



SAPIENZA
UNIVERSITÀ DI ROMA

Linear Programming

Exercises 2 — The simplex algorithm

Worked exercises on chapters 6–8: dictionaries, the simplex algorithm, cycling and Bland's rule, Phase 0. Each exercise uses a problem of the catalogue and has a complete solution.

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Simplex algorithm

6.1 Dictionaries

Exercise 6.1 — Simplex dictionaries (*P3*)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & 2 & x_1 & + & 4 & x_2 \\ \text{subject to} & - & x_1 & + & & x_2 & \leq & 2 \\ & 8 & x_1 & + & 2 & x_2 & \leq & 19 \\ & & x_1, & & & x_2 & \geq & 0 \end{array}$$

By introducing the slack variables, we get its standard form with $n + m = 4$ variables and $m = 2$ constraints:

$$\begin{array}{llllllll} \max & 2 & x_1 & + & 4 & x_2 & & & \\ \text{subject to} & - & x_1 & + & & x_2 & + & x_3 & = & 2 \\ & 8 & x_1 & + & 2 & x_2 & & & + & x_4 & = & 19 \\ & & x_1, & & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

Questions

1. Considering the following partition of the variable indices

$$\mathcal{B} = \{1, 3\} \quad \text{and} \quad \mathcal{N} = \{2, 4\}$$

verify that the matrix

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & 1 \\ 8 & 0 \end{pmatrix}$$

is a basis of the LP problem.

2. Compute the inverse matrix $\mathbf{A}_B^{-1}$ then write the associated dictionary.

A **simplex dictionary** is a system of linear equations in which the basic variables and the objective-function are written in function of the non-basic variables:

$$\begin{cases} \mathbf{x}_B = \bar{\mathbf{b}} - \bar{\mathbf{A}} \mathbf{x}_N \\ z = \bar{z} + \bar{\mathbf{c}}' \mathbf{x}_N \end{cases} \quad \text{where} \quad \begin{cases} \bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b} \\ \bar{\mathbf{A}} = \mathbf{A}_B^{-1} \mathbf{A}_N \\ \bar{z} = \mathbf{c}_B' \mathbf{A}_B^{-1} \mathbf{b} \\ \bar{\mathbf{c}}' = \mathbf{c}_N' - \mathbf{c}_B' \mathbf{A}_B^{-1} \mathbf{A}_N \end{cases}$$

Solution

1. In the LP problem the input data are:

$$\mathbf{c} = \begin{pmatrix} 2 \\ 4 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 8 & 2 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 19 \end{pmatrix}$$

With $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$, we have:

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad \mathbf{x}_N = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \quad \mathbf{A}_B = \begin{pmatrix} -1 & 1 \\ 8 & 0 \end{pmatrix} \quad \mathbf{A}_N = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \mathbf{c}_B = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \mathbf{c}_N = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

We have:

$$\mathbf{A}_B = \begin{pmatrix} -1 & 1 \\ 8 & 0 \end{pmatrix} \quad \text{and} \quad \det(\mathbf{A}_B) = -1 \cdot 0 - 1 \cdot 8 = -8 \neq 0 \quad \text{then} \quad \text{rank}(\mathbf{A}_B) = 2$$

accordingly, the matrix $\mathbf{A}_B$ is invertible and it is a basis of the LP problem.

2. We compute the inverse matrix:

$$\begin{pmatrix} -1 & 1 & | & 1 & 0 \\ 8 & 0 & | & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & | & -1 & 0 \\ 8 & 0 & | & 0 & 1 \end{pmatrix} \quad (R_1 \leftarrow -R_1)$$

$$\begin{pmatrix} 1 & -1 & | & -1 & 0 \\ 0 & 8 & | & 8 & 1 \end{pmatrix} \quad (R_2 \leftarrow R_2 - 8R_1)$$

$$\left(\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & \frac{1}{8} \end{array} \right) \quad (R_2 \leftarrow \frac{1}{8}R_2)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{8} \\ 0 & 1 & 1 & \frac{1}{8} \end{array} \right) \quad (R_1 \leftarrow R_1 + R_2)$$

Accordingly, we have:

$$\mathbf{A}_B^{-1} = \begin{pmatrix} 0 & \frac{1}{8} \\ 1 & \frac{1}{8} \end{pmatrix}$$

and we can compute:

$$\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b} = \begin{pmatrix} 0 & \frac{1}{8} \\ 1 & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 2 \\ 19 \end{pmatrix} = \begin{pmatrix} \frac{19}{8} \\ \frac{35}{8} \end{pmatrix}$$

$$\bar{\mathbf{A}} = \mathbf{A}_B^{-1} \mathbf{A}_N = \begin{pmatrix} 0 & \frac{1}{8} \\ 1 & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{5}{4} & \frac{1}{8} \end{pmatrix}$$

$$\bar{z} = \mathbf{c}'_B \mathbf{A}_B^{-1} \mathbf{b} = (2 \ 0) \begin{pmatrix} 0 & \frac{1}{8} \\ 1 & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 2 \\ 19 \end{pmatrix} = (0 \ \frac{1}{4}) \begin{pmatrix} 2 \\ 19 \end{pmatrix} = \frac{19}{4}$$

$$\bar{\mathbf{c}}' = \mathbf{c}'_N - \mathbf{c}'_B \mathbf{A}_B^{-1} \mathbf{A}_N = (4 \ 0) - (2 \ 0) \begin{pmatrix} 0 & \frac{1}{8} \\ 1 & \frac{1}{8} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$= (4 \ 0) - (0 \ \frac{1}{4}) \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$= (4 \ 0) - (\frac{1}{2} \ \frac{1}{4}) = (\frac{7}{2} \ -\frac{1}{4})$$

The dictionary is:

$$\begin{cases} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{19}{8} \\ \frac{35}{8} \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{5}{4} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \\ z = \frac{19}{4} + \begin{pmatrix} \frac{7}{2} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \end{cases}$$

or, equivalently:

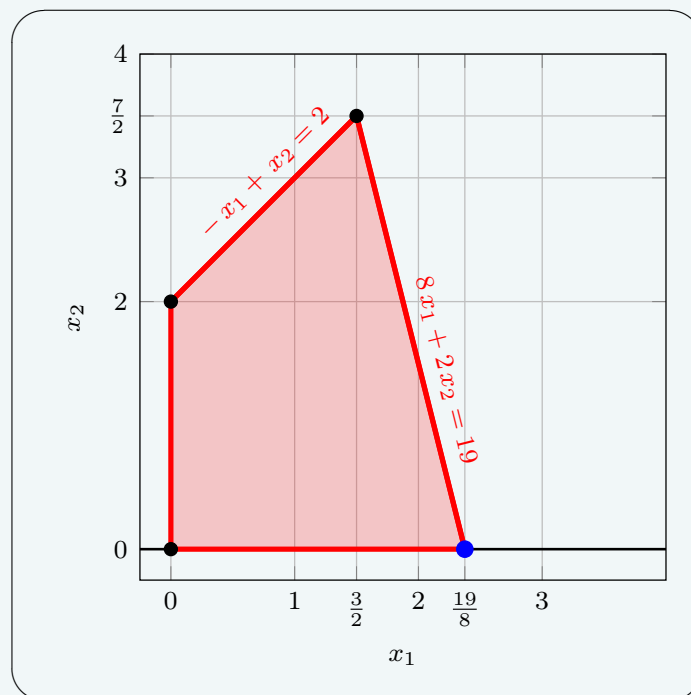
$$\begin{cases} x_1 = \frac{19}{8} - \frac{1}{4} x_2 - \frac{1}{8} x_4 \\ x_3 = \frac{35}{8} - \frac{5}{4} x_2 - \frac{1}{8} x_4 \\ z = \frac{19}{4} + \frac{7}{2} x_2 - \frac{1}{4} x_4 \end{cases}$$

where $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = \frac{19}{8}, \quad \bar{x}_3 = \frac{35}{8}, \quad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{19}{4}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



In blue the basic feasible solution associated with the basis $\mathcal{B} = \{1, 3\}$, that is the vertex:

$$\begin{pmatrix} \frac{19}{8} \\ 0 \end{pmatrix}.$$

Exercise 6.2 — Simplex dictionaries (P10)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{aligned}
 \max \quad & 5 x_1 + 2 x_2 \\
 \text{subject to} \quad & -x_1 + 3 x_2 \leq 12 \\
 & x_1 + 2 x_2 \leq 12 \\
 & 3 x_1 - 2 x_2 \leq 8 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

By introducing the slack variables we get its standard form with $n + m = 5$ variables and $m = 3$ constraints:

$$\begin{aligned}
 \max \quad & 5 x_1 + 2 x_2 \\
 \text{subject to} \quad & -x_1 + 3 x_2 + x_3 = 12 \\
 & x_1 + 2 x_2 + x_4 = 12 \\
 & 3 x_1 - 2 x_2 + x_5 = 8 \\
 & x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{aligned}$$

Questions

1. Considering the following partition of the variable indices

$$\mathcal{B} = \{2, 4, 5\} \quad \text{and} \quad \mathcal{N} = \{1, 3\}$$

verify that the matrix

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

is a basis of the LP problem.

2. Compute the inverse matrix $\mathbf{A}_{\mathcal{B}}^{-1}$ then write the associated dictionary. Is the associated basic solution feasible?

A **simplex dictionary** is a system of linear equations in which the basic variables and the objective-function are written in function of the non-basic variables:

$$\begin{cases} x_{\mathcal{B}} = \bar{b} - \bar{A} x_{\mathcal{N}} \\ z = \bar{z} + \bar{c}' x_{\mathcal{N}} \end{cases} \quad \text{where} \quad \begin{cases} \bar{b} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} \\ \bar{A} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} \\ \bar{z} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} \\ \bar{c}' = \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} \end{cases}$$

Solution

1. In the LP problem the input data are:

$$\mathbf{c} = \begin{pmatrix} 5 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix}$$

With $\mathcal{B} = \{2, 4, 5\}$, we have:

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} \det(\mathbf{A}_{\mathcal{B}}) &= 3 \cdot (1 \cdot 1 - 0 \cdot 0) - 0 \cdot (2 \cdot 1 - 0 \cdot (-2)) + 0 \cdot (2 \cdot 0 - 1 \cdot (-2)) \\ &= 3 \cdot 1 = 3 \neq 0 \end{aligned}$$

and then $\text{rank}(\mathbf{A}_{\mathcal{B}}) = 3$. Accordingly, the matrix $\mathbf{A}_{\mathcal{B}}$ is invertible and it is a basis of the LP problem.

2. We compute the inverse matrix $\mathbf{A}_{\mathcal{B}}^{-1}$ with the method of Gauss-Jordan:

$$\begin{aligned} &\left(\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \\ &\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \quad (R_1 \leftarrow \frac{1}{3}R_1) \\ &\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 & 1 \end{array} \right) \quad (R_2 \leftarrow R_2 - 2R_1, R_3 \leftarrow R_3 + 2R_1) \end{aligned}$$

Since $\mathcal{B} = \{2, 4, 5\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathbf{A}_{\mathcal{B}}^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{pmatrix}$$

accordingly $\mathcal{N} = \{1, 3\}$ and we have:

$$\mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 3 & 0 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

We can compute:

$$\bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 16 \end{pmatrix}$$

$$\bar{\mathbf{A}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{5}{3} & -\frac{2}{3} \\ \frac{7}{3} & \frac{2}{3} \end{pmatrix}$$

$$\bar{z} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} = (2 \ 0 \ 0) \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = (\frac{2}{3} \ 0 \ 0) \begin{pmatrix} 12 \\ 12 \\ 8 \end{pmatrix} = 8$$

$$\begin{aligned} \bar{\mathbf{c}}' &= \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = (5 \ 0) - (2 \ 0 \ 0) \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 3 & 0 \end{pmatrix} \\ &= (5 \ 0) - (\frac{2}{3} \ 0 \ 0) \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 3 & 0 \end{pmatrix} = (5 \ 0) - (-\frac{2}{3} \ \frac{2}{3}) = (\frac{17}{3} \ -\frac{2}{3}) \end{aligned}$$

Accordingly, the associated dictionary is:

$$\begin{cases} \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 16 \end{pmatrix} - \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{5}{3} & -\frac{2}{3} \\ \frac{7}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \\ z = 8 + (\frac{17}{3} \ -\frac{2}{3}) \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \end{cases}$$

or, equivalently:

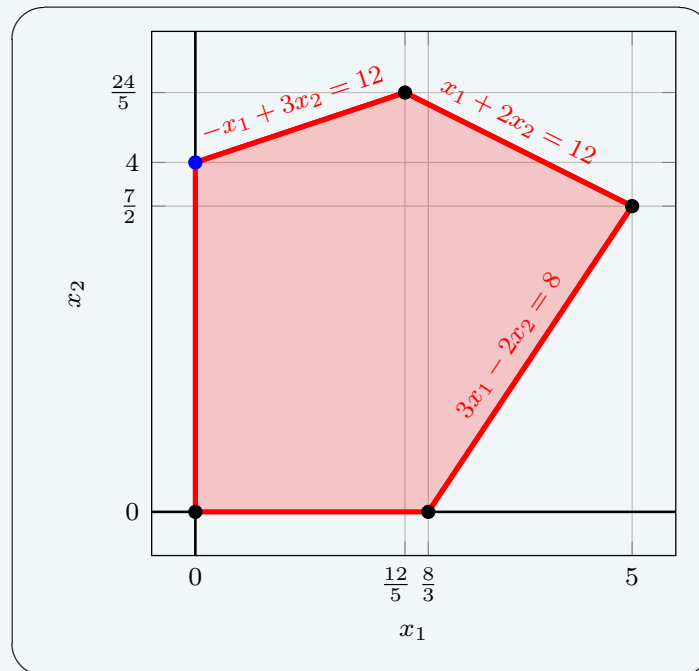
$$\begin{cases} x_2 = 4 + \frac{1}{3}x_1 - \frac{1}{3}x_3 \\ x_4 = 4 - \frac{5}{3}x_1 + \frac{2}{3}x_3 \\ x_5 = 16 - \frac{7}{3}x_1 - \frac{2}{3}x_3 \\ z = 8 + \frac{17}{3}x_1 - \frac{2}{3}x_3 \end{cases}$$

where $\mathcal{B} = \{2, 4, 5\}$ and $\mathcal{N} = \{1, 3\}$. Since $\bar{\mathbf{b}} \geq \mathbf{0}$, the associated basic solution is feasible. The basic feasible solution is:

$$\bar{x}_2 = 4, \quad \bar{x}_4 = 4, \quad \bar{x}_5 = 16, \quad \bar{x}_1 = 0, \quad \bar{x}_3 = 0 \quad \text{of value } 8$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



In blue the basic feasible solution associated with the basis $\mathcal{B} = \{2, 4, 5\}$, that is the vertex:

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

Exercise 6.3 — Simplex dictionaries (P11)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & -2 & x_1 & - & 4 & x_2 \\ \text{subject to} & - & x_1 & - & & x_2 & \leq & -2 \\ & & x_1 & - & & x_2 & \leq & 0 \\ & & x_1, & & x_2 & \geq & 0 \end{array}$$

By introducing the slack variables, we get its standard form with $n + m = 4$ variables and $m = 2$ constraints:

$$\begin{array}{llllllll} \max & -2 & x_1 & - & 4 & x_2 \\ \text{subject to} & - & x_1 & - & & x_2 & + & x_3 & = & -2 \\ & & x_1 & - & & x_2 & & + & x_4 & = & 0 \\ & & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

Questions

1. Considering the following partition of the variable indices

$$\mathcal{B} = \{1, 2\} \quad \text{and} \quad \mathcal{N} = \{3, 4\}$$

verify that the matrix

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

is a basis of the LP problem.

2. Compute the inverse matrix $\mathbf{A}_{\mathcal{B}}^{-1}$ then write the associated dictionary.

A **simplex dictionary** is a system of linear equations in which the basic variables and the objective-function are written in function of the non-basic variables:

$$\begin{cases} \mathbf{x}_{\mathcal{B}} = \bar{\mathbf{b}} - \bar{\mathbf{A}} \mathbf{x}_{\mathcal{N}} \\ z = \bar{z} + \bar{\mathbf{c}}' \mathbf{x}_{\mathcal{N}} \end{cases} \quad \text{where} \quad \begin{cases} \bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} \\ \bar{\mathbf{A}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} \\ \bar{z} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} \\ \bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} \end{cases}$$

Solution

1. In the LP problem the input data are:

$$\mathbf{c} = \begin{pmatrix} -2 \\ -4 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} -1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

With $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$, we have:

$$\mathbf{x}_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \mathbf{x}_{\mathcal{N}} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{A}_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{c}_{\mathcal{B}} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} \quad \mathbf{c}_{\mathcal{N}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We have:

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \det(\mathbf{A}_{\mathcal{B}}) = -1 \cdot (-1) - (-1) \cdot 1 = 2 \neq 0 \quad \text{then} \quad \text{rank}(\mathbf{A}_{\mathcal{B}}) = 2$$

accordingly, the matrix $\mathbf{A}_{\mathcal{B}}$ is invertible and it is a basis of the LP problem.

2. We compute the inverse matrix:

$$\begin{pmatrix} -1 & -1 & | & 1 & 0 \\ 1 & -1 & | & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & | & -1 & 0 \\ 1 & -1 & | & 0 & 1 \end{pmatrix} \quad (R_1 \leftarrow -R_1)$$

$$\begin{pmatrix} 1 & 1 & | & -1 & 0 \\ 0 & -2 & | & 1 & 1 \end{pmatrix} \quad (R_2 \leftarrow R_2 - R_1)$$

$$\begin{pmatrix} 1 & 1 & | & -1 & 0 \\ 0 & 1 & | & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad (R_2 \leftarrow -\frac{1}{2}R_2)$$

$$\begin{pmatrix} 1 & 0 & | & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & | & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad (R_1 \leftarrow R_1 - R_2)$$

Accordingly, we have:

$$\mathbf{A}_{\mathcal{B}}^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

and we can compute:

$$\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\bar{\mathbf{A}} = \mathbf{A}_B^{-1} \mathbf{A}_N = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\bar{z} = \mathbf{c}'_B \mathbf{A}_B^{-1} \mathbf{b} = (-2 \quad -4) \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} -2 \\ 0 \end{pmatrix} = (3 \quad 1) \begin{pmatrix} -2 \\ 0 \end{pmatrix} = -6$$

$$\bar{\mathbf{c}}' = \mathbf{c}'_N - \mathbf{c}'_B \mathbf{A}_B^{-1} \mathbf{A}_N = (0 \quad 0) - (-2 \quad -4) \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= (0 \quad 0) - (3 \quad 1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= (0 \quad 0) - (3 \quad 1) = (-3 \quad -1)$$

The dictionary is:

$$\begin{cases} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \\ z = -6 + (-3 \quad -1) \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \end{cases}$$

or, equivalently:

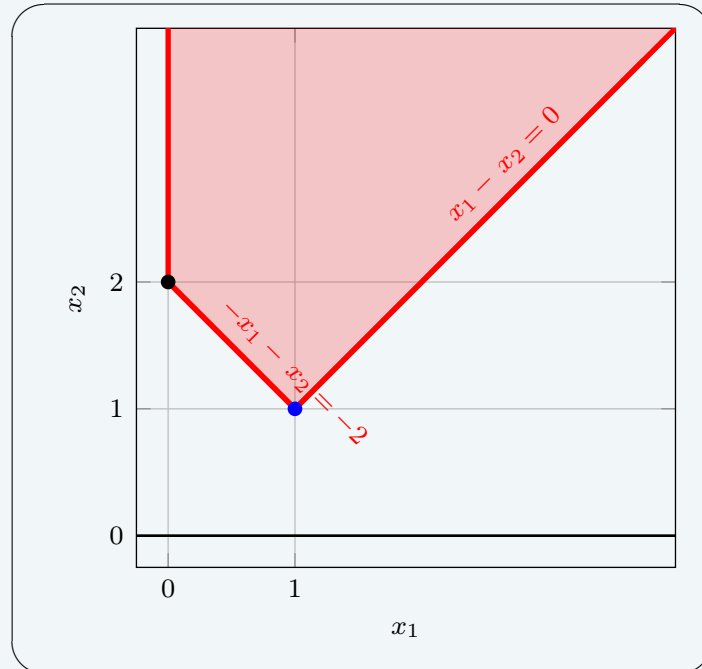
$$\begin{cases} x_1 = 1 + \frac{1}{2} x_3 - \frac{1}{2} x_4 \\ x_2 = 1 + \frac{1}{2} x_3 + \frac{1}{2} x_4 \\ z = -6 - 3 x_3 - x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = 1, \quad \bar{x}_2 = 1, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad -6$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



In blue the basic feasible solution associated with the basis $\mathcal{B} = \{1, 2\}$, that is the vertex:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Exercise 6.4 — Simplex dictionaries (P43)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & x_1 & + & 2 & x_2 & + & 7 & x_3 \\ \text{subject to} & x_1 & & & & + & 2 & x_3 & \leq & 2 \\ & & & & x_2 & + & 2 & x_3 & \leq & 2 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

By introducing the slack variables, we get its standard form with $n + m = 5$ variables and $m = 2$ constraints:

$$\begin{aligned}
 & \max \quad x_1 + 2x_2 + 7x_3 \\
 & \text{subject to} \quad x_1 + 2x_3 + x_4 = 2 \\
 & \quad \quad \quad x_2 + 2x_3 + x_5 = 2 \\
 & \quad \quad \quad x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{aligned}$$

Questions

1. Considering the following partitions of the variable indices

$$\mathcal{B} = \{2, 3\}, \mathcal{N} = \{1, 4, 5\} \quad \text{and} \quad \mathcal{B} = \{3, 5\}, \mathcal{N} = \{1, 2, 4\}$$

verify that the matrices

$$\mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_{\mathcal{B}} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

are bases of the LP problem.

2. Compute the inverse matrices $\mathbf{A}_{\mathcal{B}}^{-1}$ then write the associated dictionaries.

A **simplex dictionary** is a system of linear equations in which the basic variables and the objective-function are written in function of the non-basic variables:

$$\begin{cases} x_{\mathcal{B}} = \bar{\mathbf{b}} - \bar{\mathbf{A}} \mathbf{x}_{\mathcal{N}} \\ z = \bar{z} + \bar{\mathbf{c}}' \mathbf{x}_{\mathcal{N}} \end{cases} \quad \text{where} \quad \begin{cases} \bar{\mathbf{b}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} \\ \bar{\mathbf{A}} = \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} \\ \bar{z} = \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{b} \\ \bar{\mathbf{c}}' = \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} \end{cases}$$

Solution

In the LP problem the input data are:

$$\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 7 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

1. With

$$\mathcal{B} = \{2, 3\} \quad \text{and} \quad \mathcal{N} = \{1, 4, 5\}$$

we have:

$$\begin{aligned} \mathbf{x}_B &= \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} & \mathbf{x}_N &= \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} & \mathbf{A}_B &= \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} & \mathbf{A}_N &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ & & \mathbf{c}_B &= \begin{pmatrix} 2 \\ 7 \end{pmatrix} & \mathbf{c}_N &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

We have:

$$\mathbf{A}_B = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \det(\mathbf{A}_B) = 0 \cdot 2 - 1 \cdot 2 = -2 \neq 0$$

then $\text{rank}(\mathbf{A}_B) = 2$, accordingly the matrix $\mathbf{A}_B$ is invertible and it is a basis of the LP problem.

We compute the inverse matrix:

$$\begin{aligned} &\left(\begin{array}{cc|cc} 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right) \\ &\left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{array} \right) \quad (\text{exchange } R_1 \leftrightarrow R_2) \\ &\left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{array} \right) \quad (R_2 \leftarrow \frac{1}{2}R_2) \\ &\left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{array} \right) \quad (R_1 \leftarrow R_1 - 2R_2) \end{aligned}$$

Accordingly, we have:

$$\mathbf{A}_B^{-1} = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$$

and we can compute:

$$\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b} = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar{\mathbf{A}} = \mathbf{A}_B^{-1} \mathbf{A}_N = \mathbf{A}_B^{-1} \mathbf{A}_N = \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$\bar{z} = \mathbf{c}'_B \mathbf{A}_B^{-1} \mathbf{b} = (2 \quad 7) \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = (\frac{3}{2} \quad 2) \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 7$$

$$\begin{aligned}\bar{\mathbf{c}}' &= \mathbf{c}'_{\mathcal{N}} - \mathbf{c}'_{\mathcal{B}} \mathbf{A}_{\mathcal{B}}^{-1} \mathbf{A}_{\mathcal{N}} = (1 \ 0 \ 0) - (2 \ 7) \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= (1 \ 0 \ 0) - \left(\frac{3}{2} \ 2\right) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (1 \ 0 \ 0) - \left(\frac{3}{2} \ \frac{3}{2} \ 2\right) = \left(-\frac{1}{2} \ -\frac{3}{2} \ -2\right)\end{aligned}$$

The dictionary is:

$$\begin{cases} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 & -1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} \\ z = 7 + \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \\ x_5 \end{pmatrix} \end{cases}$$

or, equivalently:

$$\begin{cases} x_2 = 0 & + & x_1 & + & x_4 & - & x_5 \\ x_3 = 1 & - & \frac{1}{2} x_1 & - & \frac{1}{2} x_4 \\ z = 7 & - & \frac{1}{2} x_1 & - & \frac{3}{2} x_4 & - & 2 x_5 \end{cases}$$

where $\mathcal{B} = \{2, 3\}$ and $\mathcal{N} = \{1, 4, 5\}$. The basic feasible solution is:

$$\bar{x}_2 = 0, \ \bar{x}_3 = 1, \ \bar{x}_1 = 0, \ \bar{x}_4 = 0, \ \bar{x}_5 = 0 \quad \text{of value } 7$$

2. With

$$\mathcal{B} = \{3, 5\} \quad \text{and} \quad \mathcal{N} = \{1, 2, 4\}$$

we have:

$$\begin{aligned}\mathbf{x}_{\mathcal{B}} &= \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} & \mathbf{x}_{\mathcal{N}} &= \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} & \mathbf{A}_{\mathcal{B}} &= \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} & \mathbf{A}_{\mathcal{N}} &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \mathbf{c}_{\mathcal{B}} &= \begin{pmatrix} 7 \\ 0 \end{pmatrix} & \mathbf{c}_{\mathcal{N}} &= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\end{aligned}$$

We have:

$$\mathbf{A}_B = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \det(\mathbf{A}_B) = 2 \cdot 1 - 2 \cdot 0 = 2 \neq 0$$

then $\text{rank}(\mathbf{A}_B) = 2$, accordingly the matrix $\mathbf{A}_B$ is invertible and it is a basis of the LP problem.

We compute the inverse matrix:

$$\begin{pmatrix} 2 & 0 & | & 1 & 0 \\ 2 & 1 & | & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & | & \frac{1}{2} & 0 \\ 2 & 1 & | & 0 & 1 \end{pmatrix} \quad (R_1 \leftarrow \frac{1}{2}R_1)$$

$$\begin{pmatrix} 1 & 0 & | & \frac{1}{2} & 0 \\ 0 & 1 & | & -1 & 1 \end{pmatrix} \quad (R_2 \leftarrow R_2 - 2R_1)$$

Accordingly, we have:

$$\mathbf{A}_B^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix}$$

and we can compute:

$$\bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b} = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\bar{\mathbf{A}} = \mathbf{A}_B^{-1} \mathbf{A}_N = \mathbf{A}_B^{-1} \mathbf{A}_N = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 1 & -1 \end{pmatrix}$$

$$\bar{z} = \mathbf{c}'_B \mathbf{A}_B^{-1} \mathbf{b} = (7 \ 0) \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = (\frac{7}{2} \ 0) \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 7$$

$$\begin{aligned} \bar{\mathbf{c}}' &= \mathbf{c}'_N - \mathbf{c}'_B \mathbf{A}_B^{-1} \mathbf{A}_N = (1 \ 2 \ 0) - (7 \ 0) \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= (1 \ 2 \ 0) - (\frac{7}{2} \ 0) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = (1 \ 2 \ 0) - (\frac{7}{2} \ 0 \ \frac{7}{2}) = (-\frac{5}{2} \ 2 \ -\frac{7}{2}) \end{aligned}$$

The dictionary is:

$$\begin{cases} \begin{pmatrix} x_3 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \\ z = 7 + \begin{pmatrix} -\frac{5}{2} & 2 & -\frac{7}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \end{cases}$$

or, equivalently:

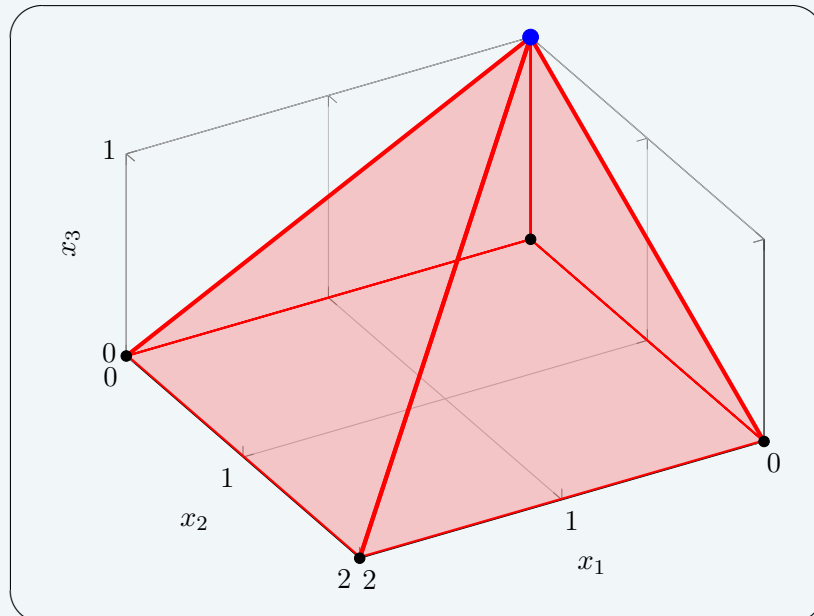
$$\begin{cases} x_3 = 1 - \frac{1}{2} x_1 - \frac{1}{2} x_4 \\ x_5 = 0 + x_1 - x_2 + x_4 \\ z = 7 - \frac{5}{2} x_1 + 2 x_2 - \frac{7}{2} x_4 \end{cases}$$

where $\mathcal{B} = \{3, 5\}$ and $\mathcal{N} = \{1, 2, 4\}$. The basic feasible solution is:

$$\bar{x}_3 = 1, \quad \bar{x}_5 = 0, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value } 7$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



In blue the basic feasible solution associated with both bases $\mathcal{B} = \{2, 3\}$ and $\mathcal{B} = \{3, 5\}$,

that is the vertex

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

of the polyhedron: the two bases are associated with the same basic solution (with $\mathcal{B} = \{2, 3\}$ the basic variable x_2 takes the value 0, with $\mathcal{B} = \{3, 5\}$ the basic variable x_5 takes the value 0).

6.2 Problems with two variables

Exercise 6.5 — Simplex algorithm (P_0)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 1$ constraint:

$$\begin{aligned} \max \quad & 2x_1 + 4x_2 \\ \text{subject to} \quad & 2x_1 + 3x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing the slack variable, we get the following starting dictionary:

$$\begin{cases} x_3 = 4 - 2x_1 - 3x_2 \\ z = 0 + 2x_1 + 4x_2 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 4 \quad \text{of value } 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- **Initialization**

Consider the starting dictionary:

$$\begin{cases} x_3 = 4 - 2x_1 - 3x_2 \\ z = 0 + 2x_1 + 4x_2 \end{cases}$$

where $\mathcal{B} = \{3\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 4, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

- **Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$x_3 = 4 - 2x_1 \quad \underbrace{4 - 2x_1 \geq 0}_{x_3 \geq 0} \quad x_1 \leq 2$$

By increasing the value of x_1 to 2, the value of x_3 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = 4 - 2x_1 - 3x_2 \quad \text{then} \quad x_1 = 2 - \frac{3}{2}x_2 - \frac{1}{2}x_3.$$

Substituting in the other equation of the dictionary, we get:

$$z = 2 \left(2 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \right) + 4x_2 = 4 + x_2 - x_3$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = 2 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \\ z = 4 + x_2 - x_3 \end{cases}$$

where $\mathcal{B} = \{1\}$ and $\mathcal{N} = \{2, 3\}$. The basic feasible solution is:

$$\bar{x}_1 = 2, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value } 4$$

- **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$x_1 = 2 - \frac{3}{2}x_2 \quad \underbrace{2 - \frac{3}{2}x_2 \geq 0}_{x_1 \geq 0} \quad x_2 \leq \frac{4}{3}$$

By increasing the value of x_2 to $\frac{4}{3}$, the value of x_1 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_1 becomes non-basic

We have:

$$x_1 = 2 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \quad \text{then} \quad x_2 = \frac{4}{3} - \frac{2}{3}x_1 - \frac{1}{3}x_3.$$

Substituting in the other equation of the dictionary, we get:

$$z = 4 + \left(\frac{4}{3} - \frac{2}{3}x_1 - \frac{1}{3}x_3 \right) - x_3 = \frac{16}{3} - \frac{2}{3}x_1 - \frac{4}{3}x_3$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

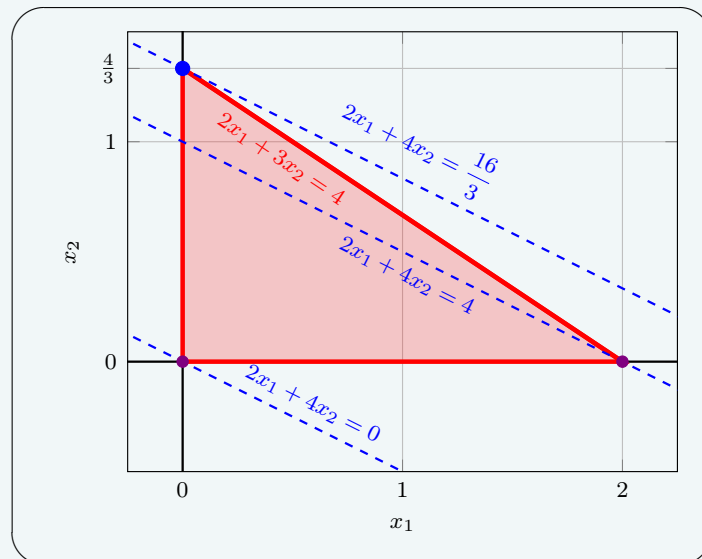
$$\begin{cases} x_2 &= \frac{4}{3} - \frac{2}{3}x_1 - \frac{1}{3}x_3 \\ z &= \frac{16}{3} - \frac{2}{3}x_1 - \frac{4}{3}x_3 \end{cases}$$

where $\mathcal{B} = \{2\}$ and $\mathcal{N} = \{1, 3\}$. The basic optimal solution is:

$$\tilde{x}_2 = \frac{4}{3}, \quad \tilde{x}_1 = 0, \quad \tilde{x}_3 = 0 \quad \text{of value} \quad \frac{16}{3}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = \frac{4}{3}$$

the optimal value $z(LP)$ is $\frac{16}{3}$.

