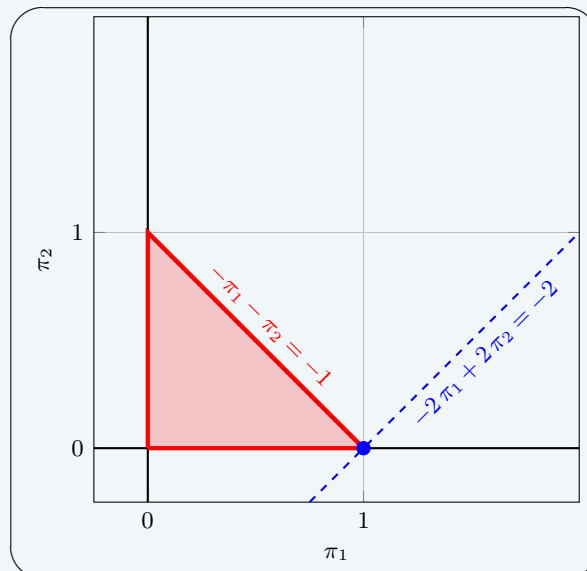
which is feasible, since $-1 + 0 = -1 \geq -1$ and $-1 - 0 = -1 \geq -1$, and whose objective value is $-2 \cdot 1 + 2 \cdot 0 = -2$.

The pair is then in the case “both problems admit an optimal solution, with equal optimal values”.

Graphical solution: the feasible region of the primal problem is the unbounded polyhedron shaded in red. The level curve $-x_1 - x_2 = -2$, drawn in blue, coincides with the line of the first constraint: all the points of the segment between the two vertices marked in blue are optimal solutions of value -2 .



The dual feasible region is the triangle shaded in red; the optimal dual solution $\tilde{\pi}_1 = 1$, $\tilde{\pi}_2 = 0$ is marked in blue. The level curve $-2\pi_1 + 2\pi_2 = -2$, drawn in blue, coincides with the line of the first dual constraint, which touches the triangle only at the optimal vertex.



Dual Simplex algorithm

10.1 Problems with two variables

Exercise 10.1 — Dual simplex algorithm (*P11*)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & -2x_1 & - & 4x_2 & & \\ \text{subject to} & -x_1 & - & x_2 & \leq & -2 \\ & x_1 & - & x_2 & \leq & 0 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$ constraints:

$$\begin{array}{llllll} \min & -2\pi_1 & & & & \\ \text{subject to} & -\pi_1 & + & \pi_2 & \geq & -2 \\ & -\pi_1 & - & \pi_2 & \geq & -4 \\ & \pi_1, & & \pi_2 & \geq & 0 \end{array}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Solve the primal LP problem with the dual simplex algorithm and, in parallel, the dual LP problem with the simplex algorithm, showing at each iteration the dual and the primal dictionaries.

Solution

• Initialization

We shift the indices of the original dual variables by $n = 2$ and we add $n = 2$ surplus variables in front. We then obtain the following standard maximization form:

$$\begin{array}{llllll} \max & & & 2 & \pi_3 & \\ \text{subject to} & -\pi_1 & & - & \pi_3 & + \pi_4 = -2 \\ & & -\pi_2 & - & \pi_3 & - \pi_4 = -4 \\ & \pi_1, & \pi_2, & \pi_3, & \pi_4 & \geq 0 \end{array}$$

The dual initial dictionary is:

$$\left\{ \begin{array}{l} \pi_1 = 2 - \pi_3 + \pi_4 \\ \pi_2 = 4 - \pi_3 - \pi_4 \\ -w = 0 + 2\pi_3 \end{array} \right.$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic dual feasible solution is:

$$\bar{\pi}_1 = 2, \bar{\pi}_2 = 4, \bar{\pi}_3 = 0, \bar{\pi}_4 = 0 \quad \text{of value } 0$$

After having added the slack variables to the primal problem, we obtain the following standard form:

$$\begin{array}{llllll} \max & -2x_1 & - & 4x_2 & & \\ \text{subject to} & -x_1 & - & x_2 & + & x_3 = -2 \\ & x_1 & - & x_2 & & + x_4 = 0 \\ & x_1, & x_2, & x_3, & x_4 & \geq 0 \end{array}$$

The primal initial dictionary is:

$$\left\{ \begin{array}{l} x_3 = -2 + x_1 + x_2 \\ x_4 = 0 - x_1 + x_2 \\ z = 0 - 2x_1 - 4x_2 \end{array} \right.$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic primal non-feasible solution is:

$$\bar{x}_3 = -2, \bar{x}_4 = 0, \bar{x}_1 = 0, \bar{x}_2 = 0 \quad \text{of value } 0$$

• **Iteration 1**

The only basic variable with negative value is x_3 , which is the pivot row. Correspondingly, in the dual dictionary we increase the variable π_3 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_1 = 2 - \pi_3 \\ \pi_2 = 4 - \pi_3 \end{cases} \quad \begin{cases} \underbrace{2 - \pi_3 \geq 0}_{\pi_1 \geq 0} \\ \underbrace{4 - \pi_3 \geq 0}_{\pi_2 \geq 0} \end{cases} \quad \begin{cases} \pi_3 \leq 2 \\ \pi_3 \leq 4 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_3 : $s_m = \min \left\{ \frac{-2}{-1}, \frac{-4}{-1} \right\} = 2$, attained by x_1 . By increasing the value of π_3 to 2, the value of π_1 decreases to 0. Then we have that:

π_3 becomes dual basic and, accordingly, x_3 becomes primal non-basic;
 π_1 becomes dual non-basic and, accordingly, x_1 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_1 = 2 - \pi_3 + \pi_4 \quad \text{then} \quad \pi_3 = 2 - \pi_1 + \pi_4.$$

Substituting in the other equations of the dual dictionary, we get:

$$\pi_2 = 4 - (2 - \pi_1 + \pi_4) - \pi_4 = 2 + \pi_1 - 2\pi_4$$

$$-w = 0 + 2(2 - \pi_1 + \pi_4) = 4 - 2\pi_1 + 2\pi_4$$

Accordingly, we now have the following dual dictionary:

$$\begin{cases} \pi_2 = 2 + \pi_1 - 2\pi_4 \\ \pi_3 = 2 - \pi_1 + \pi_4 \\ -w = 4 - 2\pi_1 + 2\pi_4 \end{cases}$$

where $\mathcal{B} = \{2, 3\}$ and $\mathcal{N} = \{1, 4\}$. The basic dual feasible solution is:

$$\bar{\pi}_2 = 2, \quad \bar{\pi}_3 = 2, \quad \bar{\pi}_1 = 0, \quad \bar{\pi}_4 = 0 \quad \text{of value} \quad -4$$

– For the primal LP problem, we have:

$$x_3 = -2 + x_1 + x_2 \quad \text{then} \quad x_1 = 2 - x_2 + x_3.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_4 = 0 - (2 + x_3 - x_2) + x_2 = -2 + 2x_2 - x_3$$

$$z = 0 - 2(2 + x_3 - x_2) - 4x_2 = -4 - 2x_2 - 2x_3$$

Accordingly, we now have the following primal dictionary:

$$\begin{cases} x_1 = 2 - x_2 + x_3 \\ x_4 = -2 + 2x_2 - x_3 \\ z = -4 - 2x_2 - 2x_3 \end{cases}$$

where $\mathcal{B} = \{1, 4\}$ and $\mathcal{N} = \{2, 3\}$. The basic primal non-feasible solution is:

$$\bar{x}_1 = 2, \quad \bar{x}_4 = -2, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad -4$$

- **Iteration 2**

The only basic variable with negative value is x_4 , which is the pivot row. Correspondingly, in the dual dictionary we increase the variable π_4 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_2 = 2 - 2\pi_4 \\ \pi_3 = 2 + \pi_4 \end{cases} \quad \begin{cases} \underbrace{2 - 2\pi_4 \geq 0}_{\pi_2 \geq 0} \\ \underbrace{2 + \pi_4 \geq 0}_{\pi_3 \geq 0} \end{cases} \quad \begin{cases} \pi_4 \leq 1 \\ \pi_4 \geq -2 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_4 : $s_m = \frac{-2}{-2} = 1$, attained by x_2 ($\bar{a}_{43} = 1 > 0$ gives no candidate). By increasing the value of π_4 to 1, the value of π_2 decreases to 0. Then we have that:

π_4 becomes dual basic and, accordingly, x_4 becomes primal non-basic;

π_2 becomes dual non-basic and, accordingly, x_2 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_2 = 2 - 2\pi_4 + \pi_1 \quad \text{then} \quad \pi_4 = 1 + \frac{1}{2}\pi_1 - \frac{1}{2}\pi_2.$$

Substituting in the other equations of the dual dictionary, we get:

$$\begin{aligned} \pi_3 &= 2 + \left(1 + \frac{1}{2}\pi_1 - \frac{1}{2}\pi_2\right) - \pi_1 = 3 - \frac{1}{2}\pi_1 - \frac{1}{2}\pi_2 \\ -w &= 4 + 2 \left(1 + \frac{1}{2}\pi_1 - \frac{1}{2}\pi_2\right) - 2\pi_1 = 6 - \pi_1 - \pi_2 \end{aligned}$$

Accordingly, we now have the following optimal dual dictionary:

$$\begin{cases} \pi_3 = 3 - \frac{1}{2} \pi_1 - \frac{1}{2} \pi_2 \\ \pi_4 = 1 + \frac{1}{2} \pi_1 - \frac{1}{2} \pi_2 \\ -w = 6 - \pi_1 - \pi_2 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. All the coefficients of the objective function are non-positive, and then the dual basic optimal solution is:

$$\tilde{\pi}_3 = 3, \quad \tilde{\pi}_4 = 1, \quad \tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0 \quad \text{of value} \quad -6$$

Recalling that the original dual variables are the shifted variables π_3, π_4 , the optimal solution of the original dual LP problem is:

$$\tilde{\pi}_1 = 3, \quad \tilde{\pi}_2 = 1, \quad \text{and} \quad z(D(LP)) = -6.$$

– For the primal LP problem, we have:

$$x_4 = -2 - x_3 + 2x_2 \quad \text{then} \quad x_2 = 1 + \frac{1}{2}x_3 + \frac{1}{2}x_4.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_1 = 2 + x_3 - \left(1 + \frac{1}{2}x_3 + \frac{1}{2}x_4\right) = 1 + \frac{1}{2}x_3 - \frac{1}{2}x_4$$

$$z = -4 - 2x_3 - 2\left(1 + \frac{1}{2}x_3 + \frac{1}{2}x_4\right) = -6 - 3x_3 - x_4$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = 1 + \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_2 = 1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ z = -6 - 3x_3 - x_4 \end{cases}$$

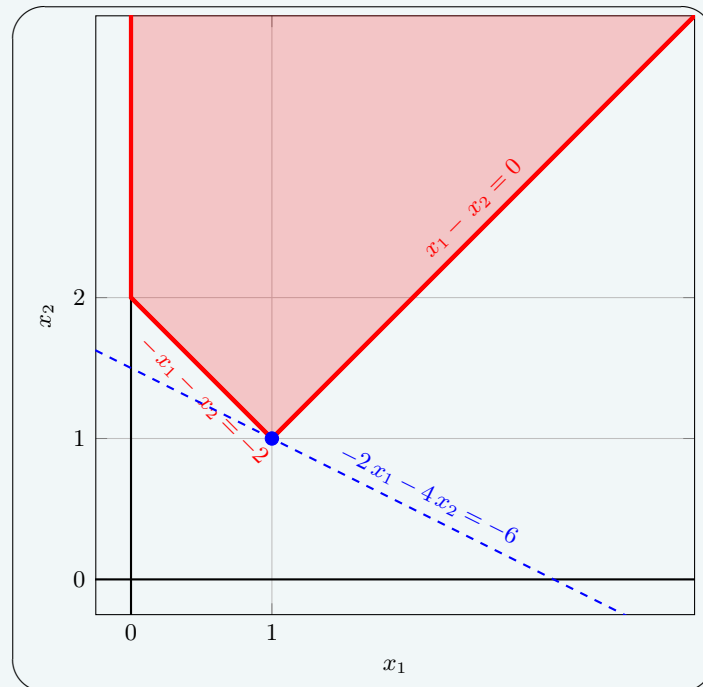
where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic solution is now primal feasible and dual feasible, and then the primal basic optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad -6,$$

in accordance with the Strong Duality theorem.

Graphical solution:

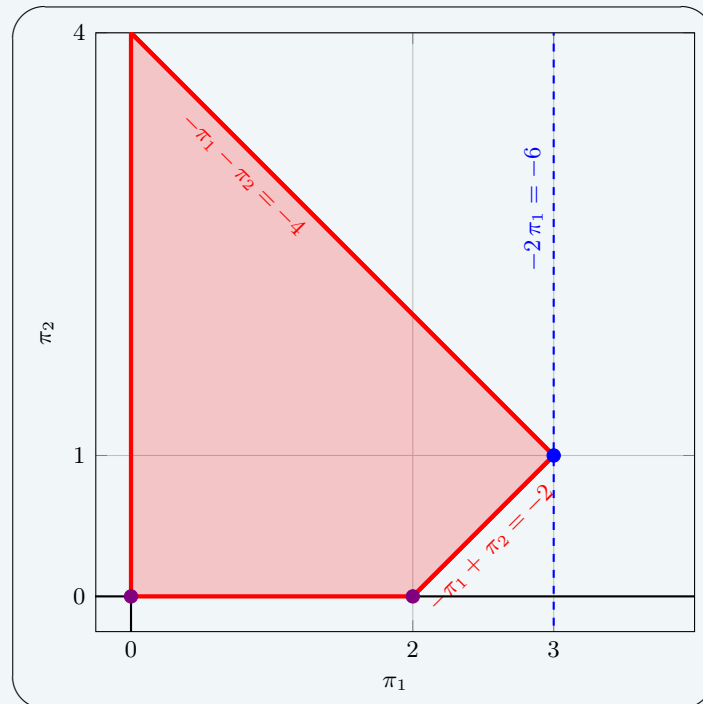
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1 \quad \text{and} \quad z(LP) = -6$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the two original dual variables π_1, π_2 , corresponding to the variables π_3, π_4 of the dual dictionaries.



An optimal dual LP solution is:

$$\tilde{\pi}_1 = 3, \quad \tilde{\pi}_2 = 1 \quad \text{and} \quad z(D(LP)) = -6.$$

In violet, we show the first and the second vertex explored by the previous execution of the dual simplex algorithm, before reaching the optimal one depicted in blue.

Exercise 10.2 — Dual simplex algorithm (*P12*)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & -x_1 - x_2 \leq -3 \\ & x_1 - 2x_2 \leq -1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$ constraints:

$$\begin{array}{llllll}
 \min & -3 & \pi_1 & - & \pi_2 & \\
 \text{subject to} & - & \pi_1 & + & \pi_2 & \geq -1 \\
 & & - & \pi_1 & - & 2 \pi_2 \geq -2 \\
 & & & \pi_1, & \pi_2 & \geq 0
 \end{array}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Solve the primal LP problem with the dual simplex algorithm and, in parallel, the dual LP problem with the simplex algorithm, showing at each iteration the dual and the primal dictionaries.

Solution

- Initialization**

We shift the indices of the original dual variables by $n = 2$ and we add $n = 2$ surplus variables in front. We then obtain the following standard maximization form:

$$\begin{array}{llllll}
 \max & & & 3 & \pi_3 & + & \pi_4 \\
 \text{subject to} & - & \pi_1 & & - & \pi_3 & + & \pi_4 & = & -1 \\
 & & & - & \pi_2 & - & \pi_3 & - & 2 & \pi_4 & = & -2 \\
 & & \pi_1, & \pi_2, & \pi_3, & \pi_4 & \geq & 0
 \end{array}$$

The dual initial dictionary is:

$$\left\{ \begin{array}{l}
 \pi_1 = 1 - \pi_3 + \pi_4 \\
 \pi_2 = 2 - \pi_3 - 2\pi_4 \\
 -w = 0 + 3\pi_3 + \pi_4
 \end{array} \right.$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic dual feasible solution is:

$$\bar{\pi}_1 = 1, \quad \bar{\pi}_2 = 2, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0$$

with $-w = 0$, i.e., $w = 0$.

