



SAPIENZA
UNIVERSITÀ DI ROMA

Linear Programming

Exercises 1 — Linear functions, polyhedra and linear programming problems

Worked exercises on half-spaces and polyhedra, linear optimization problems in one variable, the graphical method and equivalences.

Fabio Furini

fabiofurini.github.io

Department of Computer, Control and Management Engineering — Sapienza University of Rome

Online version and downloadable material:
fabiofurini.github.io/linear-programming/
github.com/fabiofurini/linear-programming

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Half-spaces and polyhedra

In the exercises of this chapter the intersection point of two supporting hyper-planes in $\mathbb{R}^2$ is computed by solving a system of two linear equations in two variables, with the following formula:

Given $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 \in \mathbb{R}$, if $(a_{11} a_{22} - a_{12} a_{21}) \neq 0$, we have:

$$\begin{cases} a_{11} x_1 + a_{12} x_2 = b_1 \\ a_{21} x_1 + a_{22} x_2 = b_2 \end{cases} \implies \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{b_1 a_{22} - a_{12} b_2}{a_{11} a_{22} - a_{12} a_{21}} \\ \frac{a_{11} b_2 - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}} \end{pmatrix}$$

Exercise 2.1 — Polyhedra (*P2*)

Consider the following half-spaces in $\mathbb{R}^2$:

$$\begin{aligned} -4 x_1 + 10 x_2 &\leq 35 \\ 3 x_1 + x_2 &\leq 12 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Questions

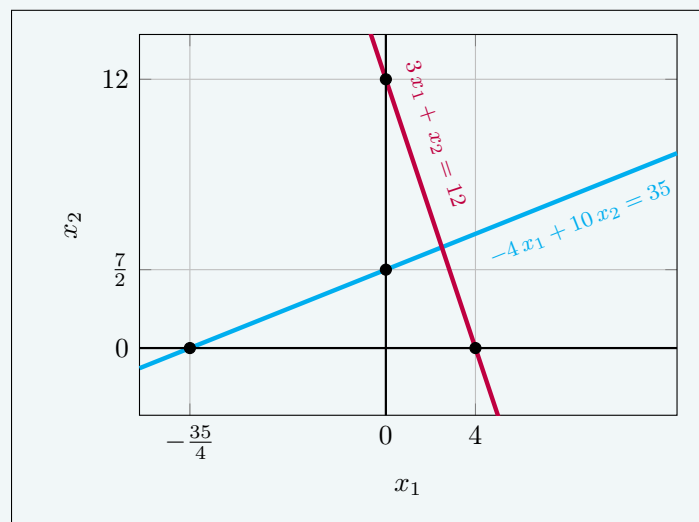
1. Draw a plot of their supporting hyper-planes.
2. Draw the polyhedron given by the intersection of the half-spaces. Is it a polytope?

Solution

1. The supporting hyper-planes associated with the half-spaces and the intersection points with the axis are:

$$\begin{array}{rclcl}
 -4 x_1 & + & 10 x_2 & = & 35 & \begin{pmatrix} -\frac{35}{4} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{7}{2} \end{pmatrix} \\
 3 x_1 & + & x_2 & = & 12 & \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 12 \end{pmatrix} \\
 x_1 & & & = & 0 & \\
 & & x_2 & = & 0 &
 \end{array}$$

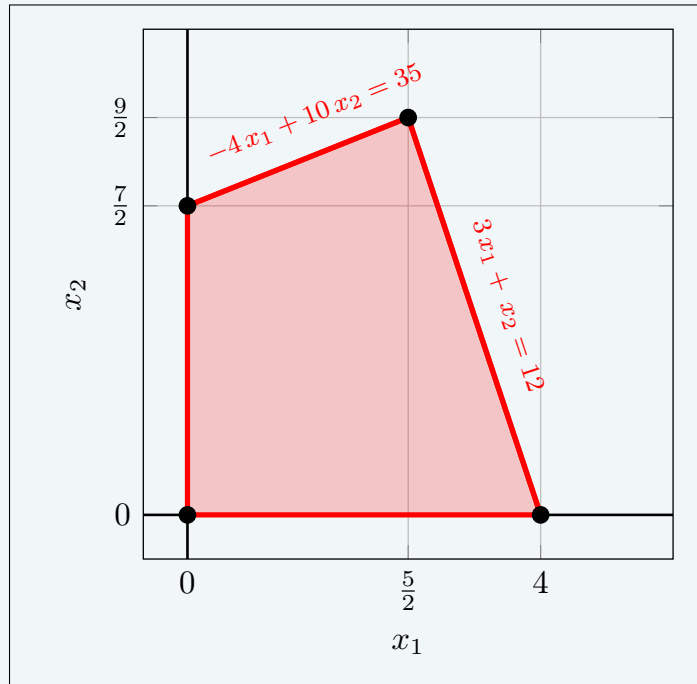
Their plot is shown below.



We compute the intersection point:

$$\begin{cases} -4 x_1 + 10 x_2 = 35 \\ 3 x_1 + x_2 = 12 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{35 \cdot 1 - 10 \cdot 12}{(-4) \cdot 1 - 10 \cdot 3} \\ \frac{(-4) \cdot 12 - 35 \cdot 3}{(-4) \cdot 1 - 10 \cdot 3} \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ \frac{9}{2} \end{pmatrix}$$

2. The polyhedron, given by the intersection of the half-spaces, is:



A polyhedron is a **polytope** if it is bounded (definitions of bounded polyhedron and of polytope in the chapter «Half-spaces and polyhedra»); equivalently, if there exist $\alpha, \beta \in \mathbb{R}^n$ such that $\alpha_j \leq x_j \leq \beta_j$ for every $\mathbf{x} \in P$ and every $j \in \{1, 2, \dots, n\}$.

This polyhedron P is bounded since for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in P$, we have for instance:

$$0 \leq x_1 \leq 4$$

$$0 \leq x_2 \leq \frac{9}{2}$$

Accordingly this polyhedron is also a polytope.

Exercise 2.2 — Polyhedra (P10)

Consider the following half-spaces in $\mathbb{R}^2$:

$$\begin{aligned}
 -x_1 + 3x_2 &\leq 12 \\
 x_1 + 2x_2 &\leq 12 \\
 3x_1 - 2x_2 &\leq 8 \\
 x_1 &\geq 0 \\
 x_2 &\geq 0
 \end{aligned}$$

Questions

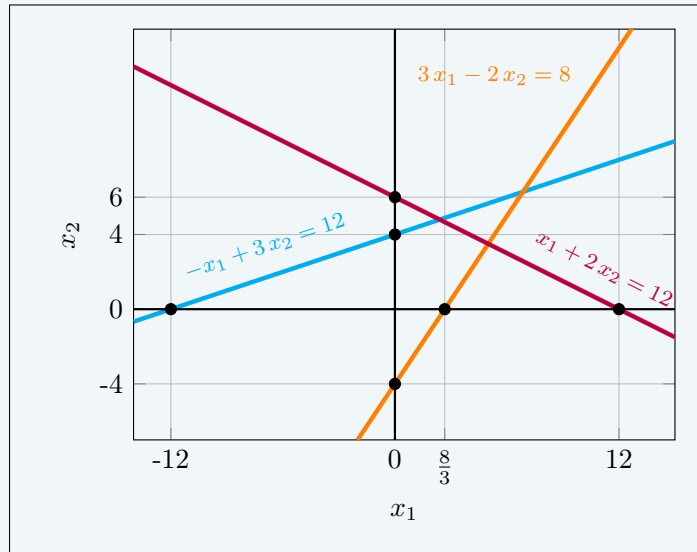
1. Draw a plot of their supporting hyper-planes.
2. Draw the polyhedron given by the intersection of the half-spaces. Is it a polytope?
3. What happens if we add the constraint $x_1 \geq 5$ to the polyhedron? What if we add the constraint $x_1 \leq 5$?

Solution

1. The supporting hyper-planes associated with the half-spaces and the intersection points with the axis are:

$$\begin{aligned}
 -x_1 + 3x_2 &= 12 & \begin{pmatrix} -12 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 4 \end{pmatrix} \\
 x_1 + 2x_2 &= 12 & \begin{pmatrix} 12 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 6 \end{pmatrix} \\
 3x_1 - 2x_2 &= 8 & \begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ -4 \end{pmatrix} \\
 x_1 &= 0 \\
 x_2 &= 0
 \end{aligned}$$

Their plot is shown below.