In violet the first and the second vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 6.6 — Simplex algorithm (P1)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 1$ constraint:

$$\begin{aligned} \max \quad & x_1 + 8x_2 \\ \text{subject to} \quad & 3x_1 + 6x_2 \leq 16 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing the slack variable, we get the following starting dictionary:

$$\begin{cases} x_3 = 16 - 3x_1 - 6x_2 \\ z = 0 + x_1 + 8x_2 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 16 \quad \text{of value } 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- **Initialization**

Consider the starting dictionary:

$$\begin{cases} x_3 = 16 - 3x_1 - 6x_2 \\ z = 0 + x_1 + 8x_2 \end{cases}$$

where $\mathcal{B} = \{3\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 16, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value} \quad 0$$

• **Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$x_3 = 16 - 3x_1 \quad \underbrace{16 - 3x_1 \geq 0}_{x_3 \geq 0} \quad x_1 \leq \frac{16}{3}$$

By increasing the value of x_1 to $\frac{16}{3}$, the value of x_3 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = 16 - 3x_1 - 6x_2 \quad \text{then} \quad x_1 = \frac{16}{3} - 2x_2 - \frac{1}{3}x_3.$$

Substituting in the other equation of the dictionary, we get:

$$z = \left(\frac{16}{3} - 2x_2 - \frac{1}{3}x_3 \right) + 8x_2 = \frac{16}{3} + 6x_2 - \frac{1}{3}x_3$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 &= \frac{16}{3} - 2x_2 - \frac{1}{3}x_3 \\ z &= \frac{16}{3} + 6x_2 - \frac{1}{3}x_3 \end{cases}$$

where $\mathcal{B} = \{1\}$ and $\mathcal{N} = \{2, 3\}$. The current basic feasible solution is:

$$\bar{x}_1 = \frac{16}{3}, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad \frac{16}{3}$$

• **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$x_1 = \frac{16}{3} - 2x_2 \quad \underbrace{\frac{16}{3} - 2x_2 \geq 0}_{x_1 \geq 0} \quad x_2 \leq \frac{8}{3}$$

By increasing the value of x_2 to $\frac{8}{3}$, the value of x_1 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_1 becomes non-basic

We have:

$$x_1 = \frac{16}{3} - 2x_2 - \frac{1}{3}x_3 \quad \text{then} \quad x_2 = \frac{8}{3} - \frac{1}{2}x_1 - \frac{1}{6}x_3.$$

Substituting in the other equation of the dictionary, we get:

$$z = \frac{16}{3} + 6 \left(\frac{8}{3} - \frac{1}{2}x_1 - \frac{1}{6}x_3 \right) - \frac{1}{3}x_3 = \frac{64}{3} - 3x_1 - \frac{4}{3}x_3$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

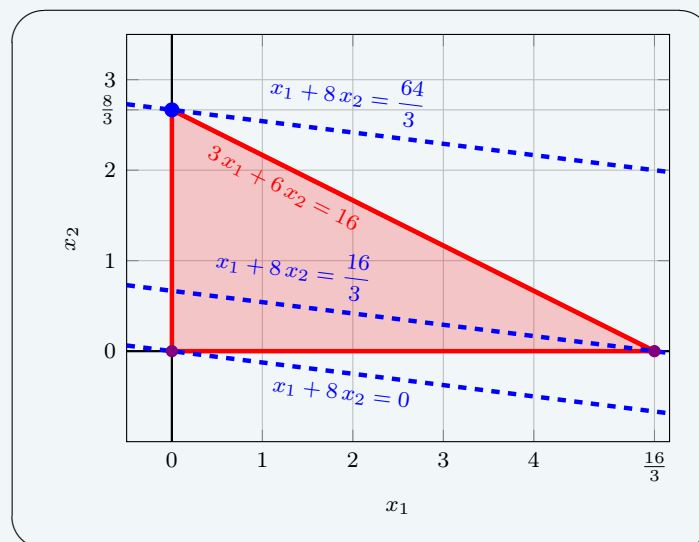
$$\begin{cases} x_2 &= \frac{8}{3} - \frac{1}{2}x_1 - \frac{1}{6}x_3 \\ z &= \frac{64}{3} - 3x_1 - \frac{4}{3}x_3 \end{cases}$$

where $\mathcal{B} = \{2\}$ and $\mathcal{N} = \{1, 3\}$. The basic optimal solution is:

$$\tilde{x}_2 = \frac{8}{3}, \quad \tilde{x}_1 = 0, \quad \tilde{x}_3 = 0 \quad \text{of value} \quad \frac{64}{3}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{x}(LP)$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = \frac{8}{3}$$

the optimal value $z(LP)$ is $\frac{64}{3}$.

In violet the first and the second vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 6.7 — Simplex algorithm (P2)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & x_1 + 3x_2 \\ \text{subject to} \quad & -4x_1 + 10x_2 \leq 35 \\ & 3x_1 + x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing the slack variables, we get the following starting dictionary:

$$\begin{cases} x_3 = 35 + 4x_1 - 10x_2 \\ x_4 = 12 - 3x_1 - x_2 \\ z = 0 + x_1 + 3x_2 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 35, \bar{x}_4 = 12 \quad \text{of value } 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- **Initialization**

Consider the starting dictionary:

$$\begin{cases} x_3 = 35 + 4x_1 - 10x_2 \\ x_4 = 12 - 3x_1 - x_2 \\ z = 0 + x_1 + 3x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 35, \quad \bar{x}_4 = 12, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

• **Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_3 = 35 + 4x_1 \\ x_4 = 12 - 3x_1 \end{cases} \quad \begin{cases} \underbrace{35 + 4x_1 \geq 0}_{x_3 \geq 0} \\ \underbrace{12 - 3x_1 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} x_1 \geq -\frac{35}{4} \\ x_1 \leq 4 \end{cases}$$

By increasing the value of x_1 to 4, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 12 - 3x_1 - x_2 \quad \text{then} \quad x_1 = 4 - \frac{1}{3}x_2 - \frac{1}{3}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_3 &= 35 + 4 \left(4 - \frac{1}{3}x_2 - \frac{1}{3}x_4 \right) - 10x_2 = 51 - \frac{34}{3}x_2 - \frac{4}{3}x_4 \\ z &= 4 - \frac{1}{3}x_2 - \frac{1}{3}x_4 + 3x_2 = 4 + \frac{8}{3}x_2 - \frac{1}{3}x_4 \end{aligned}$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = 4 - \frac{1}{3}x_2 - \frac{1}{3}x_4 \\ x_3 = 51 - \frac{34}{3}x_2 - \frac{4}{3}x_4 \\ z = 4 + \frac{8}{3}x_2 - \frac{1}{3}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = 4, \quad \bar{x}_3 = 51, \quad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value } 4$$

• **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_1 = 4 - \frac{1}{3} x_2 \\ x_3 = 51 - \frac{34}{3} x_2 \end{cases} \quad \begin{cases} \overbrace{4 - \frac{1}{3} x_2 \geq 0}^{x_1 \geq 0} \\ \overbrace{51 - \frac{34}{3} x_2 \geq 0}^{x_3 \geq 0} \end{cases} \quad \begin{cases} x_2 \leq 12 \\ x_2 \leq \frac{9}{2} \end{cases}$$

By increasing the value of x_2 to $\min \left\{ 12, \frac{9}{2} \right\} = \frac{9}{2}$, the value of x_3 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = 51 - \frac{34}{3} x_2 - \frac{4}{3} x_4 \quad \text{then} \quad x_2 = \frac{9}{2} - \frac{3}{34} x_3 - \frac{4}{34} x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_1 &= 4 - \frac{1}{3} \left(\frac{9}{2} - \frac{3}{34} x_3 - \frac{4}{34} x_4 \right) - \frac{1}{3} x_4 = \frac{5}{2} + \frac{1}{34} x_3 - \frac{5}{17} x_4 \\ z &= 4 + \frac{8}{3} \left(\frac{9}{2} - \frac{3}{34} x_3 - \frac{4}{34} x_4 \right) - \frac{1}{3} x_4 = 16 - \frac{4}{17} x_3 - \frac{11}{17} x_4 \end{aligned}$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

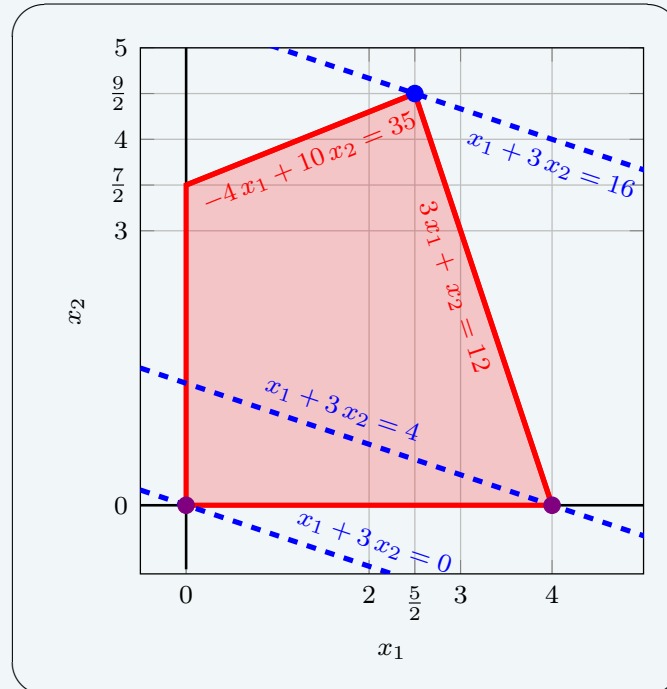
$$\begin{cases} x_1 = \frac{5}{2} + \frac{1}{34} x_3 - \frac{5}{17} x_4 \\ x_2 = \frac{9}{2} - \frac{3}{34} x_3 - \frac{4}{34} x_4 \\ z = 16 - \frac{4}{17} x_3 - \frac{11}{17} x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = \frac{9}{2}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad 16$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = \frac{9}{2}$$

the optimal value $z(LP)$ is 16.

In violet the first and the second vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 6.8 — Simplex algorithm (P3)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned}
 \max \quad & 2 x_1 + 4 x_2 \\
 \text{subject to} \quad & -x_1 + x_2 \leq 2 \\
 & 8 x_1 + 2 x_2 \leq 19 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

By introducing the slack variables, we get the following starting dictionary:

$$\begin{cases}
 x_3 = 2 + x_1 - x_2 \\
 x_4 = 19 - 8 x_1 - 2 x_2 \\
 z = 0 + 2 x_1 + 4 x_2
 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 2, \bar{x}_4 = 19 \quad \text{of value } 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- Initialization**

Consider the starting dictionary:

$$\begin{cases}
 x_3 = 2 + x_1 - x_2 \\
 x_4 = 19 - 8 x_1 - 2 x_2 \\
 z = 0 + 2 x_1 + 4 x_2
 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 2, \bar{x}_4 = 19, \bar{x}_1 = 0, \bar{x}_2 = 0 \quad \text{of value } 0$$

- Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_3 = 2 + x_1 \\ x_4 = 19 - 8x_1 \end{cases} \quad \begin{cases} \overbrace{2 + x_1 \geq 0}^{x_3 \geq 0} \\ \overbrace{19 - 8x_1 \geq 0}^{x_4 \geq 0} \end{cases} \quad \begin{cases} x_1 \geq -2 \\ x_1 \leq \frac{19}{8} \end{cases}$$

By increasing the value of x_1 to $\frac{19}{8}$, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 19 - 8x_1 - 2x_2 \quad \text{then} \quad x_1 = \frac{19}{8} - \frac{1}{4}x_2 - \frac{1}{8}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_3 &= 2 + \left(\frac{19}{8} - \frac{1}{4}x_2 - \frac{1}{8}x_4 \right) - x_2 = \frac{35}{8} - \frac{5}{4}x_2 - \frac{1}{8}x_4 \\ z &= 2 \left(\frac{19}{8} - \frac{1}{4}x_2 - \frac{1}{8}x_4 \right) + 4x_2 = \frac{19}{4} + \frac{7}{2}x_2 - \frac{1}{4}x_4 \end{aligned}$$

Accordingly, we obtain the following dictionary:

$$\begin{cases} x_1 = \frac{19}{8} - \frac{1}{4}x_2 - \frac{1}{8}x_4 \\ x_3 = \frac{35}{8} - \frac{5}{4}x_2 - \frac{1}{8}x_4 \\ z = \frac{19}{4} + \frac{7}{2}x_2 - \frac{1}{4}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = \frac{19}{8}, \quad \bar{x}_3 = \frac{35}{8}, \quad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{19}{4}$$

• Iteration 2

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_1 = \frac{19}{8} - \frac{1}{4}x_2 \\ x_3 = \frac{35}{8} - \frac{5}{4}x_2 \end{cases} \quad \begin{cases} \overbrace{\frac{19}{8} - \frac{1}{4}x_2 \geq 0}^{x_1 \geq 0} \\ \overbrace{\frac{35}{8} - \frac{5}{4}x_2 \geq 0}^{x_3 \geq 0} \end{cases} \quad \begin{cases} x_2 \leq \frac{19}{2} \\ x_2 \leq \frac{7}{2} \end{cases}$$

By increasing the value of x_2 to $\min \left\{ \frac{7}{2}, \frac{19}{2} \right\} = \frac{7}{2}$, the value of x_3 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = \frac{35}{8} - \frac{5}{4}x_2 - \frac{1}{8}x_4 \quad \text{then} \quad x_2 = \frac{7}{2} - \frac{4}{5}x_3 - \frac{1}{10}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_1 &= \frac{19}{8} - \frac{1}{4} \left(\frac{7}{2} - \frac{4}{5}x_3 - \frac{1}{10}x_4 \right) - \frac{1}{8}x_4 = \frac{3}{2} + \frac{1}{5}x_3 - \frac{1}{10}x_4 \\ z &= \frac{19}{4} + \frac{7}{2} \left(\frac{7}{2} - \frac{4}{5}x_3 - \frac{1}{10}x_4 \right) - \frac{1}{4}x_4 = 17 - \frac{14}{5}x_3 - \frac{3}{5}x_4 \end{aligned}$$

Accordingly, we have an optimal solution (no further improvement is possible) and this is an optimal dictionary:

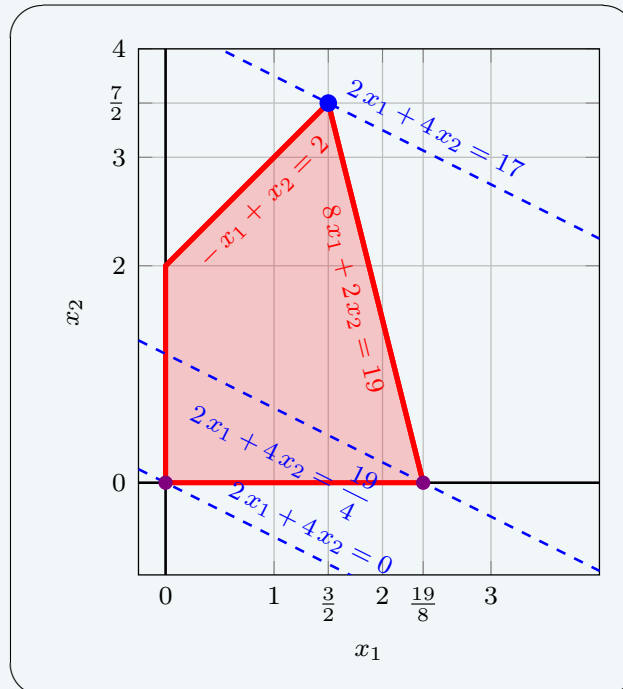
$$\begin{cases} x_1 = \frac{3}{2} & + & \frac{1}{5}x_3 & - & \frac{1}{10}x_4 \\ x_2 = \frac{7}{2} & - & \frac{4}{5}x_3 & - & \frac{1}{10}x_4 \\ z = 17 & - & \frac{14}{5}x_3 & - & \frac{3}{5}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{3}{2}, \quad \tilde{x}_2 = \frac{7}{2}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad 17$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{3}{2}, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value $z(LP)$ is 17.

In violet the first and the second vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 6.9 — Simplex algorithm (P4)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned}
 & \max \quad 5x_1 + 3x_2 \\
 & \text{subject to} \quad 2x_1 + x_2 \leq 14 \\
 & \quad \quad \quad x_1 + 8x_2 \leq 32 \\
 & \quad \quad \quad x_1, \quad x_2 \geq 0
 \end{aligned}$$

By introducing the slack variables, we get the following starting dictionary:

$$\begin{cases}
 x_3 = 14 - 2x_1 - x_2 \\
 x_4 = 32 - x_1 - 8x_2 \\
 z = 0 + 5x_1 + 3x_2
 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 14, \quad \bar{x}_4 = 32 \quad \text{of value} \quad 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- Initialization**

Consider the starting dictionary:

$$\begin{cases}
 x_3 = 14 - 2x_1 - x_2 \\
 x_4 = 32 - x_1 - 8x_2 \\
 z = 0 + 5x_1 + 3x_2
 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 14, \quad \bar{x}_4 = 32, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value} \quad 0$$

- Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_3 = 14 - 2x_1 \\ x_4 = 32 - x_1 \end{cases} \quad \begin{cases} \overbrace{14 - 2x_1 \geq 0}^{x_3 \geq 0} \\ \overbrace{32 - x_1 \geq 0}^{x_4 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq 7 \\ x_1 \leq 32 \end{cases}$$

By increasing the value of x_1 to 7, the value of x_3 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = 14 - 2x_1 - x_2 \quad \text{then} \quad x_1 = 7 - \frac{1}{2}x_2 - \frac{1}{2}x_3.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_4 &= 32 - \left(7 - \frac{1}{2}x_2 - \frac{1}{2}x_3\right) - 8x_2 = 25 - \frac{15}{2}x_2 + \frac{1}{2}x_3 \\ z &= 5 \left(7 - \frac{1}{2}x_2 - \frac{1}{2}x_3\right) + 3x_2 = 35 + \frac{1}{2}x_2 - \frac{5}{2}x_3 \end{aligned}$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = 7 - \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ x_4 = 25 - \frac{15}{2}x_2 + \frac{1}{2}x_3 \\ z = 35 + \frac{1}{2}x_2 - \frac{5}{2}x_3 \end{cases}$$

where $\mathcal{B} = \{1, 4\}$ and $\mathcal{N} = \{2, 3\}$. The basic feasible solution is:

$$\bar{x}_1 = 7, \quad \bar{x}_4 = 25, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad 35$$

- **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_1 = 7 - \frac{1}{2}x_2 \\ x_4 = 25 - \frac{15}{2}x_2 \end{cases} \quad \begin{cases} \overbrace{7 - \frac{1}{2}x_2 \geq 0}^{x_1 \geq 0} \\ \overbrace{25 - \frac{15}{2}x_2 \geq 0}^{x_4 \geq 0} \end{cases} \quad \begin{cases} x_2 \leq 14 \\ x_2 \leq \frac{10}{3} \end{cases}$$

By increasing the value of x_2 to $\min \left\{14, \frac{10}{3}\right\} = \frac{10}{3}$, the value of x_4 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 25 - \frac{15}{2}x_2 + \frac{1}{2}x_3 \quad \text{then} \quad x_2 = \frac{10}{3} + \frac{1}{15}x_3 - \frac{2}{15}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_1 &= 7 - \frac{1}{2} \left(\frac{10}{3} + \frac{1}{15}x_3 - \frac{2}{15}x_4 \right) - \frac{1}{2}x_3 = \frac{16}{3} - \frac{8}{15}x_3 + \frac{1}{15}x_4 \\ z &= 35 + \frac{1}{2} \left(\frac{10}{3} + \frac{1}{15}x_3 - \frac{2}{15}x_4 \right) - \frac{5}{2}x_3 = \frac{110}{3} - \frac{37}{15}x_3 - \frac{1}{15}x_4 \end{aligned}$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

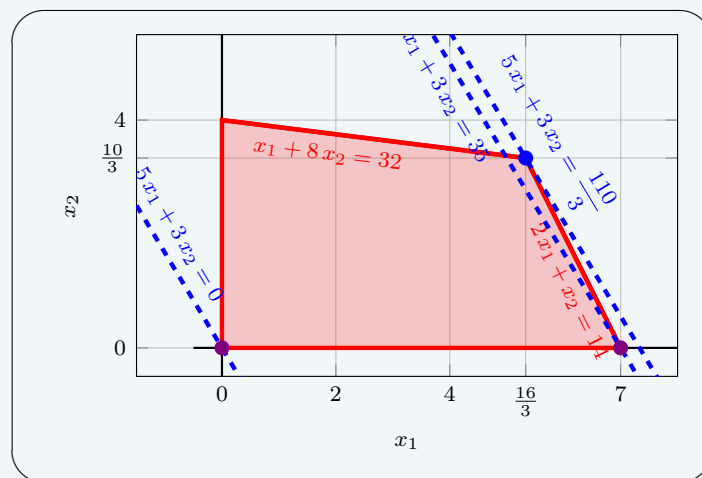
$$\begin{cases} x_1 = \frac{16}{3} - \frac{8}{15}x_3 + \frac{1}{15}x_4 \\ x_2 = \frac{10}{3} + \frac{1}{15}x_3 - \frac{2}{15}x_4 \\ z = \frac{110}{3} - \frac{37}{15}x_3 - \frac{1}{15}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{16}{3}, \quad \tilde{x}_2 = \frac{10}{3}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad \frac{110}{3}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{16}{3}, \quad \tilde{x}_2 = \frac{10}{3}$$

the optimal value $z(LP)$ is $\frac{110}{3}$.

In violet the first and the second vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 6.10 — Simplex algorithm (P5)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{subject to} \quad & 6x_1 + 21x_2 \leq 77 \\ & 8x_1 + 3x_2 \leq 36 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing the slack variables, we get the following starting dictionary:

$$\begin{cases} x_3 = 77 - 6x_1 - 21x_2 \\ x_4 = 36 - 8x_1 - 3x_2 \\ z = 0 + 2x_1 + 3x_2 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 77, \quad \bar{x}_4 = 36 \quad \text{of value} \quad 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- Initialization**

Consider the starting dictionary:

$$\begin{cases} x_3 = 77 - 6x_1 - 21x_2 \\ x_4 = 36 - 8x_1 - 3x_2 \\ z = 0 + 2x_1 + 3x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 77, \quad \bar{x}_4 = 36, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

- Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_3 = 77 - 6x_1 \\ x_4 = 36 - 8x_1 \end{cases} \quad \begin{cases} \underbrace{77 - 6x_1 \geq 0}_{x_3 \geq 0} \\ \underbrace{36 - 8x_1 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq \frac{77}{6} \\ x_1 \leq \frac{9}{2} \end{cases}$$

By increasing the value of x_1 to $\frac{9}{2}$, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 36 - 8x_1 - 3x_2 \quad \text{then} \quad x_1 = \frac{9}{2} - \frac{3}{8}x_2 - \frac{1}{8}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$x_3 = 77 - 6\left(\frac{9}{2} - \frac{3}{8}x_2 - \frac{1}{8}x_4\right) - 21x_2 = 50 - \frac{75}{4}x_2 + \frac{3}{4}x_4$$

$$z = 2\left(\frac{9}{2} - \frac{3}{8}x_2 - \frac{1}{8}x_4\right) + 3x_2 = 9 + \frac{9}{4}x_2 - \frac{1}{4}x_4$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = \frac{9}{2} - \frac{3}{8}x_2 - \frac{1}{8}x_4 \\ x_3 = 50 - \frac{75}{4}x_2 + \frac{3}{4}x_4 \\ z = 9 + \frac{9}{4}x_2 - \frac{1}{4}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = \frac{9}{2}, \quad \bar{x}_3 = 50, \quad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 9$$

- **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_1 = \frac{9}{2} - \frac{3}{8} x_2 \\ x_3 = 50 - \frac{75}{4} x_2 \end{cases} \quad \begin{cases} \overbrace{\frac{9}{2} - \frac{3}{8} x_2 \geq 0}^{x_1 \geq 0} \\ \underbrace{50 - \frac{75}{4} x_2 \geq 0}_{x_3 \geq 0} \end{cases} \quad \begin{cases} x_2 \leq 12 \\ x_2 \leq \frac{8}{3} \end{cases}$$

By increasing the value of x_2 to $\min \left\{ 12, \frac{8}{3} \right\} = \frac{8}{3}$, the value of x_3 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = 50 - \frac{75}{4} x_2 + \frac{3}{4} x_4 \quad \text{then} \quad x_2 = \frac{8}{3} - \frac{4}{75} x_3 + \frac{1}{25} x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_1 &= \frac{9}{2} - \frac{3}{8} \left(\frac{8}{3} - \frac{4}{75} x_3 + \frac{1}{25} x_4 \right) - \frac{1}{8} x_4 = \frac{7}{2} + \frac{1}{50} x_3 - \frac{7}{50} x_4 \\ z &= 9 + \frac{9}{4} \left(\frac{8}{3} - \frac{4}{75} x_3 + \frac{1}{25} x_4 \right) - \frac{1}{4} x_4 = 15 - \frac{3}{25} x_3 - \frac{4}{25} x_4 \end{aligned}$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

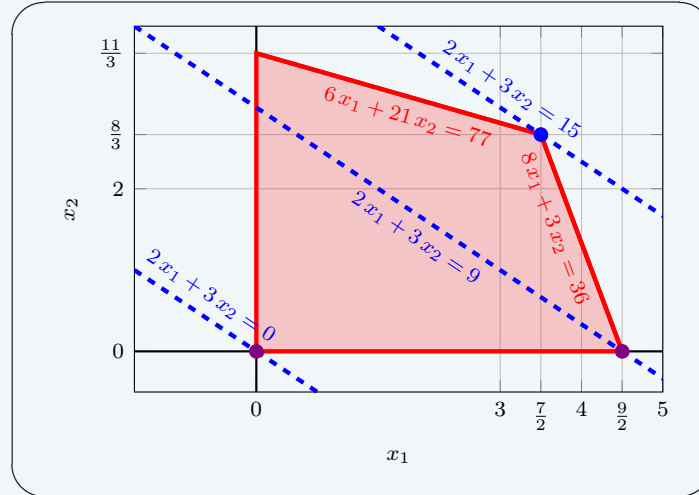
$$\begin{cases} x_1 = \frac{7}{2} + \frac{1}{50} x_3 - \frac{7}{50} x_4 \\ x_2 = \frac{8}{3} - \frac{4}{75} x_3 + \frac{1}{25} x_4 \\ z = 15 - \frac{3}{25} x_3 - \frac{4}{25} x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{7}{2}, \quad \tilde{x}_2 = \frac{8}{3}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad 15$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{7}{2}, \quad \tilde{x}_2 = \frac{8}{3}$$

the optimal value $z(LP)$ is 15.

In violet the first and the second vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 6.11 — Simplex algorithm (P6)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{subject to} \quad & 4x_1 + 15x_2 \leq 70 \\ & 12x_1 + 3x_2 \leq 52 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing the slack variables, we get the following starting dictionary:

$$\begin{cases} x_3 = 70 - 4x_1 - 15x_2 \\ x_4 = 52 - 12x_1 - 3x_2 \\ z = 0 + x_1 + x_2 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 70, \bar{x}_4 = 52 \quad \text{of value } 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- Initialization**

Consider the starting dictionary:

$$\begin{cases} x_3 = 70 - 4x_1 - 15x_2 \\ x_4 = 52 - 12x_1 - 3x_2 \\ z = 0 + x_1 + x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 70, \bar{x}_4 = 52, \quad \bar{x}_1 = 0, \bar{x}_2 = 0 \quad \text{of value } 0$$

- Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_3 = 70 - 4x_1 \\ x_4 = 52 - 12x_1 \end{cases} \quad \begin{cases} \underbrace{70 - 4x_1 \geq 0}_{x_3 \geq 0} \\ \underbrace{52 - 12x_1 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq \frac{35}{2} \\ x_1 \leq \frac{13}{3} \end{cases}$$

By increasing the value of x_1 to $\min \left\{ \frac{35}{2}, \frac{13}{3} \right\} = \frac{13}{3}$, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 52 - 12x_1 - 3x_2 \quad \text{then} \quad x_1 = \frac{13}{3} - \frac{1}{4}x_2 - \frac{1}{12}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$x_3 = 70 - 4\left(\frac{13}{3} - \frac{1}{4}x_2 - \frac{1}{12}x_4\right) - 15x_2 = \frac{158}{3} - 14x_2 + \frac{1}{3}x_4$$

$$z = \left(\frac{13}{3} - \frac{1}{4}x_2 - \frac{1}{12}x_4\right) + x_2 = \frac{13}{3} + \frac{3}{4}x_2 - \frac{1}{12}x_4$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = \frac{13}{3} - \frac{1}{4}x_2 - \frac{1}{12}x_4 \\ x_3 = \frac{158}{3} - 14x_2 + \frac{1}{3}x_4 \\ z = \frac{13}{3} + \frac{3}{4}x_2 - \frac{1}{12}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = \frac{13}{3}, \quad \bar{x}_3 = \frac{158}{3}, \quad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{13}{3}$$

• Iteration 2

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_1 = \frac{13}{3} - \frac{1}{4}x_2 \\ x_3 = \frac{158}{3} - 14x_2 \end{cases} \quad \begin{cases} \underbrace{\frac{13}{3} - \frac{1}{4}x_2}_{x_1 \geq 0} \geq 0 \\ \underbrace{\frac{158}{3} - 14x_2}_{x_3 \geq 0} \geq 0 \end{cases} \quad \begin{cases} x_2 \leq \frac{52}{3} \\ x_2 \leq \frac{79}{21} \end{cases}$$

By increasing the value of x_2 to $\min\left\{\frac{52}{3}, \frac{79}{21}\right\} = \frac{79}{21}$, the value of x_3 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = \frac{158}{3} - 14x_2 + \frac{1}{3}x_4 \quad \text{then} \quad x_2 = \frac{79}{21} - \frac{1}{14}x_3 + \frac{1}{42}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$x_1 = \frac{13}{3} - \frac{1}{4} \left(\frac{79}{21} - \frac{1}{14} x_3 + \frac{1}{42} x_4 \right) - \frac{1}{12} x_4 = \frac{95}{28} + \frac{1}{56} x_3 - \frac{5}{56} x_4$$

$$z = \frac{13}{3} + \frac{3}{4} \left(\frac{79}{21} - \frac{1}{14} x_3 + \frac{1}{42} x_4 \right) - \frac{1}{12} x_4 = \frac{601}{84} - \frac{3}{56} x_3 - \frac{11}{168} x_4$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

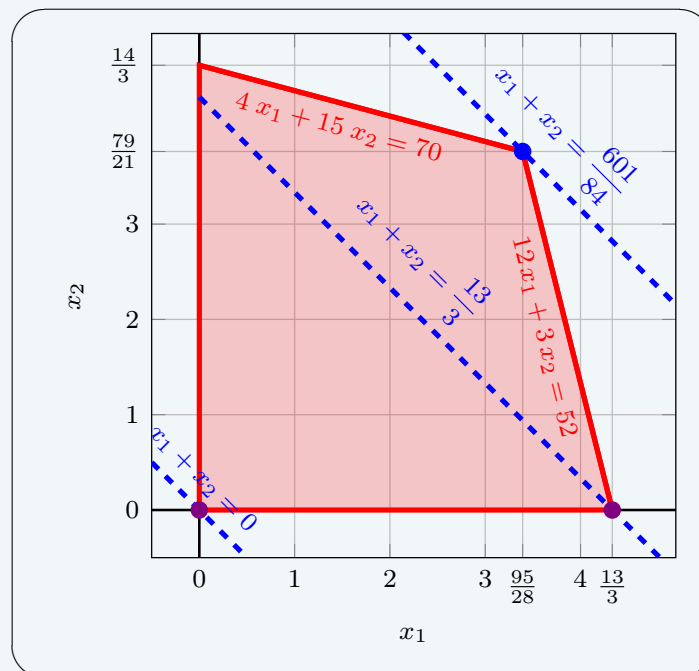
$$\begin{cases} x_1 = \frac{95}{28} + \frac{1}{56} x_3 - \frac{5}{56} x_4 \\ x_2 = \frac{79}{21} - \frac{1}{14} x_3 + \frac{1}{42} x_4 \\ z = \frac{601}{84} - \frac{3}{56} x_3 - \frac{11}{168} x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{95}{28}, \quad \tilde{x}_2 = \frac{79}{21}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad \frac{601}{84}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{x}(LP)$ is:

$$\tilde{x}_1 = \frac{95}{28}, \quad \tilde{x}_2 = \frac{79}{21}$$

the optimal value $z(LP)$ is $\frac{601}{84}$.