After having added the slack variables to the primal problem, we obtain the following standard form:

$$\begin{array}{rcllclclcl}
 \max & - & x_1 & - & 2 & x_2 & & & \\
 \text{subject to} & - & x_1 & - & & x_2 & + & x_3 & = & -3 \\
 & & x_1 & - & 2 & x_2 & & & + & x_4 & = & -1 \\
 & & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
 \end{array}$$

The primal initial dictionary is:

$$\begin{cases}
 x_3 = -3 + x_1 + x_2 \\
 x_4 = -1 - x_1 + 2x_2 \\
 z = 0 - x_1 - 2x_2
 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic primal non-feasible solution is:

$$\bar{x}_3 = -3, \quad \bar{x}_4 = -1, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value} \quad 0$$

• **Iteration 1**

In the primal dictionary, the basic variables with negative value are x_3 and x_4 : by the smallest-index rule the pivot row is x_3 . Correspondingly, in the dual dictionary we increase the variable π_3 , and we can get a better dual solution. We have:

$$\begin{cases}
 \pi_1 = 1 - \pi_3 \\
 \pi_2 = 2 - \pi_3
 \end{cases}
 \quad
 \begin{cases}
 \underbrace{1 - \pi_3 \geq 0}_{\pi_1 \geq 0} \\
 \underbrace{2 - \pi_3 \geq 0}_{\pi_2 \geq 0}
 \end{cases}
 \quad
 \begin{cases}
 \pi_3 \leq 1 \\
 \pi_3 \leq 2
 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_3 : $s_m = \min \left\{ \frac{-1}{-1}, \frac{-2}{-1} \right\} = 1$, attained by x_1 . By increasing the value of π_3 to 1, the value of π_1 decreases to 0. Then we have that:

π_3 becomes dual basic and, accordingly, x_3 becomes primal non-basic;

π_1 becomes dual non-basic and, accordingly, x_1 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_1 = 1 - \pi_3 + \pi_4 \quad \text{then} \quad \pi_3 = 1 - \pi_1 + \pi_4.$$

Substituting in the other equations of the dual dictionary, we get:

$$\pi_2 = 2 - (1 - \pi_1 + \pi_4) - 2\pi_4 = 1 + \pi_1 - 3\pi_4$$

$$-w = 3(1 - \pi_1 + \pi_4) + \pi_4 = 3 - 3\pi_1 + 4\pi_4$$

Accordingly, we now have the following dual dictionary:

$$\begin{cases} \pi_2 = 1 + \pi_1 - 3\pi_4 \\ \pi_3 = 1 - \pi_1 + \pi_4 \\ -w = 3 - 3\pi_1 + 4\pi_4 \end{cases}$$

where $\mathcal{B} = \{2, 3\}$ and $\mathcal{N} = \{1, 4\}$. The basic dual feasible solution is:

$$\bar{\pi}_2 = 1, \quad \bar{\pi}_3 = 1, \quad \bar{\pi}_1 = 0, \quad \bar{\pi}_4 = 0, \quad -w = 3, \quad w = -3.$$

– For the primal LP problem, we have:

$$x_3 = -3 + x_1 + x_2 \quad \text{then} \quad x_1 = 3 - x_2 + x_3.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_4 = -1 - (3 - x_2 + x_3) + 2x_2 = -4 + 3x_2 - x_3$$

$$z = -(3 - x_2 + x_3) - 2x_2 = -3 - x_2 - x_3$$

Accordingly, we now have the following primal dictionary:

$$\begin{cases} x_1 = 3 - x_2 + x_3 \\ x_4 = -4 + 3x_2 - x_3 \\ z = -3 - x_2 - x_3 \end{cases}$$

where $\mathcal{B} = \{1, 4\}$ and $\mathcal{N} = \{2, 3\}$. The basic primal non-feasible solution is:

$$\bar{x}_1 = 3, \quad \bar{x}_4 = -4, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad -3$$

• Iteration 2

The only basic variable with negative value is x_4 , which is the pivot row. Correspondingly, in the dual dictionary we increase the variable π_4 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_2 = 1 - 3\pi_4 \\ \pi_3 = 1 + \pi_4 \end{cases} \quad \begin{cases} \underbrace{1 - 3\pi_4}_{\pi_2 \geq 0} \geq 0 \\ \underbrace{1 + \pi_4}_{\pi_3 \geq 0} \geq 0 \end{cases} \quad \begin{cases} \pi_4 \leq \frac{1}{3} \\ \pi_4 \geq -1 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_4 : $s_m = \frac{-1}{-3} = \frac{1}{3}$, attained by x_2 ($\bar{a}_{43} = 1 > 0$ gives no candidate). By increasing the value of π_4 to $\frac{1}{3}$, the value of π_2 decreases to 0. Then we have that:

π_4 becomes dual basic and, accordingly, x_4 becomes primal non-basic;
 π_2 becomes dual non-basic and, accordingly, x_2 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_2 = 1 + \pi_1 - 3\pi_4 \quad \text{then} \quad \pi_4 = \frac{1}{3} + \frac{1}{3}\pi_1 - \frac{1}{3}\pi_2.$$

Substituting in the other equations of the dual dictionary, we get:

$$\begin{aligned} \pi_3 &= 1 - \pi_1 + \left(\frac{1}{3} + \frac{1}{3}\pi_1 - \frac{1}{3}\pi_2 \right) = \frac{4}{3} - \frac{2}{3}\pi_1 - \frac{1}{3}\pi_2 \\ -w &= 3 - 3\pi_1 + 4 \left(\frac{1}{3} + \frac{1}{3}\pi_1 - \frac{1}{3}\pi_2 \right) = \frac{13}{3} - \frac{5}{3}\pi_1 - \frac{4}{3}\pi_2. \end{aligned}$$

Accordingly, we now have the following optimal dual dictionary:

$$\begin{cases} \pi_3 = \frac{4}{3} - \frac{2}{3}\pi_1 - \frac{1}{3}\pi_2 \\ \pi_4 = \frac{1}{3} + \frac{1}{3}\pi_1 - \frac{1}{3}\pi_2 \\ -w = \frac{13}{3} - \frac{5}{3}\pi_1 - \frac{4}{3}\pi_2 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. All the coefficients of the objective function are non-positive, and then the dual basic optimal solution is:

$$\tilde{\pi}_3 = \frac{4}{3}, \quad \tilde{\pi}_4 = \frac{1}{3}, \quad \tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad -w = \frac{13}{3}, \quad w = -\frac{13}{3}.$$

Recalling that the original dual variables are the shifted variables π_3, π_4 , the optimal solution of the original dual LP problem is:

$$\tilde{\pi}_1 = \frac{4}{3}, \quad \tilde{\pi}_2 = \frac{1}{3}, \quad \text{and} \quad z(D(LP)) = -\frac{13}{3}.$$

– For the primal LP problem, we have:

$$x_4 = -4 + 3x_2 - x_3 \quad \text{then} \quad x_2 = \frac{4}{3} + \frac{1}{3}x_3 + \frac{1}{3}x_4.$$

Substituting in the other equations of the primal dictionary, we get:

$$\begin{aligned} x_1 &= 3 - \left(\frac{4}{3} + \frac{1}{3}x_3 + \frac{1}{3}x_4 \right) + x_3 = \frac{5}{3} + \frac{2}{3}x_3 - \frac{1}{3}x_4 \\ z &= -3 - \left(\frac{4}{3} + \frac{1}{3}x_3 + \frac{1}{3}x_4 \right) - x_3 = -\frac{13}{3} - \frac{4}{3}x_3 - \frac{1}{3}x_4 \end{aligned}$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = \frac{5}{3} + \frac{2}{3} x_3 - \frac{1}{3} x_4 \\ x_2 = \frac{4}{3} + \frac{1}{3} x_3 + \frac{1}{3} x_4 \\ z = -\frac{13}{3} - \frac{4}{3} x_3 - \frac{1}{3} x_4 \end{cases}$$

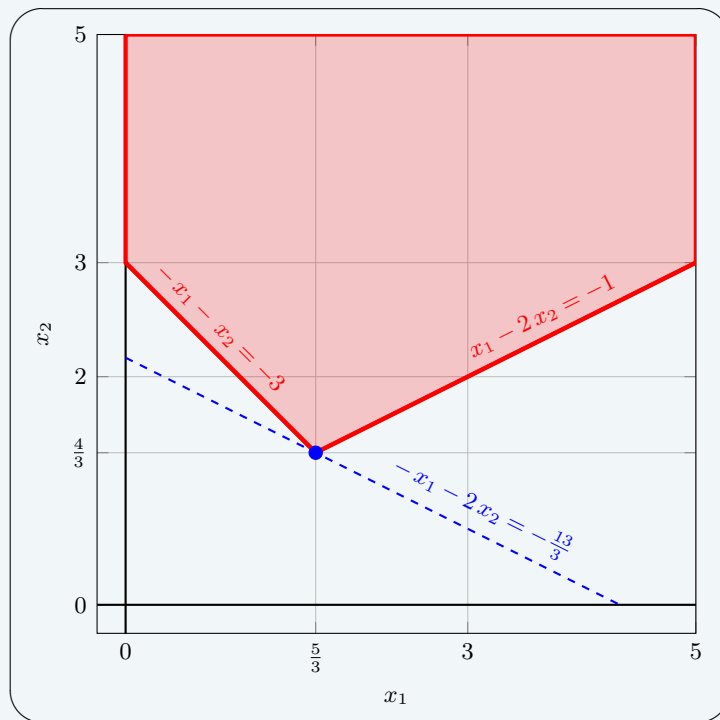
where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic solution is now primal feasible and dual feasible, and then the primal basic optimal solution is:

$$\tilde{x}_1 = \frac{5}{3}, \quad \tilde{x}_2 = \frac{4}{3}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad -\frac{13}{3},$$

in accordance with the Strong Duality theorem.

Graphical solution:

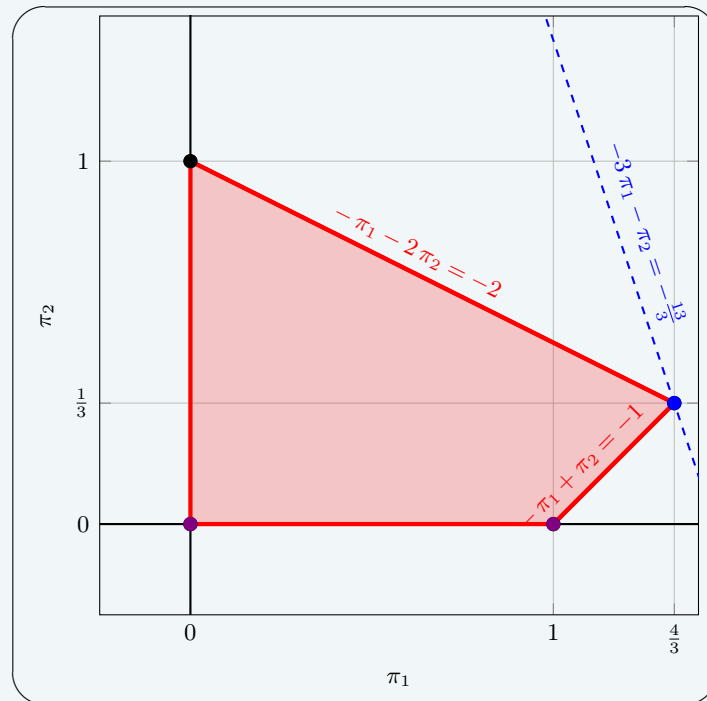
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{5}{3}, \quad \tilde{x}_2 = \frac{4}{3} \quad \text{and} \quad z(LP) = -\frac{13}{3}.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the two original dual variables π_1, π_2 , corresponding to the variables π_3, π_4 of the dual dictionaries.



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{4}{3}, \quad \tilde{\pi}_2 = \frac{1}{3}, \quad \text{and} \quad z(D(LP)) = -\frac{13}{3}.$$

In violet, we show the first and the second vertex explored by the previous execution of the dual simplex algorithm, before reaching the optimal one depicted in blue.

Exercise 10.3 — Dual simplex algorithm (P13)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{aligned} \max \quad & -x_1 - 3x_2 \\ \text{subject to} \quad & -x_1 - 2x_2 \leq -6 \\ & x_1 - 4x_2 \leq -6 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

and its dual LP problem in canonical minimization form with $m = 3$ variables and $n = 2$ constraints:

$$\begin{aligned}
 \min \quad & -6 \pi_1 - 6 \pi_2 + 3 \pi_3 \\
 \text{subject to} \quad & -\pi_1 + \pi_2 + 2 \pi_3 \geq -1 \\
 & -2 \pi_1 - 4 \pi_2 - 2 \pi_3 \geq -3 \\
 & \pi_1, \quad \pi_2, \quad \pi_3 \geq 0
 \end{aligned}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Solve the primal LP problem with the dual simplex algorithm and, in parallel, the dual LP problem with the simplex algorithm, showing at each iteration the dual and the primal dictionaries.

Solution

- **Initialization**

We shift the indices of the original dual variables by $n = 2$ and we add $n = 2$ surplus variables in front. We then obtain the following standard maximization form:

$$\begin{aligned}
 \max \quad & 6 \pi_3 + 6 \pi_4 - 3 \pi_5 \\
 \text{subject to} \quad & -\pi_1 - \pi_3 + \pi_4 + 2 \pi_5 = -1 \\
 & -\pi_2 - 2 \pi_3 - 4 \pi_4 - 2 \pi_5 = -3 \\
 & \pi_1, \quad \pi_2, \quad \pi_3, \quad \pi_4, \quad \pi_5 \geq 0
 \end{aligned}$$

The dual initial dictionary is:

$$\begin{cases}
 \pi_1 = 1 - \pi_3 + \pi_4 + 2 \pi_5 \\
 \pi_2 = 3 - 2 \pi_3 - 4 \pi_4 - 2 \pi_5 \\
 -w = 0 + 6 \pi_3 + 6 \pi_4 - 3 \pi_5
 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4, 5\}$. The basic dual feasible solution is:

$$\bar{\pi}_1 = 1, \quad \bar{\pi}_2 = 3, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0$$

with $-w = 0$, i.e., $w = 0$.