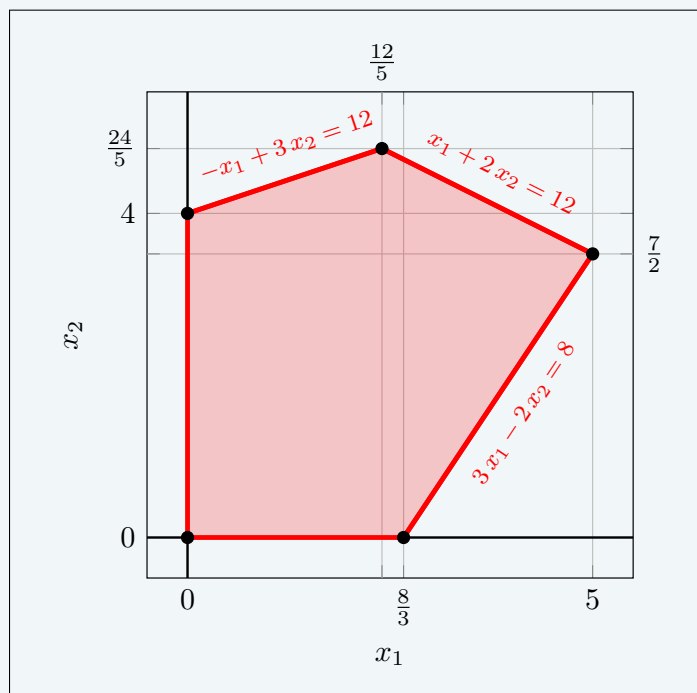


We compute the intersection points:

$$\begin{cases} -x_1 + 3x_2 = 12 \\ x_1 + 2x_2 = 12 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{12 \cdot 2 - 3 \cdot 12}{(-1) \cdot 2 - 3 \cdot 1} \\ \frac{(-1) \cdot 12 - 12 \cdot 1}{(-1) \cdot 2 - 3 \cdot 1} \end{pmatrix} = \begin{pmatrix} \frac{12}{5} \\ \frac{24}{5} \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 = 12 \\ 3x_1 - 2x_2 = 8 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{12 \cdot (-2) - 2 \cdot 8}{1 \cdot (-2) - 2 \cdot 3} \\ \frac{1 \cdot 8 - 12 \cdot 3}{1 \cdot (-2) - 2 \cdot 3} \end{pmatrix} = \begin{pmatrix} 5 \\ \frac{7}{2} \end{pmatrix}$$

2. The polyhedron, given by the intersection of the half-spaces, is:



A polyhedron is a **polytope** if it is bounded (definitions of bounded polyhedron and of polytope in the chapter «Half-spaces and polyhedra»); equivalently, if there exist $\alpha, \beta \in \mathbb{R}^n$ such that $\alpha_j \leq x_j \leq \beta_j$ for every $x \in P$ and every $j \in \{1, 2, \dots, n\}$.

This polyhedron P is bounded since for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in P$, we have for instance:

$$0 \leq x_1 \leq 5$$

$$0 \leq x_2 \leq \frac{24}{5}$$

Accordingly this polyhedron is also a polytope.

3. If we add constraint $x_1 \geq 5$ the polyhedron reduces to the point $(5, \frac{7}{2})$. If we add constraint $x_1 \leq 5$ the polyhedron does not change.

Exercise 2.3 — Polyhedra (P21)

Consider the following half-spaces in $\mathbb{R}^2$:

$$-x_1 - x_2 \leq -2$$

$$4x_1 - 2x_2 \leq 7$$

$$x_1 + 5x_2 \leq 21$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

Questions

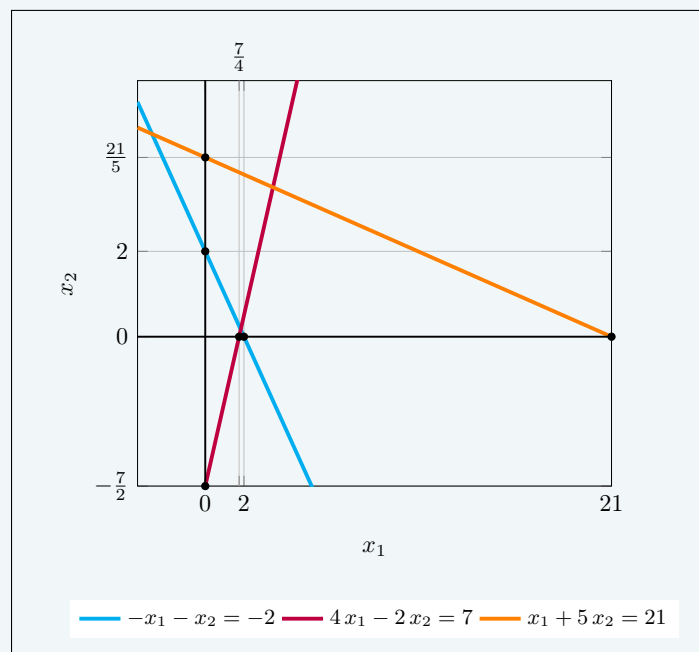
1. Draw a plot of their supporting hyper-planes.
2. Draw the polyhedron given by the intersection of the half-spaces. Is it a polytope?

Solution

1. The supporting hyper-planes associated with the half-spaces and the intersection points with the axis are:

$$\begin{array}{rclcl}
 -x_1 & - & x_2 & = & -2 & \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\
 4x_1 & - & 2x_2 & = & 7 & \begin{pmatrix} 7 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{7}{2} \end{pmatrix} \\
 x_1 & + & 5x_2 & = & 21 & \begin{pmatrix} 21 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{21}{5} \end{pmatrix} \\
 x_1 & & & = & 0 & \\
 & & x_2 & = & 0 &
 \end{array}$$

Their plot is shown below.

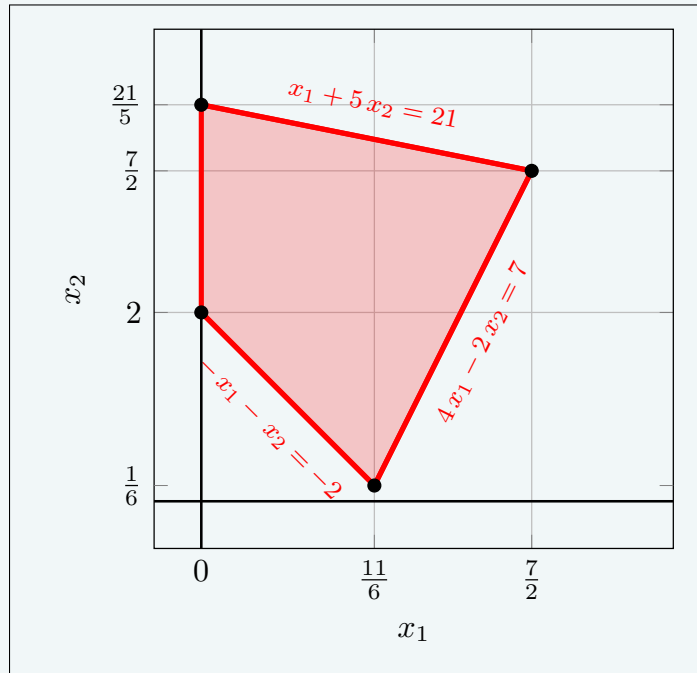


We compute the intersection points:

$$\begin{cases} x_1 + 5x_2 = 21 \\ 4x_1 - 2x_2 = 7 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{21 \cdot (-2) - 5 \cdot 7}{1 \cdot (-2) - 5 \cdot 4} \\ \frac{1 \cdot 7 - 21 \cdot 4}{1 \cdot (-2) - 5 \cdot 4} \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ \frac{7}{2} \end{pmatrix}$$

$$\begin{cases} -x_1 - x_2 = -2 \\ 4x_1 - 2x_2 = 7 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{(-2) \cdot (-2) - (-1) \cdot 7}{(-1) \cdot (-2) - (-1) \cdot 4} \\ \frac{(-1) \cdot 7 - (-2) \cdot 4}{(-1) \cdot (-2) - (-1) \cdot 4} \end{pmatrix} = \begin{pmatrix} \frac{11}{6} \\ \frac{1}{6} \end{pmatrix}$$

2. The polyhedron, given by the intersection of the half-spaces, is:



A polyhedron is a **polytope** if it is bounded (definitions of bounded polyhedron and of polytope in the chapter «Half-spaces and polyhedra»); equivalently, if there exist $\alpha, \beta \in \mathbb{R}^n$ such that $\alpha_j \leq x_j \leq \beta_j$ for every $\mathbf{x} \in P$ and every $j \in \{1, 2, \dots, n\}$.