In violet the first and the second vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 6.12 — Simplex algorithm (P7)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 3x_1 + x_2 \\ \text{subject to} \quad & x_1 + 4x_2 \leq 12 \\ & 2x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing the slack variables, we get the following starting dictionary:

$$\left\{ \begin{array}{lcl} x_3 & = & 12 - x_1 - 4x_2 \\ x_4 & = & 5 - 2x_1 - x_2 \\ z & = & 0 + 3x_1 + x_2 \end{array} \right.$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 12, \quad \bar{x}_4 = 5 \quad \text{of value} \quad 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- Initialization**

Consider the starting dictionary:

$$\begin{cases} x_3 = 12 - x_1 - 4x_2 \\ x_4 = 5 - 2x_1 - x_2 \\ z = 0 + 3x_1 + x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 12 \quad \bar{x}_4 = 5, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

- Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_3 = 12 - x_1 \\ x_4 = 5 - 2x_1 \end{cases} \quad \begin{cases} \underbrace{12 - x_1 \geq 0}_{x_3 \geq 0} \\ \underbrace{5 - 2x_1 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq 12 \\ x_1 \leq \frac{5}{2} \end{cases}$$

By increasing the value of x_1 to $\frac{5}{2}$, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 5 - 2x_1 - x_2 \quad \text{then} \quad x_1 = \frac{5}{2} - \frac{1}{2}x_2 - \frac{1}{2}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_3 &= 12 - \left(\frac{5}{2} - \frac{1}{2}x_2 - \frac{1}{2}x_4 \right) - 4x_2 = \frac{19}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_4 \\ z &= 3 \left(\frac{5}{2} - \frac{1}{2}x_2 - \frac{1}{2}x_4 \right) + x_2 = \frac{15}{2} - \frac{1}{2}x_2 - \frac{3}{2}x_4 \end{aligned}$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

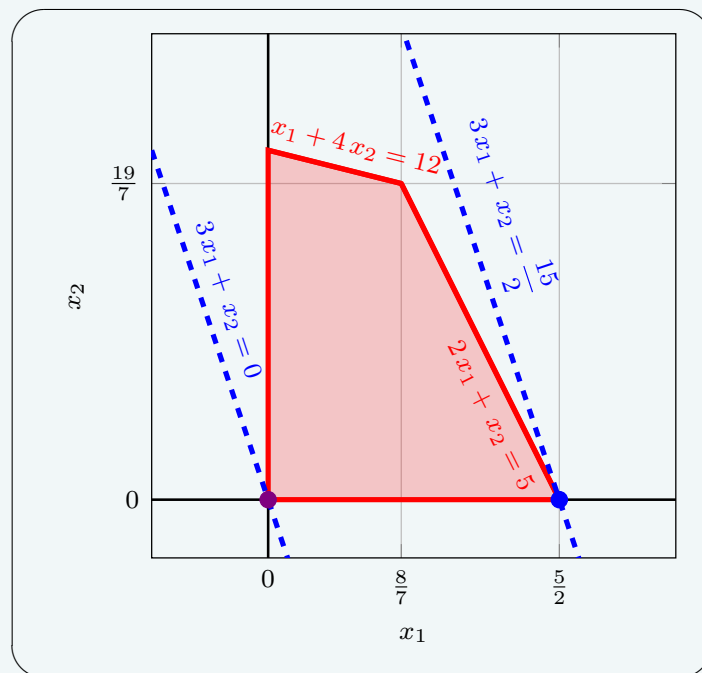
$$\begin{cases} x_1 = \frac{5}{2} - \frac{1}{2}x_2 - \frac{1}{2}x_4 \\ x_3 = \frac{19}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_4 \\ z = \frac{15}{2} - \frac{1}{2}x_2 - \frac{3}{2}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_3 = \frac{19}{2}, \quad \tilde{x}_2 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad \frac{15}{2}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = 0$$

the optimal value $z(LP)$ is $\frac{15}{2}$.

In violet the first vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the two vertices.

Exercise 6.13 — Simplex algorithm (P8)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & & x_1 & + & 2 & x_2 \\ \text{subject to} & - & x_1 & + & 2 & x_2 \leq 2 \\ & & x_1 & + & & x_2 \leq 3 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

By introducing the slack variables, we get the following starting dictionary:

$$\left\{ \begin{array}{llllll} x_3 & = & 2 & + & x_1 & - & 2x_2 \\ x_4 & = & 3 & - & x_1 & - & x_2 \\ z & = & 0 & + & x_1 & + & 2x_2 \end{array} \right.$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 2, \quad \bar{x}_4 = 3 \quad \text{of value} \quad 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

• **Initialization**

Consider the starting dictionary:

$$\left\{ \begin{array}{llllll} x_3 & = & 2 & + & x_1 & - & 2x_2 \\ x_4 & = & 3 & - & x_1 & - & x_2 \\ z & = & 0 & + & x_1 & + & 2x_2 \end{array} \right.$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 2, \quad \bar{x}_4 = 3, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value} \quad 0$$

• **Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_3 = 2 + x_1 \\ x_4 = 3 - x_1 \end{cases} \quad \begin{cases} \overbrace{2 + x_1 \geq 0}^{x_3 \geq 0} \\ \overbrace{3 - x_1 \geq 0}^{x_4 \geq 0} \end{cases} \quad \begin{cases} x_1 \geq -2 \\ x_1 \leq 3 \end{cases}$$

By increasing the value of x_1 to 3, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 3 - x_1 - x_2 \quad \text{then} \quad x_1 = 3 - x_2 - x_4.$$

Substituting in the other equations of the dictionary, we get:

$$x_3 = 2 + (3 - x_2 - x_4) - 2x_2 = 5 - 3x_2 - x_4$$

$$z = (3 - x_2 - x_4) + 2x_2 = 3 + x_2 - x_4$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = 3 - x_2 - x_4 \\ x_3 = 5 - 3x_2 - x_4 \\ z = 3 + x_2 - x_4 \end{cases}$$

where $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = 3, \quad \bar{x}_3 = 5, \quad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value } 3$$

• **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_1 = 3 - x_2 \\ x_3 = 5 - 3x_2 \end{cases} \quad \begin{cases} \overbrace{3 - x_2 \geq 0}^{x_1 \geq 0} \\ \overbrace{5 - 3x_2 \geq 0}^{x_3 \geq 0} \end{cases} \quad \begin{cases} x_2 \leq 3 \\ x_2 \leq \frac{5}{3} \end{cases}$$

By increasing the value of x_2 to $\min \left\{ 3, \frac{5}{3} \right\} = \frac{5}{3}$, the value of x_3 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = 5 - 3x_2 - x_4 \quad \text{then} \quad x_2 = \frac{5}{3} - \frac{1}{3}x_3 - \frac{1}{3}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_1 &= 3 - \left(\frac{5}{3} - \frac{1}{3}x_3 - \frac{1}{3}x_4 \right) - x_4 = \frac{4}{3} + \frac{1}{3}x_3 - \frac{2}{3}x_4 \\ z &= 3 + \left(\frac{5}{3} - \frac{1}{3}x_3 - \frac{1}{3}x_4 \right) - x_4 = \frac{14}{3} - \frac{1}{3}x_3 - \frac{4}{3}x_4 \end{aligned}$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

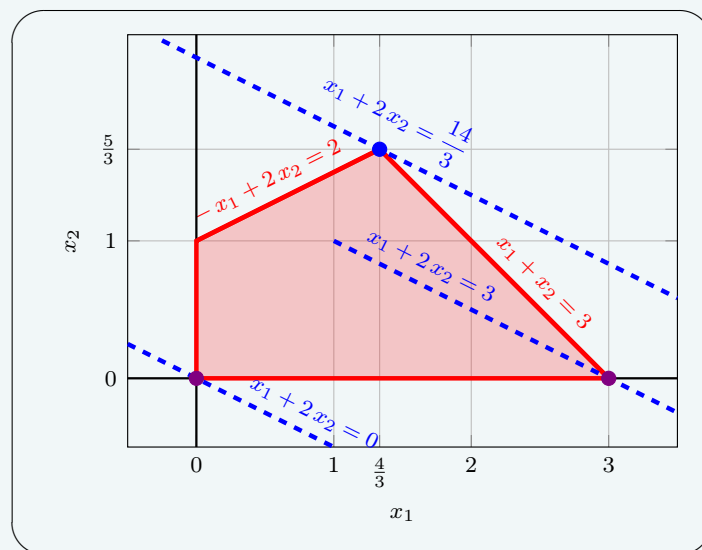
$$\begin{cases} x_1 = \frac{4}{3} + \frac{1}{3}x_3 - \frac{2}{3}x_4 \\ x_2 = \frac{5}{3} - \frac{1}{3}x_3 - \frac{1}{3}x_4 \\ z = \frac{14}{3} - \frac{1}{3}x_3 - \frac{4}{3}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{4}{3}, \quad \tilde{x}_2 = \frac{5}{3}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad \frac{14}{3}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{x}(LP)$ is:

$$\tilde{x}_1 = \frac{4}{3}, \quad \tilde{x}_2 = \frac{5}{3}$$

the optimal value $z(LP)$ is $\frac{14}{3}$.

In violet the first and the second vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 6.14 — Simplex algorithm (P9)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & 2 & x_1 & + & & x_2 \\ \text{subject to} & - & x_1 & + & & x_2 & \leq & 1 \\ & & 6 & x_1 & + & 4 & x_2 & \leq & 39 \\ & & & x_1, & & & x_2 & \geq & 0 \end{array}$$

By introducing the slack variables, we get the following starting dictionary:

$$\left\{ \begin{array}{llllll} x_3 & = & 1 & + & x_1 & - & x_2 \\ x_4 & = & 39 & - & 6x_1 & - & 4x_2 \\ z & = & 0 & + & 2x_1 & + & x_2 \end{array} \right.$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 1, \quad \bar{x}_4 = 39 \quad \text{of value} \quad 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- Initialization**

Consider the starting dictionary:

$$\begin{cases} x_3 = 1 + x_1 - x_2 \\ x_4 = 39 - 6x_1 - 4x_2 \\ z = 0 + 2x_1 + x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 1, \quad \bar{x}_4 = 39, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

- Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_3 = 1 + x_1 \\ x_4 = 39 - 6x_1 \end{cases} \quad \begin{cases} \overbrace{1 + x_1 \geq 0}^{x_3 \geq 0} \\ \overbrace{39 - 6x_1 \geq 0}^{x_4 \geq 0} \end{cases} \quad \begin{cases} x_1 \geq -1 \\ x_1 \leq \frac{13}{2} \end{cases}$$

By increasing the value of x_1 to $\frac{13}{2}$, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 39 - 6x_1 - 4x_2 \quad \text{then} \quad x_1 = \frac{13}{2} - \frac{2}{3}x_2 - \frac{1}{6}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_3 &= 1 + \left(\frac{13}{2} - \frac{2}{3}x_2 - \frac{1}{6}x_4 \right) - x_2 = \frac{15}{2} - \frac{5}{3}x_2 - \frac{1}{6}x_4 \\ z &= 2 \left(\frac{13}{2} - \frac{2}{3}x_2 - \frac{1}{6}x_4 \right) + x_2 = 13 - \frac{1}{3}x_2 - \frac{1}{3}x_4 \end{aligned}$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

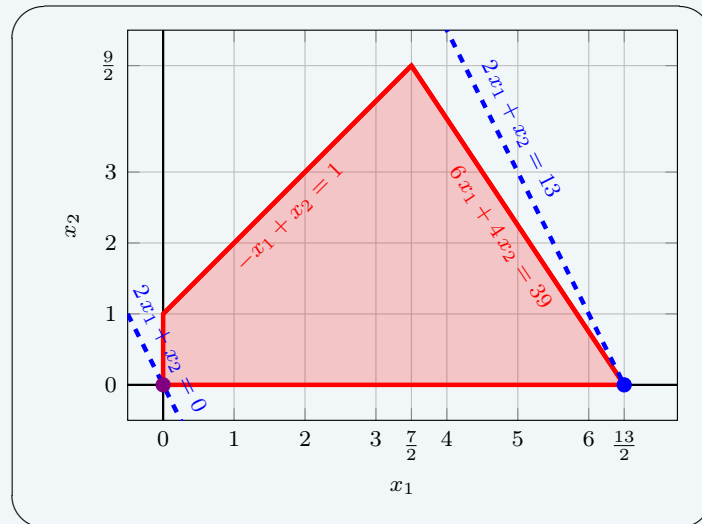
$$\begin{cases} x_1 = \frac{13}{2} - \frac{2}{3}x_2 - \frac{1}{6}x_4 \\ x_3 = \frac{15}{2} - \frac{5}{3}x_2 - \frac{1}{6}x_4 \\ z = 13 - \frac{1}{3}x_2 - \frac{1}{3}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{13}{2}, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = \frac{15}{2}, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad 13$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{13}{2}, \quad \tilde{x}_2 = 0$$

the optimal value $z(LP)$ is 13.

In violet the first vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the two vertices.

Exercise 6.15 — Simplex algorithm (P10)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{array}{llllll} \max & 5 & x_1 & + & 2 & x_2 \\ \text{subject to} & - & x_1 & + & 3 & x_2 & \leq & 12 \\ & & x_1 & + & 2 & x_2 & \leq & 12 \\ & 3 & x_1 & - & 2 & x_2 & \leq & 8 \\ & x_1, & & & x_2 & \geq & 0 \end{array}$$

By introducing the slack variables, we get the following starting dictionary:

$$\left\{ \begin{array}{llllll} x_3 & = & 12 & + & x_1 & - & 3x_2 \\ x_4 & = & 12 & - & x_1 & - & 2x_2 \\ x_5 & = & 8 & - & 3x_1 & + & 2x_2 \\ z & = & 0 & + & 5x_1 & + & 2x_2 \end{array} \right.$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 12, \quad \bar{x}_4 = 12, \quad \bar{x}_5 = 8 \quad \text{of value} \quad 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- **Initialization**

Consider the starting dictionary:

$$\begin{cases} x_3 = 12 + x_1 - 3x_2 \\ x_4 = 12 - x_1 - 2x_2 \\ x_5 = 8 - 3x_1 + 2x_2 \\ z = 0 + 5x_1 + 2x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 12, \quad \bar{x}_4 = 12, \quad \bar{x}_5 = 8, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

- **Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_3 = 12 + x_1 \\ x_4 = 12 - x_1 \\ x_5 = 8 - 3x_1 \end{cases} \quad \begin{cases} \underbrace{12 + x_1 \geq 0}_{x_3 \geq 0} \\ \underbrace{12 - x_1 \geq 0}_{x_4 \geq 0} \\ \underbrace{8 - 3x_1 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_1 \geq -12 \\ x_1 \leq 12 \\ x_1 \leq \frac{8}{3} \end{cases}$$

By increasing the value of x_1 to $\min \left\{ 12, \frac{8}{3} \right\} = \frac{8}{3}$, the value of x_5 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = 8 - 3x_1 + 2x_2 \quad \text{then} \quad x_1 = \frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5.$$

Substituting in the other equations of the dictionary, we get:

$$x_3 = 12 + \left(\frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5 \right) - 3x_2 = \frac{44}{3} - \frac{7}{3}x_2 - \frac{1}{3}x_5$$

$$x_4 = 12 - \left(\frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5 \right) - 2x_2 = \frac{28}{3} - \frac{8}{3}x_2 + \frac{1}{3}x_5$$

$$z = 5 \left(\frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5 \right) + 2x_2 = \frac{40}{3} + \frac{16}{3}x_2 - \frac{5}{3}x_5$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = \frac{8}{3} + \frac{2}{3}x_2 - \frac{1}{3}x_5 \\ x_3 = \frac{44}{3} - \frac{7}{3}x_2 - \frac{1}{3}x_5 \\ x_4 = \frac{28}{3} - \frac{8}{3}x_2 + \frac{1}{3}x_5 \\ z = \frac{40}{3} + \frac{16}{3}x_2 - \frac{5}{3}x_5 \end{cases}$$

where $\mathcal{B} = \{1, 3, 4\}$ and $\mathcal{N} = \{2, 5\}$. The basic feasible solution is:

$$\bar{x}_1 = \frac{8}{3}, \quad \bar{x}_3 = \frac{44}{3}, \quad \bar{x}_4 = \frac{28}{3}, \quad \bar{x}_2 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{40}{3}$$

• **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_1 = \frac{8}{3} + \frac{2}{3}x_2 \\ x_3 = \frac{44}{3} - \frac{7}{3}x_2 \\ x_4 = \frac{28}{3} - \frac{8}{3}x_2 \end{cases} \quad \begin{cases} \overbrace{\frac{8}{3} + \frac{2}{3}x_2 \geq 0}^{x_1 \geq 0} \\ \overbrace{\frac{44}{3} - \frac{7}{3}x_2 \geq 0}^{x_3 \geq 0} \\ \overbrace{\frac{28}{3} - \frac{8}{3}x_2 \geq 0}^{x_4 \geq 0} \end{cases} \quad \begin{cases} x_2 \geq -4 \\ x_2 \leq \frac{44}{7} \\ x_2 \leq \frac{7}{2} \end{cases}$$

By increasing the value of x_2 to $\min \left\{ \frac{44}{7}, \frac{7}{2} \right\} = \frac{7}{2}$, the value of x_4 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = \frac{28}{3} - \frac{8}{3}x_2 + \frac{1}{3}x_5 \quad \text{then} \quad x_2 = \frac{7}{2} - \frac{3}{8}x_4 + \frac{1}{8}x_5.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_1 &= \frac{8}{3} + \frac{2}{3} \left(\frac{7}{2} - \frac{3}{8}x_4 + \frac{1}{8}x_5 \right) - \frac{1}{3}x_5 = 5 - \frac{1}{4}x_4 - \frac{1}{4}x_5 \\ x_3 &= \frac{44}{3} - \frac{7}{3} \left(\frac{7}{2} - \frac{3}{8}x_4 + \frac{1}{8}x_5 \right) - \frac{1}{3}x_5 = \frac{13}{2} + \frac{7}{8}x_4 - \frac{5}{8}x_5 \\ z &= \frac{40}{3} + \frac{16}{3} \left(\frac{7}{2} - \frac{3}{8}x_4 + \frac{1}{8}x_5 \right) - \frac{5}{3}x_5 = 32 - 2x_4 - x_5 \end{aligned}$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

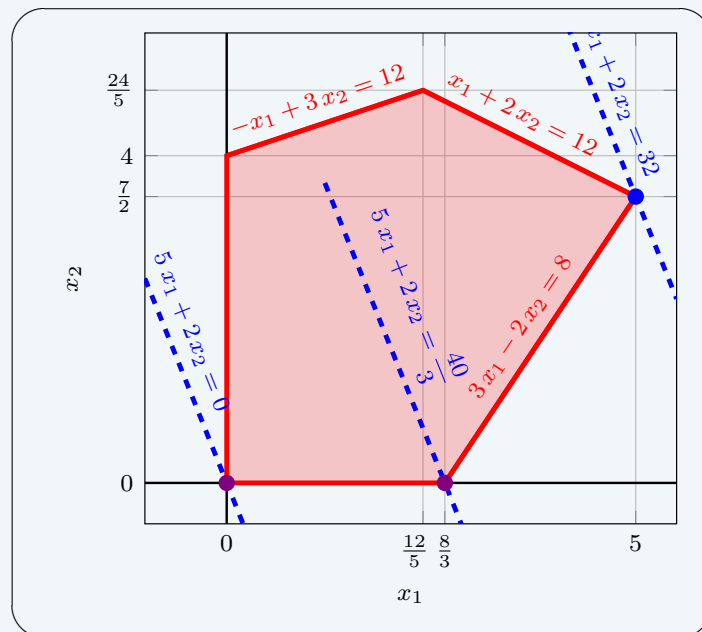
$$\left\{ \begin{array}{lclcl} x_1 = 5 & - & \frac{1}{4} x_4 & - & \frac{1}{4} x_5 \\ x_2 = \frac{7}{2} & - & \frac{3}{8} x_4 & + & \frac{1}{8} x_5 \\ x_3 = \frac{13}{2} & + & \frac{7}{8} x_4 & - & \frac{5}{8} x_5 \\ z = 32 & - & 2 x_4 & - & x_5 \end{array} \right.$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$. The basic optimal solution is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}, \quad \tilde{x}_3 = \frac{13}{2}, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad 32$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value $z(LP)$ is 32.

In violet the first and the second vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

Exercise 6.16 — Simplex algorithm with a different pivot column (P10)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{aligned} \max \quad & 5x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 3x_2 \leq 12 \\ & x_1 + 2x_2 \leq 12 \\ & 3x_1 - 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing the slack variables, we get the following starting dictionary:

$$\begin{cases} x_3 = 12 + x_1 - 3x_2 \\ x_4 = 12 - x_1 - 2x_2 \\ x_5 = 8 - 3x_1 + 2x_2 \\ z = 0 + 5x_1 + 2x_2 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 12, \bar{x}_4 = 12, \bar{x}_5 = 8 \quad \text{of value } 0$$

Questions

1. Execute the simplex algorithm to find an optimal solution of this LP problem, choosing x_2 as the pivot column at the first iteration.
2. Compare the number of iterations with the execution of the simplex algorithm that chooses x_1 as the pivot column at the first iteration (exercise *Simplex algorithm (P10)*).

Solution

 1. • **Initialization**

Consider the dictionary:

$$\begin{cases} x_3 = 12 + x_1 - 3x_2 \\ x_4 = 12 - x_1 - 2x_2 \\ x_5 = 8 - 3x_1 + 2x_2 \\ z = 0 + 5x_1 + 2x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 12, \quad \bar{x}_4 = 12, \quad \bar{x}_5 = 8, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

 • **Iteration 1**

If we increase the value of the variable x_2 (the pivot column required by the question), we can get a better solution. We have:

$$\begin{cases} x_3 = 12 - 3x_2 \\ x_4 = 12 - 2x_2 \\ x_5 = 8 + 2x_2 \end{cases} \quad \begin{cases} \underbrace{12 - 3x_2 \geq 0}_{x_3 \geq 0} \\ \underbrace{12 - 2x_2 \geq 0}_{x_4 \geq 0} \\ \underbrace{8 + 2x_2 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_2 \leq 4 \\ x_2 \leq 6 \\ x_2 \geq -4 \end{cases}$$

By increasing the value of x_2 to $\min\{4, 6\} = 4$, the value of x_3 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = 12 + x_1 - 3x_2 \quad \text{then} \quad x_2 = 4 + \frac{1}{3}x_1 - \frac{1}{3}x_3.$$

Substituting in the other equations of the dictionary, we get:

$$x_4 = 12 - x_1 - 2x_2 = 12 - x_1 - 2\left(4 + \frac{1}{3}x_1 - \frac{1}{3}x_3\right) = 4 - \frac{5}{3}x_1 + \frac{2}{3}x_3$$

$$x_5 = 8 - 3x_1 + 2x_2 = 8 - 3x_1 + 2\left(4 + \frac{1}{3}x_1 - \frac{1}{3}x_3\right) = 16 - \frac{7}{3}x_1 - \frac{2}{3}x_3$$

$$z = 5x_1 + 2x_2 = 5x_1 + 2\left(4 + \frac{1}{3}x_1 - \frac{1}{3}x_3\right) = 8 + \frac{17}{3}x_1 - \frac{2}{3}x_3$$

• **Iteration 2**

Consider the dictionary:

$$\begin{cases} x_2 = 4 + \frac{1}{3}x_1 - \frac{1}{3}x_3 \\ x_4 = 4 - \frac{5}{3}x_1 + \frac{2}{3}x_3 \\ x_5 = 16 - \frac{7}{3}x_1 - \frac{2}{3}x_3 \\ z = 8 + \frac{17}{3}x_1 - \frac{2}{3}x_3 \end{cases}$$

where $\mathcal{B} = \{2, 4, 5\}$ and $\mathcal{N} = \{1, 3\}$. The basic feasible solution is:

$$\bar{x}_2 = 4, \quad \bar{x}_4 = 4, \quad \bar{x}_5 = 16, \quad \bar{x}_1 = 0, \quad \bar{x}_3 = 0 \quad \text{of value } 8$$

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_2 = 4 + \frac{1}{3}x_1 \\ x_4 = 4 - \frac{5}{3}x_1 \\ x_5 = 16 - \frac{7}{3}x_1 \end{cases} \quad \begin{cases} \overbrace{4 + \frac{1}{3}x_1 \geq 0}^{x_2 \geq 0} \\ \overbrace{4 - \frac{5}{3}x_1 \geq 0}^{x_4 \geq 0} \\ \overbrace{16 - \frac{7}{3}x_1 \geq 0}^{x_5 \geq 0} \end{cases} \quad \begin{cases} x_1 \geq -12 \\ x_1 \leq \frac{12}{5} \\ x_1 \leq \frac{48}{7} \end{cases}$$

By increasing the value of x_1 to $\min \left\{ \frac{12}{5}, \frac{48}{7} \right\} = \frac{12}{5}$, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 4 - \frac{5}{3}x_1 + \frac{2}{3}x_3 \quad \text{then} \quad x_1 = \frac{12}{5} + \frac{2}{5}x_3 - \frac{3}{5}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_2 &= 4 + \frac{1}{3} \left(\frac{12}{5} + \frac{2}{5}x_3 - \frac{3}{5}x_4 \right) - \frac{1}{3}x_3 = \frac{24}{5} - \frac{1}{5}x_3 - \frac{1}{5}x_4 \\ x_5 &= 16 - \frac{7}{3} \left(\frac{12}{5} + \frac{2}{5}x_3 - \frac{3}{5}x_4 \right) - \frac{2}{3}x_3 = \frac{52}{5} - \frac{8}{5}x_3 + \frac{7}{5}x_4 \\ z &= 8 + \frac{17}{3} \left(\frac{12}{5} + \frac{2}{5}x_3 - \frac{3}{5}x_4 \right) - \frac{2}{3}x_3 = \frac{108}{5} + \frac{8}{5}x_3 - \frac{17}{5}x_4 \end{aligned}$$

• **Iteration 3**

Consider the dictionary:

$$\begin{cases} x_1 = \frac{12}{5} + \frac{2}{5}x_3 - \frac{3}{5}x_4 \\ x_2 = \frac{24}{5} - \frac{1}{5}x_3 - \frac{1}{5}x_4 \\ x_5 = \frac{52}{5} - \frac{8}{5}x_3 + \frac{7}{5}x_4 \\ z = \frac{108}{5} + \frac{8}{5}x_3 - \frac{17}{5}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = \frac{12}{5}, \quad \bar{x}_2 = \frac{24}{5}, \quad \bar{x}_5 = \frac{52}{5}, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{108}{5}$$

If we increase the value of the variable x_3 , we can get a better solution. We have:

$$\begin{cases} x_1 = \frac{12}{5} + \frac{2}{5}x_3 \\ x_2 = \frac{24}{5} - \frac{1}{5}x_3 \\ x_5 = \frac{52}{5} - \frac{8}{5}x_3 \end{cases} \quad \begin{cases} \overbrace{\frac{12}{5} + \frac{2}{5}x_3 \geq 0}^{x_1 \geq 0} \\ \frac{24}{5} - \frac{1}{5}x_3 \geq 0 \\ \underbrace{\frac{52}{5} - \frac{8}{5}x_3 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_3 \geq -6 \\ x_3 \leq 24 \\ x_3 \leq \frac{13}{2} \end{cases}$$

By increasing the value of x_3 to $\min\left\{24, \frac{13}{2}\right\} = \frac{13}{2}$, the value of x_5 decreases to 0.

Then, we have that:

x_3 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = \frac{52}{5} - \frac{8}{5}x_3 + \frac{7}{5}x_4 \quad \text{then} \quad x_3 = \frac{13}{2} + \frac{7}{8}x_4 - \frac{5}{8}x_5.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_1 &= \frac{12}{5} + \frac{2}{5}\left(\frac{13}{2} + \frac{7}{8}x_4 - \frac{5}{8}x_5\right) - \frac{3}{5}x_4 = 5 - \frac{1}{4}x_4 - \frac{1}{4}x_5 \\ x_2 &= \frac{24}{5} - \frac{1}{5}\left(\frac{13}{2} + \frac{7}{8}x_4 - \frac{5}{8}x_5\right) - \frac{1}{5}x_4 = \frac{7}{2} - \frac{3}{8}x_4 + \frac{1}{8}x_5 \end{aligned}$$

$$z = \frac{108}{5} + \frac{8}{5} \left(\frac{13}{2} + \frac{7}{8} x_4 - \frac{5}{8} x_5 \right) - \frac{17}{5} x_4 = 32 - 2x_4 - x_5$$

Accordingly, we obtained an optimal solution and this is the optimal dictionary:

$$\begin{cases} x_1 = 5 - \frac{1}{4} x_4 - \frac{1}{4} x_5 \\ x_2 = \frac{7}{2} - \frac{3}{8} x_4 + \frac{1}{8} x_5 \\ x_3 = \frac{13}{2} + \frac{7}{8} x_4 - \frac{5}{8} x_5 \\ z = 32 - 2x_4 - x_5 \end{cases}$$

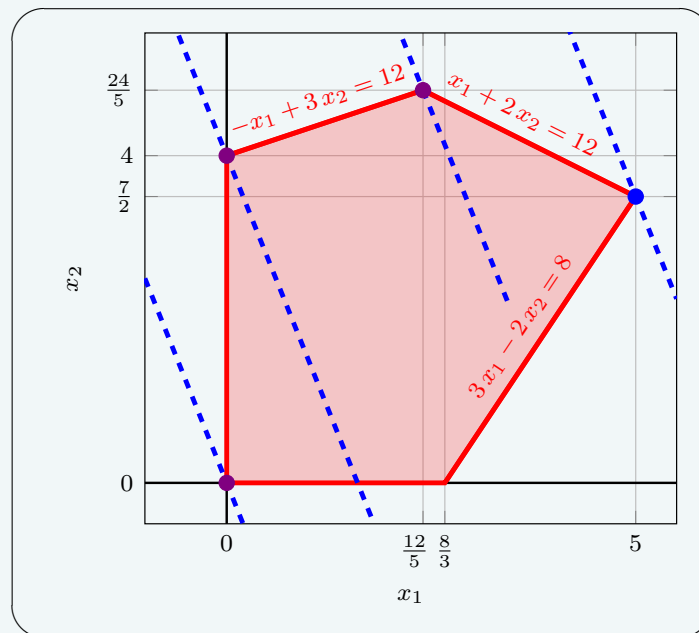
where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$. The optimal solution is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}, \quad \tilde{x}_3 = \frac{13}{2}, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad 32$$

2. The execution that chooses x_2 as the first pivot column performs three pivoting operations and visits the vertices $(0, 0)$, $(0, 4)$ and $(\frac{12}{5}, \frac{24}{5})$ before reaching the optimal one. The execution that chooses x_1 as the first pivot column performs only two pivoting operations and visits the vertices $(0, 0)$ and $(\frac{8}{3}, 0)$. Both executions reach the same optimal solution of value 32: the choice of the pivot column changes the path followed on the polyhedron, not the optimal value.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{x}(LP)$ is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value $z(LP)$ is 32.

In violet the first, second and third vertex explored by the execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines $5x_1 + 2x_2 = 0$, $5x_1 + 2x_2 = 8$, $5x_1 + 2x_2 = \frac{108}{5}$ and $5x_1 + 2x_2 = 32$ of the objective function associated with the values taken in the four vertices.

Exercise 6.17 — Simplex algorithm (B1)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 25x_1 + 16x_2 \\ \text{subject to} \quad & 50x_1 + 31x_2 \leq 250 \\ & -3x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

By introducing the slack variables, we get the following starting dictionary:

$$\begin{cases} x_3 = 250 - 50x_1 - 31x_2 \\ x_4 = 4 + 3x_1 - 2x_2 \\ z = 0 + 25x_1 + 16x_2 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 250, \quad \bar{x}_4 = 4 \quad \text{of value} \quad 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- **Initialization**

Consider the starting dictionary:

$$\begin{cases} x_3 = 250 - 50 x_1 - 31 x_2 \\ x_4 = 4 + 3 x_1 - 2 x_2 \\ z = 0 + 25 x_1 + 16 x_2 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{1, 2\}$. The basic feasible solution is:

$$\bar{x}_3 = 250, \quad \bar{x}_4 = 4, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \text{of value } 0$$

- **Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_3 = 250 - 50 x_1 \\ x_4 = 4 + 3 x_1 \end{cases} \quad \begin{cases} \underbrace{250 - 50 x_1 \geq 0}_{x_3 \geq 0} \\ \underbrace{4 + 3 x_1 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq 5 \\ x_1 \geq -\frac{4}{3} \end{cases}$$

By increasing the value of x_1 to 5, the value of x_3 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = 250 - 50 x_1 - 31 x_2 \quad \text{then} \quad x_1 = 5 - \frac{31}{50} x_2 - \frac{1}{50} x_3.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_4 &= 4 + 3 \left(5 - \frac{31}{50} x_2 - \frac{1}{50} x_3 \right) - 2 x_2 = 19 - \frac{193}{50} x_2 - \frac{3}{50} x_3 \\ z &= 25 \left(5 - \frac{31}{50} x_2 - \frac{1}{50} x_3 \right) + 16 x_2 = 125 + \frac{1}{2} x_2 - \frac{1}{2} x_3 \end{aligned}$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = 5 - \frac{31}{50} x_2 - \frac{1}{50} x_3 \\ x_4 = 19 - \frac{193}{50} x_2 - \frac{3}{50} x_3 \\ z = 125 + \frac{1}{2} x_2 - \frac{1}{2} x_3 \end{cases}$$

where $\mathcal{B} = \{1, 4\}$ and $\mathcal{N} = \{2, 3\}$. The basic feasible solution is:

$$\bar{x}_1 = 5, \quad \bar{x}_4 = 19, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad 125$$

• **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_1 = 5 - \frac{31}{50} x_2 \\ x_4 = 19 - \frac{193}{50} x_2 \end{cases} \quad \begin{cases} \overbrace{5 - \frac{31}{50} x_2 \geq 0}^{x_1 \geq 0} \\ \underbrace{19 - \frac{193}{50} x_2 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} x_2 \leq \frac{250}{31} \\ x_2 \leq \frac{950}{193} \end{cases}$$

By increasing the value of x_2 to $\min \left\{ \frac{250}{31}, \frac{950}{193} \right\} = \frac{950}{193}$, the value of x_4 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 19 - \frac{193}{50} x_2 - \frac{3}{50} x_3 \quad \text{then} \quad x_2 = \frac{950}{193} - \frac{3}{193} x_3 - \frac{50}{193} x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_1 &= 5 - \frac{31}{50} \left(\frac{950}{193} - \frac{3}{193} x_3 - \frac{50}{193} x_4 \right) - \frac{1}{50} x_3 = \frac{376}{193} - \frac{2}{193} x_3 + \frac{31}{193} x_4 \\ z &= 125 + \frac{1}{2} \left(\frac{950}{193} - \frac{3}{193} x_3 - \frac{50}{193} x_4 \right) - \frac{1}{2} x_3 = \frac{24600}{193} - \frac{98}{193} x_3 - \frac{25}{193} x_4 \end{aligned}$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

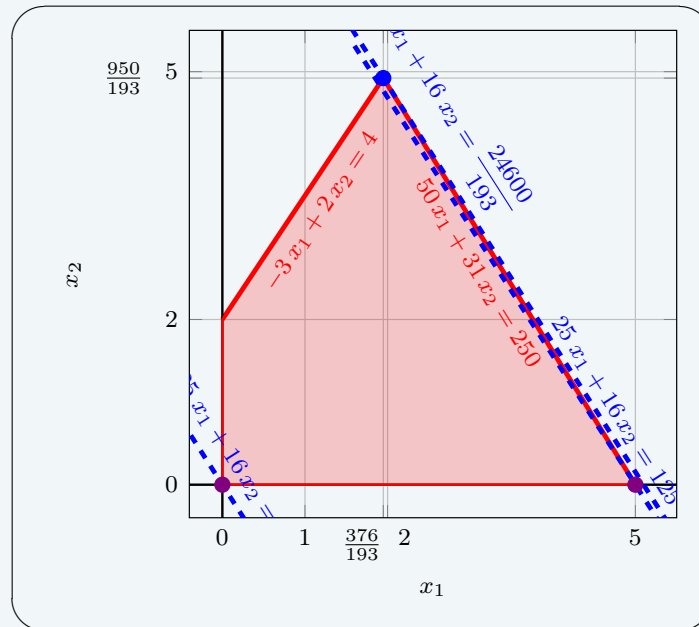
$$\begin{cases} x_1 = \frac{376}{193} - \frac{2}{193} x_3 + \frac{31}{193} x_4 \\ x_2 = \frac{950}{193} - \frac{3}{193} x_3 - \frac{50}{193} x_4 \\ z = \frac{24600}{193} - \frac{98}{193} x_3 - \frac{25}{193} x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{376}{193}, \quad \tilde{x}_2 = \frac{950}{193}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad \frac{24600}{193}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{376}{193}, \quad \tilde{x}_2 = \frac{950}{193}$$

the optimal value $z(LP)$ is $\frac{24600}{193}$.