After having added the slack variables to the primal problem, we obtain the following standard form:

$$\begin{array}{rcllclclclcl}
 \max & - & x_1 & - & 3 & x_2 & & & & & \\
 \text{subject to} & - & x_1 & - & 2 & x_2 & + & x_3 & & & = & -6 \\
 & & x_1 & - & 4 & x_2 & & & + & x_4 & & = & -6 \\
 & & 2 & x_1 & - & 2 & x_2 & & & & + & x_5 & = & 3 \\
 & & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
 \end{array}$$

The primal initial dictionary is:

$$\left\{ \begin{array}{l} x_3 = -6 + x_1 + 2x_2 \\ x_4 = -6 - x_1 + 4x_2 \\ x_5 = 3 - 2x_1 + 2x_2 \\ z = 0 - x_1 - 3x_2 \end{array} \right.$$

where $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$. The basic primal non-feasible solution is:

$$\bar{x}_3 = -6, \quad \bar{x}_4 = -6, \quad \bar{x}_5 = 3, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

• **Iteration 1**

In the primal dictionary, the basic variables with negative value are x_3 and x_4 : by the smallest-index rule the pivot row is x_3 . Correspondingly, in the dual dictionary we increase the variable π_3 , and we can get a better dual solution. We have:

$$\left\{ \begin{array}{l} \pi_1 = 1 - \pi_3 \\ \pi_2 = 3 - 2\pi_3 \end{array} \right. \quad \left\{ \begin{array}{l} 1 - \pi_3 \geq 0 \\ \pi_3 \geq 0 \\ 3 - 2\pi_3 \geq 0 \\ \pi_3 \leq 0 \end{array} \right. \quad \left\{ \begin{array}{l} \pi_3 \leq 1 \\ \pi_3 \leq \frac{3}{2} \end{array} \right.$$

These are exactly the ratios of the dual simplex algorithm on the row of x_3 : $s_m = \min \left\{ \frac{-1}{-1}, \frac{-3}{-2} \right\} = 1$, attained by x_1 . By increasing the value of π_3 to 1, the value of π_1 decreases to 0. Then we have that:

π_3 becomes dual basic and, accordingly, x_3 becomes primal non-basic;
 π_1 becomes dual non-basic and, accordingly, x_1 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_1 = 1 - \pi_3 + \pi_4 + 2\pi_5 \quad \text{then} \quad \pi_3 = 1 - \pi_1 + \pi_4 + 2\pi_5.$$

Substituting in the other equations of the dual dictionary, we get:

$$\pi_2 = 3 - 2(1 - \pi_1 + \pi_4 + 2\pi_5) - 4\pi_4 - 2\pi_5 = 1 + 2\pi_1 - 6\pi_4 - 6\pi_5$$

$$-w = 0 + 6(1 - \pi_1 + \pi_4 + 2\pi_5) + 6\pi_4 - 3\pi_5 = 6 - 6\pi_1 + 12\pi_4 + 9\pi_5$$

Accordingly, we now have the following dual dictionary:

$$\begin{cases} \pi_2 = 1 + 2\pi_1 - 6\pi_4 - 6\pi_5 \\ \pi_3 = 1 - \pi_1 + \pi_4 + 2\pi_5 \\ -w = 6 - 6\pi_1 + 12\pi_4 + 9\pi_5 \end{cases}$$

where $\mathcal{B} = \{2, 3\}$ and $\mathcal{N} = \{1, 4, 5\}$. The basic dual feasible solution is:

$$\bar{\pi}_2 = 1, \quad \bar{\pi}_3 = 1, \quad \bar{\pi}_1 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0, \quad -w = 6, \quad w = -6.$$

– For the primal LP problem, we have:

$$x_3 = -6 + x_1 + 2x_2 \quad \text{then} \quad x_1 = 6 - 2x_2 + x_3.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_4 = -6 - (6 - 2x_2 + x_3) + 4x_2 = -12 + 6x_2 - x_3$$

$$x_5 = 3 - 2(6 - 2x_2 + x_3) + 2x_2 = -9 + 6x_2 - 2x_3$$

$$z = 0 - (6 - 2x_2 + x_3) - 3x_2 = -6 - x_2 - x_3$$

Accordingly, we now have the following primal dictionary:

$$\begin{cases} x_1 = 6 - 2x_2 + x_3 \\ x_4 = -12 + 6x_2 - x_3 \\ x_5 = -9 + 6x_2 - 2x_3 \\ z = -6 - x_2 - x_3 \end{cases}$$

where $\mathcal{B} = \{1, 4, 5\}$ and $\mathcal{N} = \{2, 3\}$. The basic primal non-feasible solution is:

$$\bar{x}_1 = 6, \quad \bar{x}_4 = -12, \quad \bar{x}_5 = -9, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad -6$$

• **Iteration 2**

In the primal dictionary, the basic variables with negative value are x_4 and x_5 : by the smallest-index rule the pivot row is x_4 . Correspondingly, in the dual dictionary we increase the variable π_4 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_2 = 1 - 6\pi_4 \\ \pi_3 = 1 + \pi_4 \end{cases} \quad \begin{cases} \underbrace{1 - 6\pi_4}_{\pi_2 \geq 0} \geq 0 \\ \underbrace{1 + \pi_4}_{\pi_3 \geq 0} \geq 0 \end{cases} \quad \begin{cases} \pi_4 \leq \frac{1}{6} \\ \pi_4 \geq -1 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_4 : $s_m = \frac{-1}{-6} = \frac{1}{6}$, attained by x_2 ($\bar{a}_{43} = 1 > 0$ gives no candidate). By increasing the value of π_4 to $\frac{1}{6}$, the value of π_2 decreases to 0. Then we have that:

π_4 becomes dual basic and, accordingly, x_4 becomes primal non-basic;
 π_2 becomes dual non-basic and, accordingly, x_2 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_2 = 1 + 2\pi_1 - 6\pi_4 - 6\pi_5 \quad \text{then} \quad \pi_4 = \frac{1}{6} + \frac{1}{3}\pi_1 - \frac{1}{6}\pi_2 - \pi_5.$$

Substituting in the other equations of the dual dictionary, we get:

$$\begin{aligned} \pi_3 &= 1 - \pi_1 + \left(\frac{1}{6} + \frac{1}{3}\pi_1 - \frac{1}{6}\pi_2 - \pi_5 \right) + 2\pi_5 = \frac{7}{6} - \frac{2}{3}\pi_1 - \frac{1}{6}\pi_2 + \pi_5 \\ -w &= 6 - 6\pi_1 + 12 \left(\frac{1}{6} + \frac{1}{3}\pi_1 - \frac{1}{6}\pi_2 - \pi_5 \right) + 9\pi_5 = 8 - 2\pi_1 - 2\pi_2 - 3\pi_5. \end{aligned}$$

Accordingly, we now have the following optimal dual dictionary:

$$\begin{cases} \pi_3 = \frac{7}{6} - \frac{2}{3}\pi_1 - \frac{1}{6}\pi_2 + \pi_5 \\ \pi_4 = \frac{1}{6} + \frac{1}{3}\pi_1 - \frac{1}{6}\pi_2 - \pi_5 \\ -w = 8 - 2\pi_1 - 2\pi_2 - 3\pi_5 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2, 5\}$. All the coefficients of the objective function are non-positive, and then the dual basic optimal solution is:

$$\tilde{\pi}_3 = \frac{7}{6}, \quad \tilde{\pi}_4 = \frac{1}{6}, \quad \tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_5 = 0, \quad -w = 8, \quad w = -8.$$

Recalling that the original dual variables are the shifted variables π_3, π_4, π_5 , the optimal solution of the original dual LP problem is:

$$\tilde{\pi}_1 = \frac{7}{6}, \quad \tilde{\pi}_2 = \frac{1}{6}, \quad \tilde{\pi}_3 = 0, \quad \text{and} \quad z(D(LP)) = -8.$$

– For the primal LP problem, we have:

$$x_4 = -12 + 6x_2 - x_3 \quad \text{then} \quad x_2 = 2 + \frac{1}{6}x_3 + \frac{1}{6}x_4.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_1 = 6 - 2 \left(2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \right) + x_3 = 2 + \frac{2}{3}x_3 - \frac{1}{3}x_4$$

$$x_5 = -9 + 6 \left(2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \right) - 2x_3 = 3 - x_3 + x_4$$

$$z = -6 - \left(2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \right) - x_3 = -8 - \frac{7}{6}x_3 - \frac{1}{6}x_4$$

Accordingly, we now have the following optimal primal dictionary:

$$\left\{ \begin{array}{l} x_1 = 2 + \frac{2}{3}x_3 - \frac{1}{3}x_4 \\ x_2 = 2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \\ x_5 = 3 - x_3 + x_4 \\ z = -8 - \frac{7}{6}x_3 - \frac{1}{6}x_4 \end{array} \right.$$

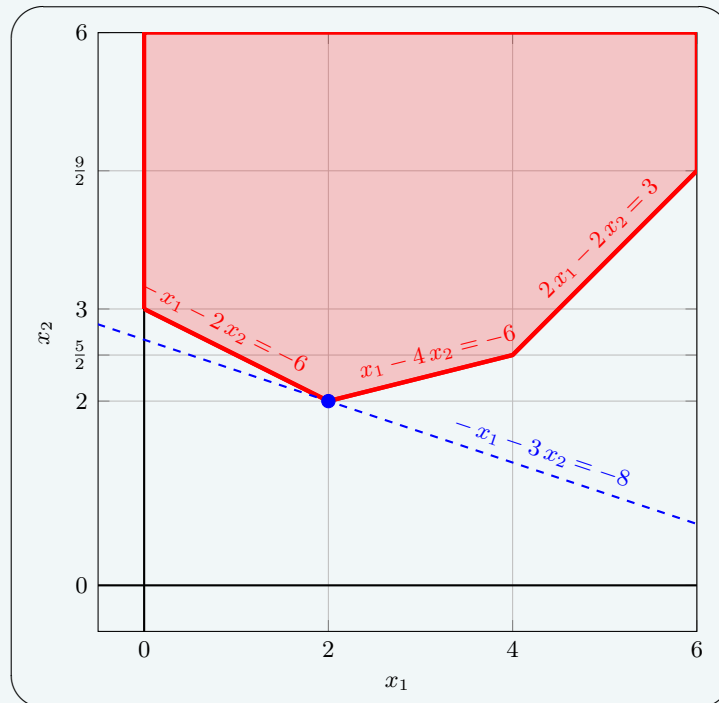
where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$. The basic solution is now primal feasible and dual feasible, and then the primal basic optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 2, \quad \tilde{x}_5 = 3, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad -8,$$

in accordance with the Strong Duality theorem.

Graphical solution:

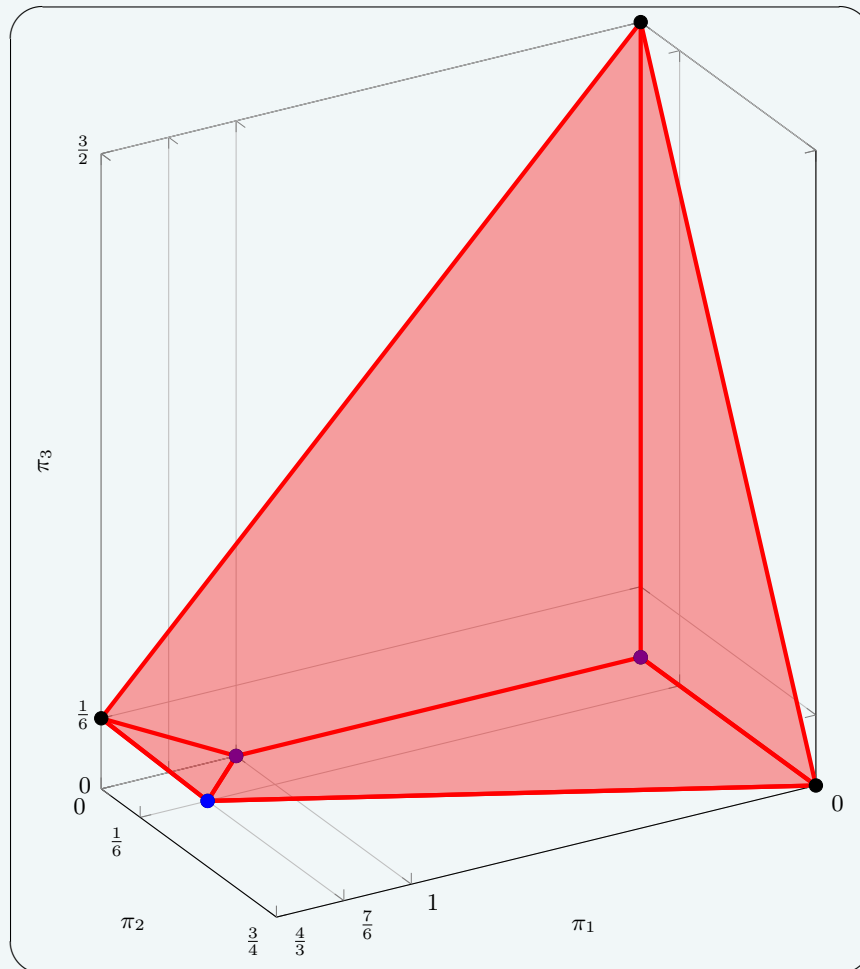
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 2 \quad \text{and} \quad z(LP) = -8.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the three original dual variables π_1, π_2, π_3 , corresponding to the variables π_3, π_4, π_5 of the dual dictionaries.



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{7}{6}, \quad \tilde{\pi}_2 = \frac{1}{6}, \quad \tilde{\pi}_3 = 0, \quad \text{and} \quad z(D(LP)) = -8.$$

In violet, we show the first and the second vertex explored by the previous execution of the dual simplex algorithm, before reaching the optimal one depicted in blue.

SageMath Code: 3D Polyhedra (<https://sagecell.sagemath.org/>)

```
# Variables are (pi_3, pi_4, pi_5), i.e., the original dual variables.
# Define the polyhedron using inequalities (Hyperplane Representation)
p = Polyhedron(ieqs=[
    (1, -1, 1, 2),      # 1 - pi_3 + pi_4 + 2*pi_5 >= 0
    (3, -2, -4, -2),    # 3 - 2*pi_3 - 4*pi_4 - 2*pi_5 >= 0
    (0, 1, 0, 0),       # pi_3 >= 0
    (0, 0, 1, 0),       # pi_4 >= 0
    (0, 0, 0, 1)        # pi_5 >= 0
])

# Display the inequalities of the polyhedron
p.Hrepresentation()

# Display the extreme points and extreme rays of the polyhedron
p.Vrepresentation()

# Plot the polyhedron (3D)
p.plot()
```

Exercise 10.4 — Dual simplex algorithm (*P14*)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{aligned}
 \max \quad & -x_1 - 2x_2 \\
 \text{subject to} \quad & -x_1 + x_2 \leq 2 \\
 & x_1 - 2x_2 \leq 2 \\
 & -x_1 - x_2 \leq -3 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

and its dual LP problem in canonical minimization form with $m = 3$ variables and $n = 2$ constraints:

$$\begin{aligned}
 \min \quad & 2\pi_1 + 2\pi_2 - 3\pi_3 \\
 \text{subject to} \quad & -\pi_1 + \pi_2 - \pi_3 \geq -1 \\
 & \pi_1 - 2\pi_2 - \pi_3 \geq -2 \\
 & \pi_1, \pi_2, \pi_3 \geq 0
 \end{aligned}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Solve the primal LP problem with the dual simplex algorithm and, in parallel, the dual LP problem with the simplex algorithm, showing at each iteration the dual and the primal dictionaries.

Solution

- **Initialization**

We shift the indices of the original dual variables by $n = 2$ and we add $n = 2$ surplus variables in front. We then obtain the following standard maximization form:

$$\begin{array}{llllllll} \max & & & -2 & \pi_3 & - & 2 & \pi_4 & + & 3 & \pi_5 \\ \text{subject to} & - & \pi_1 & & & - & & \pi_3 & + & & \pi_4 & - & & \pi_5 & = & -1 \\ & & & - & \pi_2 & + & & \pi_3 & - & 2 & \pi_4 & - & & \pi_5 & = & -2 \\ & & \pi_1, & & \pi_2, & & \pi_3, & & \pi_4, & & \pi_5 & \geq & & 0 \end{array}$$

The dual initial dictionary is:

$$\left\{ \begin{array}{l} \pi_1 = 1 - \pi_3 + \pi_4 - \pi_5 \\ \pi_2 = 2 + \pi_3 - 2\pi_4 - \pi_5 \\ -w = 0 - 2\pi_3 - 2\pi_4 + 3\pi_5 \end{array} \right.$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4, 5\}$. The basic dual feasible solution is:

$$\bar{\pi}_1 = 1, \quad \bar{\pi}_2 = 2, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0$$

with $-w = 0$, i.e., $w = 0$.