This polyhedron P is bounded since for every $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in P$, we have for instance:

$$\begin{aligned} 0 &\leq x_1 \leq \frac{7}{2} \\ 0 &\leq x_2 \leq \frac{21}{5} \end{aligned}$$

Accordingly this polyhedron is also a polytope.

Exercise 2.4 — Polyhedra (P41)

Consider the following half-spaces in $\mathbb{R}^3$:

$$\begin{aligned}
 x_1 + x_2 + x_3 &\leq 4 \\
 x_2 &\leq 2 \\
 x_3 &\leq 3 \\
 3x_1 + x_3 &\leq 6 \\
 x_1 &\geq 0 \\
 x_2 &\geq 0 \\
 x_3 &\geq 0
 \end{aligned}$$

Questions

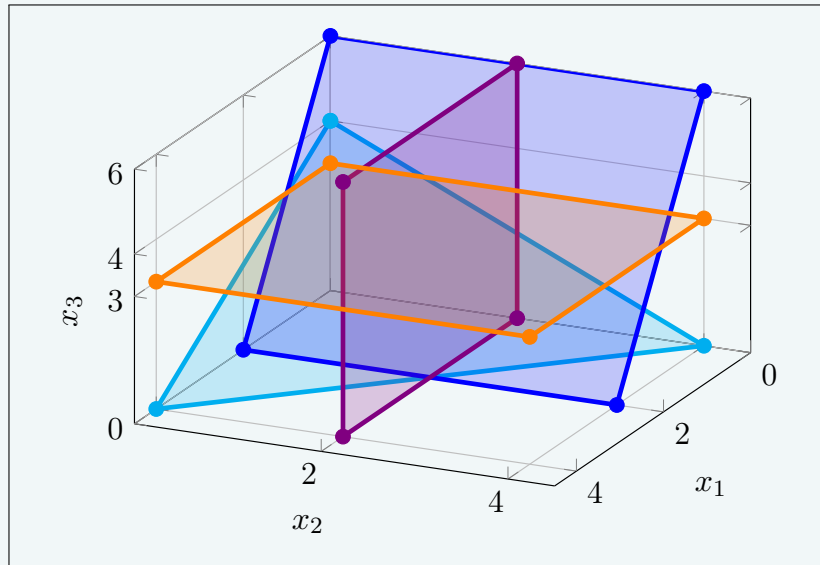
1. Draw a plot of their supporting hyper-planes.
2. Draw the polyhedron given by the intersection of the half-spaces. Is it a polytope?

Solution

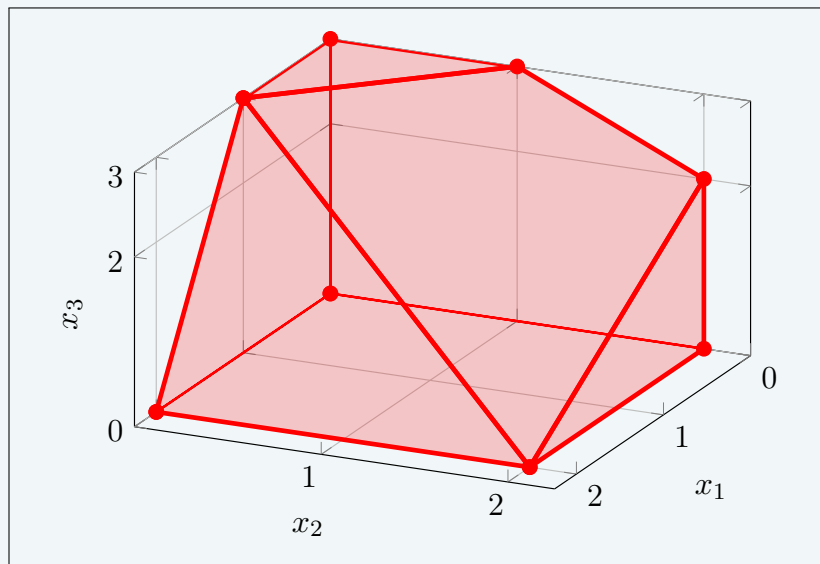
1. The supporting hyper-planes associated with the half-spaces and the intersection points with the axis are:

$$\begin{aligned}
 x_1 + x_2 + x_3 &= 4 & \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \\
 x_2 &= 2 & & \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \\
 x_3 &= 3 & & \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \\
 3x_1 + x_3 &= 6 & \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} \\
 x_1 &= 0 & & \\
 x_2 &= 0 & & \\
 x_3 &= 0 & &
 \end{aligned}$$

Their plot is shown below.



2. The polyhedron given by the intersection of the half-spaces in $\mathbb{R}^3$ is:



A polyhedron is a **polytope** if it is bounded (definitions of bounded polyhedron and of polytope in the chapter «Half-spaces and polyhedra»); equivalently, if there exist $\alpha, \beta \in \mathbb{R}^n$ such that $\alpha_j \leq x_j \leq \beta_j$ for every $\mathbf{x} \in P$ and every $j \in \{1, 2, \dots, n\}$.

This polyhedron P is bounded since for every $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in P$, we have for instance:

$$0 \leq x_1 \leq 2$$

$$0 \leq x_2 \leq 2$$

$$0 \leq x_3 \leq 3$$

Accordingly this polyhedron is also a polytope.

Exercise 2.5 — Polyhedra (P42)

Consider the following half-spaces in $\mathbb{R}^3$:

$$\begin{aligned} x_1 + x_2 + 2x_3 &\leq 4 \\ 2x_1 + 3x_3 &\leq 5 \\ 2x_1 + x_2 + 3x_3 &\leq 7 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_3 &\geq 0 \end{aligned}$$

Questions

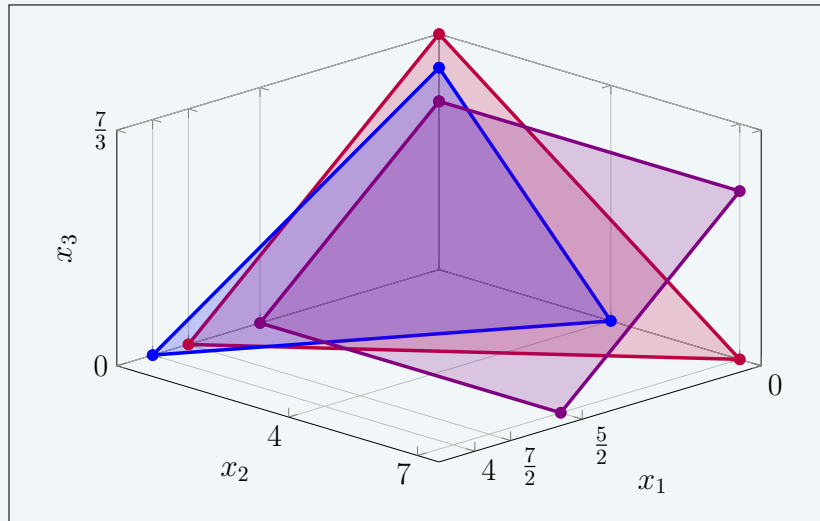
1. Draw a plot of their supporting hyper-planes.
2. Draw the polyhedron given by the intersection of the half-spaces. Is it a polytope?