In violet the first and the second vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

In dashed blue, the contour lines of the objective function associated with the values taken in the three vertices.

6.3 Problems with three variables

Exercise 6.18 — Simplex algorithm (P41)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 4$ constraints:

$$\begin{array}{llllllll} \max & 2 & x_1 & + & x_2 & + & 3 & x_3 \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & \leq & 4 \\ & & & & x_2 & & \leq & 2 \\ & & & & & x_3 & \leq & 3 \\ & 3 & x_1 & & & + & x_3 & \leq & 6 \\ & x_1, & & x_2, & & & x_3 & \geq & 0 \end{array}$$

By introducing the slack variables, we get the following starting dictionary:

$$\left\{ \begin{array}{llllll} x_4 = 4 & - & x_1 & - & x_2 & - & x_3 \\ x_5 = 2 & & & - & x_2 & & \\ x_6 = 3 & & & & & - & x_3 \\ x_7 = 6 & - & 3 & x_1 & & - & x_3 \\ z = 0 & + & 2 & x_1 & + & x_2 & + & 3 & x_3 \end{array} \right.$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 4, \quad \bar{x}_5 = 2, \quad \bar{x}_6 = 3, \quad \bar{x}_7 = 6 \quad \text{of value} \quad 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- **Initialization**

Consider the starting dictionary:

$$\left\{ \begin{array}{l} x_4 = 4 - x_1 - x_2 - x_3 \\ x_5 = 2 - x_2 \\ x_6 = 3 - x_3 \\ x_7 = 6 - 3x_1 - x_3 \\ z = 0 + 2x_1 + x_2 + 3x_3 \end{array} \right.$$

where $\mathcal{B} = \{4, 5, 6, 7\}$ and $\mathcal{N} = \{1, 2, 3\}$. The basic feasible solution is:

$$\bar{x}_4 = 4, \quad \bar{x}_5 = 2, \quad \bar{x}_6 = 3, \quad \bar{x}_7 = 6, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value } 0$$

• **Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\left\{ \begin{array}{l} x_4 = 4 - x_1 \\ x_7 = 6 - 3x_1 \end{array} \right. \quad \left\{ \begin{array}{l} \underbrace{4 - x_1 \geq 0}_{x_4 \geq 0} \\ \underbrace{6 - 3x_1 \geq 0}_{x_7 \geq 0} \end{array} \right. \quad \left\{ \begin{array}{l} x_1 \leq 4 \\ x_1 \leq 2 \end{array} \right.$$

By increasing the value of x_1 to $\min\{4, 2\} = 2$, the value of x_7 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_7 becomes non-basic

We have:

$$x_7 = 6 - 3x_1 - x_3 \quad \text{then} \quad x_1 = 2 - \frac{1}{3}x_3 - \frac{1}{3}x_7.$$

Substituting in the other equations of the dictionary, we get:

$$x_4 = 4 - \left(2 - \frac{1}{3}x_3 - \frac{1}{3}x_7\right) - x_2 - x_3 = 2 - x_2 - \frac{2}{3}x_3 + \frac{1}{3}x_7$$

$$z = 2 \left(2 - \frac{1}{3}x_3 - \frac{1}{3}x_7\right) + x_2 + 3x_3 = 4 + x_2 + \frac{7}{3}x_3 - \frac{2}{3}x_7$$

Accordingly, we now have the following dictionary:

$$\left\{ \begin{array}{lclclcl} x_1 = 2 & & - & \frac{1}{3} x_3 & - & \frac{1}{3} x_7 \\ x_4 = 2 & - & x_2 & - & \frac{2}{3} x_3 & + & \frac{1}{3} x_7 \\ x_5 = 2 & - & x_2 & & & & \\ x_6 = 3 & & & - & x_3 & & \\ z = 4 & + & x_2 & + & \frac{7}{3} x_3 & - & \frac{2}{3} x_7 \end{array} \right.$$

where $\mathcal{B} = \{1, 4, 5, 6\}$ and $\mathcal{N} = \{2, 3, 7\}$. The basic feasible solution is:

$$\bar{x}_1 = 2, \quad \bar{x}_4 = 2, \quad \bar{x}_5 = 2, \quad \bar{x}_6 = 3, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_7 = 0 \quad \text{of value} \quad 4$$

- **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\left\{ \begin{array}{l} x_4 = 2 - x_2 \\ x_5 = 2 - x_2 \end{array} \right. \quad \left\{ \begin{array}{l} \underbrace{2 - x_2 \geq 0}_{x_4 \geq 0} \\ \underbrace{2 - x_2 \geq 0}_{x_5 \geq 0} \end{array} \right. \quad \left\{ \begin{array}{l} x_2 \leq 2 \\ x_2 \leq 2 \end{array} \right.$$

By increasing the value of x_2 to $\min\{2, 2\} = 2$, the value of x_4 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 2 - x_2 - \frac{2}{3} x_3 + \frac{1}{3} x_7 \quad \text{then} \quad x_2 = 2 - \frac{2}{3} x_3 - x_4 + \frac{1}{3} x_7.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_5 &= 2 - \left(2 - \frac{2}{3} x_3 - x_4 + \frac{1}{3} x_7 \right) = \frac{2}{3} x_3 + x_4 - \frac{1}{3} x_7 \\ z &= 4 + \left(2 - \frac{2}{3} x_3 - x_4 + \frac{1}{3} x_7 \right) + \frac{7}{3} x_3 - \frac{2}{3} x_7 = 6 + \frac{5}{3} x_3 - x_4 - \frac{1}{3} x_7 \end{aligned}$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = 2 - \frac{1}{3} x_3 & - \frac{1}{3} x_7 \\ x_2 = 2 - \frac{2}{3} x_3 - x_4 + \frac{1}{3} x_7 \\ x_5 = 0 + \frac{2}{3} x_3 + x_4 - \frac{1}{3} x_7 \\ x_6 = 3 - x_3 \\ z = 6 + \frac{5}{3} x_3 - x_4 - \frac{1}{3} x_7 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5, 6\}$ and $\mathcal{N} = \{3, 4, 7\}$. The basic feasible solution is:

$$\bar{x}_1 = 2, \quad \bar{x}_2 = 2, \quad \bar{x}_5 = 0, \quad \bar{x}_6 = 3, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_7 = 0 \quad \text{of value } 6$$

• Iteration 3

If we increase the value of the variable x_3 , we can get a better solution. We have:

$$\begin{cases} x_1 = 2 - \frac{1}{3} x_3 \\ x_2 = 2 - \frac{2}{3} x_3 \\ x_5 = 0 + \frac{2}{3} x_3 \\ x_6 = 3 - x_3 \end{cases} \quad \begin{cases} \underbrace{2 - \frac{1}{3} x_3 \geq 0}_{x_1 \geq 0} \\ \underbrace{2 - \frac{2}{3} x_3 \geq 0}_{x_2 \geq 0} \\ \underbrace{0 + \frac{2}{3} x_3 \geq 0}_{x_5 \geq 0} \\ \underbrace{3 - x_3 \geq 0}_{x_6 \geq 0} \end{cases} \quad \begin{cases} x_3 \leq 6 \\ x_3 \leq 3 \\ x_3 \geq 0 \\ x_3 \leq 3 \end{cases}$$

By increasing the value of x_3 to $\min\{6, 3, 3\} = 3$, the value of x_2 decreases to 0.

Then, we have that:

x_3 becomes basic and accordingly x_2 becomes non-basic

We have:

$$x_2 = 2 - \frac{2}{3} x_3 - x_4 + \frac{1}{3} x_7 \quad \text{then} \quad x_3 = 3 - \frac{3}{2} x_2 - \frac{3}{2} x_4 + \frac{1}{2} x_7.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_1 &= 2 - \frac{1}{3} \left(3 - \frac{3}{2} x_2 - \frac{3}{2} x_4 + \frac{1}{2} x_7 \right) - \frac{1}{3} x_7 = 1 + \frac{1}{2} x_2 + \frac{1}{2} x_4 - \frac{1}{2} x_7 \\ x_5 &= \frac{2}{3} \left(3 - \frac{3}{2} x_2 - \frac{3}{2} x_4 + \frac{1}{2} x_7 \right) + x_4 - \frac{1}{3} x_7 = 2 - x_2 \end{aligned}$$

$$x_6 = 3 - \left(3 - \frac{3}{2}x_2 - \frac{3}{2}x_4 + \frac{1}{2}x_7\right) = \frac{3}{2}x_2 + \frac{3}{2}x_4 - \frac{1}{2}x_7$$

$$z = 6 + \frac{5}{3} \left(3 - \frac{3}{2}x_2 - \frac{3}{2}x_4 + \frac{1}{2}x_7\right) - x_4 - \frac{1}{3}x_7 = 11 - \frac{5}{2}x_2 - \frac{7}{2}x_4 + \frac{1}{2}x_7$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = 1 & + & \frac{1}{2}x_2 & + & \frac{1}{2}x_4 & - & \frac{1}{2}x_7 \\ x_3 = 3 & - & \frac{3}{2}x_2 & - & \frac{3}{2}x_4 & + & \frac{1}{2}x_7 \\ x_5 = 2 & - & x_2 & & & & \\ x_6 = 0 & + & \frac{3}{2}x_2 & + & \frac{3}{2}x_4 & - & \frac{1}{2}x_7 \\ z = 11 & - & \frac{5}{2}x_2 & - & \frac{7}{2}x_4 & + & \frac{1}{2}x_7 \end{cases}$$

where $\mathcal{B} = \{1, 3, 5, 6\}$ and $\mathcal{N} = \{2, 4, 7\}$. The basic feasible solution is:

$$\bar{x}_1 = 1, \quad \bar{x}_3 = 3, \quad \bar{x}_5 = 2, \quad \bar{x}_6 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_7 = 0 \quad \text{of value} \quad 11$$

• Iteration 4

If we increase the value of the variable x_7 , we can get a better solution. We have:

$$\begin{cases} x_1 = 1 - \frac{1}{2}x_7 \\ x_3 = 3 + \frac{1}{2}x_7 \\ x_6 = 0 - \frac{1}{2}x_7 \end{cases} \quad \begin{cases} \underbrace{1 - \frac{1}{2}x_7 \geq 0}_{x_1 \geq 0} \\ \underbrace{3 + \frac{1}{2}x_7 \geq 0}_{x_3 \geq 0} \\ \underbrace{0 - \frac{1}{2}x_7 \geq 0}_{x_6 \geq 0} \end{cases} \quad \begin{cases} x_7 \leq 2 \\ x_7 \geq -6 \\ x_7 \leq 0 \end{cases}$$

By increasing the value of x_7 to $\min\{2, 0\} = 0$, the value of x_6 decreases to 0.

Then, we have that:

x_7 becomes basic and accordingly x_6 becomes non-basic

We have:

$$x_6 = \frac{3}{2}x_2 + \frac{3}{2}x_4 - \frac{1}{2}x_7 \quad \text{then} \quad x_7 = 3x_2 + 3x_4 - 2x_6.$$

Substituting in the other equations of the dictionary, we get:

$$x_1 = 1 + \frac{1}{2}x_2 + \frac{1}{2}x_4 - \frac{1}{2}(3x_2 + 3x_4 - 2x_6) = 1 - x_2 - x_4 + x_6$$

$$x_3 = 3 - \frac{3}{2}x_2 - \frac{3}{2}x_4 + \frac{1}{2}(3x_2 + 3x_4 - 2x_6) = 3 - x_6$$

$$z = 11 - \frac{5}{2}x_2 - \frac{7}{2}x_4 + \frac{1}{2}(3x_2 + 3x_4 - 2x_6) = 11 - x_2 - 2x_4 - x_6$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

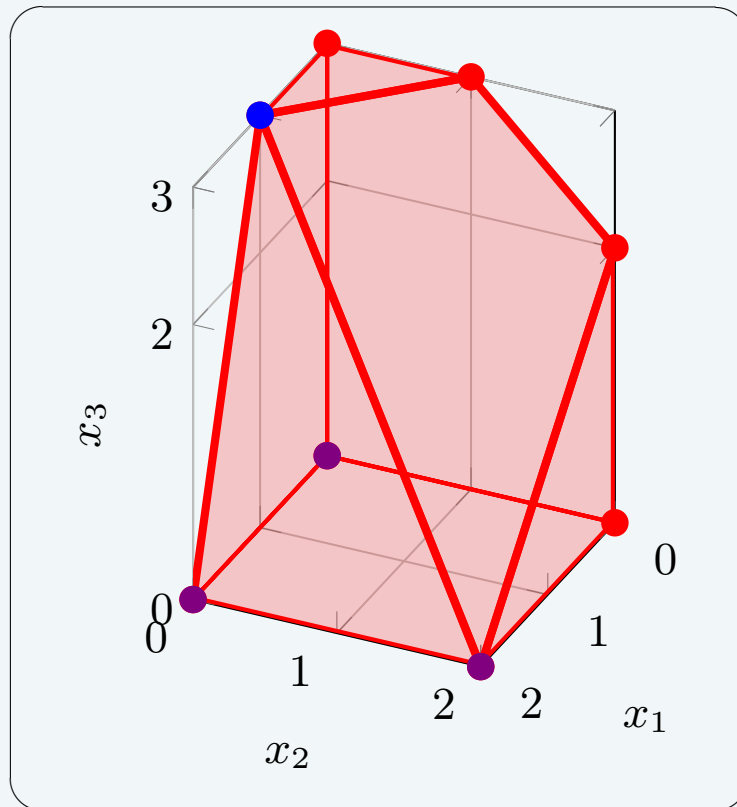
$$\left\{ \begin{array}{l} x_1 = 1 \quad - \quad x_2 \quad - \quad x_4 \quad + \quad x_6 \\ x_3 = 3 \quad \quad \quad \quad \quad \quad - \quad x_6 \\ x_5 = 2 \quad - \quad x_2 \\ x_7 = 0 \quad + \quad 3x_2 \quad + \quad 3x_4 \quad - \quad 2x_6 \\ z = 11 \quad - \quad x_2 \quad - \quad 2x_4 \quad - \quad x_6 \end{array} \right.$$

where $\mathcal{B} = \{1, 3, 5, 7\}$ and $\mathcal{N} = \{2, 4, 6\}$. The basic optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_3 = 3, \quad \tilde{x}_5 = 2, \quad \tilde{x}_7 = 0, \quad \tilde{x}_4 = 0, \quad \tilde{x}_6 = 0, \quad \tilde{x}_2 = 0 \quad \text{of value} \quad 11$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{x}(LP)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 3$$

the optimal value $z(LP)$ is 11.

In violet the first and the second vertex explored by the previous execution of the algorithm (before reaching the optimal one depicted in blue).

SageMath Code: 3D Polyhedra (<https://sagecell.sagemath.org/>)

```
# Define the polyhedron using inequalities (Hyperplane Representation)
p = Polyhedron(ieqs=[
    (4, -1, -1, -1), # (-1, -1, -1) x + 4 >= 0
    (2, 0, -1, 0), # (0, -1, 0) x + 2 >= 0
    (3, 0, 0, -1), # (0, 0, -1) x + 3 >= 0
    (6, -3, 0, -1), # (-3, 0, -1) x + 6 >= 0
    (0, 1, 0, 0), # (1, 0, 0) x + 0 >= 0
    (0, 0, 1, 0), # (0, 1, 0) x + 0 >= 0
    (0, 0, 0, 1) # (0, 0, 1) x + 0 >= 0
])

# Display the inequalities of the polyhedron
p.Hrepresentation()

# Display the extreme points and extreme rays of the polyhedron
p.Vrepresentation()

# Plot the polyhedron (3D)
p.plot()
```

Exercise 6.19 — Simplex algorithm (P42)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 3$ constraints:

$$\begin{aligned}
 &\max \quad 3x_1 + 2x_2 + 4x_3 \\
 &\text{subject to} \quad x_1 + x_2 + 2x_3 \leq 4 \\
 &\quad \quad \quad 2x_1 + 3x_3 \leq 5 \\
 &\quad \quad \quad 2x_1 + x_2 + 3x_3 \leq 7 \\
 &\quad \quad \quad x_1, x_2, x_3 \geq 0
 \end{aligned}$$

By introducing the slack variables, we get the following starting dictionary:

$$\begin{cases}
 x_4 = 4 - x_1 - x_2 - 2x_3 \\
 x_5 = 5 - 2x_1 - 3x_3 \\
 x_6 = 7 - 2x_1 - x_2 - 3x_3 \\
 z = 0 + 3x_1 + 2x_2 + 4x_3
 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 0, \bar{x}_4 = 4, \bar{x}_5 = 5, \bar{x}_6 = 7 \quad \text{of value} \quad 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- Initialization**

Consider the starting dictionary:

$$\begin{cases} x_4 = 4 - x_1 - x_2 - 2x_3 \\ x_5 = 5 - 2x_1 - 3x_3 \\ x_6 = 7 - 2x_1 - x_2 - 3x_3 \\ z = 0 + 3x_1 + 2x_2 + 4x_3 \end{cases}$$

where $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$. The basic feasible solution is:

$$\bar{x}_4 = 4, \bar{x}_5 = 5, \bar{x}_6 = 7, \bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 0 \quad \text{of value} \quad 0$$

- Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_4 = 4 - x_1 \\ x_5 = 5 - 2x_1 \\ x_6 = 7 - 2x_1 \end{cases} \quad \begin{cases} \underbrace{4 - x_1 \geq 0}_{x_4 \geq 0} \\ \underbrace{5 - 2x_1 \geq 0}_{x_5 \geq 0} \\ \underbrace{7 - 2x_1 \geq 0}_{x_6 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq 4 \\ x_1 \leq \frac{5}{2} \\ x_1 \leq \frac{7}{2} \end{cases}$$

By increasing the value of x_1 to $\min \left\{ 4, \frac{5}{2}, \frac{7}{2} \right\} = \frac{5}{2}$, the value of x_5 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = 5 - 2x_1 - 3x_3 \quad \text{then} \quad x_1 = \frac{5}{2} - \frac{3}{2}x_3 - \frac{1}{2}x_5.$$

Substituting in the other equations of the dictionary, we get:

$$x_4 = 4 - \left(\frac{5}{2} - \frac{3}{2}x_3 - \frac{1}{2}x_5 \right) - x_2 - 2x_3 = \frac{3}{2} - x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_5$$

$$x_6 = 7 - 2 \left(\frac{5}{2} - \frac{3}{2}x_3 - \frac{1}{2}x_5 \right) - x_2 - 3x_3 = 2 - x_2 + x_5$$

$$z = 3 \left(\frac{5}{2} - \frac{3}{2}x_3 - \frac{1}{2}x_5 \right) + 2x_2 + 4x_3 = \frac{15}{2} + 2x_2 - \frac{1}{2}x_3 - \frac{3}{2}x_5$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = \frac{5}{2} & - & \frac{3}{2}x_3 & - & \frac{1}{2}x_5 \\ x_4 = \frac{3}{2} & - & x_2 & - & \frac{1}{2}x_3 & + & \frac{1}{2}x_5 \\ x_6 = 2 & - & x_2 & & & + & x_5 \\ z = \frac{15}{2} & + & 2x_2 & - & \frac{1}{2}x_3 & - & \frac{3}{2}x_5 \end{cases}$$

where $\mathcal{B} = \{1, 4, 6\}$ and $\mathcal{N} = \{2, 3, 5\}$. The basic feasible solution is:

$$\bar{x}_1 = \frac{5}{2}, \quad \bar{x}_4 = \frac{3}{2}, \quad \bar{x}_6 = 2, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{15}{2}$$

• Iteration 2

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_4 = \frac{3}{2} - x_2 \\ x_6 = 2 - x_2 \end{cases} \quad \begin{cases} \underbrace{\frac{3}{2} - x_2}_{x_4 \geq 0} \geq 0 \\ \underbrace{2 - x_2}_{x_6 \geq 0} \geq 0 \end{cases} \quad \begin{cases} x_2 \leq \frac{3}{2} \\ x_2 \leq 2 \end{cases}$$

By increasing the value of x_2 to $\frac{3}{2}$, the value of x_4 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = \frac{3}{2} - x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_5 \quad \text{then} \quad x_2 = \frac{3}{2} - \frac{1}{2}x_3 - x_4 + \frac{1}{2}x_5$$

Substituting in the other equations of the dictionary, we get:

$$x_6 = 2 - \left(\frac{3}{2} - \frac{1}{2} x_3 - x_4 + \frac{1}{2} x_5 \right) + x_5 = \frac{1}{2} + \frac{1}{2} x_3 + x_4 + \frac{1}{2} x_5$$

$$z = \frac{15}{2} + 2 \left(\frac{3}{2} - \frac{1}{2} x_3 - x_4 + \frac{1}{2} x_5 \right) - \frac{1}{2} x_3 - \frac{3}{2} x_5 = \frac{21}{2} - \frac{3}{2} x_3 - 2 x_4 - \frac{1}{2} x_5$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

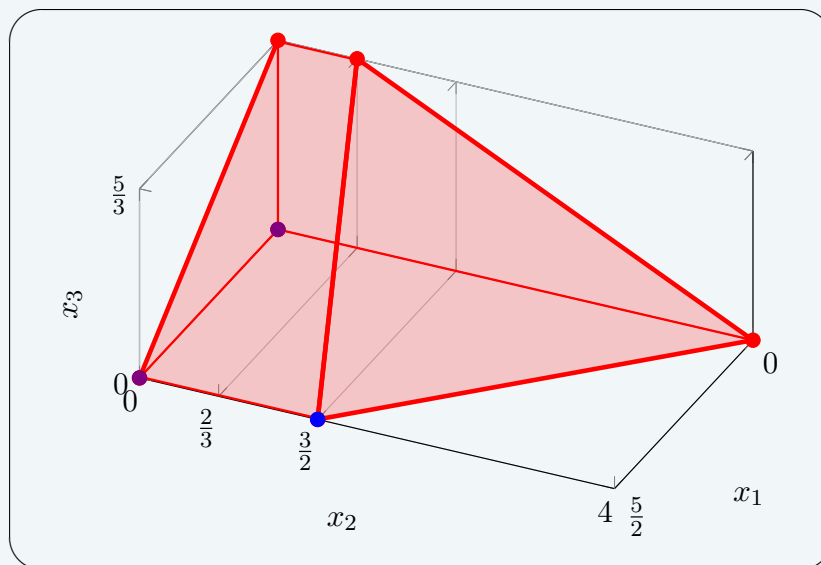
$$\begin{cases} x_1 = \frac{5}{2} - \frac{3}{2} x_3 - \frac{1}{2} x_5 \\ x_2 = \frac{3}{2} - \frac{1}{2} x_3 - x_4 + \frac{1}{2} x_5 \\ x_6 = \frac{1}{2} + \frac{1}{2} x_3 + x_4 + \frac{1}{2} x_5 \\ z = \frac{21}{2} - \frac{3}{2} x_3 - 2 x_4 - \frac{1}{2} x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 6\}$ and $\mathcal{N} = \{3, 4, 5\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = \frac{3}{2}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0, \quad \tilde{x}_6 = \frac{1}{2} \quad \text{of value} \quad \frac{21}{2}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{x}(LP)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = \frac{3}{2}, \quad \tilde{x}_3 = 0$$

the optimal value $z(LP)$ is $\frac{21}{2}$.

In violet the first and the second vertex explored by the previous execution of the algorithm (before reaching the optimal one depicted in blue).

Warning

This is another possible execution of the simplex algorithm, selecting non-basic variables with the largest reduced cost instead of following the natural order.

- Initialization**

Consider the starting dictionary:

$$\left\{ \begin{array}{l} x_4 = 4 - x_1 - x_2 - 2x_3 \\ x_5 = 5 - 2x_1 - 3x_3 \\ x_6 = 7 - 2x_1 - x_2 - 3x_3 \\ z = 0 + 3x_1 + 2x_2 + 4x_3 \end{array} \right.$$

where $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$. The basic feasible solution is:

$$\bar{x}_4 = 4, \quad \bar{x}_5 = 5, \quad \bar{x}_6 = 7, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value } 0$$

- Iteration 1**

If we increase the value of the variable x_3 , we can get a better solution. We have:

$$\left\{ \begin{array}{l} x_4 = 4 - 2x_3 \\ x_5 = 5 - 3x_3 \\ x_6 = 7 - 3x_3 \end{array} \right. \quad \left\{ \begin{array}{l} \underbrace{4 - 2x_3 \geq 0}_{x_4 \geq 0} \\ \underbrace{5 - 3x_3 \geq 0}_{x_5 \geq 0} \\ \underbrace{7 - 3x_3 \geq 0}_{x_6 \geq 0} \end{array} \right. \quad \left\{ \begin{array}{l} x_3 \leq 2 \\ x_3 \leq \frac{5}{3} \\ x_3 \leq \frac{7}{3} \end{array} \right.$$

By increasing the value of x_3 to $\frac{5}{3}$, the value of x_5 decreases to 0.

Then, we have that:

x_3 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = 5 - 2x_1 - 3x_3 \quad \text{then} \quad x_3 = \frac{5}{3} - \frac{2}{3}x_1 - \frac{1}{3}x_5.$$

Substituting in the other equations of the dictionary, we get:

$$x_4 = 4 - x_1 - x_2 - 2 \left(\frac{5}{3} - \frac{2}{3}x_1 - \frac{1}{3}x_5 \right) = \frac{2}{3} + \frac{1}{3}x_1 - x_2 + \frac{2}{3}x_5$$

$$x_6 = 7 - 2x_1 - x_2 - 3 \left(\frac{5}{3} - \frac{2}{3}x_1 - \frac{1}{3}x_5 \right) = 2 - x_2 + x_5$$

$$z = 3x_1 + 2x_2 + 4 \left(\frac{5}{3} - \frac{2}{3}x_1 - \frac{1}{3}x_5 \right) = \frac{20}{3} + \frac{1}{3}x_1 + 2x_2 - \frac{4}{3}x_5$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_3 = \frac{5}{3} & - & \frac{2}{3}x_1 & & - & \frac{1}{3}x_5 \\ x_4 = \frac{2}{3} & + & \frac{1}{3}x_1 & - & x_2 & + & \frac{2}{3}x_5 \\ x_6 = 2 & & & - & x_2 & + & x_5 \\ z = \frac{20}{3} & + & \frac{1}{3}x_1 & + & 2x_2 & - & \frac{4}{3}x_5 \end{cases}$$

where $\mathcal{B} = \{3, 4, 6\}$ and $\mathcal{N} = \{1, 2, 5\}$. The basic feasible solution is:

$$\bar{x}_3 = \frac{5}{3}, \quad \bar{x}_4 = \frac{2}{3}, \quad \bar{x}_6 = 2, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_5 = 0, \quad \text{of value} \quad \frac{20}{3}$$

• **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_4 = \frac{2}{3} - x_2 \\ x_6 = 2 - x_2 \end{cases} \quad \begin{cases} \underbrace{\frac{2}{3} - x_2 \geq 0}_{x_4 \geq 0} \\ \underbrace{2 - x_2 \geq 0}_{x_6 \geq 0} \end{cases} \quad \begin{cases} x_2 \leq \frac{2}{3} \\ x_2 \leq 2 \end{cases}$$

By increasing the value of x_2 to $\frac{2}{3}$, the value of x_4 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = \frac{2}{3} + \frac{1}{3}x_1 - x_2 + \frac{2}{3}x_5 \quad \text{then} \quad x_2 = \frac{2}{3} + \frac{1}{3}x_1 - x_4 + \frac{2}{3}x_5$$

Substituting in the other equations of the dictionary, we get:

$$x_6 = 2 - \left(\frac{2}{3} + \frac{1}{3}x_1 - x_4 + \frac{2}{3}x_5 \right) + x_5 = \frac{4}{3} - \frac{1}{3}x_1 + x_4 + \frac{1}{3}x_5$$

$$z = \frac{20}{3} + \frac{1}{3}x_1 + 2 \left(\frac{2}{3} + \frac{1}{3}x_1 - x_4 + \frac{2}{3}x_5 \right) - \frac{4}{3}x_5 = 8 + x_1 - 2x_4$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_2 = \frac{2}{3} + \frac{1}{3}x_1 - x_4 + \frac{2}{3}x_5 \\ x_3 = \frac{5}{3} - \frac{2}{3}x_1 - \frac{1}{3}x_5 \\ x_6 = \frac{4}{3} - \frac{1}{3}x_1 + x_4 + \frac{1}{3}x_5 \\ z = 8 + x_1 - 2x_4 \end{cases}$$

where $\mathcal{B} = \{2, 3, 6\}$ and $\mathcal{N} = \{1, 4, 5\}$. The basic feasible solution is:

$$\bar{x}_2 = \frac{2}{3}, \quad \bar{x}_3 = \frac{5}{3}, \quad \bar{x}_6 = \frac{4}{3}, \quad \bar{x}_1 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_5 = 0, \quad \text{of value } 8$$

• **Iteration 3**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_2 = \frac{2}{3} + \frac{1}{3}x_1 \\ x_3 = \frac{5}{3} - \frac{2}{3}x_1 \\ x_6 = \frac{4}{3} - \frac{1}{3}x_1 \end{cases} \quad \begin{cases} \underbrace{\frac{2}{3} + \frac{1}{3}x_1}_{x_2 \geq 0} \geq 0 \\ \underbrace{\frac{5}{3} - \frac{2}{3}x_1}_{x_3 \geq 0} \geq 0 \\ \underbrace{\frac{4}{3} - \frac{1}{3}x_1}_{x_6 \geq 0} \geq 0 \end{cases} \quad \begin{cases} x_1 \geq -2 \\ x_1 \leq \frac{5}{2} \\ x_1 \leq 4 \end{cases}$$

By increasing the value of x_1 to $\frac{5}{2}$, the value of x_3 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = \frac{5}{3} - \frac{2}{3}x_1 - \frac{1}{3}x_5 \quad \text{then} \quad x_1 = \frac{5}{2} - \frac{3}{2}x_3 - \frac{1}{2}x_5$$

Substituting in the other equations of the dictionary, we get:

$$x_2 = \frac{2}{3} + \frac{1}{3} \left(\frac{5}{2} - \frac{3}{2}x_3 - \frac{1}{2}x_5 \right) - x_4 + \frac{2}{3}x_5 = \frac{3}{2} - \frac{1}{2}x_3 - x_4 + \frac{1}{2}x_5$$

$$x_6 = \frac{4}{3} - \frac{1}{3} \left(\frac{5}{2} - \frac{3}{2}x_3 - \frac{1}{2}x_5 \right) + x_4 + \frac{1}{3}x_5 = \frac{1}{2} + \frac{1}{2}x_3 + x_4 + \frac{1}{2}x_5$$

$$z = 8 + \left(\frac{5}{2} - \frac{3}{2}x_3 - \frac{1}{2}x_5 \right) - 2x_4 = \frac{21}{2} - \frac{3}{2}x_3 - 2x_4 - \frac{1}{2}x_5$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

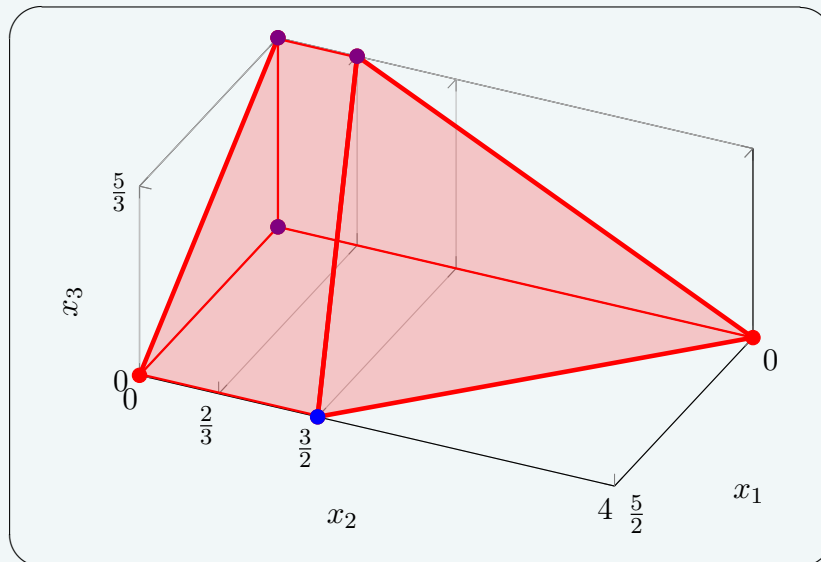
$$\left\{ \begin{array}{l} x_1 = \frac{5}{2} - \frac{3}{2}x_3 - \frac{1}{2}x_5 \\ x_2 = \frac{3}{2} - \frac{1}{2}x_3 - x_4 + \frac{1}{2}x_5 \\ x_6 = \frac{1}{2} + \frac{1}{2}x_3 + x_4 + \frac{1}{2}x_5 \\ z = \frac{21}{2} - \frac{3}{2}x_3 - 2x_4 - \frac{1}{2}x_5 \end{array} \right.$$

where $\mathcal{B} = \{1, 2, 6\}$ and $\mathcal{N} = \{3, 4, 5\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = \frac{3}{2}, \quad \tilde{x}_6 = \frac{1}{2}, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad \frac{21}{2}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = \frac{3}{2}, \quad \tilde{x}_3 = 0$$

the optimal value $z(LP)$ is $\frac{21}{2}$.

In violet the first, the second and the third vertex explored by the previous execution of the algorithm (before reaching the optimal one depicted in blue).