After having added the slack variables to the primal problem, we obtain the following standard form:

$$\begin{array}{llllllll} \max & - & x_1 & - & 2 & x_2 & & & & & & & & & & \\ \text{subject to} & - & x_1 & + & & x_2 & + & x_3 & & & & & & & = & 2 \\ & & x_1 & - & 2 & x_2 & & & + & x_4 & & & & & = & 2 \\ & - & x_1 & - & & x_2 & & & & & + & x_5 & & & = & -3 \\ & & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & & 0 \end{array}$$

The primal initial dictionary is:

$$\begin{cases} x_3 = 2 + x_1 - x_2 \\ x_4 = 2 - x_1 + 2x_2 \\ x_5 = -3 + x_1 + x_2 \\ z = 0 - x_1 - 2x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$. The basic primal non-feasible solution is:

$$\bar{x}_3 = 2, \quad \bar{x}_4 = 2, \quad \bar{x}_5 = -3, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

- **Iteration 1**

The only basic variable with negative value is x_5 , which is the pivot row. Correspondingly, in the dual dictionary we increase the variable π_5 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_1 = 1 - \pi_5 \\ \pi_2 = 2 - \pi_5 \end{cases} \quad \begin{cases} \underbrace{1 - \pi_5}_{\pi_1 \geq 0} \geq 0 \\ \underbrace{2 - \pi_5}_{\pi_2 \geq 0} \geq 0 \end{cases} \quad \begin{cases} \pi_5 \leq 1 \\ \pi_5 \leq 2 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_5 : $s_m = \min \left\{ \frac{-1}{-1}, \frac{-2}{-1} \right\} = 1$, attained by x_1 . By increasing the value of π_5 to 1, the value of π_1 decreases to 0. Then we have that:

π_5 becomes dual basic and, accordingly, x_5 becomes primal non-basic;
 π_1 becomes dual non-basic and, accordingly, x_1 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_1 = 1 - \pi_3 + \pi_4 - \pi_5 \quad \text{then} \quad \pi_5 = 1 - \pi_3 + \pi_4 - \pi_1.$$

Substituting in the other equations of the dual dictionary, we get:

$$\begin{aligned} \pi_2 &= 2 + \pi_3 - 2\pi_4 - (1 - \pi_3 + \pi_4 - \pi_1) = 1 + \pi_1 + 2\pi_3 - 3\pi_4 \\ -w &= -2\pi_3 - 2\pi_4 + 3(1 - \pi_3 + \pi_4 - \pi_1) = 3 - 3\pi_1 - 5\pi_3 + \pi_4 \end{aligned}$$

Accordingly, we now have the following dual dictionary:

$$\begin{cases} \pi_2 = 1 + \pi_1 + 2\pi_3 - 3\pi_4 \\ \pi_5 = 1 - \pi_1 - \pi_3 + \pi_4 \\ -w = 3 - 3\pi_1 - 5\pi_3 + \pi_4 \end{cases}$$

where $\mathcal{B} = \{2, 5\}$ and $\mathcal{N} = \{1, 3, 4\}$. The basic dual feasible solution is:

$$\bar{\pi}_2 = 1, \quad \bar{\pi}_5 = 1, \quad \bar{\pi}_1 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0, \quad -w = 3, \quad w = -3.$$

– For the primal LP problem, we have:

$$x_5 = -3 + x_1 + x_2 \quad \text{then} \quad x_1 = 3 - x_2 + x_5.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_3 = 2 + (3 - x_2 + x_5) - x_2 = 5 - 2x_2 + x_5$$

$$x_4 = 2 - (3 - x_2 + x_5) + 2x_2 = -1 + 3x_2 - x_5$$

$$z = -(3 - x_2 + x_5) - 2x_2 = -3 - x_2 - x_5$$

Accordingly, we now have the following primal dictionary:

$$\begin{cases} x_1 = 3 - x_2 + x_5 \\ x_3 = 5 - 2x_2 + x_5 \\ x_4 = -1 + 3x_2 - x_5 \\ z = -3 - x_2 - x_5 \end{cases}$$

where $\mathcal{B} = \{1, 3, 4\}$ and $\mathcal{N} = \{2, 5\}$. The basic primal non-feasible solution is:

$$\bar{x}_1 = 3, \quad \bar{x}_3 = 5, \quad \bar{x}_4 = -1, \quad \bar{x}_2 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad -3$$

• Iteration 2

The only basic variable with negative value is x_4 , which is the pivot row. Correspondingly, in the dual dictionary we increase the variable π_4 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_2 = 1 - 3\pi_4 \\ \pi_5 = 1 + \pi_4 \end{cases} \quad \begin{cases} \underbrace{1 - 3\pi_4}_{\pi_2 \geq 0} \geq 0 \\ \underbrace{1 + \pi_4}_{\pi_5 \geq 0} \geq 0 \end{cases} \quad \begin{cases} \pi_4 \leq \frac{1}{3} \\ \pi_4 \geq -1 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_4 : $s_m = \frac{-1}{-3} = \frac{1}{3}$, attained by x_2 ($\bar{a}_{45} = 1 > 0$ gives no candidate). By increasing the value of π_4 to $\frac{1}{3}$, the value of π_2 decreases to 0. Then we have that:

π_4 becomes dual basic and, accordingly, x_4 becomes primal non-basic;

π_2 becomes dual non-basic and, accordingly, x_2 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_2 = 1 + \pi_1 + 2\pi_3 - 3\pi_4 \quad \text{then} \quad \pi_4 = \frac{1}{3} + \frac{1}{3}\pi_1 + \frac{2}{3}\pi_3 - \frac{1}{3}\pi_2.$$

Substituting in the other equations of the dual dictionary, we get:

$$\pi_5 = 1 - \pi_1 - \pi_3 + \left(\frac{1}{3} + \frac{1}{3}\pi_1 + \frac{2}{3}\pi_3 - \frac{1}{3}\pi_2 \right) = \frac{4}{3} - \frac{2}{3}\pi_1 - \frac{1}{3}\pi_2 - \frac{1}{3}\pi_3$$

$$-w = 3 - 3\pi_1 - 5\pi_3 + \left(\frac{1}{3} + \frac{1}{3}\pi_1 + \frac{2}{3}\pi_3 - \frac{1}{3}\pi_2 \right) = \frac{10}{3} - \frac{8}{3}\pi_1 - \frac{1}{3}\pi_2 - \frac{13}{3}\pi_3.$$

Accordingly, we now have the following optimal dual dictionary:

$$\begin{cases} \pi_4 = \frac{1}{3} + \frac{1}{3}\pi_1 - \frac{1}{3}\pi_2 + \frac{2}{3}\pi_3 \\ \pi_5 = \frac{4}{3} - \frac{2}{3}\pi_1 - \frac{1}{3}\pi_2 - \frac{1}{3}\pi_3 \\ -w = \frac{10}{3} - \frac{8}{3}\pi_1 - \frac{1}{3}\pi_2 - \frac{13}{3}\pi_3 \end{cases}$$

where $\mathcal{B} = \{4, 5\}$ and $\mathcal{N} = \{1, 2, 3\}$. All the coefficients of the objective function are non-positive, and then the dual basic optimal solution is:

$$\tilde{\pi}_4 = \frac{1}{3}, \quad \tilde{\pi}_5 = \frac{4}{3}, \quad \tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0, \quad -w = \frac{10}{3}, \quad w = -\frac{10}{3}.$$

Recalling that the original dual variables are the shifted variables π_3, π_4, π_5 , the optimal solution of the original dual LP problem is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = \frac{1}{3}, \quad \tilde{\pi}_3 = \frac{4}{3}, \quad \text{and} \quad z(D(LP)) = -\frac{10}{3}.$$

– For the primal LP problem, we have:

$$x_4 = -1 + 3x_2 - x_5 \quad \text{then} \quad x_2 = \frac{1}{3} + \frac{1}{3}x_4 + \frac{1}{3}x_5.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_1 = 3 - \left(\frac{1}{3} + \frac{1}{3}x_4 + \frac{1}{3}x_5 \right) + x_5 = \frac{8}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5$$

$$x_3 = 5 - 2 \left(\frac{1}{3} + \frac{1}{3}x_4 + \frac{1}{3}x_5 \right) + x_5 = \frac{13}{3} - \frac{2}{3}x_4 + \frac{1}{3}x_5$$

$$z = -3 - \left(\frac{1}{3} + \frac{1}{3}x_4 + \frac{1}{3}x_5 \right) - x_5 = -\frac{10}{3} - \frac{1}{3}x_4 - \frac{4}{3}x_5$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = \frac{8}{3} - \frac{1}{3}x_4 + \frac{2}{3}x_5 \\ x_2 = \frac{1}{3} + \frac{1}{3}x_4 + \frac{1}{3}x_5 \\ x_3 = \frac{13}{3} - \frac{2}{3}x_4 + \frac{1}{3}x_5 \\ z = -\frac{10}{3} - \frac{1}{3}x_4 - \frac{4}{3}x_5 \end{cases}$$

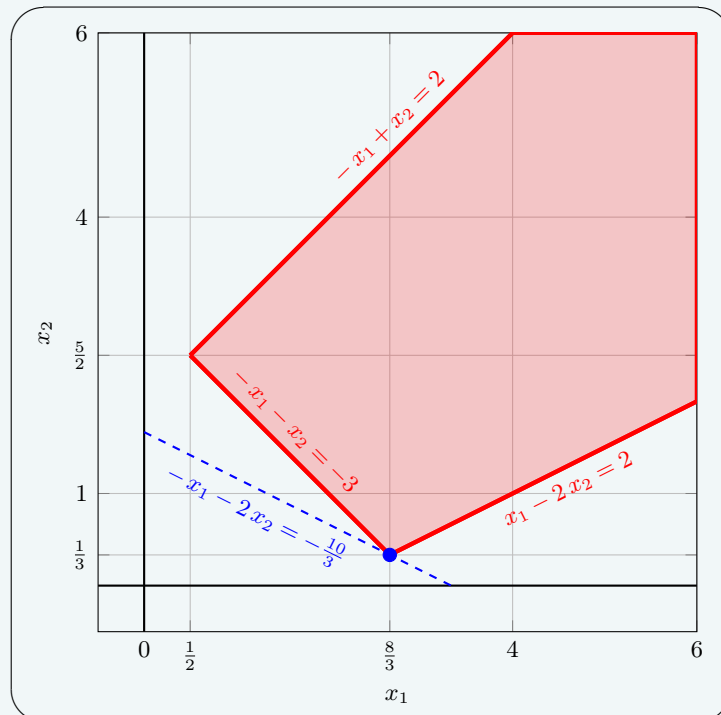
where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$. The basic solution is now primal feasible and dual feasible, and then the primal basic optimal solution is:

$$\tilde{x}_1 = \frac{8}{3}, \quad \tilde{x}_2 = \frac{1}{3}, \quad \tilde{x}_3 = \frac{13}{3}, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad -\frac{10}{3},$$

in accordance with the Strong Duality theorem.

Graphical solution:

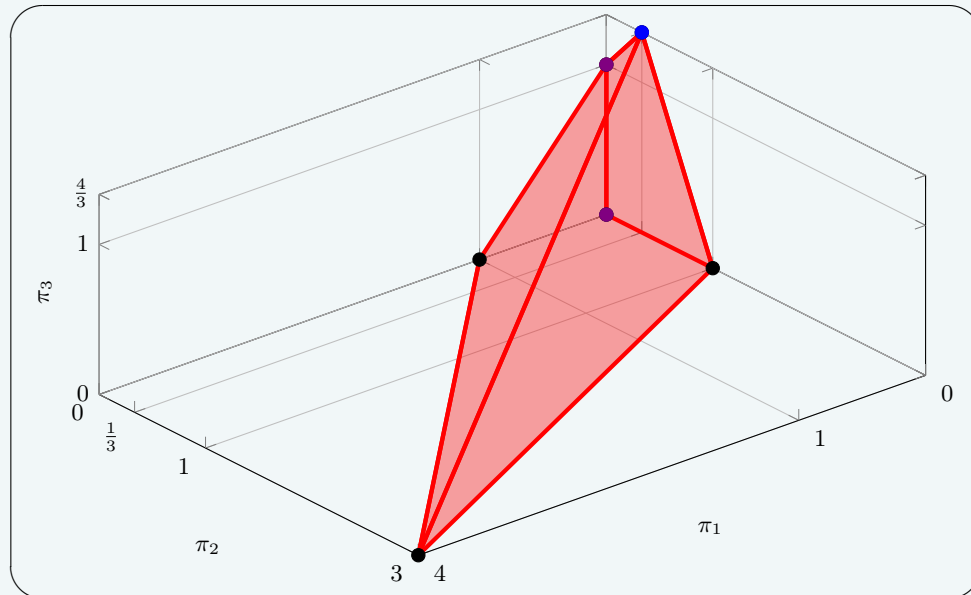
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{8}{3}, \quad \tilde{x}_2 = \frac{1}{3} \quad \text{and} \quad z(LP) = -\frac{10}{3}.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the three original dual variables π_1, π_2, π_3 , corresponding to the variables π_3, π_4, π_5 of the dual dictionaries.



An optimal dual LP solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = \frac{1}{3}, \quad \tilde{\pi}_3 = \frac{4}{3}, \quad \text{and} \quad z(D(LP)) = -\frac{10}{3}.$$

In violet, we show the first and the second vertex explored by the previous execution of the dual simplex algorithm, before reaching the optimal one depicted in blue.

SageMath Code: 3D Polyhedra (<https://sagecell.sagemath.org/>)

```
# Variables are (pi_1, pi_2, pi_3), i.e., the original dual variables.
# Define the polyhedron using inequalities (Hyperplane Representation)
p = Polyhedron(ieqs=[
    (1, -1, 1, -1), # 1 - pi_1 + pi_2 - pi_3 >= 0
    (2, 1, -2, -1), # 2 + pi_1 - 2*pi_2 - pi_3 >= 0
    (0, 1, 0, 0),   # pi_1 >= 0
    (0, 0, 1, 0),   # pi_2 >= 0
    (0, 0, 0, 1)    # pi_3 >= 0
])

# Display the inequalities of the polyhedron
p.Hrepresentation()

# Display the extreme points and extreme rays of the polyhedron
p.Vrepresentation()

# Plot the polyhedron (3D)
p.plot()
```

Exercise 10.5 — Dual simplex algorithm (*P15*)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & -3 & x_1 & - & 2 & x_2 \\ \text{subject to} & - & x_1 & - & & x_2 \leq -4 \\ & & - & x_1 & - & 3 & x_2 \leq -6 \\ & & & x_1, & & x_2 \geq 0 \end{array}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$ constraints:

$$\begin{array}{llllll} \min & -4 & \pi_1 & - & 6 & \pi_2 \\ \text{subject to} & - & \pi_1 & - & & \pi_2 \geq -3 \\ & & - & \pi_1 & - & 3 & \pi_2 \geq -2 \\ & & & \pi_1, & & \pi_2 \geq 0 \end{array}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Solve the primal LP problem with the dual simplex algorithm and, in parallel, the dual LP problem with the simplex algorithm, showing at each iteration the dual and the primal dictionaries.

Solution

- Initialization**

We shift the indices of the original dual variables by $n = 2$ and we add $n = 2$ surplus variables in front. We then obtain the following standard maximization form:

$$\begin{array}{ll} \max & 4 \pi_3 + 6 \pi_4 \\ \text{subject to} & -\pi_1 - \pi_3 - \pi_4 = -3 \\ & -\pi_2 - \pi_3 - 3 \pi_4 = -2 \\ & \pi_1, \pi_2, \pi_3, \pi_4 \geq 0 \end{array}$$

The dual initial dictionary is:

$$\left\{ \begin{array}{l} \pi_1 = 3 - \pi_3 - \pi_4 \\ \pi_2 = 2 - \pi_3 - 3 \pi_4 \\ -w = 0 + 4 \pi_3 + 6 \pi_4 \end{array} \right.$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic dual feasible solution is:

$$\bar{\pi}_1 = 3, \quad \bar{\pi}_2 = 2, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0$$

with $-w = 0$, i.e., $w = 0$.