Solution

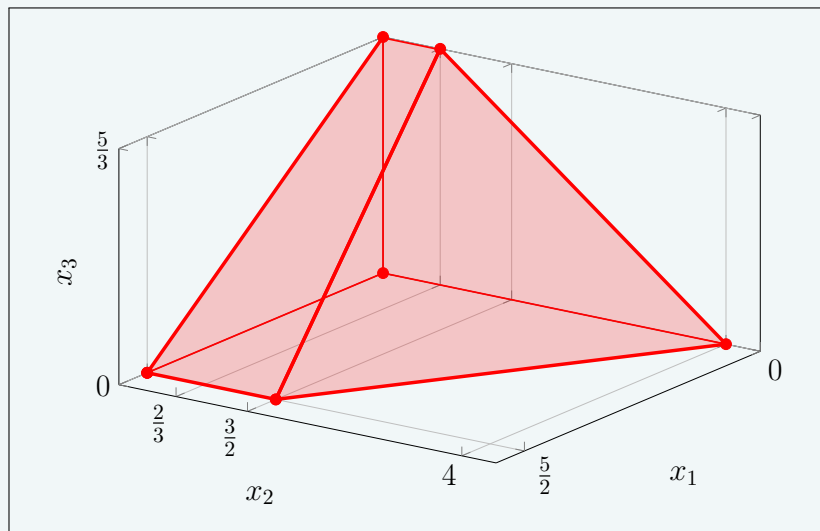
1. The supporting hyper-planes associated with the half-spaces and the intersection points with the axis are:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 4 & \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \\ 2x_1 + 3x_3 &= 5 & \begin{pmatrix} \frac{5}{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{5}{3} \end{pmatrix} \\ 2x_1 + x_2 + 3x_3 &= 7 & \begin{pmatrix} \frac{7}{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{7}{3} \end{pmatrix} \\ x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

Their plot is shown below.



2. The polyhedron given by the intersection of the half-spaces in $\mathbb{R}^3$ is:



A polyhedron is a **polytope** if it is bounded (definitions of bounded polyhedron and of polytope in the chapter «Half-spaces and polyhedra»); equivalently, if there exist $\alpha, \beta \in \mathbb{R}^n$ such that $\alpha_j \leq x_j \leq \beta_j$ for every $\mathbf{x} \in P$ and every $j \in \{1, 2, \dots, n\}$.

This polyhedron P is bounded since for every $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in P$, we have for instance:

$$0 \leq x_1 \leq 10$$

$$0 \leq x_2 \leq 10$$

$$0 \leq x_3 \leq 10$$

Accordingly this polyhedron is also a polytope.

Exercise 2.6 — Polyhedra (P44)

Consider the following half-spaces in $\mathbb{R}^3$:

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &\leq 1 \\ 2x_1 - x_2 - 3x_3 &\leq 4 \\ x_1 + 2x_2 + 5x_3 &\leq 2 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_3 &\geq 0 \end{aligned}$$

Questions

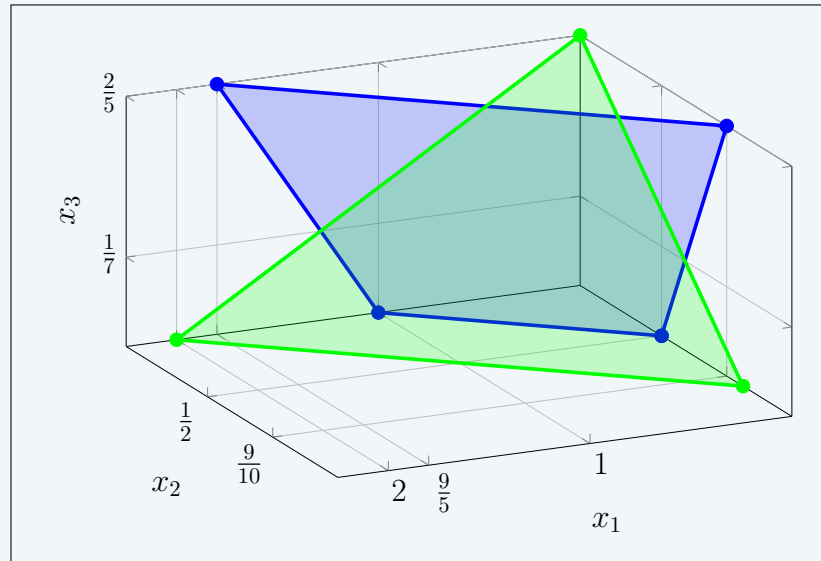
1. Draw a plot of their supporting hyper-planes.
2. Draw the polyhedron given by the intersection of the half-spaces. Is it a polytope?

Solution

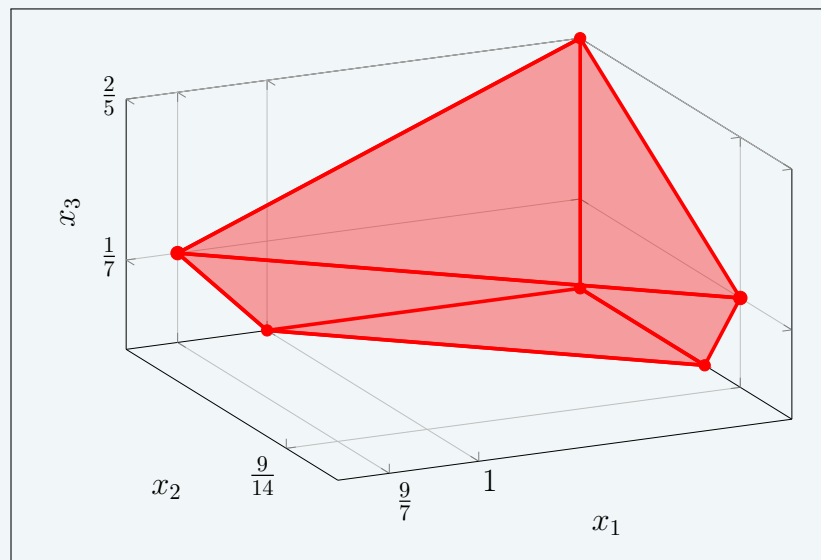
1. The supporting hyper-planes associated with the half-spaces and the intersection points with the axis are:

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 1 & \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix} \\ 2x_1 - x_2 - 3x_3 &= 4 & \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 0 \\ -\frac{4}{3} \end{pmatrix} \\ x_1 + 2x_2 + 5x_3 &= 2 & \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 0 \\ \frac{2}{5} \end{pmatrix} \\ x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

Their plot is shown below.



2. The polyhedron given by the intersection of the half-spaces in $\mathbb{R}^3$ is:



A polyhedron is a **polytope** if it is bounded (definitions of bounded polyhedron and of polytope in the chapter «Half-spaces and polyhedra»); equivalently, if there exist $\alpha, \beta \in \mathbb{R}^n$ such that $\alpha_j \leq x_j \leq \beta_j$ for every $\mathbf{x} \in P$ and every $j \in \{1, 2, \dots, n\}$.

This polyhedron P is bounded since for every $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in P$, we have for instance:

$$0 \leq x_1 \leq 10$$

$$0 \leq x_2 \leq 10$$

$$0 \leq x_3 \leq 10$$

Accordingly this polyhedron is also a polytope.

Exercise 2.7 — Polyhedra (P51)

Consider the following half-spaces in $\mathbb{R}^3$:

$$\begin{aligned} x_1 - x_2 - 3x_3 &\leq -5 \\ -3x_1 - 2x_2 + 2x_3 &\leq -2 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_3 &\geq 0 \end{aligned}$$

Questions

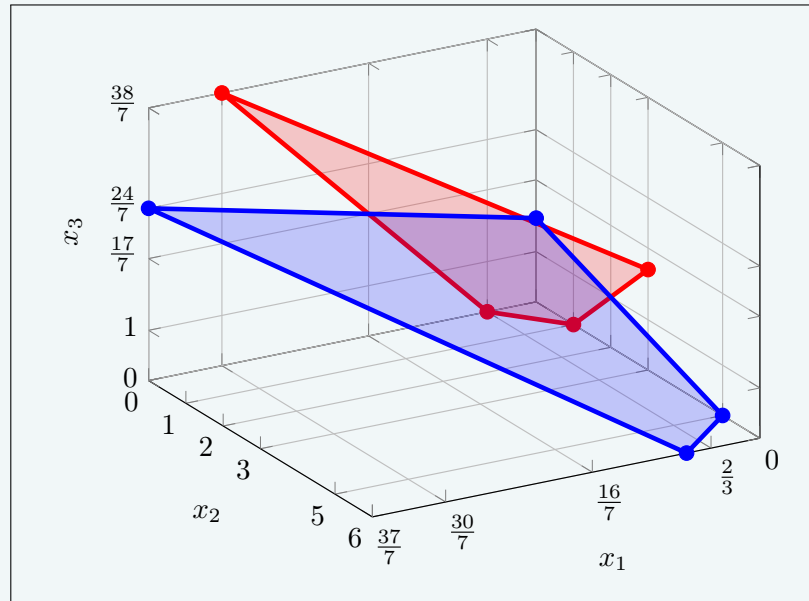
1. Draw a plot of their supporting hyper-planes.
2. Draw the polyhedron given by the intersection of the half-spaces. Is it a polytope?