SageMath Code: 3D Polyhedra (<https://sagecell.sagemath.org/>)

```
# Define the polyhedron using inequalities (Hyperplane Representation)
p = Polyhedron(ieqs=[
    (4, -1, -1, -2), # (-1, -1, -2) x + 4 >= 0
    (5, -2, 0, -3), # (-2, 0, -3) x + 5 >= 0
    (7, -2, -1, -3), # (-2, -1, -3) x + 7 >= 0
    (0, 1, 0, 0), # (1, 0, 0) x + 0 >= 0
    (0, 0, 1, 0), # (0, 1, 0) x + 0 >= 0
    (0, 0, 0, 1) # (0, 0, 1) x + 0 >= 0
])

# Display the inequalities of the polyhedron
p.Hrepresentation()

# Display the extreme points and extreme rays of the polyhedron
p.Vrepresentation()

# Plot the polyhedron (3D)
p.plot()
```

Exercise 6.20 — Simplex algorithm (P43)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 2$ constraints:

$$\begin{array}{llllll}
 \max & x_1 & + & 2x_2 & + & 7x_3 \\
 \text{subject to} & x_1 & & & + & 2x_3 \leq 2 \\
 & & & x_2 & + & 2x_3 \leq 2 \\
 & x_1, & & x_2, & & x_3 \geq 0
 \end{array}$$

By introducing the slack variables, we get the following starting dictionary:

$$\begin{cases}
 x_4 = 2 - x_1 - 2x_3 \\
 x_5 = 2 - x_2 - 2x_3 \\
 z = 0 + x_1 + 2x_2 + 7x_3
 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 0, \bar{x}_4 = 2, \bar{x}_5 = 2 \text{ of value } 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- Initialization**

Consider the starting dictionary:

$$\begin{cases} x_4 = 2 - x_1 - 2x_3 \\ x_5 = 2 - x_2 - 2x_3 \\ z = 0 + x_1 + 2x_2 + 7x_3 \end{cases}$$

where $\mathcal{B} = \{4, 5\}$ and $\mathcal{N} = \{1, 2, 3\}$. The basic feasible solution is:

$$\bar{x}_4 = 2, \bar{x}_5 = 2, \bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 0 \quad \text{of value } 0$$

- Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_4 = 2 - x_1 \\ \underbrace{2 - x_1 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq 2 \end{cases}$$

By increasing the value of x_1 to 2, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 2 - x_1 - 2x_3 \quad \text{then} \quad x_1 = 2 - 2x_3 - x_4.$$

Substituting in the other equations of the dictionary, we get:

$$z = (2 - 2x_3 - x_4) + 2x_2 + 7x_3 = 2 + 2x_2 + 5x_3 - x_4$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = 2 - 2x_3 - x_4 \\ x_5 = 2 - x_2 - 2x_3 \\ z = 2 + 2x_2 + 5x_3 - x_4 \end{cases}$$

where $\mathcal{B} = \{1, 5\}$ and $\mathcal{N} = \{2, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = 2, \quad \bar{x}_5 = 2, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 2$$

• **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_5 = 2 - x_2 \\ \underbrace{2 - x_2}_{x_5 \geq 0} \geq 0 \end{cases} \quad \begin{cases} x_2 \leq 2 \end{cases}$$

By increasing the value of x_2 to 2, the value of x_5 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = 2 - x_2 - 2x_3 \quad \text{then} \quad x_2 = 2 - 2x_3 - x_5.$$

Substituting in the other equations of the dictionary, we get:

$$z = 2 + 2(2 - 2x_3 - x_5) + 5x_3 - x_4 = 6 + x_3 - x_4 - 2x_5$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = 2 & - & 2x_3 & - & x_4 \\ x_2 = 2 & - & 2x_3 & & - & x_5 \\ z = 6 & + & x_3 & - & x_4 & - & 2x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4, 5\}$. The basic feasible solution is:

$$\bar{x}_1 = 2, \quad \bar{x}_2 = 2, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad 6$$

• **Iteration 3**

If we increase the value of the variable x_3 , we can get a better solution. We have:

$$\begin{cases} x_1 = 2 - 2x_3 \\ x_2 = 2 - 2x_3 \end{cases} \quad \begin{cases} \underbrace{2 - 2x_3}_{x_1 \geq 0} \geq 0 \\ \underbrace{2 - 2x_3}_{x_2 \geq 0} \geq 0 \end{cases} \quad \begin{cases} x_3 \leq 1 \\ x_3 \leq 1 \end{cases}$$

By increasing the value of x_3 to 1, the value of both x_1 and x_2 decreases to 0. We choose x_1 as the leaving variable.

Then, we have that:

x_3 becomes basic and accordingly x_1 becomes non-basic

We have:

$$x_1 = 2 - 2x_3 - x_4 \quad \text{then} \quad x_3 = 1 - \frac{1}{2}x_1 - \frac{1}{2}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_2 &= 2 - 2 \left(1 - \frac{1}{2}x_1 - \frac{1}{2}x_4 \right) - x_5 = x_1 + x_4 - x_5 \\ z &= 6 + \left(1 - \frac{1}{2}x_1 - \frac{1}{2}x_4 \right) - x_4 - 2x_5 = 7 - \frac{1}{2}x_1 - \frac{3}{2}x_4 - 2x_5 \end{aligned}$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

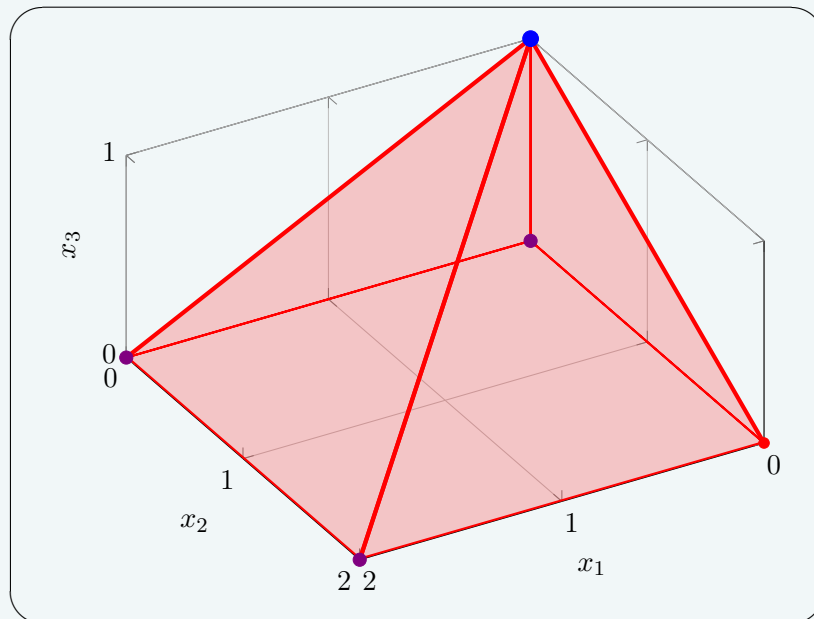
$$\begin{cases} x_2 = 0 & + & x_1 & + & x_4 & - & x_5 \\ x_3 = 1 & - & \frac{1}{2}x_1 & - & \frac{1}{2}x_4 & & \\ z = 7 & - & \frac{1}{2}x_1 & - & \frac{3}{2}x_4 & - & 2x_5 \end{cases}$$

where $\mathcal{B} = \{2, 3\}$ and $\mathcal{N} = \{1, 4, 5\}$. The basic optimal solution is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 1, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad 7$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{x}(LP)$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = 1$$

the optimal value $z(LP)$ is 7.

In violet the first, the second and the third vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

SageMath Code: 3D Polyhedra (<https://sagecell.sagemath.org/>)

```
# Define the polyhedron using inequalities (Hyperplane Representation)
p = Polyhedron(ieqs=[
    (2, -1, 0, -2), # (-1, 0, -2) x + 2 >= 0
    (2, 0, -1, -2), # (0, -1, -2) x + 2 >= 0
    (0, 1, 0, 0), # (1, 0, 0) x + 0 >= 0
    (0, 0, 1, 0), # (0, 1, 0) x + 0 >= 0
    (0, 0, 0, 1) # (0, 0, 1) x + 0 >= 0
])

# Display the inequalities of the polyhedron
p.Hrepresentation()

# Display the extreme points and extreme rays of the polyhedron
p.Vrepresentation()

# Plot the polyhedron (3D)
p.plot()
```

Exercise 6.21 — Simplex algorithm (P44)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 3$ constraints:

$$\begin{array}{llllll} \max & x_1 & + & 2x_2 & + & 3x_3 \\ \text{subject to} & x_1 & + & 2x_2 & - & 2x_3 & \leq & 1 \\ & 2x_1 & - & x_2 & - & 3x_3 & \leq & 4 \\ & x_1 & + & 2x_2 & + & 5x_3 & \leq & 2 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

By introducing the slack variables, we get the following starting dictionary:

$$\begin{cases} x_4 = 1 & - & x_1 & - & 2 x_2 & + & 2 x_3 \\ x_5 = 4 & - & 2 x_1 & + & x_2 & + & 3 x_3 \\ x_6 = 2 & - & x_1 & - & 2 x_2 & - & 5 x_3 \\ z = 0 & + & x_1 & + & 2 x_2 & + & 3 x_3 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 0, \bar{x}_4 = 1, \bar{x}_5 = 4, \bar{x}_6 = 2 \quad \text{of value } 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- Initialization**

Consider the starting dictionary:

$$\begin{cases} x_4 = 1 & - & x_1 & - & 2 x_2 & + & 2 x_3 \\ x_5 = 4 & - & 2 x_1 & + & x_2 & + & 3 x_3 \\ x_6 = 2 & - & x_1 & - & 2 x_2 & - & 5 x_3 \\ z = 0 & + & x_1 & + & 2 x_2 & + & 3 x_3 \end{cases}$$

where $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$. The basic feasible solution is:

$$\bar{x}_4 = 1, \bar{x}_5 = 4, \bar{x}_6 = 2, \quad \bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 0 \quad \text{of value } 0$$

- Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_4 = 1 - x_1 \\ x_5 = 4 - 2 x_1 \\ x_6 = 2 - x_1 \end{cases} \quad \begin{cases} \underbrace{1 - x_1 \geq 0}_{x_4 \geq 0} \\ \underbrace{4 - 2 x_1 \geq 0}_{x_5 \geq 0} \\ \underbrace{2 - x_1 \geq 0}_{x_6 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq 1 \\ x_1 \leq 2 \\ x_1 \leq 2 \end{cases}$$

By increasing the value of x_1 to 1, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 1 - x_1 - 2x_2 + 2x_3 \quad \text{then} \quad x_1 = 1 - 2x_2 + 2x_3 - x_4.$$

Substituting in the other equations of the dictionary, we get:

$$x_5 = 4 - 2(1 - 2x_2 + 2x_3 - x_4) + x_2 + 3x_3 = 2 + 5x_2 - x_3 + 2x_4$$

$$x_6 = 2 - (1 - 2x_2 + 2x_3 - x_4) - 2x_2 - 5x_3 = 1 - 7x_3 + x_4$$

$$z = (1 - 2x_2 + 2x_3 - x_4) + 2x_2 + 3x_3 = 1 + 5x_3 - x_4$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = 1 - 2x_2 + 2x_3 - x_4 \\ x_5 = 2 + 5x_2 - x_3 + 2x_4 \\ x_6 = 1 - 7x_3 + x_4 \\ z = 1 + 5x_3 - x_4 \end{cases}$$

where $\mathcal{B} = \{1, 5, 6\}$ and $\mathcal{N} = \{2, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = 1, \quad \bar{x}_5 = 2, \quad \bar{x}_6 = 1, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 1$$

• Iteration 2

If we increase the value of the variable x_3 , we can get a better solution. We have:

$$\begin{cases} x_1 = 1 + 2x_3 \\ x_5 = 2 - x_3 \\ x_6 = 1 - 7x_3 \end{cases} \quad \begin{cases} \overbrace{1 + 2x_3 \geq 0}^{x_1 \geq 0} \\ \overbrace{2 - x_3 \geq 0}^{x_5 \geq 0} \\ \overbrace{1 - 7x_3 \geq 0}^{x_6 \geq 0} \end{cases} \quad \begin{cases} x_3 \geq -\frac{1}{2} \\ x_3 \leq 2 \\ x_3 \leq \frac{1}{7} \end{cases}$$

By increasing the value of x_3 to $\frac{1}{7}$, the value of x_6 decreases to 0.

Then, we have that:

x_3 becomes basic and accordingly x_6 becomes non-basic

We have:

$$x_6 = 1 - 7x_3 + x_4 \quad \text{then} \quad x_3 = \frac{1}{7} + \frac{1}{7}x_4 - \frac{1}{7}x_6.$$

Substituting in the other equations of the dictionary, we get:

$$x_1 = 1 - 2x_2 + 2\left(\frac{1}{7} + \frac{1}{7}x_4 - \frac{1}{7}x_6\right) - x_4 = \frac{9}{7} - 2x_2 - \frac{5}{7}x_4 - \frac{2}{7}x_6$$

$$x_5 = 2 + 5x_2 - \left(\frac{1}{7} + \frac{1}{7}x_4 - \frac{1}{7}x_6\right) + 2x_4 = \frac{13}{7} + 5x_2 + \frac{13}{7}x_4 + \frac{1}{7}x_6$$

$$z = 1 + 5\left(\frac{1}{7} + \frac{1}{7}x_4 - \frac{1}{7}x_6\right) - x_4 = \frac{12}{7} - \frac{2}{7}x_4 - \frac{5}{7}x_6$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

$$\left\{ \begin{array}{l} x_1 = \frac{9}{7} - 2x_2 - \frac{5}{7}x_4 - \frac{2}{7}x_6 \\ x_3 = \frac{1}{7} + \frac{1}{7}x_4 - \frac{1}{7}x_6 \\ x_5 = \frac{13}{7} + 5x_2 + \frac{13}{7}x_4 + \frac{1}{7}x_6 \\ z = \frac{12}{7} - \frac{2}{7}x_4 - \frac{5}{7}x_6 \end{array} \right.$$

where $\mathcal{B} = \{1, 3, 5\}$ and $\mathcal{N} = \{2, 4, 6\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{9}{7}, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = \frac{1}{7}, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = \frac{13}{7}, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad \frac{12}{7}$$

- **Multiple optimal solutions**

In the optimal dictionary the non-basic variable x_2 has reduced cost $\bar{c}_2 = 0$. Accordingly, increasing the value of x_2 does not change the value of z . We have:

$$x_1 = \frac{9}{7} - 2x_2 \quad \underbrace{\frac{9}{7} - 2x_2}_{x_1 \geq 0} \geq 0 \quad x_2 \leq \frac{9}{14}$$

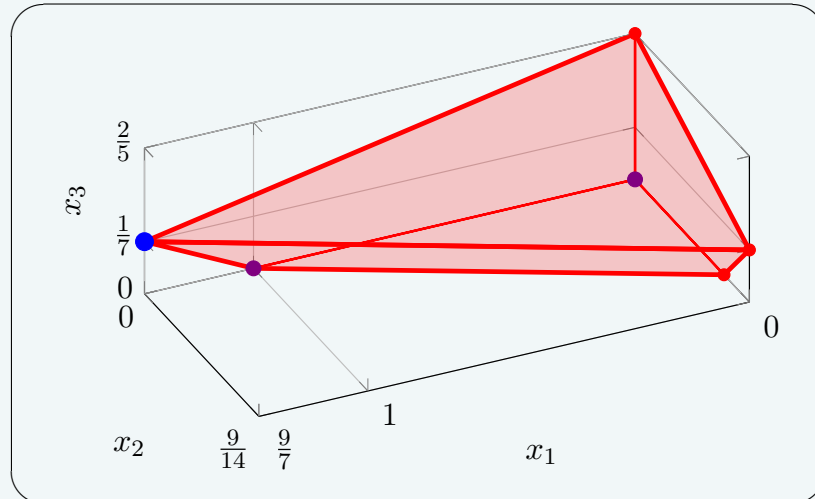
($x_3 = \frac{1}{7}$ does not change and x_5 increases). By increasing the value of x_2 to $\frac{9}{14}$, the value of x_1 decreases to 0 and we obtain the vertex

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = \frac{9}{14}, \quad \tilde{x}_3 = \frac{1}{7} \quad \text{of value} \quad \frac{12}{7}$$

which is optimal as well. Then, the optimal solution is **not unique**: every point of the edge joining $\left(\frac{9}{7}, 0, \frac{1}{7}\right)$ and $\left(0, \frac{9}{14}, \frac{1}{7}\right)$ is optimal, of value $\frac{12}{7}$.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{9}{7}, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = \frac{1}{7}$$

the optimal value $z(LP)$ is $\frac{12}{7}$.

In violet the first and the second vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

SageMath Code: 3D Polyhedra (<https://sagecell.sagemath.org/>)

```
# Define the polyhedron using inequalities (Hyperplane Representation)
p = Polyhedron(ieqs=[
    (1, -1, -2, 2),    # 1 - x1 - 2*x2 + 2*x3 >= 0
    (4, -2, 1, 3),    # 4 - 2*x1 + x2 + 3*x3 >= 0
    (2, -1, -2, -5),  # 2 - x1 - 2*x2 - 5*x3 >= 0
    (0, 1, 0, 0),     # x1 >= 0
    (0, 0, 1, 0),     # x2 >= 0
    (0, 0, 0, 1)      # x3 >= 0
])

# Display the inequalities of the polyhedron
p.Hrepresentation()

# Display the extreme points and extreme rays of the polyhedron
p.Vrepresentation()

# Plot the polyhedron (3D)
p.plot()
```

Exercise 6.22 — Simplex algorithm (P45)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 2$ constraints:

$$\begin{array}{llllll}
 \max & 4 & x_1 & - & 13 & x_2 & + & 7 & x_3 \\
 \text{subject to} & 3 & x_1 & + & 2 & x_2 & + & 5 & x_3 & \leq & 5 \\
 & & x_1 & - & 3 & x_2 & + & 5 & x_3 & \leq & 3 \\
 & & x_1, & & x_2, & & x_3 & \geq & 0
 \end{array}$$

By introducing the slack variables, we get the following starting dictionary:

$$\begin{cases}
 x_4 = 5 - 3x_1 - 2x_2 - 5x_3 \\
 x_5 = 3 - x_1 + 3x_2 - 5x_3 \\
 z = 0 + 4x_1 - 13x_2 + 7x_3
 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 5, \quad \bar{x}_5 = 3 \quad \text{of value} \quad 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- Initialization**

Consider the starting dictionary:

$$\begin{cases} x_4 = 5 - 3x_1 - 2x_2 - 5x_3 \\ x_5 = 3 - x_1 + 3x_2 - 5x_3 \\ z = 0 + 4x_1 - 13x_2 + 7x_3 \end{cases}$$

where $\mathcal{B} = \{4, 5\}$ and $\mathcal{N} = \{1, 2, 3\}$. The basic feasible solution is:

$$\bar{x}_4 = 5, \quad \bar{x}_5 = 3, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value } 0$$

- Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_4 = 5 - 3x_1 \\ x_5 = 3 - x_1 \end{cases} \quad \begin{cases} \underbrace{5 - 3x_1 \geq 0}_{x_4 \geq 0} \\ \underbrace{3 - x_1 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq \frac{5}{3} \\ x_1 \leq 3 \end{cases}$$

By increasing the value of x_1 to $\frac{5}{3}$, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 5 - 3x_1 - 2x_2 - 5x_3 \quad \text{then} \quad x_1 = \frac{5}{3} - \frac{2}{3}x_2 - \frac{5}{3}x_3 - \frac{1}{3}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$x_5 = 3 - \left(\frac{5}{3} - \frac{2}{3}x_2 - \frac{5}{3}x_3 - \frac{1}{3}x_4 \right) + 3x_2 - 5x_3 = \frac{4}{3} + \frac{11}{3}x_2 - \frac{10}{3}x_3 + \frac{1}{3}x_4$$

$$z = 4 \left(\frac{5}{3} - \frac{2}{3}x_2 - \frac{5}{3}x_3 - \frac{1}{3}x_4 \right) - 13x_2 + 7x_3 = \frac{20}{3} - \frac{47}{3}x_2 + \frac{1}{3}x_3 - \frac{4}{3}x_4$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = \frac{5}{3} - \frac{2}{3}x_2 - \frac{5}{3}x_3 - \frac{1}{3}x_4 \\ x_5 = \frac{4}{3} + \frac{11}{3}x_2 - \frac{10}{3}x_3 + \frac{1}{3}x_4 \\ z = \frac{20}{3} - \frac{47}{3}x_2 + \frac{1}{3}x_3 - \frac{4}{3}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 5\}$ and $\mathcal{N} = \{2, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = \frac{5}{3}, \quad \bar{x}_5 = \frac{4}{3}, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{20}{3}$$

• **Iteration 2**

If we increase the value of the variable x_3 , we can get a better solution. We have:

$$\begin{cases} x_1 = \frac{5}{3} - \frac{5}{3}x_3 \\ x_5 = \frac{4}{3} - \frac{10}{3}x_3 \end{cases} \quad \begin{cases} \underbrace{\frac{5}{3} - \frac{5}{3}x_3 \geq 0}_{x_1 \geq 0} \\ \underbrace{\frac{4}{3} - \frac{10}{3}x_3 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_3 \leq 1 \\ x_3 \leq \frac{2}{5} \end{cases}$$

By increasing the value of x_3 to $\frac{2}{5}$, the value of x_5 decreases to 0.

Then, we have that:

x_3 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = \frac{4}{3} + \frac{11}{3}x_2 - \frac{10}{3}x_3 + \frac{1}{3}x_4 \quad \text{then} \quad x_3 = \frac{2}{5} + \frac{11}{10}x_2 + \frac{1}{10}x_4 - \frac{3}{10}x_5.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_1 &= \frac{5}{3} - \frac{2}{3}x_2 - \frac{5}{3}\left(\frac{2}{5} + \frac{11}{10}x_2 + \frac{1}{10}x_4 - \frac{3}{10}x_5\right) - \frac{1}{3}x_4 = 1 - \frac{5}{2}x_2 - \frac{1}{2}x_4 + \frac{1}{2}x_5 \\ z &= \frac{20}{3} - \frac{47}{3}x_2 + \frac{1}{3}\left(\frac{2}{5} + \frac{11}{10}x_2 + \frac{1}{10}x_4 - \frac{3}{10}x_5\right) - \frac{4}{3}x_4 = \frac{34}{5} - \frac{153}{10}x_2 - \frac{13}{10}x_4 - \frac{1}{10}x_5 \end{aligned}$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

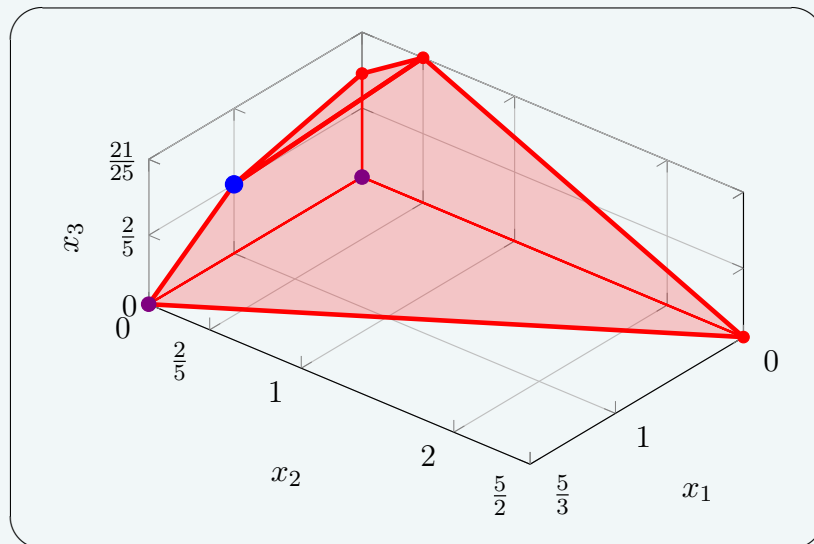
$$\begin{cases} x_1 = 1 & - & \frac{5}{2} x_2 & - & \frac{1}{2} x_4 & + & \frac{1}{2} x_5 \\ x_3 = \frac{2}{5} & + & \frac{11}{10} x_2 & + & \frac{1}{10} x_4 & - & \frac{3}{10} x_5 \\ z = \frac{34}{5} & - & \frac{153}{10} x_2 & - & \frac{13}{10} x_4 & - & \frac{1}{10} x_5 \end{cases}$$

where $\mathcal{B} = \{1, 3\}$ and $\mathcal{N} = \{2, 4, 5\}$. The basic optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = \frac{2}{5}, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad \frac{34}{5}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 0, \quad \tilde{x}_3 = \frac{2}{5}$$

the optimal value $z(LP)$ is $\frac{34}{5}$.

In violet the first and the second vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

SageMath Code: 3D Polyhedra (<https://sagecell.sagemath.org/>)

```
# Define the polyhedron using inequalities (Hyperplane Representation)
p = Polyhedron(ieqs=[
    (5, -3, -2, -5), # 5 - 3*x1 - 2*x2 - 5*x3 >= 0
    (3, -1, 3, -5), # 3 - x1 + 3*x2 - 5*x3 >= 0
    (0, 1, 0, 0), # x1 >= 0
    (0, 0, 1, 0), # x2 >= 0
    (0, 0, 0, 1) # x3 >= 0
])

# Display the inequalities of the polyhedron
p.Hrepresentation()

# Display the extreme points and extreme rays of the polyhedron
p.Vrepresentation()

# Plot the polyhedron (3D)
p.plot()
```

Exercise 6.23 — Simplex algorithm (P46)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 3$ constraints:

$$\begin{aligned}
 \max \quad & 5x_1 + 6x_2 + 10x_3 \\
 \text{subject to} \quad & 9x_1 + 2x_2 + 2x_3 \leq 11 \\
 & 2x_1 + 6x_2 + 6x_3 \leq 16 \\
 & 4x_1 + 2x_2 + 7x_3 \leq 10 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

By introducing the slack variables, we get the following starting dictionary:

$$\begin{cases}
 x_4 = 11 - 9x_1 - 2x_2 - 2x_3 \\
 x_5 = 16 - 2x_1 - 6x_2 - 6x_3 \\
 x_6 = 10 - 4x_1 - 2x_2 - 7x_3 \\
 z = 0 + 5x_1 + 6x_2 + 10x_3
 \end{cases}$$

The associated feasible solution is:

$$\bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 0, \bar{x}_4 = 11, \bar{x}_5 = 16, \bar{x}_6 = 10 \quad \text{of value} \quad 0$$

Questions

Execute the simplex algorithm to find an optimal solution of this LP problem.

Solution

- Initialization**

Consider the starting dictionary:

$$\left\{ \begin{array}{l} x_4 = 11 - 9x_1 - 2x_2 - 2x_3 \\ x_5 = 16 - 2x_1 - 6x_2 - 6x_3 \\ x_6 = 10 - 4x_1 - 2x_2 - 7x_3 \\ z = 0 + 5x_1 + 6x_2 + 10x_3 \end{array} \right.$$

where $\mathcal{B} = \{4, 5, 6\}$ and $\mathcal{N} = \{1, 2, 3\}$. The basic feasible solution is:

$$\bar{x}_4 = 11, \bar{x}_5 = 16, \bar{x}_6 = 10, \bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 0 \quad \text{of value} \quad 0$$

- Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\left\{ \begin{array}{l} x_4 = 11 - 9x_1 \\ x_5 = 16 - 2x_1 \\ x_6 = 10 - 4x_1 \end{array} \right. \quad \left\{ \begin{array}{l} \underbrace{11 - 9x_1 \geq 0}_{x_4 \geq 0} \\ \underbrace{16 - 2x_1 \geq 0}_{x_5 \geq 0} \\ \underbrace{10 - 4x_1 \geq 0}_{x_6 \geq 0} \end{array} \right. \quad \left\{ \begin{array}{l} x_1 \leq \frac{11}{9} \\ x_1 \leq 8 \\ x_1 \leq \frac{5}{2} \end{array} \right.$$

By increasing the value of x_1 to $\frac{11}{9}$, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 11 - 9x_1 - 2x_2 - 2x_3 \quad \text{then} \quad x_1 = \frac{11}{9} - \frac{2}{9}x_2 - \frac{2}{9}x_3 - \frac{1}{9}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$x_5 = 16 - 2\left(\frac{11}{9} - \frac{2}{9}x_2 - \frac{2}{9}x_3 - \frac{1}{9}x_4\right) - 6x_2 - 6x_3 = \frac{122}{9} - \frac{50}{9}x_2 - \frac{50}{9}x_3 + \frac{2}{9}x_4$$

$$x_6 = 10 - 4\left(\frac{11}{9} - \frac{2}{9}x_2 - \frac{2}{9}x_3 - \frac{1}{9}x_4\right) - 2x_2 - 7x_3 = \frac{46}{9} - \frac{10}{9}x_2 - \frac{55}{9}x_3 + \frac{4}{9}x_4$$

$$z = 5\left(\frac{11}{9} - \frac{2}{9}x_2 - \frac{2}{9}x_3 - \frac{1}{9}x_4\right) + 6x_2 + 10x_3 = \frac{55}{9} + \frac{44}{9}x_2 + \frac{80}{9}x_3 - \frac{5}{9}x_4$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = \frac{11}{9} - \frac{2}{9}x_2 - \frac{2}{9}x_3 - \frac{1}{9}x_4 \\ x_5 = \frac{122}{9} - \frac{50}{9}x_2 - \frac{50}{9}x_3 + \frac{2}{9}x_4 \\ x_6 = \frac{46}{9} - \frac{10}{9}x_2 - \frac{55}{9}x_3 + \frac{4}{9}x_4 \\ z = \frac{55}{9} + \frac{44}{9}x_2 + \frac{80}{9}x_3 - \frac{5}{9}x_4 \end{cases}$$

where $\mathcal{B} = \{1, 5, 6\}$ and $\mathcal{N} = \{2, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = \frac{11}{9}, \quad \bar{x}_5 = \frac{122}{9}, \quad \bar{x}_6 = \frac{46}{9}, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad \frac{55}{9}$$

• Iteration 2

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_1 = \frac{11}{9} - \frac{2}{9}x_2 \\ x_5 = \frac{122}{9} - \frac{50}{9}x_2 \\ x_6 = \frac{46}{9} - \frac{10}{9}x_2 \end{cases} \quad \begin{cases} \underbrace{\frac{11}{9} - \frac{2}{9}x_2}_{x_1 \geq 0} \geq 0 \\ \underbrace{\frac{122}{9} - \frac{50}{9}x_2}_{x_5 \geq 0} \geq 0 \\ \underbrace{\frac{46}{9} - \frac{10}{9}x_2}_{x_6 \geq 0} \geq 0 \end{cases} \quad \begin{cases} x_2 \leq \frac{11}{2} \\ x_2 \leq \frac{61}{25} \\ x_2 \leq \frac{23}{5} \end{cases}$$

By increasing the value of x_2 to $\frac{61}{25}$, the value of x_5 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = \frac{122}{9} - \frac{50}{9} x_2 - \frac{50}{9} x_3 + \frac{2}{9} x_4 \quad \text{then} \quad x_2 = \frac{61}{25} - x_3 + \frac{1}{25} x_4 - \frac{9}{50} x_5.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_1 &= \frac{11}{9} - \frac{2}{9} \left(\frac{61}{25} - x_3 + \frac{1}{25} x_4 - \frac{9}{50} x_5 \right) - \frac{2}{9} x_3 - \frac{1}{9} x_4 = \frac{17}{25} - \frac{3}{25} x_4 + \frac{1}{25} x_5 \\ x_6 &= \frac{46}{9} - \frac{10}{9} \left(\frac{61}{25} - x_3 + \frac{1}{25} x_4 - \frac{9}{50} x_5 \right) - \frac{55}{9} x_3 + \frac{4}{9} x_4 = \frac{12}{5} - 5 x_3 + \frac{2}{5} x_4 + \frac{1}{5} x_5 \\ z &= \frac{55}{9} + \frac{44}{9} \left(\frac{61}{25} - x_3 + \frac{1}{25} x_4 - \frac{9}{50} x_5 \right) + \frac{80}{9} x_3 - \frac{5}{9} x_4 = \frac{451}{25} + 4 x_3 - \frac{9}{25} x_4 - \frac{22}{25} x_5 \end{aligned}$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_1 = \frac{17}{25} & - \frac{3}{25} x_4 + \frac{1}{25} x_5 \\ x_2 = \frac{61}{25} & - x_3 + \frac{1}{25} x_4 - \frac{9}{50} x_5 \\ x_6 = \frac{12}{5} & - 5 x_3 + \frac{2}{5} x_4 + \frac{1}{5} x_5 \\ z = \frac{451}{25} & + 4 x_3 - \frac{9}{25} x_4 - \frac{22}{25} x_5 \end{cases}$$

where $\mathcal{B} = \{1, 2, 6\}$ and $\mathcal{N} = \{3, 4, 5\}$. The basic feasible solution is:

$$\bar{x}_1 = \frac{17}{25}, \quad \bar{x}_2 = \frac{61}{25}, \quad \bar{x}_6 = \frac{12}{5}, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad \frac{451}{25}$$

• Iteration 3

If we increase the value of the variable x_3 , we can get a better solution. We have:

$$\begin{cases} x_2 = \frac{61}{25} - x_3 \\ x_6 = \frac{12}{5} - 5 x_3 \end{cases} \quad \begin{cases} \underbrace{\frac{61}{25} - x_3 \geq 0}_{x_2 \geq 0} \\ \underbrace{\frac{12}{5} - 5 x_3 \geq 0}_{x_6 \geq 0} \end{cases} \quad \begin{cases} x_3 \leq \frac{61}{25} \\ x_3 \leq \frac{12}{25} \end{cases}$$

By increasing the value of x_3 to $\frac{12}{25}$, the value of x_6 decreases to 0.