After having added the slack variables to the primal problem, we obtain the following standard form:

$$\begin{array}{ll} \max & -3 x_1 - 2 x_2 \\ \text{subject to} & -x_1 - x_2 + x_3 = -4 \\ & -x_1 - 3 x_2 + x_4 = -6 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

The primal initial dictionary is:

$$\begin{cases} x_3 = -4 + x_1 + x_2 \\ x_4 = -6 + x_1 + 3x_2 \\ z = 0 - 3x_1 - 2x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic primal non-feasible solution is:

$$\bar{x}_3 = -4, \quad \bar{x}_4 = -6, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

- **Iteration 1**

In the primal dictionary, the basic variables with negative value are x_3 and x_4 : by the smallest-index rule the pivot row is x_3 . Correspondingly, in the dual dictionary we increase the variable π_3 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_1 = 3 - \pi_3 \\ \pi_2 = 2 - \pi_3 \end{cases} \quad \begin{cases} \underbrace{3 - \pi_3}_{\pi_1 \geq 0} \geq 0 \\ \underbrace{2 - \pi_3}_{\pi_2 \geq 0} \geq 0 \end{cases} \quad \begin{cases} \pi_3 \leq 3 \\ \pi_3 \leq 2 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_3 : $s_m = \min \left\{ \frac{-3}{-1}, \frac{-2}{-1} \right\} = 2$, attained by x_2 . By increasing the value of π_3 to 2, the value of π_2 decreases to 0. Then we have that:

π_3 becomes dual basic and, accordingly, x_3 becomes primal non-basic;
 π_2 becomes dual non-basic and, accordingly, x_2 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_2 = 2 - \pi_3 - 3\pi_4 \quad \text{then} \quad \pi_3 = 2 - \pi_2 - 3\pi_4.$$

Substituting in the other equations of the dual dictionary, we get:

$$\pi_1 = 3 - (2 - \pi_2 - 3\pi_4) - \pi_4 = 1 + \pi_2 + 2\pi_4$$

$$-w = 4(2 - \pi_2 - 3\pi_4) + 6\pi_4 = 8 - 4\pi_2 - 6\pi_4$$

Accordingly, we now have the following optimal dual dictionary:

$$\begin{cases} \pi_1 = 1 + \pi_2 + 2\pi_4 \\ \pi_3 = 2 - \pi_2 - 3\pi_4 \\ -w = 8 - 4\pi_2 - 6\pi_4 \end{cases}$$

where $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$. All the coefficients of the objective function are non-positive, and then the dual basic optimal solution is:

$$\tilde{\pi}_1 = 1, \quad \tilde{\pi}_3 = 2, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \quad -w = 8, \quad w = -8.$$

Recalling that the original dual variables are the shifted variables π_3, π_4 , the optimal solution of the original dual LP problem is:

$$\tilde{\pi}_1 = 2, \quad \tilde{\pi}_2 = 0, \quad \text{and} \quad z(D(LP)) = -8.$$

– For the primal LP problem, we have:

$$x_3 = -4 + x_1 + x_2 \quad \text{then} \quad x_2 = 4 - x_1 + x_3.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_4 = -6 + x_1 + 3(4 - x_1 + x_3) = 6 - 2x_1 + 3x_3$$

$$z = -3x_1 - 2(4 - x_1 + x_3) = -8 - x_1 - 2x_3$$

Accordingly, we now have the following optimal primal dictionary:

$$\left\{ \begin{array}{l} x_2 = 4 - x_1 + x_3 \\ x_4 = 6 - 2x_1 + 3x_3 \\ z = -8 - x_1 - 2x_3 \end{array} \right.$$

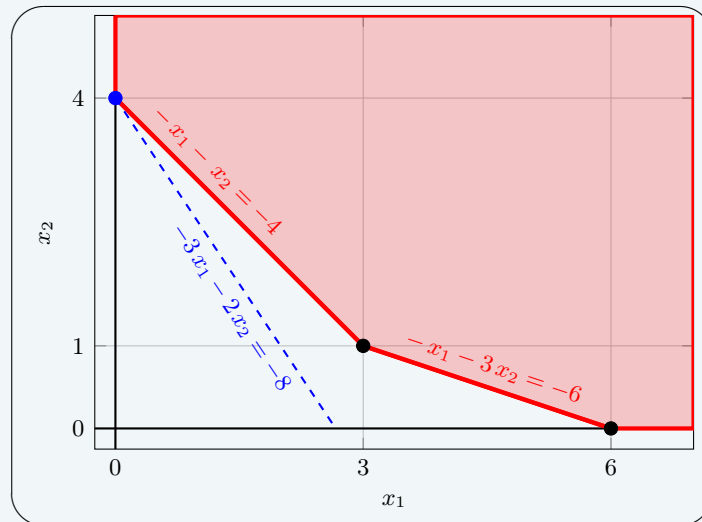
where $\mathcal{B} = \{2, 4\}$ and $\mathcal{N} = \{1, 3\}$. The basic solution is now primal feasible and dual feasible, and then the primal basic optimal solution is:

$$\tilde{x}_2 = 4, \quad \tilde{x}_4 = 6, \quad \tilde{x}_1 = 0, \quad \tilde{x}_3 = 0 \quad \text{of value} \quad -8,$$

in accordance with the Strong Duality theorem.

Graphical solution:

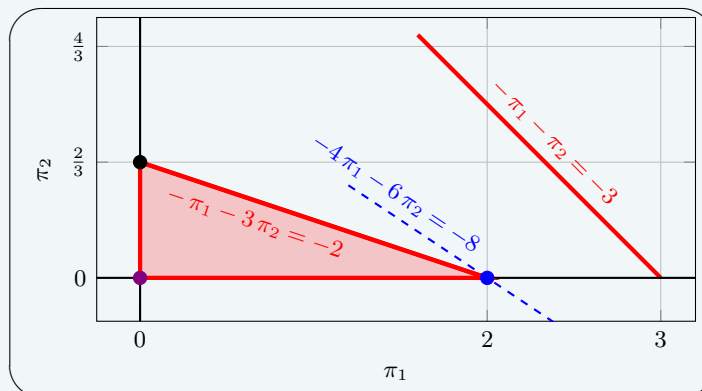
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 4 \quad \text{and} \quad z(LP) = -8.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the two original dual variables π_1, π_2 , corresponding to the variables π_3, π_4 of the dual dictionaries. The constraint $-\pi_1 - \pi_2 \geq -3$ is redundant.



An optimal dual LP solution is:

$$\tilde{\pi}_1 = 2, \quad \tilde{\pi}_2 = 0, \quad \text{and} \quad z(D(LP)) = -8.$$

In violet, we show the first vertex explored by the previous execution of the dual simplex algorithm, before reaching the optimal one depicted in blue.

Exercise 10.6 — Dual simplex algorithm (P16)

Consider the following primal LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & & - & x_1 & - & 2 & x_2 \\ \text{subject to} & -2 & x_1 & - & & x_2 & \leq & -4 \\ & & - & x_1 & - & 3 & x_2 & \leq & -6 \\ & & & x_1, & & x_2 & \geq & 0 \end{array}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 2$ constraints:

$$\begin{array}{llllll} \min & & -4 & \pi_1 & - & 6 & \pi_2 \\ \text{subject to} & -2 & \pi_1 & - & & \pi_2 & \geq & -1 \\ & & - & \pi_1 & - & 3 & \pi_2 & \geq & -2 \\ & & & \pi_1, & & \pi_2 & \geq & 0 \end{array}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Solve the primal LP problem with the dual simplex algorithm and, in parallel, the dual LP problem with the simplex algorithm, showing at each iteration the dual and the primal dictionaries.

Solution

- Initialization**

We shift the indices of the original dual variables by $n = 2$ and we add $n = 2$ surplus variables in front. We then obtain the following standard maximization form:

$$\begin{array}{llllllll} \max & & & & 4 & \pi_3 & + & 6 & \pi_4 \\ \text{subject to} & - & \pi_1 & & & - & 2 & \pi_3 & - & & \pi_4 & = & -1 \\ & & & - & \pi_2 & - & & \pi_3 & - & 3 & \pi_4 & = & -2 \\ & & \pi_1, & \pi_2, & \pi_3, & \pi_4 & \geq & 0 \end{array}$$

The dual initial dictionary is:

$$\begin{cases} \pi_1 = 1 - 2\pi_3 - \pi_4 \\ \pi_2 = 2 - \pi_3 - 3\pi_4 \\ -w = 0 + 4\pi_3 + 6\pi_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic dual feasible solution is:

$$\bar{\pi}_1 = 1, \quad \bar{\pi}_2 = 2, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0$$

with $-w = 0$, i.e., $w = 0$.

After having added the slack variables to the primal problem, we obtain the following standard form:

$$\begin{aligned} \max \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & -2x_1 - x_2 + x_3 = -4 \\ & -x_1 - 3x_2 + x_4 = -6 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The primal initial dictionary is:

$$\begin{cases} x_3 = -4 + 2x_1 + x_2 \\ x_4 = -6 + x_1 + 3x_2 \\ z = 0 - x_1 - 2x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic primal non-feasible solution is:

$$\bar{x}_3 = -4, \quad \bar{x}_4 = -6, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

• Iteration 1

In the primal dictionary, the basic variables with negative value are x_3 and x_4 : by the smallest-index rule the pivot row is x_3 . Correspondingly, in the dual dictionary we increase the variable π_3 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_1 = 1 - 2\pi_3 \\ \pi_2 = 2 - \pi_3 \end{cases} \quad \begin{cases} \underbrace{1 - 2\pi_3 \geq 0}_{\pi_1 \geq 0} \\ \underbrace{2 - \pi_3 \geq 0}_{\pi_2 \geq 0} \end{cases} \quad \begin{cases} \pi_3 \leq \frac{1}{2} \\ \pi_3 \leq 2 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_3 : $s_m = \min \left\{ \frac{-1}{-2}, \frac{-2}{-1} \right\} = \frac{1}{2}$, attained by x_1 . By increasing the value of π_3 to $\frac{1}{2}$, the value of π_1 decreases to 0. Then we have that:

π_3 becomes dual basic and, accordingly, x_3 becomes primal non-basic;
 π_1 becomes dual non-basic and, accordingly, x_1 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_1 = 1 - 2\pi_3 - \pi_4 \quad \text{then} \quad \pi_3 = \frac{1}{2} - \frac{1}{2}\pi_1 - \frac{1}{2}\pi_4.$$

Substituting in the other equations of the dual dictionary, we get:

$$\pi_2 = 2 - \left(\frac{1}{2} - \frac{1}{2}\pi_1 - \frac{1}{2}\pi_4 \right) - 3\pi_4 = \frac{3}{2} + \frac{1}{2}\pi_1 - \frac{5}{2}\pi_4$$

$$-w = 4 \left(\frac{1}{2} - \frac{1}{2}\pi_1 - \frac{1}{2}\pi_4 \right) + 6\pi_4 = 2 - 2\pi_1 + 4\pi_4$$

Accordingly, we now have the following dual dictionary:

$$\begin{cases} \pi_2 = \frac{3}{2} + \frac{1}{2}\pi_1 - \frac{5}{2}\pi_4 \\ \pi_3 = \frac{1}{2} - \frac{1}{2}\pi_1 - \frac{1}{2}\pi_4 \\ -w = 2 - 2\pi_1 + 4\pi_4 \end{cases}$$

where $\mathcal{B} = \{2, 3\}$ and $\mathcal{N} = \{1, 4\}$. The basic dual feasible solution is:

$$\bar{\pi}_2 = \frac{3}{2}, \quad \bar{\pi}_3 = \frac{1}{2}, \quad \bar{\pi}_1 = 0, \quad \bar{\pi}_4 = 0, \quad -w = 2, \quad w = -2.$$

– For the primal LP problem, we have:

$$x_3 = -4 + 2x_1 + x_2 \quad \text{then} \quad x_1 = 2 - \frac{1}{2}x_2 + \frac{1}{2}x_3.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_4 = -6 + \left(2 - \frac{1}{2}x_2 + \frac{1}{2}x_3 \right) + 3x_2 = -4 + \frac{5}{2}x_2 + \frac{1}{2}x_3$$

$$z = - \left(2 - \frac{1}{2}x_2 + \frac{1}{2}x_3 \right) - 2x_2 = -2 - \frac{3}{2}x_2 - \frac{1}{2}x_3$$

Accordingly, we now have the following primal dictionary:

$$\begin{cases} x_1 = 2 - \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ x_4 = -4 + \frac{5}{2}x_2 + \frac{1}{2}x_3 \\ z = -2 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \end{cases}$$

where $\mathcal{B} = \{1, 4\}$ and $\mathcal{N} = \{2, 3\}$. The basic primal non-feasible solution is:

$$\bar{x}_1 = 2, \quad \bar{x}_4 = -4, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad -2$$

• **Iteration 2**

The only basic variable with negative value is x_4 , which is the pivot row. Correspondingly, in the dual dictionary we increase the variable π_4 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_2 = \frac{3}{2} - \frac{5}{2} \pi_4 \\ \pi_3 = \frac{1}{2} - \frac{1}{2} \pi_4 \end{cases} \quad \begin{cases} \underbrace{\frac{3}{2} - \frac{5}{2} \pi_4}_{\pi_2 \geq 0} \geq 0 \\ \underbrace{\frac{1}{2} - \frac{1}{2} \pi_4}_{\pi_3 \geq 0} \geq 0 \end{cases} \quad \begin{cases} \pi_4 \leq \frac{3}{5} \\ \pi_4 \leq 1 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_4 : $s_m = \min \left\{ \frac{-3/2}{-5/2}, \frac{-1/2}{-1/2} \right\} = \frac{3}{5}$, attained by x_2 . By increasing the value of π_4 to $\frac{3}{5}$, the value of π_2 decreases to 0. Then we have that:

π_4 becomes dual basic and, accordingly, x_4 becomes primal non-basic;
 π_2 becomes dual non-basic and, accordingly, x_2 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_2 = \frac{3}{2} + \frac{1}{2} \pi_1 - \frac{5}{2} \pi_4 \quad \text{then} \quad \pi_4 = \frac{3}{5} + \frac{1}{5} \pi_1 - \frac{2}{5} \pi_2.$$

Substituting in the other equations of the dual dictionary, we get:

$$\begin{aligned} \pi_3 &= \frac{1}{2} - \frac{1}{2} \pi_1 - \frac{1}{2} \left(\frac{3}{5} + \frac{1}{5} \pi_1 - \frac{2}{5} \pi_2 \right) = \frac{1}{5} - \frac{3}{5} \pi_1 + \frac{1}{5} \pi_2 \\ -w &= 2 - 2 \pi_1 + 4 \left(\frac{3}{5} + \frac{1}{5} \pi_1 - \frac{2}{5} \pi_2 \right) = \frac{22}{5} - \frac{6}{5} \pi_1 - \frac{8}{5} \pi_2. \end{aligned}$$

Accordingly, we now have the following optimal dual dictionary:

$$\begin{cases} \pi_3 = \frac{1}{5} - \frac{3}{5} \pi_1 + \frac{1}{5} \pi_2 \\ \pi_4 = \frac{3}{5} + \frac{1}{5} \pi_1 - \frac{2}{5} \pi_2 \\ -w = \frac{22}{5} - \frac{6}{5} \pi_1 - \frac{8}{5} \pi_2 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. All the coefficients of the objective function are non-positive, and then the dual basic optimal solution is:

$$\tilde{\pi}_3 = \frac{1}{5}, \quad \tilde{\pi}_4 = \frac{3}{5}, \quad \tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad -w = \frac{22}{5}, \quad w = -\frac{22}{5}.$$