Solution

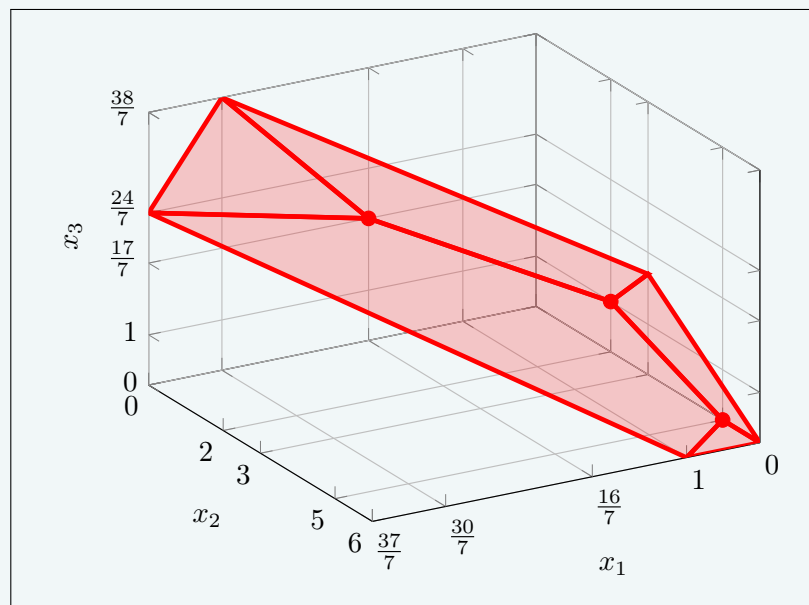
1. The supporting hyper-planes associated with the half-spaces and the intersection points with the axis are:

$$\begin{aligned} x_1 - x_2 - 3x_3 &= -5 & \begin{pmatrix} -5 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 0 \\ \frac{5}{3} \end{pmatrix} \\ -3x_1 - 2x_2 + 2x_3 &= -2 & \begin{pmatrix} \frac{2}{3} \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \\ x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

Their plot is shown below.



2. The polyhedron given by the intersection of the half-spaces in $\mathbb{R}^3$ is:



A polyhedron is a **polytope** if it is bounded (definitions of bounded polyhedron and of polytope in the chapter «Half-spaces and polyhedra»); equivalently, if there exist $\alpha, \beta \in \mathbb{R}^n$ such that $\alpha_j \leq x_j \leq \beta_j$ for every $x \in P$ and every $j \in \{1, 2, \dots, n\}$.

This polyhedron P is not bounded, accordingly it is not a polytope.

2.1 Further reading: drawing polyhedra with SageMath

- The polyhedra of the exercises can be drawn with SageMath, which can be used online at <https://sagecell.sagemath.org/>, without installing anything.
- A polyhedron is given by the list of its half-spaces (the **H-representation**). SageMath writes every half-space in the form

$$b + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq 0$$

and describes it by the tuple $(b, a_1, a_2, \dots, a_n)$. A constraint $\mathbf{a}'\mathbf{x} \leq b$ is first written as $b - \mathbf{a}'\mathbf{x} \geq 0$, so its tuple is $(b, -a_1, \dots, -a_n)$; a non-negativity constraint $x_j \geq 0$ has the tuple with 0 in the first position and 1 in position $j + 1$.

- The method `Hrepresentation()` lists the half-spaces, `Vrepresentation()` lists the extreme points and the extreme rays, and `plot()` draws the polyhedron (in $\mathbb{R}^2$ or in $\mathbb{R}^3$).

Example in $\mathbb{R}^2$ (Exercise with the polyhedron $P2$).

SageMath code: a polyhedron in $\mathbb{R}^2$

```
# Define the polyhedron using inequalities (Hyperplane Representation)
p = Polyhedron(ieqs=[
    (35, 4, -10),      # ( 4, -10) x + 35 >= 0
    (12, -3, -1),      # (-3, -1) x + 12 >= 0
    (0, 1, 0),         # ( 1, 0) x + 0 >= 0
    (0, 0, 1)         # ( 0, 1) x + 0 >= 0
])

# Display the inequalities of the polyhedron
p.Hrepresentation()

# Display the extreme points and extreme rays of the polyhedron
p.Vrepresentation()

# Plot the polyhedron (2D)
p.plot()
```

Example in $\mathbb{R}^3$ (Exercise with the polyhedron $P41$).

SageMath code: a polyhedron in $\mathbb{R}^3$

```
# Define the polyhedron using inequalities (Hyperplane Representation)
p = Polyhedron(ieqs=[
    (4, -1, -1, -1),   # (-1, -1, -1) x + 4 >= 0
    (2, 0, -1, 0),     # (0, -1, 0) x + 2 >= 0
    (3, 0, 0, -1),     # (0, 0, -1) x + 3 >= 0
    (6, -3, 0, -1),    # (-3, 0, -1) x + 6 >= 0
])
```

```
(0, 1, 0, 0), # (1, 0, 0) x + 0 >= 0
(0, 0, 1, 0), # (0, 1, 0) x + 0 >= 0
(0, 0, 0, 1)  # (0, 0, 1) x + 0 >= 0
])

# Display the inequalities of the polyhedron
p.Hrepresentation()

# Display the extreme points and extreme rays of the polyhedron
p.Vrepresentation()

# Plot the polyhedron (3D)
p.plot()
```

The other polyhedra of this chapter are drawn in the same way, by changing the list of tuples.

Linear optimization problems

Exercise 3.1 — Linear optimization problem (E1)

Consider the following linear objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto 3x - 2 \quad \text{and} \quad F = \left\{ x \in \mathbb{R} : -\frac{1}{2} \leq x \leq \frac{9}{2} \right\} = \left[-\frac{1}{2}, \frac{9}{2} \right].$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms.

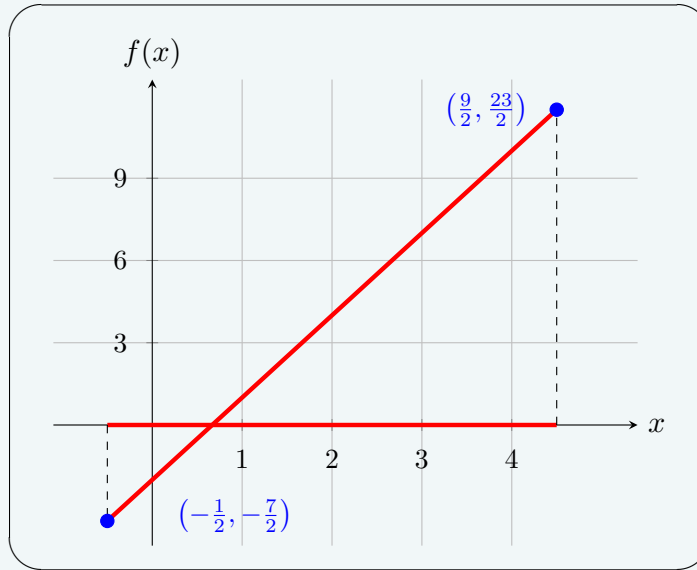
Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ 3x - 2 \text{ subject to } -\frac{1}{2} \leq x \leq \frac{9}{2} \right\}$$

$$\max \left\{ 3x - 2 \text{ subject to } -\frac{1}{2} \leq x \leq \frac{9}{2} \right\}$$

2. The graph of the objective function over the feasible region is red; the feasible region is the red part of the x -axis and the optimal solutions are marked in blue.