Then, we have that:

x_3 becomes basic and accordingly x_6 becomes non-basic

We have:

$$x_6 = \frac{12}{5} - 5x_3 + \frac{2}{5}x_4 + \frac{1}{5}x_5 \quad \text{then} \quad x_3 = \frac{12}{25} + \frac{2}{25}x_4 + \frac{1}{25}x_5 - \frac{1}{5}x_6.$$

Substituting in the other equations of the dictionary, we get:

$$x_2 = \frac{61}{25} - \left(\frac{12}{25} + \frac{2}{25}x_4 + \frac{1}{25}x_5 - \frac{1}{5}x_6 \right) + \frac{1}{25}x_4 - \frac{9}{50}x_5 = \frac{49}{25} - \frac{1}{25}x_4 - \frac{11}{50}x_5 + \frac{1}{5}x_6$$

$$z = \frac{451}{25} + 4 \left(\frac{12}{25} + \frac{2}{25}x_4 + \frac{1}{25}x_5 - \frac{1}{5}x_6 \right) - \frac{9}{25}x_4 - \frac{22}{25}x_5 = \frac{499}{25} - \frac{1}{25}x_4 - \frac{18}{25}x_5 - \frac{4}{5}x_6$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

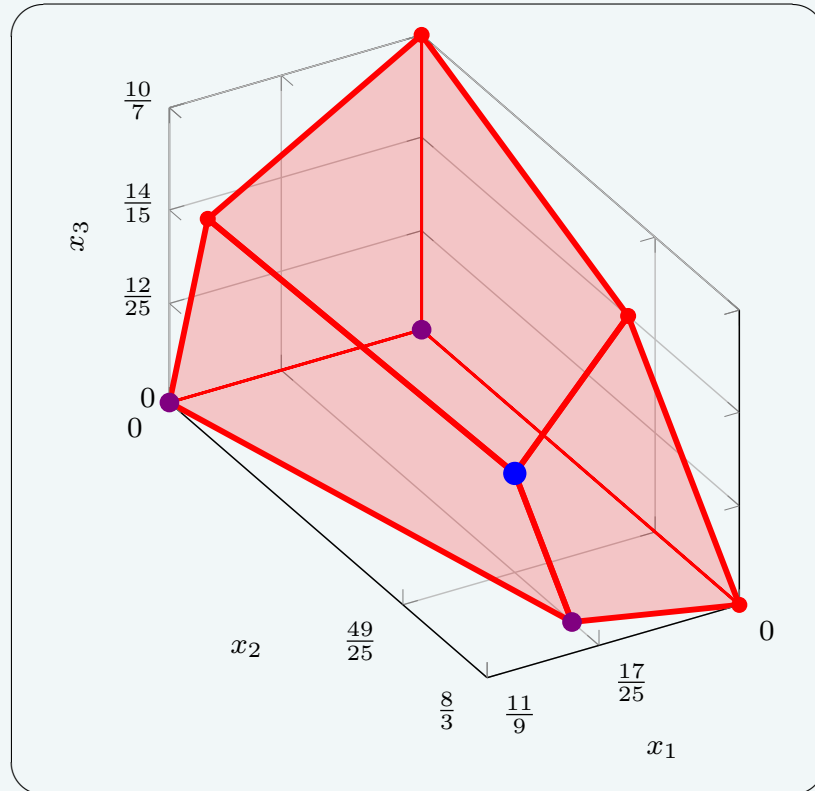
$$\begin{cases} x_1 = \frac{17}{25} & - \frac{3}{25}x_4 & + \frac{1}{25}x_5 \\ x_2 = \frac{49}{25} & - \frac{1}{25}x_4 & - \frac{11}{50}x_5 & + \frac{1}{5}x_6 \\ x_3 = \frac{12}{25} & + \frac{2}{25}x_4 & + \frac{1}{25}x_5 & - \frac{1}{5}x_6 \\ z = \frac{499}{25} & - \frac{1}{25}x_4 & - \frac{18}{25}x_5 & - \frac{4}{5}x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5, 6\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{17}{25}, \quad \tilde{x}_2 = \frac{49}{25}, \quad \tilde{x}_3 = \frac{12}{25}, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0, \quad \tilde{x}_6 = 0 \quad \text{of value} \quad \frac{499}{25}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{x}(LP)$ is:

$$\tilde{x}_1 = \frac{17}{25}, \quad \tilde{x}_2 = \frac{49}{25}, \quad \tilde{x}_3 = \frac{12}{25}$$

the optimal value $z(LP)$ is $\frac{499}{25}$.

In violet the first, the second and the third vertex explored by the previous execution of the simplex algorithm (before reaching the optimal one depicted in blue).

SageMath Code: 3D Polyhedra (<https://sagecell.sagemath.org/>)

```
# Define the polyhedron using inequalities (Hyperplane Representation)
p = Polyhedron(ieqs=[
    (11, -9, -2, -2), # 11 - 9*x1 - 2*x2 - 2*x3 >= 0
    (16, -2, -6, -6), # 16 - 2*x1 - 6*x2 - 6*x3 >= 0
    (10, -4, -2, -7), # 10 - 4*x1 - 2*x2 - 7*x3 >= 0
    (0, 1, 0, 0), # x1 >= 0
    (0, 0, 1, 0), # x2 >= 0
    (0, 0, 0, 1) # x3 >= 0
])

# Display the inequalities of the polyhedron
p.Hrepresentation()

# Display the extreme points and extreme rays of the polyhedron
p.Vrepresentation()

# Plot the polyhedron (3D)
p.plot()
```

Optimality conditions and pivot rules

Exercise 7.1 — Simplex algorithm (C1)

Consider the following LP problem in canonical maximization form with $n = 4$ variables and $m = 3$ constraints:

$$\begin{array}{ll}
 \max & 53 x_1 + 41 x_2 - 204 x_3 - 20 x_4 \\
 \text{subject to} & -11 x_1 - 5 x_2 + 18 x_3 + 2 x_4 \leq 0 \\
 & 4 x_1 + 2 x_2 - 8 x_3 - x_4 \leq 0 \\
 & 11 x_1 + 5 x_2 - 18 x_3 - 2 x_4 \leq 1 \\
 & x_1, \quad x_2, \quad x_3, \quad x_4 \geq 0
 \end{array}$$

By introducing the slack variables, we get the following starting dictionary:

$$\left\{ \begin{array}{l}
 x_5 = 0 + 11 x_1 + 5 x_2 - 18 x_3 - 2 x_4 \\
 x_6 = 0 - 4 x_1 - 2 x_2 + 8 x_3 + x_4 \\
 x_7 = 1 - 11 x_1 - 5 x_2 + 18 x_3 + 2 x_4 \\
 z = 0 + 53 x_1 + 41 x_2 - 204 x_3 - 20 x_4
 \end{array} \right.$$

The associated basic feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_5 = 0, \quad \bar{x}_6 = 0, \quad \bar{x}_7 = 1 \quad \text{of value } 0$$

Questions

1. Execute the simplex algorithm to find an optimal solution of this LP problem. Choose the pivot column as the one with the largest reduced cost. When two or more pivot

rows exist, use the basic variable associated with the constraint of smallest index (in each pivoting operation the pivot column takes the constraint of the pivot row; initially x_5 , x_6 and x_7 are associated with the constraints 1, 2 and 3, respectively).

2. Execute the simplex algorithm to find an optimal solution of this LP problem. Choose the pivot column and the pivot row using the Bland's pivot rule.

Solution

1. • Initialization

Consider the starting dictionary:

$$\begin{cases} x_5 = 0 & + & 11 x_1 & + & 5 x_2 & - & 18 x_3 & - & 2 x_4 \\ x_6 = 0 & - & 4 x_1 & - & 2 x_2 & + & 8 x_3 & + & x_4 \\ x_7 = 1 & - & 11 x_1 & - & 5 x_2 & + & 18 x_3 & + & 2 x_4 \\ z = 0 & + & 53 x_1 & + & 41 x_2 & - & 204 x_3 & - & 20 x_4 \end{cases}$$

where $\mathcal{B} = \{5, 6, 7\}$ and $\mathcal{N} = \{1, 2, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_5 = 0, \bar{x}_6 = 0, \bar{x}_7 = 1, \quad \bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 0, \bar{x}_4 = 0 \quad \text{of value } 0$$

• Iteration 1

The largest reduced cost is the one associated with the non-basic variable x_1 . Thus, if we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_5 = 0 + 11 x_1 \\ x_6 = 0 - 4 x_1 \\ x_7 = 1 - 11 x_1 \end{cases} \quad \begin{cases} \underbrace{0 + 11 x_1 \geq 0}_{x_5 \geq 0} \\ \underbrace{0 - 4 x_1 \geq 0}_{x_6 \geq 0} \\ \underbrace{1 - 11 x_1 \geq 0}_{x_7 \geq 0} \end{cases} \quad \begin{cases} x_1 \geq 0 \\ x_1 \leq 0 \\ x_1 \leq \frac{1}{11} \end{cases}$$

By increasing the value of x_1 to $\min\left\{0, \frac{1}{11}\right\} = 0$, the value of x_6 decreases to 0. This is a degenerate pivot, since the value of the objective function does not increase.

Then, we have that:

x_1 becomes basic and accordingly x_6 becomes non-basic

We have:

$$x_6 = 0 - 4 x_1 - 2 x_2 + 8 x_3 + x_4 \quad \text{then} \quad x_1 = -\frac{1}{2} x_2 + 2 x_3 + \frac{1}{4} x_4 - \frac{1}{4} x_6.$$

Substituting in the other equations of the dictionary, we get the following dictionary:

$$\begin{aligned}
 x_5 &= 11 \left(-\frac{1}{2}x_2 + 2x_3 + \frac{1}{4}x_4 - \frac{1}{4}x_6 \right) + 5x_2 - 18x_3 - 2x_4 \\
 &= -\frac{1}{2}x_2 + 4x_3 + \frac{3}{4}x_4 - \frac{11}{4}x_6 \\
 x_7 &= 1 - 11 \left(-\frac{1}{2}x_2 + 2x_3 + \frac{1}{4}x_4 - \frac{1}{4}x_6 \right) - 5x_2 + 18x_3 + 2x_4 \\
 &= 1 + \frac{1}{2}x_2 - 4x_3 - \frac{3}{4}x_4 + \frac{11}{4}x_6 \\
 z &= 53 \left(-\frac{1}{2}x_2 + 2x_3 + \frac{1}{4}x_4 - \frac{1}{4}x_6 \right) + 41x_2 - 204x_3 - 20x_4 \\
 &= \frac{29}{2}x_2 - 98x_3 - \frac{27}{4}x_4 - \frac{53}{4}x_6
 \end{aligned}$$

$$\begin{cases}
 x_1 = 0 & - & \frac{1}{2} x_2 & + & 2 x_3 & + & \frac{1}{4} x_4 & - & \frac{1}{4} x_6 \\
 x_5 = 0 & - & \frac{1}{2} x_2 & + & 4 x_3 & + & \frac{3}{4} x_4 & - & \frac{11}{4} x_6 \\
 x_7 = 1 & + & \frac{1}{2} x_2 & - & 4 x_3 & - & \frac{3}{4} x_4 & + & \frac{11}{4} x_6 \\
 z = 0 & + & \frac{29}{2} x_2 & - & 98 x_3 & - & \frac{27}{4} x_4 & - & \frac{53}{4} x_6
 \end{cases}$$

where $\mathcal{B} = \{1, 5, 7\}$ and $\mathcal{N} = \{2, 3, 4, 6\}$. The basic feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_5 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad 0$$

- **Iteration 2**

The largest reduced cost is now the one associated with x_2 . We have:

$$\begin{cases}
 x_1 = 0 - \frac{1}{2} x_2 \\
 x_5 = 0 - \frac{1}{2} x_2 \\
 x_7 = 1 + \frac{1}{2} x_2
 \end{cases}
 \begin{cases}
 \underbrace{0 - \frac{1}{2} x_2 \geq 0}_{x_1 \geq 0} \\
 \underbrace{0 - \frac{1}{2} x_2 \geq 0}_{x_5 \geq 0} \\
 \underbrace{1 + \frac{1}{2} x_2 \geq 0}_{x_7 \geq 0}
 \end{cases}
 \begin{cases}
 x_2 \leq 0 \\
 x_2 \leq 0 \\
 x_2 \geq -2
 \end{cases}$$

By increasing the value of x_2 to 0, both x_1 and x_5 remain equal to 0. Since two pivot rows exist, we use the basic variable associated with the constraint of smallest index: x_5 is associated with constraint 1, while x_1 is associated with constraint 2 (it replaced x_6). Therefore x_5 is the pivot row.

Then, we have that:

x_2 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = 0 - \frac{1}{2}x_2 + 4x_3 + \frac{3}{4}x_4 - \frac{11}{4}x_6 \quad \text{then} \quad x_2 = -2x_5 + 8x_3 + \frac{3}{2}x_4 - \frac{11}{2}x_6.$$

Substituting in the other equations of the dictionary, we get the following dictionary:

$$\begin{aligned} x_1 &= -\frac{1}{2} \left(-2x_5 + 8x_3 + \frac{3}{2}x_4 - \frac{11}{2}x_6 \right) + 2x_3 + \frac{1}{4}x_4 - \frac{1}{4}x_6 \\ &= x_5 - 2x_3 - \frac{1}{2}x_4 + \frac{5}{2}x_6 \\ x_7 &= 1 + \frac{1}{2} \left(-2x_5 + 8x_3 + \frac{3}{2}x_4 - \frac{11}{2}x_6 \right) - 4x_3 - \frac{3}{4}x_4 + \frac{11}{4}x_6 \\ &= 1 - x_5 \\ z &= \frac{29}{2} \left(-2x_5 + 8x_3 + \frac{3}{2}x_4 - \frac{11}{2}x_6 \right) - 98x_3 - \frac{27}{4}x_4 - \frac{53}{4}x_6 \\ &= -29x_5 + 18x_3 + 15x_4 - 93x_6 \end{aligned}$$

$$\begin{cases} x_1 = 0 & + & x_5 & - & 2x_3 & - & \frac{1}{2}x_4 & + & \frac{5}{2}x_6 \\ x_2 = 0 & - & 2x_5 & + & 8x_3 & + & \frac{3}{2}x_4 & - & \frac{11}{2}x_6 \\ x_7 = 1 & - & x_5 \\ z = 0 & - & 29x_5 & + & 18x_3 & + & 15x_4 & - & 93x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 7\}$ and $\mathcal{N} = \{3, 4, 5, 6\}$. The basic feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_5 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad 0$$

• **Iteration 3**

The largest reduced cost is now the one associated with x_3 . We have:

$$\begin{cases} x_1 = 0 - 2x_3 \\ x_2 = 0 + 8x_3 \end{cases} \quad \begin{cases} \underbrace{0 - 2x_3}_{x_1 \geq 0} \geq 0 \\ \underbrace{0 + 8x_3}_{x_2 \geq 0} \geq 0 \end{cases} \quad \begin{cases} x_3 \leq 0 \\ x_3 \geq 0 \end{cases}$$

By increasing the value of x_3 to 0, the value of x_1 decreases to 0. This is again a degenerate pivot.

Then, we have that:

x_3 becomes basic and accordingly x_1 becomes non-basic

We have:

$$x_1 = 0 + x_5 - 2x_3 - \frac{1}{2}x_4 + \frac{5}{2}x_6 \quad \text{then} \quad x_3 = \frac{1}{2}x_5 - \frac{1}{2}x_1 - \frac{1}{4}x_4 + \frac{5}{4}x_6.$$

Substituting in the other equations of the dictionary, we get the following dictionary:

$$\begin{aligned} x_2 &= -2x_5 + 8 \left(\frac{1}{2}x_5 - \frac{1}{2}x_1 - \frac{1}{4}x_4 + \frac{5}{4}x_6 \right) + \frac{3}{2}x_4 - \frac{11}{2}x_6 \\ &= 2x_5 - 4x_1 - \frac{1}{2}x_4 + \frac{9}{2}x_6 \\ z &= -29x_5 + 18 \left(\frac{1}{2}x_5 - \frac{1}{2}x_1 - \frac{1}{4}x_4 + \frac{5}{4}x_6 \right) + 15x_4 - 93x_6 \\ &= -20x_5 - 9x_1 + \frac{21}{2}x_4 - \frac{141}{2}x_6 \end{aligned}$$

$$\begin{cases} x_2 = 0 & + & 2x_5 & - & 4x_1 & - & \frac{1}{2}x_4 & + & \frac{9}{2}x_6 \\ x_3 = 0 & + & \frac{1}{2}x_5 & - & \frac{1}{2}x_1 & - & \frac{1}{4}x_4 & + & \frac{5}{4}x_6 \\ x_7 = 1 & - & x_5 \\ z = 0 & - & 20x_5 & - & 9x_1 & + & \frac{21}{2}x_4 & - & \frac{141}{2}x_6 \end{cases}$$

where $\mathcal{B} = \{2, 3, 7\}$ and $\mathcal{N} = \{1, 4, 5, 6\}$. The basic feasible solution is:

$$\bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_1 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_5 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad 0$$

- **Iteration 4**

The largest reduced cost is now the one associated with x_4 . We have:

$$\begin{cases} x_2 = 0 - \frac{1}{2}x_4 \\ x_3 = 0 - \frac{1}{4}x_4 \end{cases} \quad \begin{cases} \underbrace{0 - \frac{1}{2}x_4 \geq 0}_{x_2 \geq 0} \\ \underbrace{0 - \frac{1}{4}x_4 \geq 0}_{x_3 \geq 0} \end{cases} \quad \begin{cases} x_4 \leq 0 \\ x_4 \leq 0 \end{cases}$$

By increasing the value of x_4 to 0, both x_2 and x_3 remain equal to 0. The basic variable x_2 is associated with constraint 1 (it replaced x_5), while x_3 is associated with constraint 2 (it replaced x_1). Therefore x_2 is the pivot row.

Then, we have that:

x_4 becomes basic and accordingly x_2 becomes non-basic

We have:

$$x_2 = 0 + 2x_5 - 4x_1 - \frac{1}{2}x_4 + \frac{9}{2}x_6 \quad \text{then} \quad x_4 = 4x_5 - 8x_1 - 2x_2 + 9x_6.$$

Substituting in the other equations of the dictionary, we get the following dictionary:

$$\begin{aligned} x_3 &= \frac{1}{2}x_5 - \frac{1}{2}x_1 - \frac{1}{4}(4x_5 - 8x_1 - 2x_2 + 9x_6) + \frac{5}{4}x_6 \\ &= -\frac{1}{2}x_5 + \frac{3}{2}x_1 + \frac{1}{2}x_2 - x_6 \\ z &= -20x_5 - 9x_1 + \frac{21}{2}(4x_5 - 8x_1 - 2x_2 + 9x_6) - \frac{141}{2}x_6 \\ &= 22x_5 - 93x_1 - 21x_2 + 24x_6 \end{aligned}$$

$$\left\{ \begin{array}{l} x_3 = 0 - \frac{1}{2}x_5 + \frac{3}{2}x_1 + \frac{1}{2}x_2 - x_6 \\ x_4 = 0 + 4x_5 - 8x_1 - 2x_2 + 9x_6 \\ x_7 = 1 - x_5 \\ z = 0 + 22x_5 - 93x_1 - 21x_2 + 24x_6 \end{array} \right.$$

where $\mathcal{B} = \{3, 4, 7\}$ and $\mathcal{N} = \{1, 2, 5, 6\}$. The basic feasible solution is:

$$\bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_5 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad 0$$

• **Iteration 5**

The largest reduced cost is now the one associated with x_6 . We have:

$$\left\{ \begin{array}{l} x_3 = 0 - x_6 \\ x_4 = 0 + 9x_6 \end{array} \right. \quad \left\{ \begin{array}{l} 0 - x_6 \geq 0 \\ x_3 \geq 0 \\ 0 + 9x_6 \geq 0 \\ x_4 \geq 0 \end{array} \right. \quad \left\{ \begin{array}{l} x_6 \leq 0 \\ x_6 \geq 0 \end{array} \right.$$

By increasing the value of x_6 to 0, the value of x_3 decreases to 0. This is a degenerate pivot.

Then, we have that:

x_6 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = 0 - \frac{1}{2}x_5 + \frac{3}{2}x_1 + \frac{1}{2}x_2 - x_6 \quad \text{then} \quad x_6 = -\frac{1}{2}x_5 + \frac{3}{2}x_1 + \frac{1}{2}x_2 - x_3.$$

Substituting in the other equations of the dictionary, we get the following dictionary:

$$\begin{aligned}
 x_4 &= 4x_5 - 8x_1 - 2x_2 + 9 \left(-\frac{1}{2}x_5 + \frac{3}{2}x_1 + \frac{1}{2}x_2 - x_3 \right) \\
 &= -\frac{1}{2}x_5 + \frac{11}{2}x_1 + \frac{5}{2}x_2 - 9x_3 \\
 z &= 22x_5 - 93x_1 - 21x_2 + 24 \left(-\frac{1}{2}x_5 + \frac{3}{2}x_1 + \frac{1}{2}x_2 - x_3 \right) \\
 &= 10x_5 - 57x_1 - 9x_2 - 24x_3
 \end{aligned}$$

$$\begin{cases}
 x_4 = 0 & - & \frac{1}{2} & x_5 & + & \frac{11}{2} & x_1 & + & \frac{5}{2} & x_2 & - & 9 & x_3 \\
 x_6 = 0 & - & \frac{1}{2} & x_5 & + & \frac{3}{2} & x_1 & + & \frac{1}{2} & x_2 & - & & x_3 \\
 x_7 = 1 & - & & x_5 & & & & & & & & & \\
 z = 0 & + & 10 & x_5 & - & 57 & x_1 & - & 9 & x_2 & - & 24 & x_3
 \end{cases}$$

where $\mathcal{B} = \{4, 6, 7\}$ and $\mathcal{N} = \{1, 2, 3, 5\}$. The basic feasible solution is:

$$\bar{x}_4 = 0, \quad \bar{x}_6 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_5 = 0 \quad \text{of value } 0$$

• Iteration 6

The largest reduced cost is now the one associated with x_5 . We have:

$$\begin{cases}
 x_4 = 0 - \frac{1}{2} x_5 \\
 x_6 = 0 - \frac{1}{2} x_5 \\
 x_7 = 1 - x_5
 \end{cases}
 \quad
 \begin{cases}
 \underbrace{0 - \frac{1}{2} x_5 \geq 0}_{x_4 \geq 0} \\
 \underbrace{0 - \frac{1}{2} x_5 \geq 0}_{x_6 \geq 0} \\
 \underbrace{1 - x_5 \geq 0}_{x_7 \geq 0}
 \end{cases}
 \quad
 \begin{cases}
 x_5 \leq 0 \\
 x_5 \leq 0 \\
 x_5 \leq 1
 \end{cases}$$

By increasing the value of x_5 to 0, both x_4 and x_6 remain equal to 0. The basic variable x_4 is associated with constraint 1 (it replaced x_2), while x_6 is associated with constraint 2 (it replaced x_3). Therefore x_4 is the pivot row.

Then, we have that:

x_5 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 0 - \frac{1}{2} x_5 + \frac{11}{2} x_1 + \frac{5}{2} x_2 - 9 x_3 \quad \text{then} \quad x_5 = 11 x_1 + 5 x_2 - 18 x_3 - 2 x_4.$$

Substituting in the other equations of the dictionary, we get the following dictionary:

$$x_6 = -\frac{1}{2}(11x_1 + 5x_2 - 18x_3 - 2x_4) + \frac{3}{2}x_1 + \frac{1}{2}x_2 - x_3$$

$$= -4x_1 - 2x_2 + 8x_3 + x_4$$

$$x_7 = 1 - (11x_1 + 5x_2 - 18x_3 - 2x_4)$$

$$= 1 - 11x_1 - 5x_2 + 18x_3 + 2x_4$$

$$z = 10(11x_1 + 5x_2 - 18x_3 - 2x_4) - 57x_1 - 9x_2 - 24x_3$$

$$= 53x_1 + 41x_2 - 204x_3 - 20x_4$$

$$\left\{ \begin{array}{l} x_5 = 0 \quad + \quad 11x_1 \quad + \quad 5x_2 \quad - \quad 18x_3 \quad - \quad 2x_4 \\ x_6 = 0 \quad - \quad 4x_1 \quad - \quad 2x_2 \quad + \quad 8x_3 \quad + \quad x_4 \\ x_7 = 1 \quad - \quad 11x_1 \quad - \quad 5x_2 \quad + \quad 18x_3 \quad + \quad 2x_4 \\ z = 0 \quad + \quad 53x_1 \quad + \quad 41x_2 \quad - \quad 204x_3 \quad - \quad 20x_4 \end{array} \right.$$

where $\mathcal{B} = \{5, 6, 7\}$ and $\mathcal{N} = \{1, 2, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_5 = 0, \quad \bar{x}_6 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 0$$

This dictionary is the same as the initial one and the simplex algorithm cycles.

2. We now execute the simplex algorithm by using Bland's pivot rule. Thus, among the non-basic variables with positive reduced cost, we choose the one with the smallest index. If more than one leaving variable is possible, we choose the basic variable with the smallest index.

- **Iteration 1**

At the initial dictionary, the positive reduced costs are associated with x_1 and x_2 . Bland's rule selects x_1 . As in the previous execution, the ratio test gives x_6 as leaving variable.

Then, we have that:

x_1 becomes basic and accordingly x_6 becomes non-basic

We have:

$$x_6 = 0 - 4x_1 - 2x_2 + 8x_3 + x_4 \quad \text{then} \quad x_1 = -\frac{1}{2}x_2 + 2x_3 + \frac{1}{4}x_4 - \frac{1}{4}x_6.$$

Substituting in the other equations of the dictionary, we get the following dictionary:

$$\begin{aligned}
 x_5 &= 11 \left(-\frac{1}{2}x_2 + 2x_3 + \frac{1}{4}x_4 - \frac{1}{4}x_6 \right) + 5x_2 - 18x_3 - 2x_4 \\
 &= -\frac{1}{2}x_2 + 4x_3 + \frac{3}{4}x_4 - \frac{11}{4}x_6 \\
 x_7 &= 1 - 11 \left(-\frac{1}{2}x_2 + 2x_3 + \frac{1}{4}x_4 - \frac{1}{4}x_6 \right) - 5x_2 + 18x_3 + 2x_4 \\
 &= 1 + \frac{1}{2}x_2 - 4x_3 - \frac{3}{4}x_4 + \frac{11}{4}x_6 \\
 z &= 53 \left(-\frac{1}{2}x_2 + 2x_3 + \frac{1}{4}x_4 - \frac{1}{4}x_6 \right) + 41x_2 - 204x_3 - 20x_4 \\
 &= \frac{29}{2}x_2 - 98x_3 - \frac{27}{4}x_4 - \frac{53}{4}x_6
 \end{aligned}$$

$$\begin{cases}
 x_1 = 0 & - & \frac{1}{2} x_2 & + & 2 x_3 & + & \frac{1}{4} x_4 & - & \frac{1}{4} x_6 \\
 x_5 = 0 & - & \frac{1}{2} x_2 & + & 4 x_3 & + & \frac{3}{4} x_4 & - & \frac{11}{4} x_6 \\
 x_7 = 1 & + & \frac{1}{2} x_2 & - & 4 x_3 & - & \frac{3}{4} x_4 & + & \frac{11}{4} x_6 \\
 z = 0 & + & \frac{29}{2} x_2 & - & 98 x_3 & - & \frac{27}{4} x_4 & - & \frac{53}{4} x_6
 \end{cases}$$

where $\mathcal{B} = \{1, 5, 7\}$ and $\mathcal{N} = \{2, 3, 4, 6\}$. The basic feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_5 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad 0$$

• Iteration 2

The only positive reduced cost is the one associated with x_2 . We have:

$$\begin{cases}
 x_1 = 0 - \frac{1}{2} x_2 \\
 x_5 = 0 - \frac{1}{2} x_2 \\
 x_7 = 1 + \frac{1}{2} x_2
 \end{cases}
 \begin{cases}
 \underbrace{0 - \frac{1}{2} x_2 \geq 0}_{x_1 \geq 0} \\
 \underbrace{0 - \frac{1}{2} x_2 \geq 0}_{x_5 \geq 0} \\
 \underbrace{1 + \frac{1}{2} x_2 \geq 0}_{x_7 \geq 0}
 \end{cases}
 \begin{cases}
 x_2 \leq 0 \\
 x_2 \leq 0 \\
 x_2 \geq -2
 \end{cases}$$

By increasing the value of x_2 to 0, both x_1 and x_5 remain equal to 0. Bland's rule selects x_1 as leaving variable, since it has the smallest index among the possible leaving variables.

Then, we have that:

x_2 becomes basic and accordingly x_1 becomes non-basic

We have:

$$x_1 = 0 - \frac{1}{2}x_2 + 2x_3 + \frac{1}{4}x_4 - \frac{1}{4}x_6 \quad \text{then} \quad x_2 = -2x_1 + 4x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_6.$$

Substituting in the other equations of the dictionary, we get the following dictionary:

$$\begin{aligned} x_5 &= -\frac{1}{2} \left(-2x_1 + 4x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_6 \right) + 4x_3 + \frac{3}{4}x_4 - \frac{11}{4}x_6 \\ &= x_1 + 2x_3 + \frac{1}{2}x_4 - \frac{5}{2}x_6 \\ x_7 &= 1 + \frac{1}{2} \left(-2x_1 + 4x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_6 \right) - 4x_3 - \frac{3}{4}x_4 + \frac{11}{4}x_6 \\ &= 1 - x_1 - 2x_3 - \frac{1}{2}x_4 + \frac{5}{2}x_6 \\ z &= \frac{29}{2} \left(-2x_1 + 4x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_6 \right) - 98x_3 - \frac{27}{4}x_4 - \frac{53}{4}x_6 \\ &= -29x_1 - 40x_3 + \frac{1}{2}x_4 - \frac{41}{2}x_6 \end{aligned}$$

$$\left\{ \begin{array}{l} x_2 = 0 - 2x_1 + 4x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_6 \\ x_5 = 0 + x_1 + 2x_3 + \frac{1}{2}x_4 - \frac{5}{2}x_6 \\ x_7 = 1 - x_1 - 2x_3 - \frac{1}{2}x_4 + \frac{5}{2}x_6 \\ z = 0 - 29x_1 - 40x_3 + \frac{1}{2}x_4 - \frac{41}{2}x_6 \end{array} \right.$$

where $\mathcal{B} = \{2, 5, 7\}$ and $\mathcal{N} = \{1, 3, 4, 6\}$. The basic feasible solution is:

$$\bar{x}_2 = 0, \quad \bar{x}_5 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_1 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad 0$$

• Iteration 3

The only positive reduced cost is the one associated with x_4 . We have:

$$\left\{ \begin{array}{l} x_2 = 0 + \frac{1}{2}x_4 \\ x_5 = 0 + \frac{1}{2}x_4 \\ x_7 = 1 - \frac{1}{2}x_4 \end{array} \right. \quad \left\{ \begin{array}{l} \underbrace{0 + \frac{1}{2}x_4 \geq 0}_{x_2 \geq 0} \\ \underbrace{0 + \frac{1}{2}x_4 \geq 0}_{x_5 \geq 0} \\ \underbrace{1 - \frac{1}{2}x_4 \geq 0}_{x_7 \geq 0} \end{array} \right. \quad \left\{ \begin{array}{l} x_4 \geq 0 \\ x_4 \geq 0 \\ x_4 \leq 2 \end{array} \right.$$

By increasing the value of x_4 to 2, the value of x_7 decreases to 0.

Then, we have that:

x_4 becomes basic and accordingly x_7 becomes non-basic

We have:

$$x_7 = 1 - x_1 - 2x_3 - \frac{1}{2}x_4 + \frac{5}{2}x_6 \quad \text{then} \quad x_4 = 2 - 2x_1 - 4x_3 + 5x_6 - 2x_7.$$

Substituting in the other equations of the dictionary, we obtain an optimal dictionary:

$$\begin{aligned} x_2 &= -2x_1 + 4x_3 + \frac{1}{2}(2 - 2x_1 - 4x_3 + 5x_6 - 2x_7) - \frac{1}{2}x_6 \\ &= 1 - 3x_1 + 2x_3 + 2x_6 - x_7 \\ x_5 &= x_1 + 2x_3 + \frac{1}{2}(2 - 2x_1 - 4x_3 + 5x_6 - 2x_7) - \frac{5}{2}x_6 \\ &= 1 - x_7 \\ z &= -29x_1 - 40x_3 + \frac{1}{2}(2 - 2x_1 - 4x_3 + 5x_6 - 2x_7) - \frac{41}{2}x_6 \\ &= 1 - 30x_1 - 42x_3 - 18x_6 - x_7 \end{aligned}$$

$$\left\{ \begin{array}{l} x_2 = 1 - 3x_1 + 2x_3 + 2x_6 - x_7 \\ x_4 = 2 - 2x_1 - 4x_3 + 5x_6 - 2x_7 \\ x_5 = 1 - x_7 \\ z = 1 - 30x_1 - 42x_3 - 18x_6 - x_7 \end{array} \right.$$

where $\mathcal{B} = \{2, 4, 5\}$ and $\mathcal{N} = \{1, 3, 6, 7\}$. Since all the reduced costs are non-positive, no further improvement is possible. The basic optimal solution is:

$$\tilde{x}_2 = 1, \quad \tilde{x}_4 = 2, \quad \tilde{x}_5 = 1, \quad \tilde{x}_1 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_6 = 0, \quad \tilde{x}_7 = 0 \quad \text{of value} \quad 1$$

Exercise 7.2 — Simplex algorithm (C2)

Consider the following LP problem in canonical maximization form with $n = 4$ variables and $m = 3$ constraints:

$$\begin{aligned} \max \quad & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} \quad & \frac{1}{2}x_1 - \frac{11}{2}x_2 - \frac{5}{2}x_3 + 9x_4 \leq 0 \\ & \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 + x_4 \leq 0 \\ & x_1 \leq 1 \\ & x_1, \quad x_2, \quad x_3, \quad x_4 \geq 0 \end{aligned}$$

By introducing the slack variables, we get the following starting dictionary:

$$\left\{ \begin{array}{l} x_5 = 0 - \frac{1}{2} x_1 + \frac{11}{2} x_2 + \frac{5}{2} x_3 - 9 x_4 \\ x_6 = 0 - \frac{1}{2} x_1 + \frac{3}{2} x_2 + \frac{1}{2} x_3 - x_4 \\ x_7 = 1 - x_1 \\ z = 0 + 10 x_1 - 57 x_2 - 9 x_3 - 24 x_4 \end{array} \right.$$

The associated basic feasible solution is:

$$\bar{x}_5 = 0, \bar{x}_6 = 0, \bar{x}_7 = 1, \quad \bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 0, \bar{x}_4 = 0 \quad \text{of value } 0$$

Questions

1. Execute the simplex algorithm to find an optimal solution of this LP problem. Choose the pivot column as the one with the largest reduced cost. When two or more pivot rows exist, use the one with the smallest index.
2. Execute the simplex algorithm to find an optimal solution of this LP problem. Choose the pivot column and the pivot row using the Bland's pivot rule.

Solution

1. • Initialization

Consider the starting dictionary:

$$\left\{ \begin{array}{l} x_5 = 0 - \frac{1}{2} x_1 + \frac{11}{2} x_2 + \frac{5}{2} x_3 - 9 x_4 \\ x_6 = 0 - \frac{1}{2} x_1 + \frac{3}{2} x_2 + \frac{1}{2} x_3 - x_4 \\ x_7 = 1 - x_1 \\ z = 0 + 10 x_1 - 57 x_2 - 9 x_3 - 24 x_4 \end{array} \right.$$

where $\mathcal{B} = \{5, 6, 7\}$ and $\mathcal{N} = \{1, 2, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_5 = 0, \bar{x}_6 = 0, \bar{x}_7 = 1, \quad \bar{x}_1 = 0, \bar{x}_2 = 0, \bar{x}_3 = 0, \bar{x}_4 = 0 \quad \text{of value } 0$$

• Iteration 1

The largest reduced cost is the one associated with the non-basic variable x_1 . Thus, if we increase the value of the variable x_1 , we can get a better solution.