Recalling that the original dual variables are the shifted variables π_3, π_4 , the optimal solution of the original dual LP problem is:

$$\tilde{\pi}_1 = \frac{1}{5}, \quad \tilde{\pi}_2 = \frac{3}{5}, \quad \text{and} \quad z(D(LP)) = -\frac{22}{5}.$$

– For the primal LP problem, we have:

$$x_4 = -4 + \frac{5}{2}x_2 + \frac{1}{2}x_3 \quad \text{then} \quad x_2 = \frac{8}{5} - \frac{1}{5}x_3 + \frac{2}{5}x_4.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_1 = 2 - \frac{1}{2} \left(\frac{8}{5} - \frac{1}{5}x_3 + \frac{2}{5}x_4 \right) + \frac{1}{2}x_3 = \frac{6}{5} + \frac{3}{5}x_3 - \frac{1}{5}x_4$$

$$z = -2 - \frac{3}{2} \left(\frac{8}{5} - \frac{1}{5}x_3 + \frac{2}{5}x_4 \right) - \frac{1}{2}x_3 = -\frac{22}{5} - \frac{1}{5}x_3 - \frac{3}{5}x_4$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = \frac{6}{5} + \frac{3}{5}x_3 - \frac{1}{5}x_4 \\ x_2 = \frac{8}{5} - \frac{1}{5}x_3 + \frac{2}{5}x_4 \\ z = -\frac{22}{5} - \frac{1}{5}x_3 - \frac{3}{5}x_4 \end{cases}$$

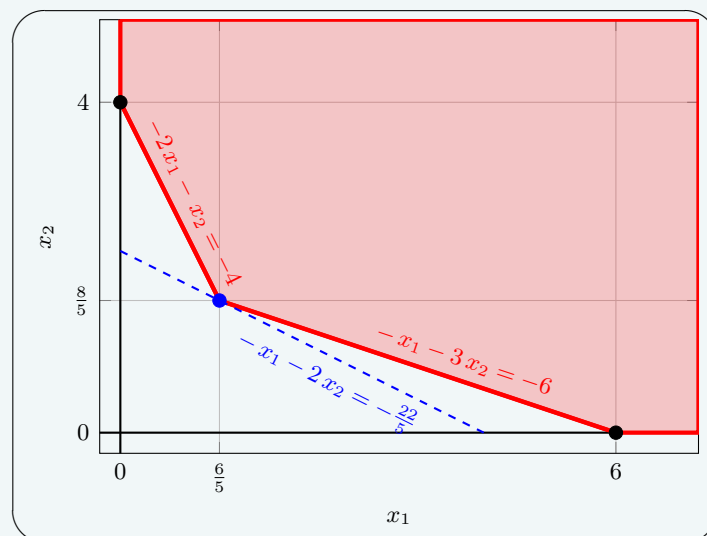
where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic solution is now primal feasible and dual feasible, and then the primal basic optimal solution is:

$$\tilde{x}_1 = \frac{6}{5}, \quad \tilde{x}_2 = \frac{8}{5}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad -\frac{22}{5},$$

in accordance with the Strong Duality theorem.

Graphical solution:

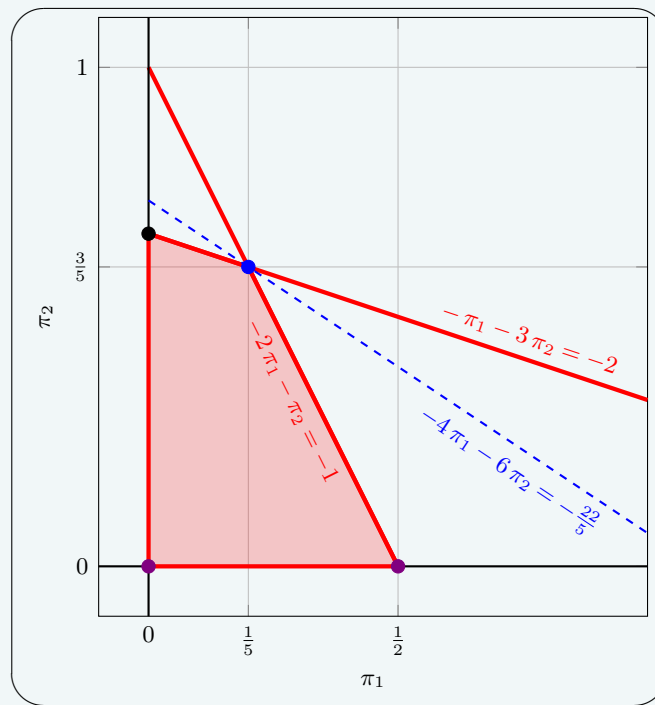
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = \frac{6}{5}, \quad \tilde{x}_2 = \frac{8}{5} \quad \text{and} \quad z(LP) = -\frac{22}{5}.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the two original dual variables π_1, π_2 , corresponding to the variables π_3, π_4 of the dual dictionaries.



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{1}{5}, \quad \tilde{\pi}_2 = \frac{3}{5}, \quad \text{and} \quad z(D(LP)) = -\frac{22}{5}.$$

In violet, we show the first and the second vertex explored by the previous execution of the dual simplex algorithm, before reaching the optimal one depicted in blue.

10.2 Problems with three variables

Exercise 10.7 — Dual simplex algorithm (P51)

Consider the following primal LP problem in canonical maximization form with $n = 3$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & - & 2x_1 & - & x_2 & \\ \text{subject to} & & x_1 & - & x_2 & - 3x_3 \leq -5 \\ & & -3x_1 & - & 2x_2 & + 2x_3 \leq -2 \\ & & x_1, & & x_2, & & x_3 \geq 0 \end{array}$$

and its dual LP problem in canonical minimization form with $m = 2$ variables and $n = 3$ constraints:

$$\begin{array}{llllll} \min & -5\pi_1 & - & 2\pi_2 & \\ \text{subject to} & & \pi_1 & - & 3\pi_2 \geq -2 \\ & & -\pi_1 & - & 2\pi_2 \geq -1 \\ & & -3\pi_1 & + & 2\pi_2 \geq 0 \\ & & \pi_1, & & \pi_2 \geq 0 \end{array}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Solve the primal LP problem with the dual simplex algorithm and, in parallel, the dual LP problem with the simplex algorithm, showing at each iteration the dual and the primal dictionaries.

Solution

- **Initialization**

We shift the indices of the original dual variables by $n = 3$ and we add $n = 3$ surplus variables in front. We then obtain the following standard maximization form:

$$\begin{array}{llllll}
 \max & & & 5\pi_4 & + & 2\pi_5 \\
 \text{subject to} & - & \pi_1 & & + & \pi_4 & - & 3\pi_5 & = & -2 \\
 & & & - & \pi_2 & & - & \pi_4 & - & 2\pi_5 & = & -1 \\
 & & & & & - & \pi_3 & - & 3\pi_4 & + & 2\pi_5 & = & 0 \\
 & & \pi_1, & \pi_2, & \pi_3, & \pi_4, & \pi_5 & \geq & 0
 \end{array}$$

The dual initial dictionary is:

$$\left\{ \begin{array}{l}
 \pi_1 = 2 + \pi_4 - 3\pi_5 \\
 \pi_2 = 1 - \pi_4 - 2\pi_5 \\
 \pi_3 = 0 - 3\pi_4 + 2\pi_5 \\
 -w = 0 + 5\pi_4 + 2\pi_5
 \end{array} \right.$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$. The basic dual feasible solution is:

$$\bar{\pi}_1 = 2, \quad \bar{\pi}_2 = 1, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0 \quad \text{of value } 0$$

After having added the slack variables to the primal problem, we obtain the following standard form:

$$\begin{array}{llllll}
 \max & -2x_1 & - & x_2 & & & \\
 \text{subject to} & x_1 & - & x_2 & - & 3x_3 & + & x_4 & = & -5 \\
 & -3x_1 & - & 2x_2 & + & 2x_3 & & & + & x_5 & = & -2 \\
 & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0
 \end{array}$$

The primal initial dictionary is:

$$\left\{ \begin{array}{l}
 x_4 = -5 - x_1 + x_2 + 3x_3 \\
 x_5 = -2 + 3x_1 + 2x_2 - 2x_3 \\
 z = 0 - 2x_1 - x_2
 \end{array} \right.$$

where $\mathcal{B} = \{4, 5\}$ and $\mathcal{N} = \{1, 2, 3\}$. The basic primal non-feasible solution is:

$$\bar{x}_4 = -5, \quad \bar{x}_5 = -2, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value } 0$$

• **Iteration 1**

In the primal dictionary, the basic variables with negative value are x_4 and x_5 : by the smallest-index rule the pivot row is x_4 . Correspondingly, in the dual dictionary we increase the variable π_4 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_1 = 2 + \pi_4 \\ \pi_2 = 1 - \pi_4 \\ \pi_3 = -3\pi_4 \end{cases} \quad \begin{cases} \underbrace{2 + \pi_4 \geq 0}_{\pi_1 \geq 0} \\ \underbrace{1 - \pi_4 \geq 0}_{\pi_2 \geq 0} \\ \underbrace{-3\pi_4 \geq 0}_{\pi_3 \geq 0} \end{cases} \quad \begin{cases} \pi_4 \geq -2 \\ \pi_4 \leq 1 \\ \pi_4 \leq 0 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_4 : $s_m = \min \left\{ \frac{-1}{-1}, \frac{0}{-3} \right\} = 0$, attained by x_3 ($\bar{a}_{41} = 1 > 0$ gives no candidate). By increasing the value of π_4 to 0, the value of π_3 remains equal to 0. Then we have that:

π_4 becomes dual basic and, accordingly, x_4 becomes primal non-basic;
 π_3 becomes dual non-basic and, accordingly, x_3 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_3 = -3\pi_4 + 2\pi_5 \quad \text{then} \quad \pi_4 = -\frac{1}{3}\pi_3 + \frac{2}{3}\pi_5.$$

Substituting in the other equations of the dual dictionary, we get:

$$\begin{aligned} \pi_1 &= 2 - \frac{1}{3}\pi_3 + \frac{2}{3}\pi_5 - 3\pi_5 = 2 - \frac{1}{3}\pi_3 - \frac{7}{3}\pi_5 \\ \pi_2 &= 1 + \frac{1}{3}\pi_3 - \frac{2}{3}\pi_5 - 2\pi_5 = 1 + \frac{1}{3}\pi_3 - \frac{8}{3}\pi_5 \\ -w &= 5 \left(-\frac{1}{3}\pi_3 + \frac{2}{3}\pi_5 \right) + 2\pi_5 = -\frac{5}{3}\pi_3 + \frac{16}{3}\pi_5 \end{aligned}$$

Accordingly, we now have the following dual dictionary:

$$\begin{cases} \pi_1 = 2 & - & \frac{1}{3} \pi_3 & - & \frac{7}{3} \pi_5 \\ \pi_2 = 1 & + & \frac{1}{3} \pi_3 & - & \frac{8}{3} \pi_5 \\ \pi_4 = 0 & - & \frac{1}{3} \pi_3 & + & \frac{2}{3} \pi_5 \\ -w = 0 & - & \frac{5}{3} \pi_3 & + & \frac{16}{3} \pi_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5\}$. The basic dual feasible solution is:

$$\bar{\pi}_1 = 2, \quad \bar{\pi}_2 = 1, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_3 = 0, \quad \bar{\pi}_5 = 0 \quad \text{of value} \quad 0$$

– For the primal LP problem, we have:

$$x_4 = -5 - x_1 + x_2 + 3x_3 \quad \text{then} \quad x_3 = \frac{5}{3} + \frac{1}{3}x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_4.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_5 = -2 + 3x_1 + 2x_2 - 2\left(\frac{5}{3} + \frac{1}{3}x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_4\right) = -\frac{16}{3} + \frac{7}{3}x_1 + \frac{8}{3}x_2 - \frac{2}{3}x_4$$

$$z = -2x_1 - x_2$$

Accordingly, we now have the following primal dictionary:

$$\begin{cases} x_3 = & \frac{5}{3} & + & \frac{1}{3}x_1 & - & \frac{1}{3}x_2 & + & \frac{1}{3}x_4 \\ x_5 = & -\frac{16}{3} & + & \frac{7}{3}x_1 & + & \frac{8}{3}x_2 & - & \frac{2}{3}x_4 \\ z = & 0 & - & 2x_1 & - & x_2 \end{cases}$$

where $\mathcal{B} = \{3, 5\}$ and $\mathcal{N} = \{1, 2, 4\}$. The basic primal non-feasible solution is:

$$\bar{x}_3 = \frac{5}{3}, \quad \bar{x}_5 = -\frac{16}{3}, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 0$$

• Iteration 2

The only basic variable with negative value is x_5 , which is the pivot row. Correspondingly, in the dual dictionary we increase the variable π_5 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_1 = & 2 - \frac{7}{3}\pi_5 \\ \pi_2 = & 1 - \frac{8}{3}\pi_5 \\ \pi_4 = & \frac{2}{3}\pi_5 \end{cases} \quad \begin{cases} \underbrace{2 - \frac{7}{3}\pi_5}_{\pi_1 \geq 0} \geq 0 \\ \underbrace{1 - \frac{8}{3}\pi_5}_{\pi_2 \geq 0} \geq 0 \\ \underbrace{\frac{2}{3}\pi_5}_{\pi_4 \geq 0} \geq 0 \end{cases} \quad \begin{cases} \pi_5 \leq \frac{6}{7} \\ \pi_5 \leq \frac{3}{8} \\ \pi_5 \geq 0 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_5 : $s_m = \min \left\{ \frac{-2}{-7/3}, \frac{-1}{-8/3} \right\} = \frac{3}{8}$, attained by x_2 ($\bar{a}_{54} = \frac{2}{3} > 0$ gives no candidate). By increasing the value of π_5 to $\frac{3}{8}$, the value of π_2 decreases to 0. Then we have that:

π_5 becomes dual basic and, accordingly, x_5 becomes primal non-basic;

π_2 becomes dual non-basic and, accordingly, x_2 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_2 = 1 + \frac{1}{3} \pi_3 - \frac{8}{3} \pi_5 \quad \text{then} \quad \pi_5 = \frac{3}{8} + \frac{1}{8} \pi_3 - \frac{3}{8} \pi_2.$$

Substituting in the other equations of the dual dictionary, we get:

$$\pi_1 = 2 - \frac{1}{3} \pi_3 - \frac{7}{3} \left(\frac{3}{8} + \frac{1}{8} \pi_3 - \frac{3}{8} \pi_2 \right) = \frac{9}{8} + \frac{7}{8} \pi_2 - \frac{5}{8} \pi_3$$

$$\pi_4 = -\frac{1}{3} \pi_3 + \frac{2}{3} \left(\frac{3}{8} + \frac{1}{8} \pi_3 - \frac{3}{8} \pi_2 \right) = \frac{1}{4} - \frac{1}{4} \pi_2 - \frac{1}{4} \pi_3$$

$$-w = -\frac{5}{3} \pi_3 + \frac{16}{3} \left(\frac{3}{8} + \frac{1}{8} \pi_3 - \frac{3}{8} \pi_2 \right) = 2 - 2 \pi_2 - \pi_3$$