3. The feasible region is the interval $F = [-\frac{1}{2}, \frac{9}{2}]$ and the linear function f is strictly increasing: if x increases by 1, $f(x)$ increases by 3.

- For the minimization form, since for every feasible x we have $x \geq -\frac{1}{2}$ and then $f(x) = 3x - 2 \geq -\frac{3}{2} - 2 = -\frac{7}{2} = f(-\frac{1}{2})$, the unique optimal solution and the optimal value are:

$$\tilde{x}(LP) = \left(-\frac{1}{2}\right) \quad \text{and} \quad z(LP) = -\frac{7}{2}.$$

- For the maximization form, since for every feasible x we have $x \leq \frac{9}{2}$ and then $f(x) = 3x - 2 \leq \frac{27}{2} - 2 = \frac{23}{2} = f(\frac{9}{2})$, the unique optimal solution and the optimal value are:

$$\tilde{x}(LP) = \left(\frac{9}{2}\right) \quad \text{and} \quad z(LP) = \frac{23}{2}.$$

Exercise 3.2 — Linear optimization problem (E2)

Consider the following linear objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto 3 - \frac{1}{2}x \quad \text{and} \quad F = \{x \in \mathbb{R} : x \geq -2\} = [-2, +\infty).$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms.

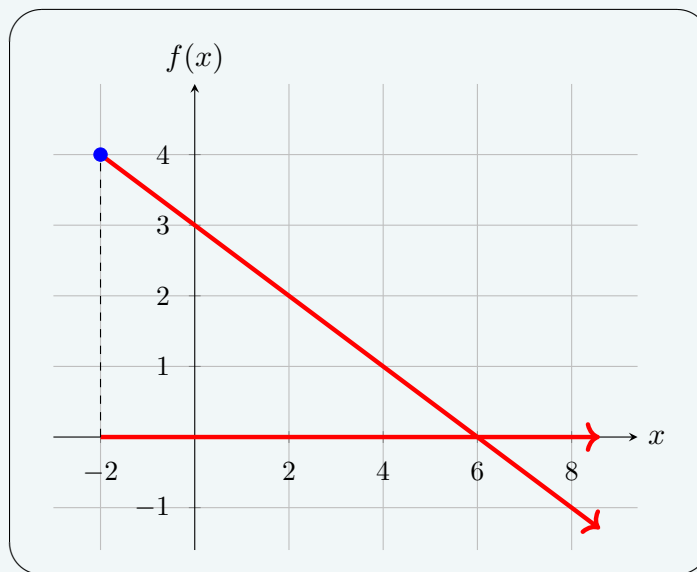
Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ 3 - \frac{1}{2}x \text{ subject to } x \geq -2 \right\}$$

$$\max \left\{ 3 - \frac{1}{2}x \text{ subject to } x \geq -2 \right\}$$

2. The graph of the objective function over the feasible region is red; the feasible region is the red part of the x -axis and the optimal solutions are marked in blue. The feasible region is a half-line: the arrows mean that the graph and the feasible region continue to the right.



3. The feasible region is the half-line $F = [-2, +\infty)$ and the linear function f is strictly decreasing: if x increases by 2, $f(x)$ decreases by 1.

- For the minimization form, the objective function decreases indefinitely on the feasible region: for every $k \in \{1, 2, 3, \dots\}$ the point $x = 2k$ is feasible and $f(2k) = 3 - k$. The problem is **unbounded**, there is no optimal solution and

$$\arg \min \left\{ 3 - \frac{1}{2}x \text{ subject to } x \geq -2 \right\} = \emptyset \quad \text{and} \quad z(LP) = -\infty.$$

- For the maximization form, since for every feasible x we have $x \geq -2$ and then $f(x) = 3 - \frac{1}{2}x \leq 3 + 1 = 4 = f(-2)$, the unique optimal solution and the optimal value are:

$$\tilde{x}(LP) = (-2) \quad \text{and} \quad z(LP) = 4.$$

Exercise 3.3 — Linear optimization problem (E3)

Consider the following linear objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto 2x + 1 \quad \text{and} \quad F = \mathbb{R}.$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms.

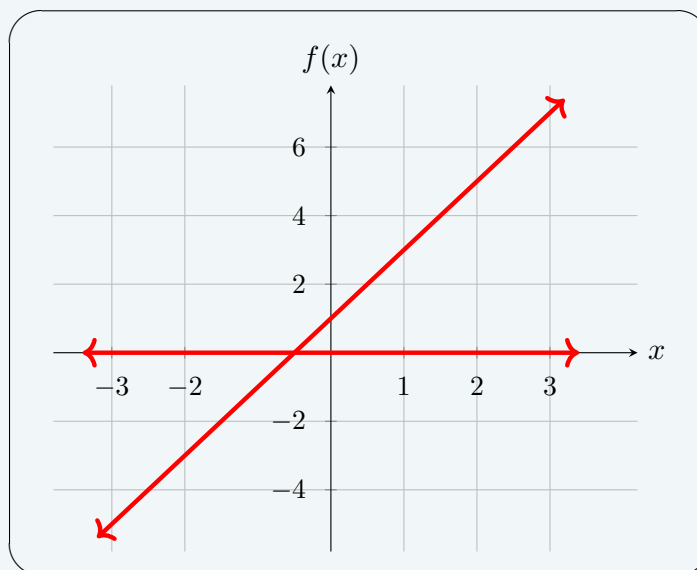
Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ 2x + 1 \text{ subject to } x \in \mathbb{R} \right\}$$

$$\max \left\{ 2x + 1 \text{ subject to } x \in \mathbb{R} \right\}$$

2. The graph of the objective function over the feasible region is red; the feasible region is the whole x -axis (red). The arrows mean that the graph and the feasible region continue in both directions.



3. The feasible region is the whole line $F = \mathbb{R}$ and the linear function f is strictly increasing: if x increases by 1, $f(x)$ increases by 2.

- For the minimization form, the objective function decreases indefinitely on the feasible region: for every $k \in \{1, 2, 3, \dots\}$ the point $x = -k$ is feasible and $f(-k) = 1 - 2k$. The problem is **unbounded**, there is no optimal solution and

$$\arg \min \{ 2x + 1 \text{ subject to } x \in \mathbb{R} \} = \emptyset \quad \text{and} \quad z(LP) = -\infty.$$

- For the maximization form, the objective function increases indefinitely on the feasible region: for every $k \in \{1, 2, 3, \dots\}$ the point $x = k$ is feasible and $f(k) = 2k + 1$. The problem is **unbounded**, there is no optimal solution and

$$\arg \max \{ 2x + 1 \text{ subject to } x \in \mathbb{R} \} = \emptyset \quad \text{and} \quad z(LP) = +\infty.$$

Exercise 3.4 — Linear optimization problem (E4)

Consider the following linear objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto 3 \quad \text{and} \quad F = \{ x \in \mathbb{R} : 1 \leq x \leq 4 \} = [1, 4].$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms.

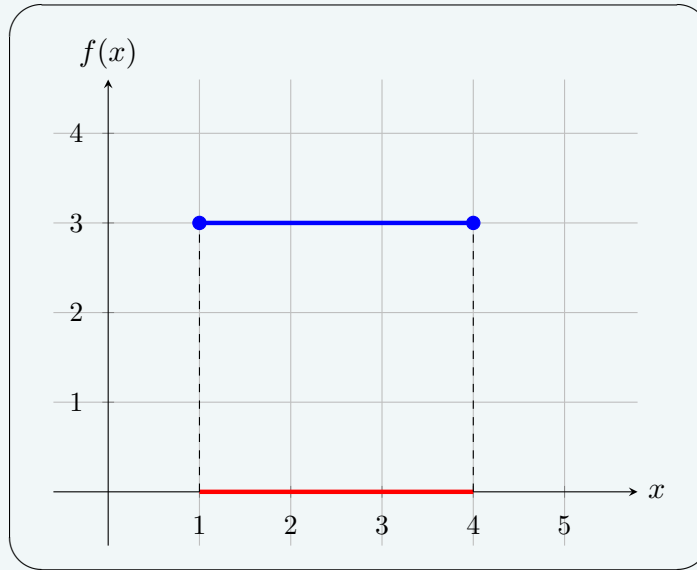
Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ 3 \text{ subject to } 1 \leq x \leq 4 \right\}$$

$$\max \left\{ 3 \text{ subject to } 1 \leq x \leq 4 \right\}$$

2. The graph of the objective function over the feasible region is a horizontal segment, blue since every feasible point is an optimal solution; the feasible region is the red part of the x -axis.