We have:

$$\begin{cases} x_5 = 0 - \frac{1}{2} x_1 \\ x_6 = 0 - \frac{1}{2} x_1 \\ x_7 = 1 - x_1 \end{cases} \quad \begin{cases} \underbrace{0 - \frac{1}{2} x_1 \geq 0}_{x_5 \geq 0} \\ \underbrace{0 - \frac{1}{2} x_1 \geq 0}_{x_6 \geq 0} \\ \underbrace{1 - x_1 \geq 0}_{x_7 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq 0 \\ x_1 \leq 0 \\ x_1 \leq 1 \end{cases}$$

By increasing the value of x_1 to 0, both x_5 and x_6 remain equal to 0. Since two pivot rows exist, we use the one with the smallest index, and therefore x_5 is the pivot row. This is a degenerate pivot, since the value of the objective function does not increase.

Then, we have that:

x_1 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = -\frac{1}{2}x_1 + \frac{11}{2}x_2 + \frac{5}{2}x_3 - 9x_4 \quad \text{then} \quad x_1 = 11x_2 + 5x_3 - 18x_4 - 2x_5.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_6 &= -\frac{1}{2}(11x_2 + 5x_3 - 18x_4 - 2x_5) + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4 \\ &= -4x_2 - 2x_3 + 8x_4 + x_5 \\ x_7 &= 1 - (11x_2 + 5x_3 - 18x_4 - 2x_5) = 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5 \\ z &= 10(11x_2 + 5x_3 - 18x_4 - 2x_5) - 57x_2 - 9x_3 - 24x_4 \\ &= 53x_2 + 41x_3 - 204x_4 - 20x_5 \end{aligned}$$

The resulting dictionary is:

$$\begin{cases} x_1 = 0 & + & 11 & x_2 & + & 5 & x_3 & - & 18 & x_4 & - & 2 & x_5 \\ x_6 = 0 & - & 4 & x_2 & - & 2 & x_3 & + & 8 & x_4 & + & & x_5 \\ x_7 = 1 & - & 11 & x_2 & - & 5 & x_3 & + & 18 & x_4 & + & 2 & x_5 \\ z = 0 & + & 53 & x_2 & + & 41 & x_3 & - & 204 & x_4 & - & 20 & x_5 \end{cases}$$

where $\mathcal{B} = \{1, 6, 7\}$ and $\mathcal{N} = \{2, 3, 4, 5\}$. The basic feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_6 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value} \quad 0$$

• **Iteration 2**

The largest reduced cost is now the one associated with x_2 . We have:

$$\begin{cases} x_1 = 0 + 11x_2 \\ x_6 = 0 - 4x_2 \\ x_7 = 1 - 11x_2 \end{cases} \quad \begin{cases} \underbrace{0 + 11x_2 \geq 0}_{x_1 \geq 0} \\ \underbrace{0 - 4x_2 \geq 0}_{x_6 \geq 0} \\ \underbrace{1 - 11x_2 \geq 0}_{x_7 \geq 0} \end{cases} \quad \begin{cases} x_2 \geq 0 \\ x_2 \leq 0 \\ x_2 \leq \frac{1}{11} \end{cases}$$

By increasing the value of x_2 to $\min\left\{0, \frac{1}{11}\right\} = 0$, the value of x_6 remains equal to 0, and x_6 is the only pivot row. This is again a degenerate pivot.

Then, we have that:

x_2 becomes basic and accordingly x_6 becomes non-basic

We have:

$$x_6 = -4x_2 - 2x_3 + 8x_4 + x_5 \quad \text{then} \quad x_2 = -\frac{1}{2}x_3 + 2x_4 + \frac{1}{4}x_5 - \frac{1}{4}x_6.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_1 &= 11 \left(-\frac{1}{2}x_3 + 2x_4 + \frac{1}{4}x_5 - \frac{1}{4}x_6 \right) + 5x_3 - 18x_4 - 2x_5 \\ &= -\frac{1}{2}x_3 + 4x_4 + \frac{3}{4}x_5 - \frac{11}{4}x_6 \\ x_7 &= 1 - 11 \left(-\frac{1}{2}x_3 + 2x_4 + \frac{1}{4}x_5 - \frac{1}{4}x_6 \right) - 5x_3 + 18x_4 + 2x_5 \\ &= 1 + \frac{1}{2}x_3 - 4x_4 - \frac{3}{4}x_5 + \frac{11}{4}x_6 \\ z &= 53 \left(-\frac{1}{2}x_3 + 2x_4 + \frac{1}{4}x_5 - \frac{1}{4}x_6 \right) + 41x_3 - 204x_4 - 20x_5 \\ &= \frac{29}{2}x_3 - 98x_4 - \frac{27}{4}x_5 - \frac{53}{4}x_6 \end{aligned}$$

The resulting dictionary is:

$$\begin{cases} x_1 = 0 & - & \frac{1}{2}x_3 & + & 4x_4 & + & \frac{3}{4}x_5 & - & \frac{11}{4}x_6 \\ x_2 = 0 & - & \frac{1}{2}x_3 & + & 2x_4 & + & \frac{1}{4}x_5 & - & \frac{1}{4}x_6 \\ x_7 = 1 & + & \frac{1}{2}x_3 & - & 4x_4 & - & \frac{3}{4}x_5 & + & \frac{11}{4}x_6 \\ z = 0 & + & \frac{29}{2}x_3 & - & 98x_4 & - & \frac{27}{4}x_5 & - & \frac{53}{4}x_6 \end{cases}$$

where $\mathcal{B} = \{1, 2, 7\}$ and $\mathcal{N} = \{3, 4, 5, 6\}$. The basic feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_5 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad 0$$

• **Iteration 3**

The only positive reduced cost is the one associated with x_3 . We have:

$$\begin{cases} x_1 = 0 - \frac{1}{2} x_3 \\ x_2 = 0 - \frac{1}{2} x_3 \\ x_7 = 1 + \frac{1}{2} x_3 \end{cases} \quad \begin{cases} \underbrace{0 - \frac{1}{2} x_3 \geq 0}_{x_1 \geq 0} \\ \underbrace{0 - \frac{1}{2} x_3 \geq 0}_{x_2 \geq 0} \\ \underbrace{1 + \frac{1}{2} x_3 \geq 0}_{x_7 \geq 0} \end{cases} \quad \begin{cases} x_3 \leq 0 \\ x_3 \leq 0 \\ x_3 \geq -2 \end{cases}$$

By increasing the value of x_3 to 0, both x_1 and x_2 remain equal to 0. Since two pivot rows exist, we use the one with the smallest index, and therefore x_1 is the pivot row.

Then, we have that:

x_3 becomes basic and accordingly x_1 becomes non-basic

We have:

$$x_1 = -\frac{1}{2}x_3 + 4x_4 + \frac{3}{4}x_5 - \frac{11}{4}x_6 \quad \text{then} \quad x_3 = -2x_1 + 8x_4 + \frac{3}{2}x_5 - \frac{11}{2}x_6.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_2 &= -\frac{1}{2} \left(-2x_1 + 8x_4 + \frac{3}{2}x_5 - \frac{11}{2}x_6 \right) + 2x_4 + \frac{1}{4}x_5 - \frac{1}{4}x_6 \\ &= x_1 - 2x_4 - \frac{1}{2}x_5 + \frac{5}{2}x_6 \\ z &= \frac{29}{2} \left(-2x_1 + 8x_4 + \frac{3}{2}x_5 - \frac{11}{2}x_6 \right) - 98x_4 - \frac{27}{4}x_5 - \frac{53}{4}x_6 \\ &= -29x_1 + 18x_4 + 15x_5 - 93x_6 \end{aligned}$$

The resulting dictionary is:

$$\begin{cases} x_2 = 0 & + & x_1 & - & 2x_4 & - & \frac{1}{2}x_5 & + & \frac{5}{2}x_6 \\ x_3 = 0 & - & 2x_1 & + & 8x_4 & + & \frac{3}{2}x_5 & - & \frac{11}{2}x_6 \\ x_7 = 1 & - & x_1 & & & & & & \\ z = 0 & - & 29x_1 & + & 18x_4 & + & 15x_5 & - & 93x_6 \end{cases}$$

where $\mathcal{B} = \{2, 3, 7\}$ and $\mathcal{N} = \{1, 4, 5, 6\}$. The basic feasible solution is:

$$\bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_1 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_5 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad 0$$

• **Iteration 4**

The largest reduced cost is now the one associated with x_4 . We have:

$$\begin{cases} x_2 = 0 - 2x_4 \\ x_3 = 0 + 8x_4 \end{cases} \quad \begin{cases} \underbrace{0 - 2x_4 \geq 0}_{x_2 \geq 0} \\ \underbrace{0 + 8x_4 \geq 0}_{x_3 \geq 0} \end{cases} \quad \begin{cases} x_4 \leq 0 \\ x_4 \geq 0 \end{cases}$$

By increasing the value of x_4 to 0, the value of x_2 remains equal to 0, and x_2 is the only pivot row (x_7 does not depend on x_4).

Then, we have that:

x_4 becomes basic and accordingly x_2 becomes non-basic

We have:

$$x_2 = x_1 - 2x_4 - \frac{1}{2}x_5 + \frac{5}{2}x_6 \quad \text{then} \quad x_4 = \frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{4}x_5 + \frac{5}{4}x_6.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_3 &= -2x_1 + 8 \left(\frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{4}x_5 + \frac{5}{4}x_6 \right) + \frac{3}{2}x_5 - \frac{11}{2}x_6 \\ &= 2x_1 - 4x_2 - \frac{1}{2}x_5 + \frac{9}{2}x_6 \\ z &= -29x_1 + 18 \left(\frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{4}x_5 + \frac{5}{4}x_6 \right) + 15x_5 - 93x_6 \\ &= -20x_1 - 9x_2 + \frac{21}{2}x_5 - \frac{141}{2}x_6 \end{aligned}$$

The resulting dictionary is:

$$\begin{cases} x_3 = 0 & + & 2x_1 & - & 4x_2 & - & \frac{1}{2}x_5 & + & \frac{9}{2}x_6 \\ x_4 = 0 & + & \frac{1}{2}x_1 & - & \frac{1}{2}x_2 & - & \frac{1}{4}x_5 & + & \frac{5}{4}x_6 \\ x_7 = 1 & - & x_1 \\ z = 0 & - & 20x_1 & - & 9x_2 & + & \frac{21}{2}x_5 & - & \frac{141}{2}x_6 \end{cases}$$

where $\mathcal{B} = \{3, 4, 7\}$ and $\mathcal{N} = \{1, 2, 5, 6\}$. The basic feasible solution is:

$$\bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_5 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad 0$$

• **Iteration 5**

The only positive reduced cost is the one associated with x_5 . We have:

$$\begin{cases} x_3 = 0 - \frac{1}{2} x_5 \\ x_4 = 0 - \frac{1}{4} x_5 \end{cases} \quad \begin{cases} \underbrace{0 - \frac{1}{2} x_5}_{x_3 \geq 0} \geq 0 \\ \underbrace{0 - \frac{1}{4} x_5}_{x_4 \geq 0} \geq 0 \end{cases} \quad \begin{cases} x_5 \leq 0 \\ x_5 \leq 0 \end{cases}$$

By increasing the value of x_5 to 0, both x_3 and x_4 remain equal to 0 (x_7 does not depend on x_5). Since two pivot rows exist, we use the one with the smallest index, and therefore x_3 is the pivot row.

Then, we have that:

x_5 becomes basic and accordingly x_3 becomes non-basic

We have:

$$x_3 = 2x_1 - 4x_2 - \frac{1}{2}x_5 + \frac{9}{2}x_6 \quad \text{then} \quad x_5 = 4x_1 - 8x_2 - 2x_3 + 9x_6.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_4 &= \frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{4}(4x_1 - 8x_2 - 2x_3 + 9x_6) + \frac{5}{4}x_6 \\ &= -\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_6 \\ z &= -20x_1 - 9x_2 + \frac{21}{2}(4x_1 - 8x_2 - 2x_3 + 9x_6) - \frac{141}{2}x_6 \\ &= 22x_1 - 93x_2 - 21x_3 + 24x_6 \end{aligned}$$

The resulting dictionary is:

$$\begin{cases} x_4 = 0 & - & \frac{1}{2} x_1 & + & \frac{3}{2} x_2 & + & \frac{1}{2} x_3 & - & x_6 \\ x_5 = 0 & + & 4 x_1 & - & 8 x_2 & - & 2 x_3 & + & 9 x_6 \\ x_7 = 1 & - & x_1 \\ z = 0 & + & 22 x_1 & - & 93 x_2 & - & 21 x_3 & + & 24 x_6 \end{cases}$$

where $\mathcal{B} = \{4, 5, 7\}$ and $\mathcal{N} = \{1, 2, 3, 6\}$. The basic feasible solution is:

$$\bar{x}_4 = 0, \quad \bar{x}_5 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad 0$$

• Iteration 6

The largest reduced cost is now the one associated with x_6 . We have:

$$\begin{cases} x_4 = 0 - x_6 \\ x_5 = 0 + 9 x_6 \end{cases} \quad \begin{cases} \underbrace{0 - x_6}_{x_4 \geq 0} \geq 0 \\ \underbrace{0 + 9 x_6}_{x_5 \geq 0} \geq 0 \end{cases} \quad \begin{cases} x_6 \leq 0 \\ x_6 \geq 0 \end{cases}$$

By increasing the value of x_6 to 0, the value of x_4 remains equal to 0, and x_4 is the only pivot row (x_7 does not depend on x_6).

Then, we have that:

x_6 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = -\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_6 \quad \text{then} \quad x_6 = -\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_5 &= 4x_1 - 8x_2 - 2x_3 + 9 \left(-\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4 \right) \\ &= -\frac{1}{2}x_1 + \frac{11}{2}x_2 + \frac{5}{2}x_3 - 9x_4 \\ z &= 22x_1 - 93x_2 - 21x_3 + 24 \left(-\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_4 \right) \\ &= 10x_1 - 57x_2 - 9x_3 - 24x_4 \end{aligned}$$

The resulting dictionary is:

$$\begin{cases} x_5 = 0 & - & \frac{1}{2} x_1 & + & \frac{11}{2} x_2 & + & \frac{5}{2} x_3 & - & 9 x_4 \\ x_6 = 0 & - & \frac{1}{2} x_1 & + & \frac{3}{2} x_2 & + & \frac{1}{2} x_3 & - & x_4 \\ x_7 = 1 & - & x_1 \\ z = 0 & + & 10 x_1 & - & 57 x_2 & - & 9 x_3 & - & 24 x_4 \end{cases}$$

where $\mathcal{B} = \{5, 6, 7\}$ and $\mathcal{N} = \{1, 2, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_5 = 0, \quad \bar{x}_6 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad 0$$

This dictionary is the same as the initial one and the simplex algorithm cycles.

2. We now execute the simplex algorithm by using Bland's pivot rule. Thus, among the non-basic variables with positive reduced cost, we choose the one with the smallest index. If more than one leaving variable is possible, we choose the basic variable with the smallest index.

Iterations 1–5 are the same as in the previous execution: in each of them, the non-basic variable with the largest reduced cost is also the one with the smallest index among those with positive reduced cost. We start from the dictionary obtained at Iteration 5, where $\mathcal{B} = \{4, 5, 7\}$ and $\mathcal{N} = \{1, 2, 3, 6\}$.

• **Iteration 6**

At this dictionary, the positive reduced costs are associated with x_1 and x_6 . Bland's rule selects x_1 . We have:

$$\begin{cases} x_4 = 0 - \frac{1}{2} x_1 \\ x_5 = 0 + 4 x_1 \\ x_7 = 1 - x_1 \end{cases} \quad \begin{cases} \underbrace{0 - \frac{1}{2} x_1 \geq 0}_{x_4 \geq 0} \\ \underbrace{0 + 4 x_1 \geq 0}_{x_5 \geq 0} \\ \underbrace{1 - x_1 \geq 0}_{x_7 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq 0 \\ x_1 \geq 0 \\ x_1 \leq 1 \end{cases}$$

By increasing the value of x_1 to $\min\{0, 1\} = 0$, the value of x_4 remains equal to 0, and x_4 is the only pivot row. This is again a degenerate pivot.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = -\frac{1}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 - x_6 \quad \text{then} \quad x_1 = 3x_2 + x_3 - 2x_4 - 2x_6.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_5 &= 4(3x_2 + x_3 - 2x_4 - 2x_6) - 8x_2 - 2x_3 + 9x_6 \\ &= 4x_2 + 2x_3 - 8x_4 + x_6 \\ x_7 &= 1 - (3x_2 + x_3 - 2x_4 - 2x_6) = 1 - 3x_2 - x_3 + 2x_4 + 2x_6 \\ z &= 22(3x_2 + x_3 - 2x_4 - 2x_6) - 93x_2 - 21x_3 + 24x_6 \\ &= -27x_2 + x_3 - 44x_4 - 20x_6 \end{aligned}$$

The resulting dictionary is:

$$\begin{cases} x_1 = 0 + 3x_2 + x_3 - 2x_4 - 2x_6 \\ x_5 = 0 + 4x_2 + 2x_3 - 8x_4 + x_6 \\ x_7 = 1 - 3x_2 - x_3 + 2x_4 + 2x_6 \\ z = 0 - 27x_2 + x_3 - 44x_4 - 20x_6 \end{cases}$$

where $\mathcal{B} = \{1, 5, 7\}$ and $\mathcal{N} = \{2, 3, 4, 6\}$. The basic feasible solution is:

$$\bar{x}_1 = 0, \quad \bar{x}_5 = 0, \quad \bar{x}_7 = 1, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_6 = 0 \quad \text{of value} \quad 0$$

• **Iteration 7**

The only positive reduced cost is the one associated with x_3 . We have:

$$\begin{cases} x_1 = 0 + x_3 \\ x_5 = 0 + 2x_3 \\ x_7 = 1 - x_3 \end{cases} \quad \begin{cases} 0 + x_3 \geq 0 \\ x_1 \geq 0 \\ 0 + 2x_3 \geq 0 \\ x_5 \geq 0 \\ 1 - x_3 \geq 0 \\ x_7 \geq 0 \end{cases} \quad \begin{cases} x_3 \geq 0 \\ x_3 \geq 0 \\ x_3 \leq 1 \end{cases}$$

By increasing the value of x_3 to 1, the value of x_7 decreases to 0.

Then, we have that:

x_3 becomes basic and accordingly x_7 becomes non-basic

We have:

$$x_7 = 1 - 3x_2 - x_3 + 2x_4 + 2x_6 \quad \text{then} \quad x_3 = 1 - 3x_2 + 2x_4 + 2x_6 - x_7.$$

Substituting in the other equations of the dictionary, we obtain an optimal dictionary:

$$\begin{aligned} x_1 &= 3x_2 + (1 - 3x_2 + 2x_4 + 2x_6 - x_7) - 2x_4 - 2x_6 = 1 - x_7 \\ x_5 &= 4x_2 + 2(1 - 3x_2 + 2x_4 + 2x_6 - x_7) - 8x_4 + x_6 \\ &= 2 - 2x_2 - 4x_4 + 5x_6 - 2x_7 \\ z &= -27x_2 + (1 - 3x_2 + 2x_4 + 2x_6 - x_7) - 44x_4 - 20x_6 \\ &= 1 - 30x_2 - 42x_4 - 18x_6 - x_7 \end{aligned}$$

The resulting dictionary is:

$$\begin{cases} x_1 = 1 & & & & & & - & x_7 \\ x_3 = 1 & - & 3 & x_2 & + & 2 & x_4 & + & 2 & x_6 & - & x_7 \\ x_5 = 2 & - & 2 & x_2 & - & 4 & x_4 & + & 5 & x_6 & - & 2 & x_7 \\ z = 1 & - & 30 & x_2 & - & 42 & x_4 & - & 18 & x_6 & - & x_7 \end{cases}$$

where $\mathcal{B} = \{1, 3, 5\}$ and $\mathcal{N} = \{2, 4, 6, 7\}$. The basic optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_3 = 1, \quad \tilde{x}_5 = 2, \quad \tilde{x}_2 = 0, \quad \tilde{x}_4 = 0, \quad \tilde{x}_6 = 0, \quad \tilde{x}_7 = 0 \quad \text{of value} \quad 1$$

Phase 0 of the simplex algorithm

Exercise 8.1 — Phase zero of the simplex algorithm (P11)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & -2 & x_1 & - & 4 & x_2 \\ \text{subject to} & - & x_1 & - & & x_2 & \leq & -2 \\ & & x_1 & - & & x_2 & \leq & 0 \\ & & x_1, & & x_2 & \geq & 0 \end{array}$$

Questions

1. Execute the phase zero of the simplex algorithm to obtain a feasible solution and the associated dictionary for the LP problem.
2. Starting from the obtained dictionary, execute the simplex algorithm to solve the LP problem.

Solution

1. First phase of the simplex algorithm

We write the auxiliary problem in standard form:

$$\begin{array}{llllllll}
 \max & - & x_0 & & & & & \\
 \text{subject to} & - & x_0 & - & x_1 & - & x_2 & + & x_3 & & = & -2 \\
 & & - & x_0 & + & x_1 & - & x_2 & & & + & x_4 & = & 0 \\
 & & & x_0, & & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
 \end{array}$$

and its starting dictionary is:

$$\begin{cases}
 x_3 = -2 + x_0 + x_1 + x_2 \\
 x_4 = 0 + x_0 - x_1 + x_2 \\
 z = 0 - x_0
 \end{cases}$$

where $\mathcal{B} = \{3, 4\}$ and $\mathcal{N} = \{0, 1, 2\}$. The basic **non-feasible** solution is:

$$\bar{x}_3 = -2, \quad \bar{x}_4 = 0, \quad \bar{x}_0 = 0, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

- **Initialization**

To obtain a feasible solution of the auxiliary problem we increase the value of x_0 . We have:

$$\begin{cases}
 x_3 = -2 + x_0 \\
 x_4 = x_0
 \end{cases}
 \quad
 \begin{cases}
 \underbrace{-2 + x_0}_{x_3 \geq 0} \geq 0 \\
 \underbrace{x_0}_{x_4 \geq 0} \geq 0
 \end{cases}
 \quad
 \begin{cases}
 x_0 \geq 2 \\
 x_0 \geq 0
 \end{cases}$$

By increasing the value of x_0 to 2, the value of x_3 increases to 0, so x_3 leaves the basis.

We have:

$$x_3 = -2 + x_0 + x_1 + x_2 \quad \text{then} \quad x_0 = 2 - x_1 - x_2 + x_3$$

Substituting in the other equations of the dictionary, we get:

$$x_4 = (2 - x_1 - x_2 + x_3) - x_1 + x_2 = 2 - 2x_1 + x_3$$

$$z = -(2 - x_1 - x_2 + x_3) = -2 + x_1 + x_2 - x_3$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_0 = 2 - x_1 - x_2 + x_3 \\ x_4 = 2 - 2x_1 + x_3 \\ z = -2 + x_1 + x_2 - x_3 \end{cases}$$

where $\mathcal{B} = \{0, 4\}$ and $\mathcal{N} = \{1, 2, 3\}$. The basic feasible solution is:

$$\bar{x}_0 = 2, \quad \bar{x}_4 = 2, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad -2$$

• **Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_0 = 2 - x_1 \\ x_4 = 2 - 2x_1 \end{cases} \quad \begin{cases} \underbrace{2 - x_1 \geq 0}_{x_0 \geq 0} \\ \underbrace{2 - 2x_1 \geq 0}_{x_4 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq 2 \\ x_1 \leq 1 \end{cases}$$

By increasing the value of x_1 to $\min\{2, 1\} = 1$, the value of x_4 decreases to 0. Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 2 - 2x_1 + x_3 \quad \text{then} \quad x_1 = 1 + \frac{1}{2}x_3 - \frac{1}{2}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_0 &= 2 - \left(1 + \frac{1}{2}x_3 - \frac{1}{2}x_4\right) - x_2 + x_3 = 1 - x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ z &= -2 + \left(1 + \frac{1}{2}x_3 - \frac{1}{2}x_4\right) + x_2 - x_3 = -1 + x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \end{aligned}$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_0 = 1 - x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_1 = 1 + \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ z = -1 + x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \end{cases}$$

where $\mathcal{B} = \{0, 1\}$ and $\mathcal{N} = \{2, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_0 = 1, \quad \bar{x}_1 = 1, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad -1$$

• **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_0 = 1 - x_2 \\ x_1 = 1 \end{cases} \quad \begin{cases} \underbrace{1 - x_2 \geq 0}_{x_0 \geq 0} \\ \underbrace{x_1 = 1 \geq 0}_{x_1 \geq 0} \end{cases} \quad \begin{cases} x_2 \leq 1 \end{cases}$$

By increasing the value of x_2 to 1, the value of x_0 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_0 becomes non-basic

We have:

$$x_0 = 1 - x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \quad \text{then} \quad x_2 = 1 - x_0 + \frac{1}{2}x_3 + \frac{1}{2}x_4$$

Substituting in the other equation of the dictionary, we get:

$$z = -1 + \left(1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 - x_0\right) - \frac{1}{2}x_3 - \frac{1}{2}x_4 = -x_0$$

Accordingly, we obtained an optimal solution of the auxiliary problem (no further improvement is possible) and this is an optimal dictionary:

$$\begin{cases} x_1 = 1 & + \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_2 = 1 - x_0 & + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ z = 0 - x_0 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{0, 3, 4\}$. The basic optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1, \quad \tilde{x}_0 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad 0$$

Warning

Given an optimal dictionary of the auxiliary problem with optimal value zero, we can construct a dictionary of the original LP that corresponds to a feasible solution of the original LP problem.

By removing the auxiliary variable x_0 , we obtain:

$$\begin{aligned} x_1 &= 1 + \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_2 &= 1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \end{aligned}$$

Substituting in the objective function of the original problem, we can express it as a function of the non-basic variables x_3 and x_4 :

$$z = -2x_1 - 4x_2 = -2 \left(1 + \frac{1}{2}x_3 - \frac{1}{2}x_4 \right) - 4 \left(1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \right) = -6 - 3x_3 - x_4$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

$$\begin{cases} x_1 = 1 & + \frac{1}{2} x_3 & - \frac{1}{2} x_4 \\ x_2 = 1 & + \frac{1}{2} x_3 & + \frac{1}{2} x_4 \\ z = -6 & - 3 x_3 & - x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2\}$ and $\mathcal{N} = \{3, 4\}$. The basic optimal solution is:

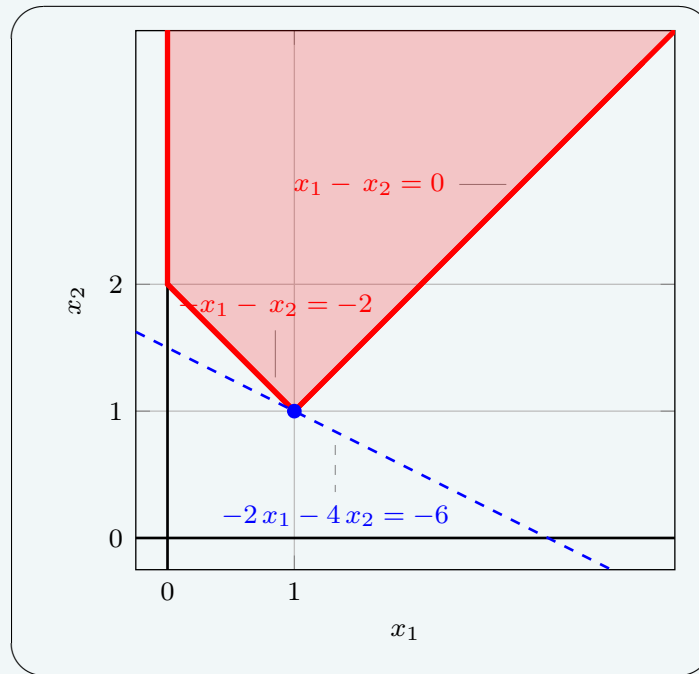
$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value} \quad -6$$

2. Second phase of the simplex algorithm

An optimal dictionary and an optimal solution of the LP problem have been determined in the first phase of the simplex algorithm, so the second phase is not executed.

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1$$

the optimal value $z(LP)$ is -6 .

In blue the optimal vertex obtained by the first phase of the simplex algorithm. In dashed blue, the contour line of the objective function associated with the value taken in the vertex.

Exercise 8.2 — Phase zero of the simplex algorithm (P21)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{array}{llllll}
 \max & x_1 & + & 2 & x_2 & \\
 \text{subject to} & -x_1 & - & & x_2 & \leq -2 \\
 & 4 & x_1 & - & 2 & x_2 \leq 7 \\
 & & x_1 & + & 5 & x_2 \leq 21 \\
 & x_1, & & & x_2 & \geq 0
 \end{array}$$

Questions

1. Execute the phase zero of the simplex algorithm to obtain a feasible solution and the associated dictionary for the LP problem.
2. Starting from the obtained dictionary, execute the simplex algorithm to solve the LP problem.

Solution

1. First phase of the simplex algorithm

We write the auxiliary problem in standard form:

$$\begin{array}{llllllllll}
 \max & - & x_0 & & & & & & & \\
 \text{subject to} & - & x_0 & - & x_1 & - & x_2 & + & x_3 & = & -2 \\
 & - & x_0 & + & 4 & x_1 & - & 2 & x_2 & + & x_4 = 7 \\
 & - & x_0 & + & x_1 & + & 5 & x_2 & & + & x_5 = 21 \\
 & & x_0, & & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
 \end{array}$$

and its starting dictionary is:

$$\left\{ \begin{array}{l}
 x_3 = -2 + x_0 + x_1 + x_2 \\
 x_4 = 7 + x_0 - 4x_1 + 2x_2 \\
 x_5 = 21 + x_0 - x_1 - 5x_2 \\
 z = 0 - x_0
 \end{array} \right.$$

where $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{0, 1, 2\}$. The basic **non-feasible** solution is:

$$\bar{x}_3 = -2, \quad \bar{x}_4 = 7, \quad \bar{x}_5 = 21, \quad \bar{x}_0 = 0, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value} \quad 0$$

• **Initialization**

To obtain a feasible solution of the auxiliary problem we increase the value of x_0 . We have:

$$\begin{cases} x_3 = -2 + x_0 \\ x_4 = 7 + x_0 \\ x_5 = 21 + x_0 \end{cases} \quad \begin{cases} \underbrace{-2 + x_0}_{x_3 \geq 0} \geq 0 \\ \underbrace{7 + x_0}_{x_4 \geq 0} \geq 0 \\ \underbrace{21 + x_0}_{x_5 \geq 0} \geq 0 \end{cases} \quad \begin{cases} x_0 \geq 2 \\ x_0 \geq -7 \\ x_0 \geq -21 \end{cases}$$

By increasing the value of x_0 to 2, the value of x_3 increases to 0, so x_3 leaves the basis.

We have:

$$x_3 = -2 + x_0 + x_1 + x_2 \quad \text{then} \quad x_0 = 2 - x_1 - x_2 + x_3$$

Substituting in the other equations of the dictionary, we get:

$$x_4 = 7 + (2 - x_1 - x_2 + x_3) - 4x_1 + 2x_2 = 9 - 5x_1 + x_2 + x_3$$

$$x_5 = 21 + (2 - x_1 - x_2 + x_3) - x_1 - 5x_2 = 23 - 2x_1 - 6x_2 + x_3$$

$$z = -(2 - x_1 - x_2 + x_3) = -2 + x_1 + x_2 - x_3$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_0 = 2 - x_1 - x_2 + x_3 \\ x_4 = 9 - 5x_1 + x_2 + x_3 \\ x_5 = 23 - 2x_1 - 6x_2 + x_3 \\ z = -2 + x_1 + x_2 - x_3 \end{cases}$$

where $\mathcal{B} = \{0, 4, 5\}$ and $\mathcal{N} = \{1, 2, 3\}$. The basic feasible solution is:

$$\bar{x}_0 = 2, \quad \bar{x}_4 = 9, \quad \bar{x}_5 = 23, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value} \quad -2$$

• **Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_0 = 2 - x_1 \\ x_4 = 9 - 5x_1 \\ x_5 = 23 - 2x_1 \end{cases} \quad \begin{cases} \underbrace{2 - x_1 \geq 0}_{x_0 \geq 0} \\ \underbrace{9 - 5x_1 \geq 0}_{x_4 \geq 0} \\ \underbrace{23 - 2x_1 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq 2 \\ x_1 \leq \frac{9}{5} \\ x_1 \leq \frac{23}{2} \end{cases}$$

By increasing the value of x_1 to $\min\left\{2, \frac{9}{5}, \frac{23}{2}\right\} = \frac{9}{5}$, the value of x_4 decreases to 0.