Accordingly, we now have the following optimal dual dictionary:

$$\left\{ \begin{array}{l} \pi_1 = \frac{9}{8} + \frac{7}{8} \pi_2 - \frac{5}{8} \pi_3 \\ \pi_4 = \frac{1}{4} - \frac{1}{4} \pi_2 - \frac{1}{4} \pi_3 \\ \pi_5 = \frac{3}{8} - \frac{3}{8} \pi_2 + \frac{1}{8} \pi_3 \\ -w = 2 - 2 \pi_2 - \pi_3 \end{array} \right.$$

where $\mathcal{B} = \{1, 4, 5\}$ and $\mathcal{N} = \{2, 3\}$. All the coefficients of the objective function are non-positive, and then the dual basic optimal solution is:

$$\tilde{\pi}_1 = \frac{9}{8}, \quad \tilde{\pi}_4 = \frac{1}{4}, \quad \tilde{\pi}_5 = \frac{3}{8}, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_3 = 0 \quad \text{of value} \quad -2$$

Recalling that the original dual variables are the shifted variables π_4, π_5 , the optimal solution of the original dual LP problem is:

$$\tilde{\pi}_1 = \frac{1}{4}, \quad \tilde{\pi}_2 = \frac{3}{8}, \quad \text{and} \quad z(D(LP)) = -2.$$

– For the primal LP problem, we have:

$$x_5 = -\frac{16}{3} + \frac{7}{3} x_1 + \frac{8}{3} x_2 - \frac{2}{3} x_4 \quad \text{then} \quad x_2 = 2 - \frac{7}{8} x_1 + \frac{1}{4} x_4 + \frac{3}{8} x_5.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_3 = \frac{5}{3} + \frac{1}{3} x_1 - \frac{1}{3} \left(2 - \frac{7}{8} x_1 + \frac{1}{4} x_4 + \frac{3}{8} x_5 \right) + \frac{1}{3} x_4 = 1 + \frac{5}{8} x_1 + \frac{1}{4} x_4 - \frac{1}{8} x_5$$

$$z = -2x_1 - \left(2 - \frac{7}{8}x_1 + \frac{1}{4}x_4 + \frac{3}{8}x_5\right) = -2 - \frac{9}{8}x_1 - \frac{1}{4}x_4 - \frac{3}{8}x_5$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_2 = 2 - \frac{7}{8}x_1 + \frac{1}{4}x_4 + \frac{3}{8}x_5 \\ x_3 = 1 + \frac{5}{8}x_1 + \frac{1}{4}x_4 - \frac{1}{8}x_5 \\ z = -2 - \frac{9}{8}x_1 - \frac{1}{4}x_4 - \frac{3}{8}x_5 \end{cases}$$

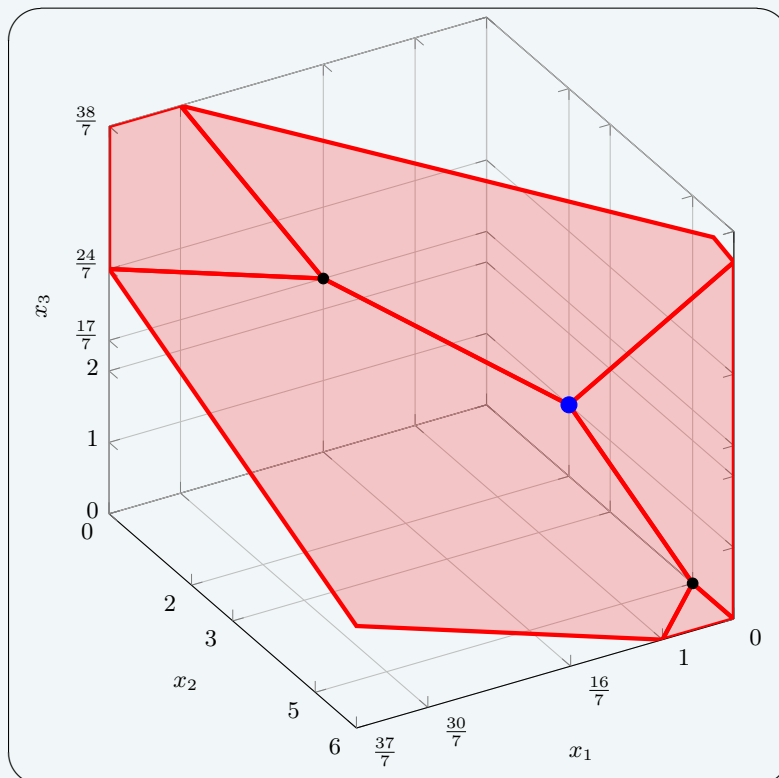
where $\mathcal{B} = \{2, 3\}$ and $\mathcal{N} = \{1, 4, 5\}$. The basic solution is now primal feasible and dual feasible, and then the primal basic optimal solution is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 2, \quad \tilde{x}_3 = 1, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad -2,$$

in accordance with the Strong Duality theorem.

Graphical solution:

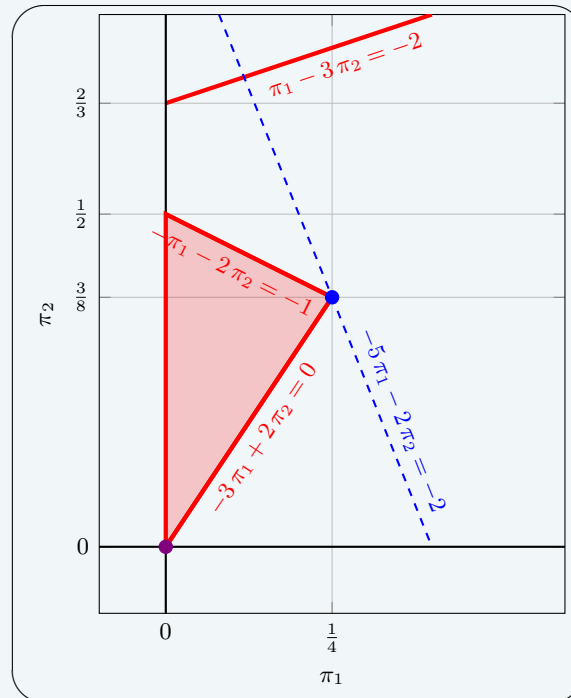
The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded.



An optimal primal LP solution is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 2, \quad \tilde{x}_3 = 1 \quad \text{and} \quad z(LP) = -2.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the two original dual variables π_1, π_2 , corresponding to the variables π_4, π_5 of the dual dictionaries.



An optimal dual LP solution is:

$$\tilde{\pi}_1 = \frac{1}{4}, \quad \tilde{\pi}_2 = \frac{3}{8} \quad \text{and} \quad z(D(LP)) = -2.$$

In violet, we show the first vertex explored by the previous execution of the dual simplex algorithm, before reaching the optimal one depicted in blue.

Exercise 10.8 — Dual simplex algorithm (P53)

Consider the following primal LP problem in canonical maximization form with $n = 3$ variables and $m = 3$ constraints:

$$\begin{array}{llllll}
 \max & - & x_1 & - & 2x_2 & - & 4x_3 \\
 \text{subject to} & - & x_1 & - & x_2 & & \leq -2 \\
 & & & & - & x_2 & - & x_3 \leq -3 \\
 & - & x_1 & & & - & x_3 \leq -2 \\
 & & x_1, & & x_2, & & x_3 \geq 0
 \end{array}$$

and its dual LP problem in canonical minimization form with $m = 3$ variables and $n = 3$ constraints:

$$\begin{array}{llllll}
 \min & - & 2\pi_1 & - & 3\pi_2 & - & 2\pi_3 \\
 \text{subject to} & - & \pi_1 & & & - & \pi_3 \geq -1 \\
 & - & \pi_1 & - & \pi_2 & & \geq -2 \\
 & & & - & \pi_2 & - & \pi_3 \geq -4 \\
 & & \pi_1, & & \pi_2, & & \pi_3 \geq 0
 \end{array}$$

Since $\mathbf{c} \leq \mathbf{0}$, the basis of the m slack variables is **dual feasible** but it is not primal feasible.

Questions

Solve the primal LP problem with the dual simplex algorithm and, in parallel, the dual LP problem with the simplex algorithm, showing at each iteration the dual and the primal dictionaries.

Solution

- Initialization**

We shift the indices of the original dual variables by $n = 3$ and we add $n = 3$ surplus variables in front. We then obtain the following standard maximization form:

$$\begin{array}{llllllll}
 \max & & & & 2\pi_4 & + & 3\pi_5 & + & 2\pi_6 \\
 \text{subject to} & - & \pi_1 & & & - & \pi_4 & & - & \pi_6 = -1 \\
 & & & - & \pi_2 & & & - & \pi_4 & - & \pi_5 = -2 \\
 & & & & & - & \pi_3 & & - & \pi_5 & - & \pi_6 = -4 \\
 & & \pi_1, & & \pi_2, & & \pi_3, & & \pi_4, & & \pi_5, & & \pi_6 \geq 0
 \end{array}$$

The dual initial dictionary is:

$$\left\{ \begin{array}{rclclcl} \pi_1 & = & 1 & - & \pi_4 & & - & \pi_6 \\ \pi_2 & = & 2 & - & \pi_4 & - & \pi_5 & \\ \pi_3 & = & 4 & & & - & \pi_5 & - & \pi_6 \\ -w & = & 0 & + & 2\pi_4 & + & 3\pi_5 & + & 2\pi_6 \end{array} \right.$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5, 6\}$. The basic dual feasible solution is:

$$\bar{\pi}_1 = 1, \quad \bar{\pi}_2 = 2, \quad \bar{\pi}_3 = 4, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_5 = 0, \quad \bar{\pi}_6 = 0$$

with $-w = 0$, i.e., $w = 0$.

After having added the slack variables to the primal problem, we obtain the following standard form:

$$\begin{array}{rclclclclcl} \max & - & x_1 & - & 2x_2 & - & 4x_3 & & & \\ \text{subject to} & - & x_1 & - & x_2 & & & + & x_4 & = & -2 \\ & & & - & x_2 & - & x_3 & & + & x_5 & = & -3 \\ & & - & x_1 & & - & x_3 & & & + & x_6 & = & -2 \\ & & & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0 \end{array}$$

The primal initial dictionary is:

$$\left\{ \begin{array}{rclclcl} x_4 & = & -2 & + & x_1 & + & x_2 \\ x_5 & = & -3 & & & + & x_2 & + & x_3 \\ x_6 & = & -2 & + & x_1 & & & + & x_3 \\ z & = & 0 & - & x_1 & - & 2x_2 & - & 4x_3 \end{array} \right.$$

where $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$. The basic primal non-feasible solution is:

$$\bar{x}_4 = -2, \quad \bar{x}_5 = -3, \quad \bar{x}_6 = -2, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value } 0$$

• Iteration 1

In the primal dictionary, the basic variables with negative value are x_4 , x_5 and x_6 : by the smallest-index rule the pivot row is x_4 . Correspondingly, in the dual dictionary

we increase the variable π_4 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_1 = 1 - \pi_4 \\ \pi_2 = 2 - \pi_4 \\ \pi_3 = 4 \end{cases} \quad \begin{cases} \underbrace{1 - \pi_4 \geq 0}_{\pi_1 \geq 0} \\ \underbrace{2 - \pi_4 \geq 0}_{\pi_2 \geq 0} \\ \underbrace{4 \geq 0}_{\pi_3 \geq 0} \end{cases} \quad \begin{cases} \pi_4 \leq 1 \\ \pi_4 \leq 2 \\ (\text{no condition}) \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_4 : $s_m = \min \left\{ \frac{-1}{-1}, \frac{-2}{-1} \right\} = 1$, attained by x_1 . By increasing the value of π_4 to 1, the value of π_1 decreases to 0. Then we have that:

π_4 becomes dual basic and, accordingly, x_4 becomes primal non-basic;
 π_1 becomes dual non-basic and, accordingly, x_1 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_1 = 1 - \pi_4 - \pi_6 \quad \text{then} \quad \pi_4 = 1 - \pi_1 - \pi_6.$$

Substituting in the other equations of the dual dictionary, we get:

$$\pi_2 = 2 - (1 - \pi_1 - \pi_6) - \pi_5 = 1 + \pi_1 - \pi_5 + \pi_6$$

$$\pi_3 = 4 - \pi_5 - \pi_6$$

$$-w = 2(1 - \pi_1 - \pi_6) + 3\pi_5 + 2\pi_6 = 2 - 2\pi_1 + 3\pi_5$$

Accordingly, we now have the following dual dictionary:

$$\begin{cases} \pi_2 = 1 + \pi_1 - \pi_5 + \pi_6 \\ \pi_3 = 4 - \pi_5 - \pi_6 \\ \pi_4 = 1 - \pi_1 - \pi_6 \\ -w = 2 - 2\pi_1 + 3\pi_5 \end{cases}$$

where $\mathcal{B} = \{2, 3, 4\}$ and $\mathcal{N} = \{1, 5, 6\}$. The basic dual feasible solution is:

$$\bar{\pi}_2 = 1, \quad \bar{\pi}_3 = 4, \quad \bar{\pi}_4 = 1, \quad \bar{\pi}_1 = 0, \quad \bar{\pi}_5 = 0, \quad \bar{\pi}_6 = 0, \quad -w = 2, \quad w = -2.$$

– For the primal LP problem, we have:

$$x_4 = -2 + x_1 + x_2 \quad \text{then} \quad x_1 = 2 - x_2 + x_4.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_5 = -3 + x_2 + x_3$$

$$x_6 = -2 + (2 - x_2 + x_4) + x_3 = 0 - x_2 + x_3 + x_4$$

$$z = -(2 - x_2 + x_4) - 2x_2 - 4x_3 = -2 - x_2 - 4x_3 - x_4$$

Accordingly, we now have the following primal dictionary:

$$\begin{cases} x_1 = 2 - x_2 + x_4 \\ x_5 = -3 + x_2 + x_3 \\ x_6 = 0 - x_2 + x_3 + x_4 \\ z = -2 - x_2 - 4x_3 - x_4 \end{cases}$$

where $\mathcal{B} = \{1, 5, 6\}$ and $\mathcal{N} = \{2, 3, 4\}$. The basic primal non-feasible solution is:

$$\bar{x}_1 = 2, \quad \bar{x}_5 = -3, \quad \bar{x}_6 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad -2$$

• Iteration 2

The only basic variable with negative value is x_5 , which is the pivot row. Correspondingly, in the dual dictionary we increase the variable π_5 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_2 = 1 - \pi_5 \\ \pi_3 = 4 - \pi_5 \\ \pi_4 = 1 \end{cases} \quad \begin{cases} \underbrace{1 - \pi_5 \geq 0}_{\pi_2 \geq 0} \\ \underbrace{4 - \pi_5 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{1 \geq 0}_{\pi_4 \geq 0} \end{cases} \quad \begin{cases} \pi_5 \leq 1 \\ \pi_5 \leq 4 \\ \text{(no condition)} \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_5 : $s_m = \min \left\{ \frac{-1}{-1}, \frac{-4}{-1} \right\} = 1$, attained by x_2 . By increasing the value of π_5 to 1, the value of π_2 decreases to 0. Then we have that:

π_5 becomes dual basic and, accordingly, x_5 becomes primal non-basic;
 π_2 becomes dual non-basic and, accordingly, x_2 becomes primal basic.