3. The feasible region is the interval $F = [1, 4]$ and the objective function is constant: $f(x) = 3$ for every $x \in F$ (a linear function whose coefficient of x is 0).

- For the minimization form, since $f(x) = 3 \leq 3 = f(y)$ for every $x, y \in F$, every feasible solution is optimal: the problem has **infinitely many** optimal solutions and

$$\arg \min \{ 3 \text{ subject to } 1 \leq x \leq 4 \} = [1, 4] \quad \text{and} \quad z(LP) = 3.$$

- For the maximization form, since $f(x) = 3 \geq 3 = f(y)$ for every $x, y \in F$, every feasible solution is optimal: the problem has **infinitely many** optimal solutions and

$$\arg \max \{ 3 \text{ subject to } 1 \leq x \leq 4 \} = [1, 4] \quad \text{and} \quad z(LP) = 3.$$

- Every point of $[1, 4]$ is at the same time a point of minimum and a point of maximum of f in F , and the minimum and the maximum of f in F coincide.

Exercise 3.5 — Linear optimization problem (E5)

Consider the following linear objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f : x \mapsto -1 \quad \text{and} \quad F = \{ x \in \mathbb{R} : x \geq 2 \} = [2, +\infty).$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms.

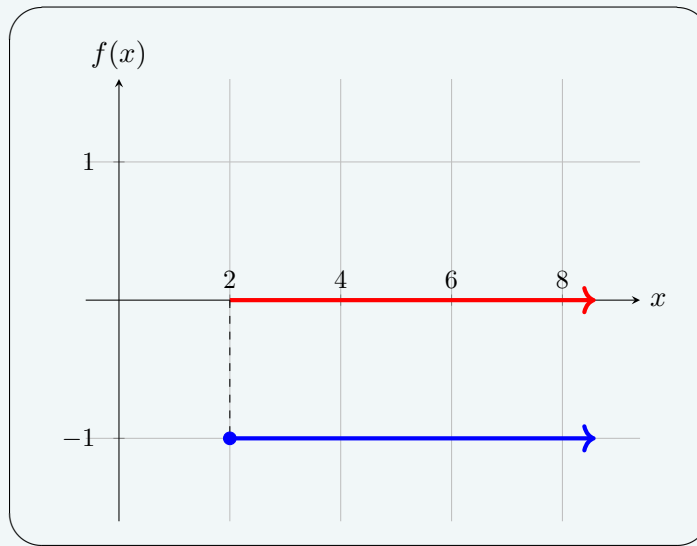
Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ -1 \text{ subject to } x \geq 2 \right\}$$

$$\max \left\{ -1 \text{ subject to } x \geq 2 \right\}$$

2. The graph of the objective function over the feasible region is a horizontal half-line, blue since every feasible point is an optimal solution; the feasible region is the red part of the x -axis. The arrows mean that the graph and the feasible region continue to the right.



3. The feasible region is the half-line $F = [2, +\infty)$ and the objective function is constant: $f(x) = -1$ for every $x \in F$. Although the feasible region is unbounded, the objective function neither decreases nor increases indefinitely.

- For the minimization form, since $f(x) = -1 \leq -1 = f(y)$ for every $x, y \in F$, every feasible solution is optimal: the problem has **infinitely many** optimal solutions and

$$\arg \min \left\{ -1 \text{ subject to } x \geq 2 \right\} = [2, +\infty) \quad \text{and} \quad z(LP) = -1.$$

- For the maximization form, since $f(x) = -1 \geq -1 = f(y)$ for every $x, y \in F$, every feasible solution is optimal: the problem has **infinitely many** optimal solutions and

$$\arg \max \left\{ -1 \text{ subject to } x \geq 2 \right\} = [2, +\infty) \quad \text{and} \quad z(LP) = -1.$$

- The same holds with the feasible region $F = \mathbb{R}$: every $x \in \mathbb{R}$ is optimal in both forms and $z(LP) = -1$. Compare with the previous two exercises, where a non-constant linear objective function on a half-line or on the whole line makes the problem unbounded.

Exercise 3.6 — Linear optimization problem (E6)

Consider the following linear objective function and feasible region:

$$f : \mathbb{R} \rightarrow \mathbb{R}, f : x \mapsto x + 1 \quad \text{and} \quad F = \{x \in \mathbb{R} : x \geq 3 \text{ and } x \leq 1\}.$$

Questions

1. Write the associated optimization problems in minimization and maximization forms.
2. Draw the graph of the objective function over the feasible region.
3. Find all optimal solutions and the optimal value for both forms.

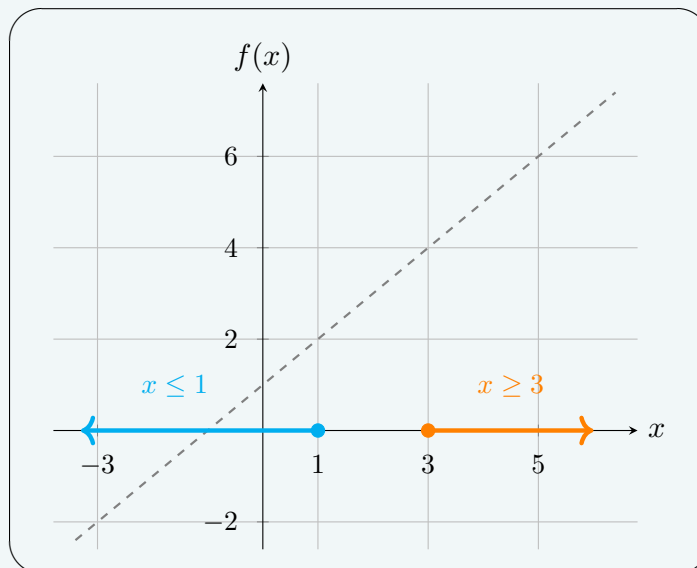
Solution

1. The associated optimization problems in minimization and maximization forms are:

$$\min \left\{ x + 1 \text{ subject to } x \geq 3, x \leq 1 \right\}$$

$$\max \left\{ x + 1 \text{ subject to } x \geq 3, x \leq 1 \right\}$$

2. The two constraints define the half-lines $[3, +\infty)$ (orange) and $(-\infty, 1]$ (cyan) on the x -axis: they have no common point, so the feasible region is empty and the graph of the objective function over the feasible region is empty too. The dashed line is the graph of f on the whole line, which is never reached by a feasible point.



3. The feasible region is empty: no $x \in \mathbb{R}$ satisfies at the same time $x \geq 3$ and $x \leq 1$ (it would give $3 \leq x \leq 1$). Hence $F = \emptyset$.

- For the minimization form, the problem is **infeasible**: there is no feasible solution, hence no optimal solution, and

$$\arg \min \{ x + 1 \text{ subject to } x \geq 3, x \leq 1 \} = \emptyset \quad \text{and} \quad z(LP) \text{ is not defined.}$$

- For the maximization form, the problem is **infeasible**: there is no feasible solution, hence no optimal solution, and

$$\arg \max \{ x + 1 \text{ subject to } x \geq 3, x \leq 1 \} = \emptyset \quad \text{and} \quad z(LP) \text{ is not defined.}$$

Linear Programming problems

Exercise 4.1 — Graphical solution method (P_0)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 2x_1 + 4x_2 \\ \text{subject to} \quad & 2x_1 + 3x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Questions

1. Write the objective function of the LP problem and compute its gradient.
2. Write the feasible region of the LP problem.
3. Find graphically an optimal solution of this LP problem. Determine the optimal value.