Then, we have that:

x_1 becomes basic and accordingly x_4 becomes non-basic

We have:

$$x_4 = 9 - 5x_1 + x_2 + x_3 \quad \text{then} \quad x_1 = \frac{9}{5} + \frac{1}{5}x_2 + \frac{1}{5}x_3 - \frac{1}{5}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_0 &= 2 - \left(\frac{9}{5} + \frac{1}{5}x_2 + \frac{1}{5}x_3 - \frac{1}{5}x_4\right) - x_2 + x_3 = \frac{1}{5} - \frac{6}{5}x_2 + \frac{4}{5}x_3 + \frac{1}{5}x_4 \\ x_5 &= 23 - 2\left(\frac{9}{5} + \frac{1}{5}x_2 + \frac{1}{5}x_3 - \frac{1}{5}x_4\right) - 6x_2 + x_3 = \frac{97}{5} - \frac{32}{5}x_2 + \frac{3}{5}x_3 + \frac{2}{5}x_4 \\ z &= -2 + \left(\frac{9}{5} + \frac{1}{5}x_2 + \frac{1}{5}x_3 - \frac{1}{5}x_4\right) + x_2 - x_3 = -\frac{1}{5} + \frac{6}{5}x_2 - \frac{4}{5}x_3 - \frac{1}{5}x_4 \end{aligned}$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_0 = \frac{1}{5} - \frac{6}{5}x_2 + \frac{4}{5}x_3 + \frac{1}{5}x_4 \\ x_1 = \frac{9}{5} + \frac{1}{5}x_2 + \frac{1}{5}x_3 - \frac{1}{5}x_4 \\ x_5 = \frac{97}{5} - \frac{32}{5}x_2 + \frac{3}{5}x_3 + \frac{2}{5}x_4 \\ z = -\frac{1}{5} + \frac{6}{5}x_2 - \frac{4}{5}x_3 - \frac{1}{5}x_4 \end{cases}$$

where $\mathcal{B} = \{0, 1, 5\}$ and $\mathcal{N} = \{2, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_0 = \frac{1}{5}, \quad \bar{x}_1 = \frac{9}{5}, \quad \bar{x}_5 = \frac{97}{5}, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad -\frac{1}{5}$$

• **Iteration 2**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_0 = \frac{1}{5} - \frac{6}{5}x_2 \\ x_1 = \frac{9}{5} + \frac{1}{5}x_2 \\ x_5 = \frac{97}{5} - \frac{32}{5}x_2 \end{cases} \quad \begin{cases} \underbrace{\frac{1}{5} - \frac{6}{5}x_2 \geq 0}_{x_0 \geq 0} \\ \underbrace{\frac{9}{5} + \frac{1}{5}x_2 \geq 0}_{x_1 \geq 0} \\ \underbrace{\frac{97}{5} - \frac{32}{5}x_2 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_2 \leq \frac{1}{6} \\ x_2 \geq -9 \\ x_2 \leq \frac{97}{32} \end{cases}$$

By increasing the value of x_2 to $\min\left\{\frac{1}{6}, \frac{97}{32}\right\} = \frac{1}{6}$, the value of x_0 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_0 becomes non-basic

We have:

$$x_0 = \frac{1}{5} - \frac{6}{5}x_2 + \frac{4}{5}x_3 + \frac{1}{5}x_4 \quad \text{then} \quad x_2 = \frac{1}{6} - \frac{5}{6}x_0 + \frac{2}{3}x_3 + \frac{1}{6}x_4.$$

Substituting in the other equations of the dictionary, we get:

$$x_1 = \frac{9}{5} + \frac{1}{5} \left(\frac{1}{6} - \frac{5}{6}x_0 + \frac{2}{3}x_3 + \frac{1}{6}x_4 \right) + \frac{1}{5}x_3 - \frac{1}{5}x_4 = \frac{11}{6} - \frac{1}{6}x_0 + \frac{1}{3}x_3 - \frac{1}{6}x_4$$

$$x_5 = \frac{97}{5} - \frac{32}{5} \left(\frac{1}{6} - \frac{5}{6}x_0 + \frac{2}{3}x_3 + \frac{1}{6}x_4 \right) + \frac{3}{5}x_3 + \frac{2}{5}x_4 = \frac{55}{3} + \frac{16}{3}x_0 - \frac{11}{3}x_3 - \frac{2}{3}x_4$$

$$z = -\frac{1}{5} + \frac{6}{5} \left(\frac{1}{6} - \frac{5}{6}x_0 + \frac{2}{3}x_3 + \frac{1}{6}x_4 \right) - \frac{4}{5}x_3 - \frac{1}{5}x_4 = -x_0$$

Accordingly, we obtained an optimal solution of the auxiliary problem (no further improvement is possible) and this is an optimal dictionary:

$$\begin{cases} x_1 = \frac{11}{6} - \frac{1}{6}x_0 + \frac{1}{3}x_3 - \frac{1}{6}x_4 \\ x_2 = \frac{1}{6} - \frac{5}{6}x_0 + \frac{2}{3}x_3 + \frac{1}{6}x_4 \\ x_5 = \frac{55}{3} + \frac{16}{3}x_0 - \frac{11}{3}x_3 - \frac{2}{3}x_4 \\ z = 0 - x_0 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{0, 3, 4\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{11}{6}, \quad \tilde{x}_2 = \frac{1}{6}, \quad \tilde{x}_5 = \frac{55}{3}, \quad \tilde{x}_0 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value } 0$$

Warning

Given an optimal dictionary of the auxiliary problem with optimal value zero, we can construct a dictionary of the original LP that corresponds to a feasible solution of the original LP problem.

By removing the auxiliary variable x_0 , we obtain:

$$x_1 = \frac{11}{6} + \frac{1}{3}x_3 - \frac{1}{6}x_4$$

$$x_2 = \frac{1}{6} + \frac{2}{3}x_3 + \frac{1}{6}x_4$$

$$x_5 = \frac{55}{3} - \frac{11}{3}x_3 - \frac{2}{3}x_4$$

Substituting in the objective function of the original problem, we can express it as a function of the non-basic variables x_3 and x_4 :

$$z = x_1 + 2x_2 = \frac{11}{6} + \frac{1}{3}x_3 - \frac{1}{6}x_4 + 2\left(\frac{1}{6} + \frac{2}{3}x_3 + \frac{1}{6}x_4\right) = \frac{13}{6} + \frac{5}{3}x_3 + \frac{1}{6}x_4$$

Accordingly, we have the following dictionary of the original LP problem:

$$\left\{ \begin{array}{l} x_1 = \frac{11}{6} + \frac{1}{3}x_3 - \frac{1}{6}x_4 \\ x_2 = \frac{1}{6} + \frac{2}{3}x_3 + \frac{1}{6}x_4 \\ x_5 = \frac{55}{3} - \frac{11}{3}x_3 - \frac{2}{3}x_4 \\ z = \frac{13}{6} + \frac{5}{3}x_3 + \frac{1}{6}x_4 \end{array} \right.$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = \frac{11}{6}, \quad \bar{x}_2 = \frac{1}{6}, \quad \bar{x}_5 = \frac{55}{3}, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value } \frac{13}{6}$$

2. Second phase of the simplex algorithm

We execute the simplex algorithm for the original LP problem.

- **Initialization**

We start from the dictionary obtained by the first phase of the simplex algorithm.

- **Iteration 1**

If we increase the value of the variable x_3 , we can get a better solution. We have:

$$\left\{ \begin{array}{l} x_1 = \frac{11}{6} + \frac{1}{3} x_3 \\ x_2 = \frac{1}{6} + \frac{2}{3} x_3 \\ x_5 = \frac{55}{3} - \frac{11}{3} x_3 \end{array} \right. \quad \left\{ \begin{array}{l} \underbrace{\frac{11}{6} + \frac{1}{3} x_3}_{x_1 \geq 0} \geq 0 \\ \underbrace{\frac{1}{6} + \frac{2}{3} x_3}_{x_2 \geq 0} \geq 0 \\ \underbrace{\frac{55}{3} - \frac{11}{3} x_3}_{x_5 \geq 0} \geq 0 \end{array} \right. \quad \left\{ \begin{array}{l} x_3 \geq -\frac{11}{3} \\ x_3 \geq -\frac{1}{4} \\ x_3 \leq 5 \end{array} \right.$$

By increasing the value of x_3 to 5, the value of x_5 decreases to 0.

Then, we have that:

x_3 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = \frac{55}{3} - \frac{11}{3} x_3 - \frac{2}{3} x_4 \quad \text{then} \quad x_3 = 5 - \frac{2}{11} x_4 - \frac{3}{11} x_5 .$$

Substituting in the other equations of the dictionary, we get:

$$x_1 = \frac{11}{6} + \frac{1}{3} \left(5 - \frac{2}{11} x_4 - \frac{3}{11} x_5 \right) - \frac{1}{6} x_4 = \frac{7}{2} - \frac{5}{22} x_4 - \frac{1}{11} x_5$$

$$x_2 = \frac{1}{6} + \frac{2}{3} \left(5 - \frac{2}{11} x_4 - \frac{3}{11} x_5 \right) + \frac{1}{6} x_4 = \frac{7}{2} + \frac{1}{22} x_4 - \frac{2}{11} x_5$$

$$z = \frac{13}{6} + \frac{5}{3} \left(5 - \frac{2}{11} x_4 - \frac{3}{11} x_5 \right) + \frac{1}{6} x_4 = \frac{21}{2} - \frac{3}{22} x_4 - \frac{5}{11} x_5$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

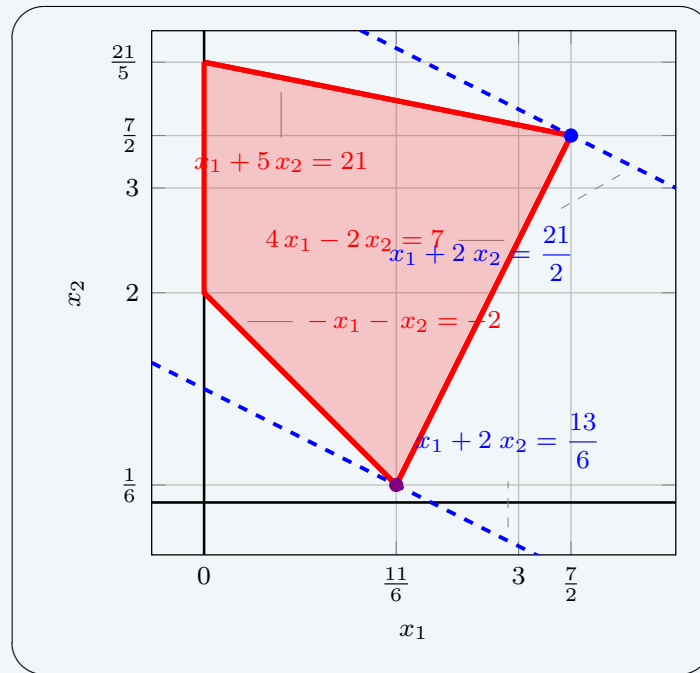
$$\left\{ \begin{array}{l} x_1 = \frac{7}{2} - \frac{5}{22} x_4 - \frac{1}{11} x_5 \\ x_2 = \frac{7}{2} + \frac{1}{22} x_4 - \frac{2}{11} x_5 \\ x_3 = 5 - \frac{2}{11} x_4 - \frac{3}{11} x_5 \\ z = \frac{21}{2} - \frac{3}{22} x_4 - \frac{5}{11} x_5 \end{array} \right.$$

where $\mathcal{B} = \{1, 2, 3\}$ and $\mathcal{N} = \{4, 5\}$. The basic optimal solution is:

$$\tilde{x}_1 = \frac{7}{2}, \quad \tilde{x}_2 = \frac{7}{2}, \quad \tilde{x}_3 = 5, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad \frac{21}{2}$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{7}{2}, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value $z(LP)$ is $\frac{21}{2}$.

In violet the first vertex obtained by the first phase of the simplex algorithm, in blue the optimal one. In dashed blue, the contour lines of the objective function associated with the values taken in the two vertices.

Exercise 8.3 — Phase zero of the simplex algorithm (P22)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{array}{llllll}
 \max & 3x_1 & + & x_2 & & \\
 \text{subject to} & x_1 & - & x_2 & \leq & -1 \\
 & -x_1 & - & x_2 & \leq & -3 \\
 & 2x_1 & + & x_2 & \leq & 4 \\
 & x_1, & & x_2 & \geq & 0
 \end{array}$$

Questions

1. Execute the phase zero of the simplex algorithm to obtain a feasible solution and the associated dictionary for the LP problem.
2. Starting from the obtained dictionary, execute the simplex algorithm to solve the LP problem.

Solution

1. First phase of the simplex algorithm

We write the auxiliary problem in standard form:

$$\begin{array}{llllllllll}
 \max & - & x_0 & & & & & & & \\
 \text{subject to} & - & x_0 & + & x_1 & - & x_2 & + & x_3 & = & -1 \\
 & - & x_0 & - & x_1 & - & x_2 & & + & x_4 & = & -3 \\
 & - & x_0 & + & 2x_1 & + & x_2 & & & + & x_5 & = & 4 \\
 & x_0, & & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
 \end{array}$$

and its starting dictionary is:

$$\left\{ \begin{array}{l}
 x_3 = -1 + x_0 - x_1 + x_2 \\
 x_4 = -3 + x_0 + x_1 + x_2 \\
 x_5 = 4 + x_0 - 2x_1 - x_2 \\
 z = 0 - x_0
 \end{array} \right.$$

where $\mathcal{B} = \{3, 4, 5\}$ and $\mathcal{N} = \{0, 1, 2\}$. The basic **non-feasible** solution is:

$$\bar{x}_3 = -1, \quad \bar{x}_4 = -3, \quad \bar{x}_5 = 4, \quad \bar{x}_0 = 0, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0 \quad \text{of value } 0$$

- **Initialization**

To obtain a feasible solution of the auxiliary problem we increase the value of x_0 . We have:

$$\begin{cases} x_3 = -1 + x_0 \\ x_4 = -3 + x_0 \\ x_5 = 4 + x_0 \end{cases} \quad \begin{cases} \underbrace{-1 + x_0 \geq 0}_{x_3 \geq 0} \\ \underbrace{-3 + x_0 \geq 0}_{x_4 \geq 0} \\ \underbrace{4 + x_0 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_0 \geq 1 \\ x_0 \geq 3 \\ x_0 \geq -4 \end{cases}$$

By increasing the value of x_0 to 3, the value of x_4 increases to 0, so x_4 leaves the basis.

We have:

$$x_4 = -3 + x_0 + x_1 + x_2 \quad \text{then} \quad x_0 = 3 - x_1 - x_2 + x_4$$

Substituting in the other equations of the dictionary, we get:

$$x_3 = -1 + (3 - x_1 - x_2 + x_4) - x_1 + x_2 = 2 - 2x_1 + x_4$$

$$x_5 = 4 + (3 - x_1 - x_2 + x_4) - 2x_1 - x_2 = 7 - 3x_1 - 2x_2 + x_4$$

$$z = -(3 - x_1 - x_2 + x_4) = -3 + x_1 + x_2 - x_4$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_0 = 3 - x_1 - x_2 + x_4 \\ x_3 = 2 - 2x_1 + x_4 \\ x_5 = 7 - 3x_1 - 2x_2 + x_4 \\ z = -3 + x_1 + x_2 - x_4 \end{cases}$$

where $\mathcal{B} = \{0, 3, 5\}$ and $\mathcal{N} = \{1, 2, 4\}$. The basic feasible solution is:

$$\bar{x}_0 = 3, \quad \bar{x}_3 = 2, \quad \bar{x}_5 = 7, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_4 = 0 \quad \text{of value } -3$$

- **Iteration 1**

If we increase the value of the variable x_1 , we can get a better solution. We have:

$$\begin{cases} x_0 = 3 - x_1 \\ x_3 = 2 - 2x_1 \\ x_5 = 7 - 3x_1 \end{cases} \quad \begin{cases} \underbrace{3 - x_1 \geq 0}_{x_0 \geq 0} \\ \underbrace{2 - 2x_1 \geq 0}_{x_3 \geq 0} \\ \underbrace{7 - 3x_1 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_1 \leq 3 \\ x_1 \leq 1 \\ x_1 \leq \frac{7}{3} \end{cases}$$

By increasing the value of x_1 to 1, the value of x_3 decreases to 0, so x_3 leaves the basis.

$$x_3 = 2 - 2x_1 + x_4 \quad \text{then} \quad x_1 = 1 + \frac{1}{2}x_4 - \frac{1}{2}x_3$$

Substituting in the other equations of the dictionary, we get:

$$x_0 = 3 - \left(1 + \frac{1}{2}x_4 - \frac{1}{2}x_3\right) - x_2 + x_4 = 2 - x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4$$

$$x_5 = 7 - 3\left(1 + \frac{1}{2}x_4 - \frac{1}{2}x_3\right) - 2x_2 + x_4 = 4 - 2x_2 + \frac{3}{2}x_3 - \frac{1}{2}x_4$$

$$z = -3 + \left(1 + \frac{1}{2}x_4 - \frac{1}{2}x_3\right) + x_2 - x_4 = -2 + x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_0 = 2 - x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_1 = 1 - \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_5 = 4 - 2x_2 + \frac{3}{2}x_3 - \frac{1}{2}x_4 \\ z = -2 + x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \end{cases}$$

where $\mathcal{B} = \{0, 1, 5\}$ and $\mathcal{N} = \{2, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_0 = 2, \quad \bar{x}_1 = 1, \quad \bar{x}_5 = 4, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value } -2$$

• Iteration 2

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_0 = 2 - x_2 \\ x_5 = 4 - 2x_2 \end{cases} \quad \begin{cases} \underbrace{2 - x_2 \geq 0}_{x_0 \geq 0} \\ \underbrace{4 - 2x_2 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_2 \leq 2 \\ x_2 \leq 2 \end{cases}$$

By increasing the value of x_2 to $\min\{2, 2\} = 2$, the value of both x_0 and x_5 decreases to 0. We choose x_0 as the leaving variable, so that the auxiliary variable leaves the basis (accordingly, x_5 stays basic with value 0).

Then, we have that:

x_2 becomes basic and accordingly x_0 becomes non-basic

We have:

$$x_0 = 2 - x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \quad \text{then} \quad x_2 = 2 - x_0 + \frac{1}{2}x_3 + \frac{1}{2}x_4$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_5 &= 4 - 2 \left(2 - x_0 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \right) + \frac{3}{2}x_3 - \frac{1}{2}x_4 = 2x_0 + \frac{1}{2}x_3 - \frac{3}{2}x_4 \\ z &= -2 + \left(2 - x_0 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \right) - \frac{1}{2}x_3 - \frac{1}{2}x_4 = -x_0 \end{aligned}$$

Accordingly, we obtained an optimal solution of the auxiliary problem (no further improvement is possible) and this is an optimal dictionary:

$$\left\{ \begin{array}{lclclcl} x_1 & = & 1 & & -\frac{1}{2}x_3 & + & \frac{1}{2}x_4 \\ x_2 & = & 2 & - & x_0 & + & \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_5 & = & 0 & + & 2x_0 & + & \frac{1}{2}x_3 - \frac{3}{2}x_4 \\ z & = & 0 & - & x_0 & & \end{array} \right.$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{0, 3, 4\}$. The basic optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 2, \quad \tilde{x}_5 = 0, \quad \tilde{x}_0 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0, \quad \text{of value } 0$$

Warning

Given an optimal dictionary of the auxiliary problem with optimal value zero, we can construct a dictionary of the original LP that corresponds to a feasible solution of the original LP problem.

By removing the auxiliary variable x_0 , while keeping the basic variables x_1, x_2, x_5 , we obtain:

$$\begin{aligned} x_1 &= 1 - \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_2 &= 2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_5 &= \frac{1}{2}x_3 - \frac{3}{2}x_4 \end{aligned}$$

Substituting in the objective function of the original problem, we can express it as a function of the non-basic variables x_3 and x_4 :

$$z = 3 \left(1 - \frac{1}{2}x_3 + \frac{1}{2}x_4 \right) + \left(2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \right) = 5 - x_3 + 2x_4$$

Accordingly, we have the following dictionary of the original LP problem:

$$\begin{cases} x_1 = 1 - \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_2 = 2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ x_5 = 0 + \frac{1}{2}x_3 - \frac{3}{2}x_4 \\ z = 5 - x_3 + 2x_4 \end{cases}$$

where $\mathcal{B} = \{1, 2, 5\}$ and $\mathcal{N} = \{3, 4\}$. The basic feasible solution is:

$$\bar{x}_1 = 1, \quad \bar{x}_2 = 2, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0, \quad \bar{x}_5 = 0 \quad \text{of value } 5$$

2. Second phase of the simplex algorithm

We execute the simplex algorithm for the original LP problem.

- **Initialization**

We start from the dictionary obtained by the first phase of the simplex algorithm.

- **Iteration 1**

If we increase the value of the variable x_4 , we can get a better solution. We have:

$$\begin{cases} x_1 = 1 + \frac{1}{2}x_4 \\ x_2 = 2 + \frac{1}{2}x_4 \\ x_5 = -\frac{3}{2}x_4 \end{cases} \quad \begin{cases} \underbrace{1 + \frac{1}{2}x_4 \geq 0}_{x_1 \geq 0} \\ \underbrace{2 + \frac{1}{2}x_4 \geq 0}_{x_2 \geq 0} \\ \underbrace{-\frac{3}{2}x_4 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_4 \geq -2 \\ x_4 \geq -4 \\ x_4 \leq 0 \end{cases}$$

By increasing the value of x_4 to 0 (a **degenerate** pivoting operation, $t_m = 0$), the value of x_5 remains 0: the only bound on x_4 comes from $x_5 \geq 0$, so x_5 leaves the basis.

Then, we have that:

x_4 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = \frac{1}{2}x_3 - \frac{3}{2}x_4 \quad \text{then} \quad x_4 = \frac{1}{3}x_3 - \frac{2}{3}x_5$$

Substituting in the other equations of the dictionary, we get:

$$x_1 = 1 - \frac{1}{2}x_3 + \frac{1}{2} \left(\frac{1}{3}x_3 - \frac{2}{3}x_5 \right) = 1 - \frac{1}{3}x_3 - \frac{1}{3}x_5$$

$$x_2 = 2 + \frac{1}{2}x_3 + \frac{1}{2} \left(\frac{1}{3}x_3 - \frac{2}{3}x_5 \right) = 2 + \frac{2}{3}x_3 - \frac{1}{3}x_5$$

$$z = 5 - x_3 + 2 \left(\frac{1}{3}x_3 - \frac{2}{3}x_5 \right) = 5 - \frac{1}{3}x_3 - \frac{4}{3}x_5$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

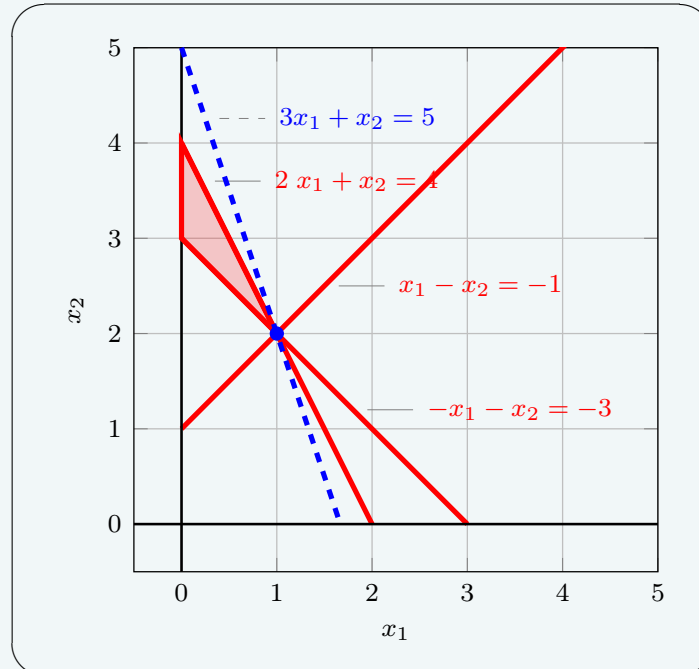
$$\left\{ \begin{array}{rcll} x_1 & = & 1 & - \frac{1}{3} x_3 - \frac{1}{3} x_5 \\ x_2 & = & 2 & + \frac{2}{3} x_3 - \frac{1}{3} x_5 \\ x_4 & = & 0 & + \frac{1}{3} x_3 - \frac{2}{3} x_5 \\ z & = & 5 & - \frac{1}{3} x_3 - \frac{4}{3} x_5 \end{array} \right.$$

where $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{N} = \{3, 5\}$. The basic optimal solution is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 2, \quad \tilde{x}_4 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value } 5$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 2$$

the optimal value $z(LP)$ is 5.

In blue the optimal vertex obtained by the first phase of the simplex algorithm. In dashed blue, the contour line of the objective function associated with the value taken in the vertex.

Exercise 8.4 — Phase zero of the simplex algorithm (P51)

Consider the following LP problem in canonical maximization form with $n = 3$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & -2x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 - 3x_3 \leq -5 \\ & -3x_1 - 2x_2 + 2x_3 \leq -2 \\ & x_1, \quad x_2, \quad x_3 \geq 0 \end{aligned}$$

Questions

1. Execute the phase zero of the simplex algorithm to obtain a feasible solution and the associated dictionary for the LP problem.
2. Starting from the obtained dictionary, execute the simplex algorithm to solve the LP problem.

Solution

1. First phase of the simplex algorithm

We write the auxiliary problem in standard form:

$$\begin{aligned}
 \max \quad & -x_0 \\
 \text{subject to} \quad & -x_0 + x_1 - x_2 - 3x_3 + x_4 = -5 \\
 & -x_0 - 3x_1 - 2x_2 + 2x_3 + x_5 = -2 \\
 & x_0, x_1, x_2, x_3, x_4, x_5 \geq 0
 \end{aligned}$$

and its starting dictionary is:

$$\begin{cases}
 x_4 = -5 + x_0 - x_1 + x_2 + 3x_3 \\
 x_5 = -2 + x_0 + 3x_1 + 2x_2 - 2x_3 \\
 z = 0 - x_0
 \end{cases}$$

where $\mathcal{B} = \{4, 5\}$ and $\mathcal{N} = \{0, 1, 2, 3\}$. The basic **non-feasible** solution is:

$$\bar{x}_4 = -5, \quad \bar{x}_5 = -2, \quad \bar{x}_0 = 0, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0 \quad \text{of value } 0$$

- **Initialization**

To obtain a feasible basic solution of the auxiliary problem we increase the value of x_0 . We have:

$$\begin{cases}
 x_4 = -5 + x_0 \\
 x_5 = -2 + x_0
 \end{cases}
 \quad
 \begin{cases}
 \underbrace{-5 + x_0 \geq 0}_{x_4 \geq 0} \\
 \underbrace{-2 + x_0 \geq 0}_{x_5 \geq 0}
 \end{cases}
 \quad
 \begin{cases}
 x_0 \geq 5 \\
 x_0 \geq 2
 \end{cases}$$

By increasing the value of x_0 to 5, the value of x_4 increases to 0, so x_4 leaves the basis. This variable is the slack variable associated with the largest negative RHS. We have:

$$x_4 = -5 + x_0 - x_1 + x_2 + 3x_3 \quad \text{then} \quad x_0 = 5 + x_1 - x_2 - 3x_3 + x_4$$

Substituting in the other equations of the dictionary, we get:

$$x_5 = -2 + (5 + x_1 - x_2 - 3x_3 + x_4) + 3x_1 + 2x_2 - 2x_3 = 3 + 4x_1 + x_2 - 5x_3 + x_4$$

$$z = 0 - (5 + x_1 - x_2 - 3x_3 + x_4) = -5 - x_1 + x_2 + 3x_3 - x_4$$

Accordingly, we now have the following dictionary:

$$\begin{cases} x_0 = 5 + x_1 - x_2 - 3x_3 + x_4 \\ x_5 = 3 + 4x_1 + x_2 - 5x_3 + x_4 \\ z = -5 - x_1 + x_2 + 3x_3 - x_4 \end{cases}$$

where $\mathcal{B} = \{0, 5\}$ and $\mathcal{N} = \{1, 2, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_0 = 5, \quad \bar{x}_5 = 3, \quad \bar{x}_1 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value} \quad -5$$

- **Iteration 1**

If we increase the value of the variable x_2 , we can get a better solution. We have:

$$\begin{cases} x_0 = 5 - x_2 \\ x_5 = 3 + x_2 \end{cases} \quad \begin{cases} \underbrace{5 - x_2 \geq 0}_{x_0 \geq 0} \\ \underbrace{3 + x_2 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_2 \leq 5 \\ x_2 \geq -3 \end{cases}$$

By increasing the value of x_2 to 5, the value of x_0 decreases to 0.

Then, we have that:

x_2 becomes basic and accordingly x_0 becomes non-basic

We have:

$$x_0 = 5 + x_1 - x_2 - 3x_3 + x_4 \quad \text{then} \quad x_2 = 5 - x_0 + x_1 - 3x_3 + x_4$$

Substituting in the other equations of the dictionary, we get:

$$x_5 = 3 + 4x_1 + (5 - x_0 + x_1 - 3x_3 + x_4) - 5x_3 + x_4 = 8 - x_0 + 5x_1 - 8x_3 + 2x_4$$

$$z = -5 - x_1 + (5 - x_0 + x_1 - 3x_3 + x_4) + 3x_3 - x_4 = -x_0$$

Accordingly, we obtained an optimal solution of the auxiliary problem (no further improvement is possible) and this is an optimal dictionary:

$$\begin{cases} x_2 = 5 - x_0 + x_1 - 3x_3 + x_4 \\ x_5 = 8 - x_0 + 5x_1 - 8x_3 + 2x_4 \\ z = 0 - x_0 \end{cases}$$

where $\mathcal{B} = \{2, 5\}$ and $\mathcal{N} = \{0, 1, 3, 4\}$. The basic optimal solution is:

$$\tilde{x}_2 = 5, \quad \tilde{x}_5 = 8, \quad \tilde{x}_0 = 0, \quad \tilde{x}_1 = 0, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = 0 \quad \text{of value } 0$$

Warning

Given an optimal dictionary of the auxiliary problem with optimal value zero, we can construct a dictionary of the original LP that corresponds to a feasible solution of the original LP problem.

By removing the auxiliary variable x_0 , we obtain:

$$\begin{aligned} x_2 &= 5 + x_1 - 3x_3 + x_4 \\ x_5 &= 8 + 5x_1 - 8x_3 + 2x_4 \end{aligned}$$

Substituting in the objective function of the original problem, we can express it as a function of the non-basic variables x_1 , x_3 and x_4 :

$$z = -2x_1 - (5 + x_1 - 3x_3 + x_4) = -5 - 3x_1 + 3x_3 - x_4$$

Accordingly, we have the following dictionary of the original LP problem:

$$\begin{cases} x_2 = 5 + x_1 - 3x_3 + x_4 \\ x_5 = 8 + 5x_1 - 8x_3 + 2x_4 \\ z = -5 - 3x_1 + 3x_3 - x_4 \end{cases}$$

where $\mathcal{B} = \{2, 5\}$ and $\mathcal{N} = \{1, 3, 4\}$. The basic feasible solution is:

$$\bar{x}_2 = 5, \quad \bar{x}_5 = 8, \quad \bar{x}_1 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_4 = 0 \quad \text{of value } -5$$

2. Second phase of the simplex algorithm

We execute the simplex algorithm for the original LP problem.

- **Initialization**

We start from the dictionary obtained by the first phase of the simplex algorithm.

- **Iteration 1**

If we increase the value of the variable x_3 , we can get a better solution. We have:

$$\begin{cases} x_2 = 5 - 3x_3 \\ x_5 = 8 - 8x_3 \end{cases} \quad \begin{cases} \underbrace{5 - 3x_3 \geq 0}_{x_2 \geq 0} \\ \underbrace{8 - 8x_3 \geq 0}_{x_5 \geq 0} \end{cases} \quad \begin{cases} x_3 \leq \frac{5}{3} \\ x_3 \leq 1 \end{cases}$$

By increasing the value of x_3 to 1, the value of x_5 decreases to 0.

Then, we have that:

x_3 becomes basic and accordingly x_5 becomes non-basic

We have:

$$x_5 = 8 + 5x_1 - 8x_3 + 2x_4 \quad \text{then} \quad x_3 = 1 + \frac{5}{8}x_1 + \frac{1}{4}x_4 - \frac{1}{8}x_5$$

Substituting in the other equations of the dictionary, we get:

$$\begin{aligned} x_2 &= 5 + x_1 - 3 \left(1 + \frac{5}{8}x_1 + \frac{1}{4}x_4 - \frac{1}{8}x_5 \right) + x_4 = 2 - \frac{7}{8}x_1 + \frac{1}{4}x_4 + \frac{3}{8}x_5 \\ z &= -5 - 3x_1 + 3 \left(1 + \frac{5}{8}x_1 + \frac{1}{4}x_4 - \frac{1}{8}x_5 \right) - x_4 = -2 - \frac{9}{8}x_1 - \frac{1}{4}x_4 - \frac{3}{8}x_5 \end{aligned}$$

Accordingly, we obtained an optimal solution (no further improvement is possible) and this is an optimal dictionary:

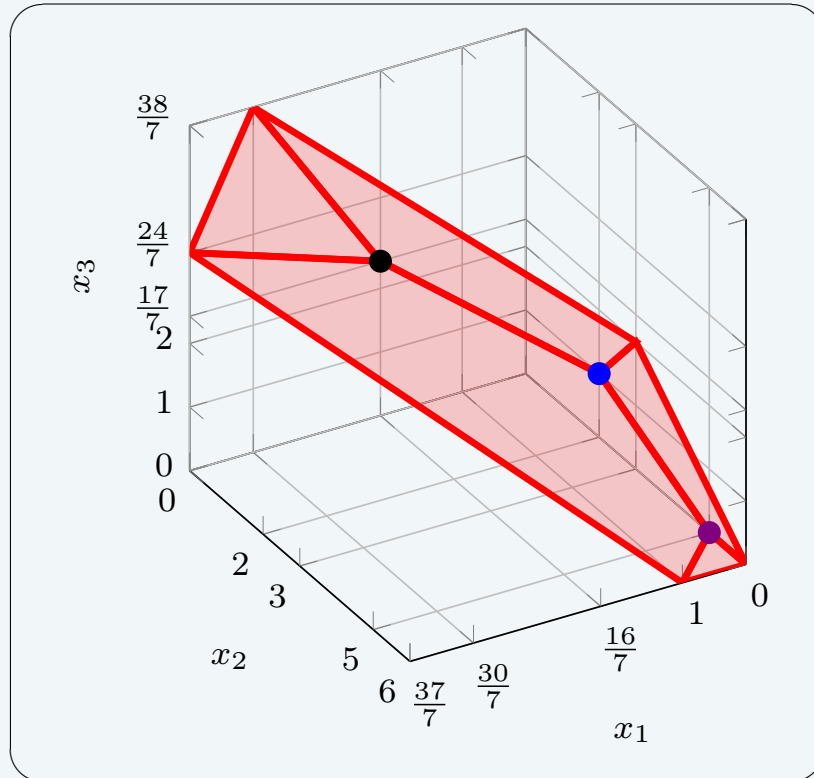
$$\begin{cases} x_2 = 2 - \frac{7}{8}x_1 + \frac{1}{4}x_4 + \frac{3}{8}x_5 \\ x_3 = 1 + \frac{5}{8}x_1 + \frac{1}{4}x_4 - \frac{1}{8}x_5 \\ z = -2 - \frac{9}{8}x_1 - \frac{1}{4}x_4 - \frac{3}{8}x_5 \end{cases}$$

where $\mathcal{B} = \{2, 3\}$ and $\mathcal{N} = \{1, 4, 5\}$. The basic optimal solution is:

$$\tilde{x}_2 = 2, \quad \tilde{x}_3 = 1, \quad \tilde{x}_1 = 0, \quad \tilde{x}_4 = 0, \quad \tilde{x}_5 = 0 \quad \text{of value} \quad -2$$

Graphical solution:

The feasible region of the LP problem (in canonical form) is the polyhedron shaded in red:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = 2, \quad \tilde{x}_3 = 1$$

the optimal value $z(LP)$ is -2 .

In violet the first vertex obtained by the first phase of the simplex algorithm, in blue the optimal one.

SageMath Code: 3D Polyhedra (<https://sagecell.sagemath.org/>)

```
# Define the polyhedron using inequalities (Hyperplane Representation)
p = Polyhedron(ieqs=[
    (-5, -1, 1, 3),      # (-1, 1, 3) x - 5 >= 0
    (-2, 3, 2, -2),      # (3, 2, -2) x - 2 >= 0
    (0, 1, 0, 0),        # (1, 0, 0) x + 0 >= 0
    (0, 0, 1, 0),        # (0, 1, 0) x + 0 >= 0
    (0, 0, 0, 1)         # (0, 0, 1) x + 0 >= 0
])

# Display the inequalities of the polyhedron
p.Hrepresentation()

# Display the extreme points and extreme rays of the polyhedron
p.Vrepresentation()

# Plot the polyhedron (3D)
p.plot()
```