- For the dual LP problem, we have:

$$\pi_2 = 1 + \pi_1 - \pi_5 + \pi_6 \quad \text{then} \quad \pi_5 = 1 + \pi_1 - \pi_2 + \pi_6.$$

Substituting in the other equations of the dual dictionary, we get:

$$\pi_3 = 4 - (1 + \pi_1 - \pi_2 + \pi_6) - \pi_6 = 3 - \pi_1 + \pi_2 - 2\pi_6$$

$$\pi_4 = 1 - \pi_1 - \pi_6$$

$$-w = 2 - 2\pi_1 + 3(1 + \pi_1 - \pi_2 + \pi_6) = 5 + \pi_1 - 3\pi_2 + 3\pi_6$$

Accordingly, we now have the following dual dictionary:

$$\left\{ \begin{array}{rclclcl} \pi_3 & = & 3 & - & \pi_1 & + & \pi_2 & - & 2\pi_6 \\ \pi_4 & = & 1 & - & \pi_1 & & & - & \pi_6 \\ \pi_5 & = & 1 & + & \pi_1 & - & \pi_2 & + & \pi_6 \\ -w & = & 5 & + & \pi_1 & - & 3\pi_2 & + & 3\pi_6 \end{array} \right.$$

where $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2, 6\}$. The basic dual feasible solution is:

$$\bar{\pi}_3 = 3, \quad \bar{\pi}_4 = 1, \quad \bar{\pi}_5 = 1, \quad \bar{\pi}_1 = 0, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_6 = 0, \quad -w = 5, \quad w = -5.$$

- For the primal LP problem, we have:

$$x_5 = -3 + x_2 + x_3 \quad \text{then} \quad x_2 = 3 - x_3 + x_5.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_1 = 2 - (3 - x_3 + x_5) + x_4 = -1 + x_3 + x_4 - x_5$$

$$x_6 = -(3 - x_3 + x_5) + x_3 + x_4 = -3 + 2x_3 + x_4 - x_5$$

$$z = -2 - (3 - x_3 + x_5) - 4x_3 - x_4 = -5 - 3x_3 - x_4 - x_5$$

Accordingly, we now have the following primal dictionary:

$$\left\{ \begin{array}{rclclcl} x_1 & = & -1 & + & x_3 & + & x_4 & - & x_5 \\ x_2 & = & 3 & - & x_3 & & & + & x_5 \\ x_6 & = & -3 & + & 2x_3 & + & x_4 & - & x_5 \\ z & = & -5 & - & 3x_3 & - & x_4 & - & x_5 \end{array} \right.$$

where $\mathcal{B} = \{1, 2, 6\}$ and $\mathcal{N} = \{3, 4, 5\}$. The basic primal non-feasible solution is:

$$\bar{x}_1 = -1, \quad \bar{x}_2 = 3, \quad \bar{x}_6 = -3, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad -5$$

• **Iteration 3**

In the primal dictionary, the basic variables with negative value are x_1 and x_6 : by the smallest-index rule the pivot row is x_1 (even if $\bar{x}_6 = -3$ is more negative). Correspondingly, in the dual dictionary we increase the variable π_1 , i.e., the variable with positive coefficient in the objective function having the smallest index, and we can get a better dual solution. We have:

$$\begin{cases} \pi_3 = 3 - \pi_1 \\ \pi_4 = 1 - \pi_1 \\ \pi_5 = 1 + \pi_1 \end{cases} \quad \begin{cases} \underbrace{3 - \pi_1 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{1 - \pi_1 \geq 0}_{\pi_4 \geq 0} \\ \underbrace{1 + \pi_1 \geq 0}_{\pi_5 \geq 0} \end{cases} \quad \begin{cases} \pi_1 \leq 3 \\ \pi_1 \leq 1 \\ \pi_1 \geq -1 \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_1 : $s_m = \min \left\{ \frac{-3}{-1}, \frac{-1}{-1} \right\} = 1$, attained by x_4 ($\bar{a}_{15} = 1 > 0$ gives no candidate). By increasing the value of π_1 to 1, the value of π_4 decreases to 0. Then we have that:

π_1 becomes dual basic and, accordingly, x_1 becomes primal non-basic;
 π_4 becomes dual non-basic and, accordingly, x_4 becomes primal basic.

– For the dual LP problem, we have:

$$\pi_4 = 1 - \pi_1 - \pi_6 \quad \text{then} \quad \pi_1 = 1 - \pi_4 - \pi_6.$$

Substituting in the other equations of the dual dictionary, we get:

$$\pi_3 = 3 - (1 - \pi_4 - \pi_6) + \pi_2 - 2\pi_6 = 2 + \pi_2 + \pi_4 - \pi_6$$

$$\pi_5 = 1 + (1 - \pi_4 - \pi_6) - \pi_2 + \pi_6 = 2 - \pi_2 - \pi_4$$

$$-w = 5 + (1 - \pi_4 - \pi_6) - 3\pi_2 + 3\pi_6 = 6 - 3\pi_2 - \pi_4 + 2\pi_6$$

Accordingly, we now have the following dual dictionary:

$$\begin{cases} \pi_1 = 1 & - \pi_4 & - \pi_6 \\ \pi_3 = 2 & + \pi_2 & + \pi_4 & - \pi_6 \\ \pi_5 = 2 & - \pi_2 & - \pi_4 \\ -w = 6 & - 3\pi_2 & - \pi_4 & + 2\pi_6 \end{cases}$$

where $\mathcal{B} = \{1, 3, 5\}$ and $\mathcal{N} = \{2, 4, 6\}$. The basic dual feasible solution is:

$$\bar{\pi}_1 = 1, \quad \bar{\pi}_3 = 2, \quad \bar{\pi}_5 = 2, \quad \bar{\pi}_2 = 0, \quad \bar{\pi}_4 = 0, \quad \bar{\pi}_6 = 0, \quad -w = 6, \quad w = -6.$$

– For the primal LP problem, we have:

$$x_1 = -1 + x_3 + x_4 - x_5 \quad \text{then} \quad x_4 = 1 + x_1 - x_3 + x_5.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_2 = 3 - x_3 + x_5$$

$$x_6 = -3 + 2x_3 + (1 + x_1 - x_3 + x_5) - x_5 = -2 + x_1 + x_3$$

$$z = -5 - 3x_3 - (1 + x_1 - x_3 + x_5) - x_5 = -6 - x_1 - 2x_3 - 2x_5$$

Accordingly, we now have the following primal dictionary:

$$\begin{cases} x_2 = 3 & - & x_3 & + & x_5 \\ x_4 = 1 & + & x_1 & - & x_3 & + & x_5 \\ x_6 = -2 & + & x_1 & + & x_3 \\ z = -6 & - & x_1 & - & 2x_3 & - & 2x_5 \end{cases}$$

where $\mathcal{B} = \{2, 4, 6\}$ and $\mathcal{N} = \{1, 3, 5\}$. The basic primal non-feasible solution is:

$$\bar{x}_2 = 3, \quad \bar{x}_4 = 1, \quad \bar{x}_6 = -2, \quad \bar{x}_1 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad -6$$

• Iteration 4

The only basic variable with negative value is x_6 , which is the pivot row. Correspondingly, in the dual dictionary we increase the variable π_6 , and we can get a better dual solution. We have:

$$\begin{cases} \pi_1 = 1 - \pi_6 \\ \pi_3 = 2 - \pi_6 \\ \pi_5 = 2 \end{cases} \quad \begin{cases} \underbrace{1 - \pi_6 \geq 0}_{\pi_1 \geq 0} \\ \underbrace{2 - \pi_6 \geq 0}_{\pi_3 \geq 0} \\ \underbrace{2 \geq 0}_{\pi_5 \geq 0} \end{cases} \quad \begin{cases} \pi_6 \leq 1 \\ \pi_6 \leq 2 \\ (\text{no condition}) \end{cases}$$

These are exactly the ratios of the dual simplex algorithm on the row of x_6 : $s_m = \min \left\{ \frac{-1}{-1}, \frac{-2}{-1} \right\} = 1$, attained by x_1 . By increasing the value of π_6 to 1, the value of π_1 decreases to 0. Then we have that:

π_6 becomes dual basic and, accordingly, x_6 becomes primal non-basic;
 π_1 becomes dual non-basic and, accordingly, x_1 becomes primal basic.

- For the dual LP problem, we have:

$$\pi_1 = 1 - \pi_4 - \pi_6 \quad \text{then} \quad \pi_6 = 1 - \pi_1 - \pi_4.$$

Substituting in the other equations of the dual dictionary, we get:

$$\pi_3 = 2 + \pi_2 + \pi_4 - (1 - \pi_1 - \pi_4) = 1 + \pi_1 + \pi_2 + 2\pi_4$$

$$\pi_5 = 2 - \pi_2 - \pi_4$$

$$-w = 6 - 3\pi_2 - \pi_4 + 2(1 - \pi_1 - \pi_4) = 8 - 2\pi_1 - 3\pi_2 - 3\pi_4$$

Accordingly, we now have the following optimal dual dictionary:

$$\left\{ \begin{array}{rclclcl} \pi_3 & = & 1 & + & \pi_1 & + & \pi_2 & + & 2\pi_4 \\ \pi_5 & = & 2 & & & - & \pi_2 & - & \pi_4 \\ \pi_6 & = & 1 & - & \pi_1 & & & - & \pi_4 \\ -w & = & 8 & - & 2\pi_1 & - & 3\pi_2 & - & 3\pi_4 \end{array} \right.$$

where $\mathcal{B} = \{3, 5, 6\}$ and $\mathcal{N} = \{1, 2, 4\}$. All the coefficients of the objective function are non-positive, and then the dual basic optimal solution is:

$$\tilde{\pi}_3 = 1, \quad \tilde{\pi}_5 = 2, \quad \tilde{\pi}_6 = 1, \quad \tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 0, \quad \tilde{\pi}_4 = 0, \quad -w = 8, \quad w = -8.$$

Recalling that the original dual variables are the shifted variables π_4, π_5, π_6 , the optimal solution of the original dual LP problem is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 2, \quad \tilde{\pi}_3 = 1, \quad \text{and} \quad z(D(LP)) = -8.$$

- For the primal LP problem, we have:

$$x_6 = -2 + x_1 + x_3 \quad \text{then} \quad x_1 = 2 - x_3 + x_6.$$

Substituting in the other equations of the primal dictionary, we get:

$$x_2 = 3 - x_3 + x_5$$

$$x_4 = 1 + (2 - x_3 + x_6) - x_3 + x_5 = 3 - 2x_3 + x_5 + x_6$$

$$z = -6 - (2 - x_3 + x_6) - 2x_3 - 2x_5 = -8 - x_3 - 2x_5 - x_6$$

Accordingly, we now have the following optimal primal dictionary:

$$\begin{cases} x_1 = 2 - x_3 + x_6 \\ x_2 = 3 - x_3 + x_5 \\ x_4 = 3 - 2x_3 + x_5 + x_6 \\ z = -8 - x_3 - 2x_5 - x_6 \end{cases}$$

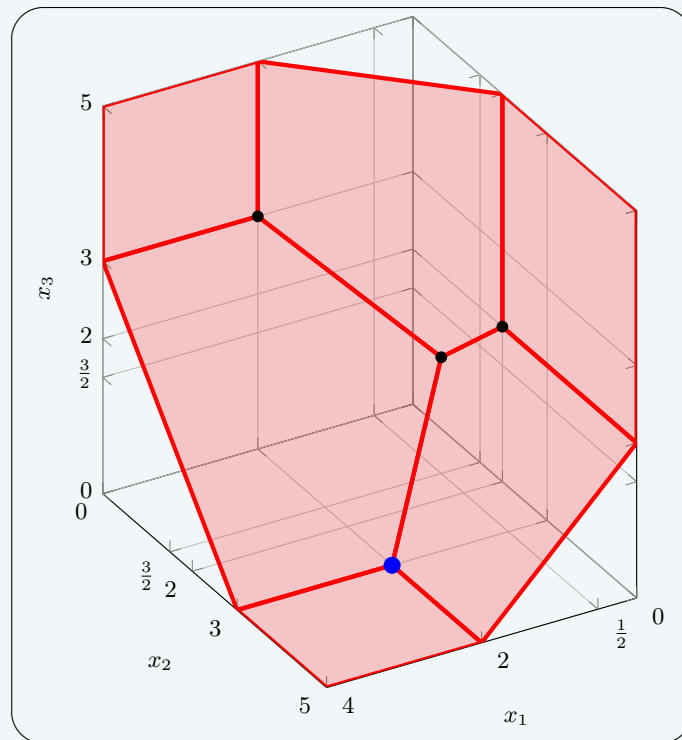
where $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5, 6\}$. The basic solution is now primal feasible and dual feasible, and then the primal basic optimal solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 3, \quad \tilde{x}_4 = 3, \quad \tilde{x}_3 = 0, \quad \tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad -8,$$

in accordance with the Strong Duality theorem.

Graphical solution:

The feasible region of the LP problem in canonical form is the polyhedron shaded in red. The feasible region is unbounded: we draw only its faces, cut at $x_1 = 4$, $x_2 = 5$ and $x_3 = 5$.

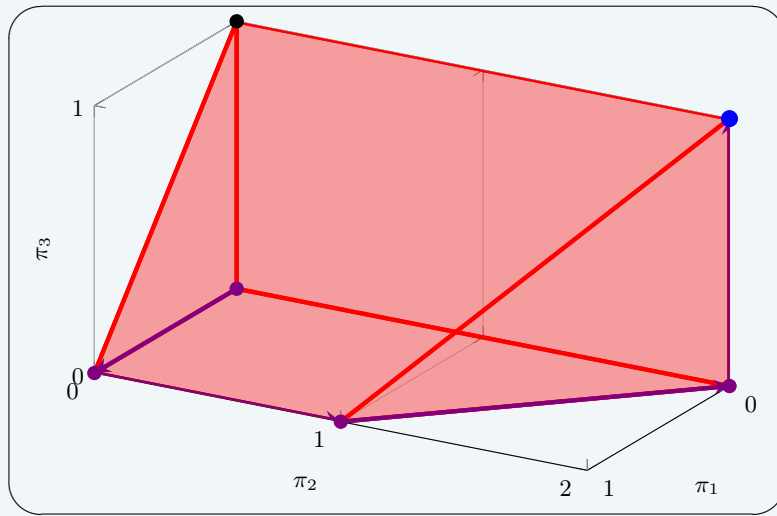


An optimal primal LP solution is:

$$\tilde{x}_1 = 2, \quad \tilde{x}_2 = 3, \quad \tilde{x}_3 = 0 \quad \text{and} \quad z(LP) = -8.$$

The feasible region of the dual LP problem in canonical minimization form is the polyhedron shaded in red. In the following figure we plot the three original dual variables

π_1, π_2, π_3 , corresponding to the variables π_4, π_5, π_6 of the dual dictionaries. The constraint $-\pi_2 - \pi_3 \geq -4$ is redundant.



An optimal dual LP solution is:

$$\tilde{\pi}_1 = 0, \quad \tilde{\pi}_2 = 2, \quad \tilde{\pi}_3 = 1, \quad \text{and} \quad z(D(LP)) = -8.$$

In violet, we show the four vertices $(0,0,0) \rightarrow (1,0,0) \rightarrow (1,1,0) \rightarrow (0,2,0)$ explored by the previous execution of the dual simplex algorithm, before reaching the optimal one $(0,2,1)$ depicted in blue. The dual objective values are $0 \rightarrow -2 \rightarrow -5 \rightarrow -6 \rightarrow -8$.

SageMath Code: 3D Polyhedra (<https://sagecell.sagemath.org/>)

```
# Variables are (pi_1, pi_2, pi_3), i.e., the original dual variables.
# Define the polyhedron using inequalities (Hyperplane Representation)
p = Polyhedron(ieqs=[
    (1, -1, 0, -1),    # 1 - pi_1 - pi_3 >= 0
    (2, -1, -1, 0),    # 2 - pi_1 - pi_2 >= 0
    (4, 0, -1, -1),    # 4 - pi_2 - pi_3 >= 0
    (0, 1, 0, 0),      # pi_1 >= 0
    (0, 0, 1, 0),      # pi_2 >= 0
    (0, 0, 0, 1)       # pi_3 >= 0
])

# Display the inequalities of the polyhedron
p.Hrepresentation()

# Display the extreme points and extreme rays of the polyhedron
p.Vrepresentation()

# Plot the polyhedron (3D)
p.plot()
```