Solution

- The objective function of the LP problem is the scalar linear function:

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto 2x_1 + 4x_2 \quad \text{its gradient is} \quad \nabla f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

- The feasible region of the LP problem is the polyhedron:

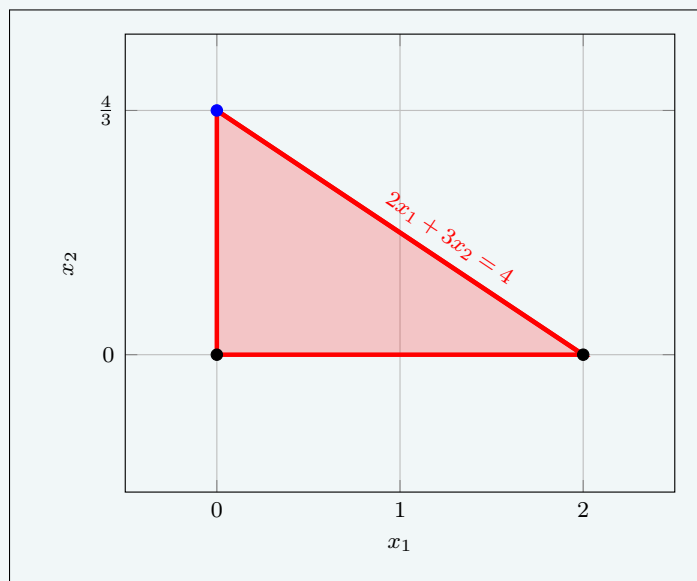
$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : 2x_1 + 3x_2 \leq 4, \quad x_1 \geq 0, \quad x_2 \geq 0 \right\}.$$

It is a bounded polyhedron, since, for instance, we have:

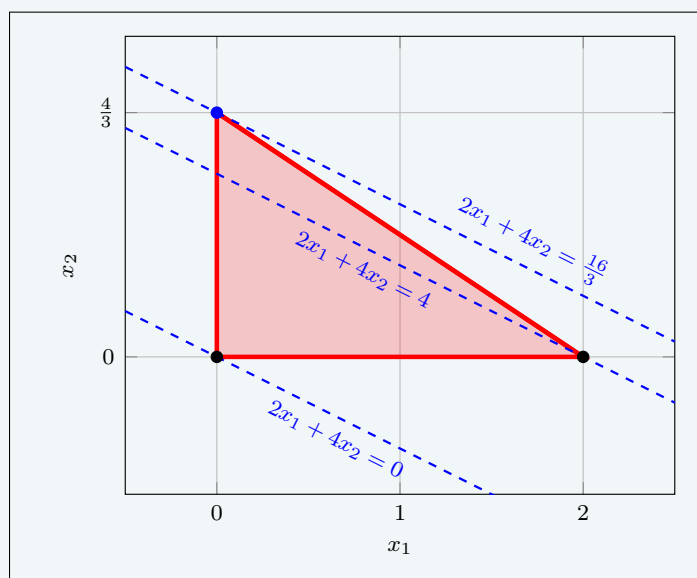
$$\begin{aligned} 0 &\leq x_1 \leq 2, \quad \forall \mathbf{x} \in F \\ 0 &\leq x_2 \leq \frac{4}{3}, \quad \forall \mathbf{x} \in F \end{aligned}$$

then it is also a polytope.

- This is the polyhedron of the feasible region:



These are the contour lines at the vertices of the polyhedron:



An optimal solution $\tilde{x}(LP)$ is:

$$\tilde{x}_1 = 0, \quad \tilde{x}_2 = \frac{4}{3}$$

the optimal value $z(LP)$ is $\frac{16}{3}$.

Exercise 4.2 — Graphical solution method (P2)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & & x_1 & + & 3 & x_2 \\ \text{subject to} & -4 & x_1 & + & 10 & x_2 \leq 35 \\ & & 3 & x_1 & + & x_2 \leq 12 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

Questions

1. Write the objective function of the LP problem and compute its gradient.
2. Write the feasible region of the LP problem.
3. Find graphically an optimal solution of this LP problem. Determine the optimal value.

Solution

- The objective function of the LP problem is the scalar linear function:

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 + 3x_2 \quad \text{its gradient is} \quad \nabla f \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

- The feasible region of the LP problem is the polyhedron:

$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -4x_1 + 10x_2 \leq 35, \quad 3x_1 + x_2 \leq 12, \quad x_1 \geq 0, \quad x_2 \geq 0 \right\}.$$

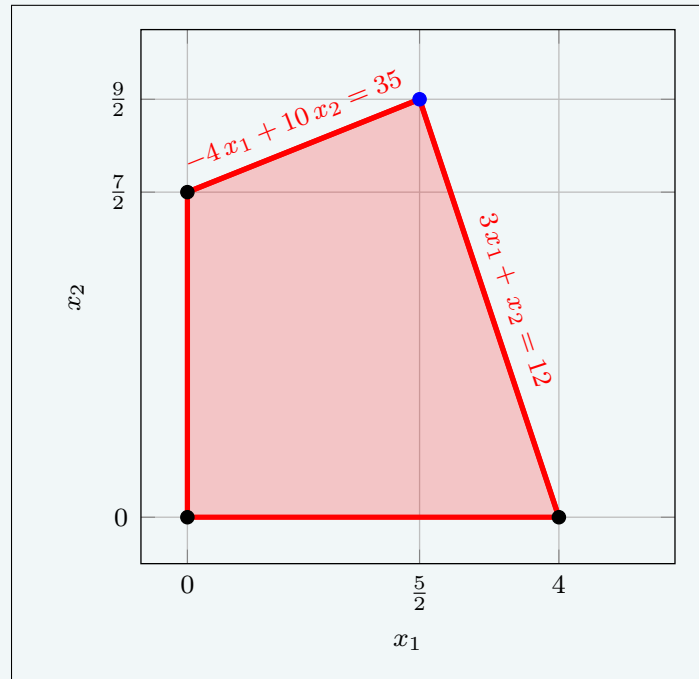
It is a bounded polyhedron, since, for instance, we have:

$$0 \leq x_1 \leq 4, \quad \forall \mathbf{x} \in F$$

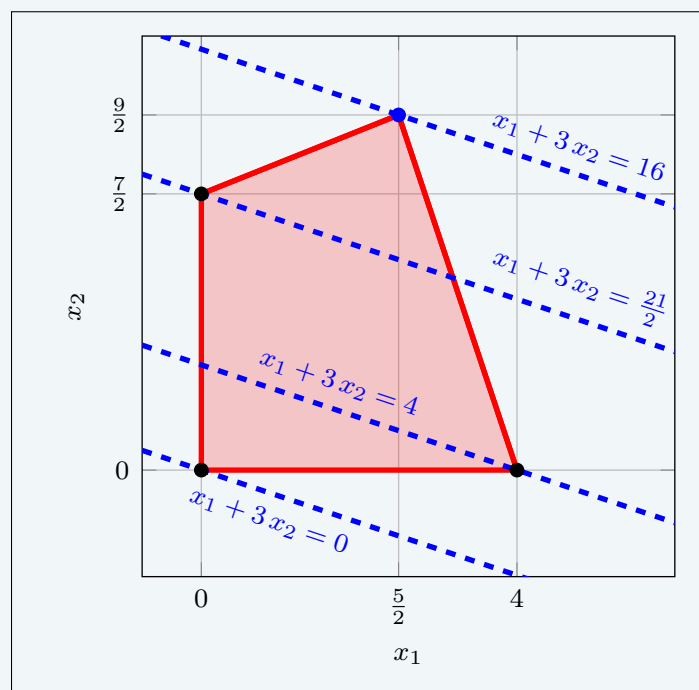
$$0 \leq x_2 \leq 5, \quad \forall \mathbf{x} \in F$$

then it is also a polytope.

- This is the polyhedron of the feasible region:



These are the contour lines at the vertices of the polyhedron:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{5}{2}, \quad \tilde{x}_2 = \frac{9}{2}$$

the optimal value $z(LP)$ is 16.

Exercise 4.3 — Graphical solution method (P3)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & 2x_1 + 4x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 2 \\ & 8x_1 + 2x_2 \leq 19 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Questions

1. Write the objective function of the LP problem and compute its gradient.
2. Write the feasible region of the LP problem.
3. Find graphically an optimal solution of this LP problem. Determine the optimal value.

Solution

- The objective function of the LP problem is the scalar linear function:

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto 2x_1 + 4x_2 \quad \text{its gradient is} \quad \nabla f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

- The feasible region of the LP problem is the polyhedron:

$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -x_1 + x_2 \leq 2, \quad 8x_1 + 2x_2 \leq 19, \quad x_1 \geq 0, \quad x_2 \geq 0 \right\}.$$

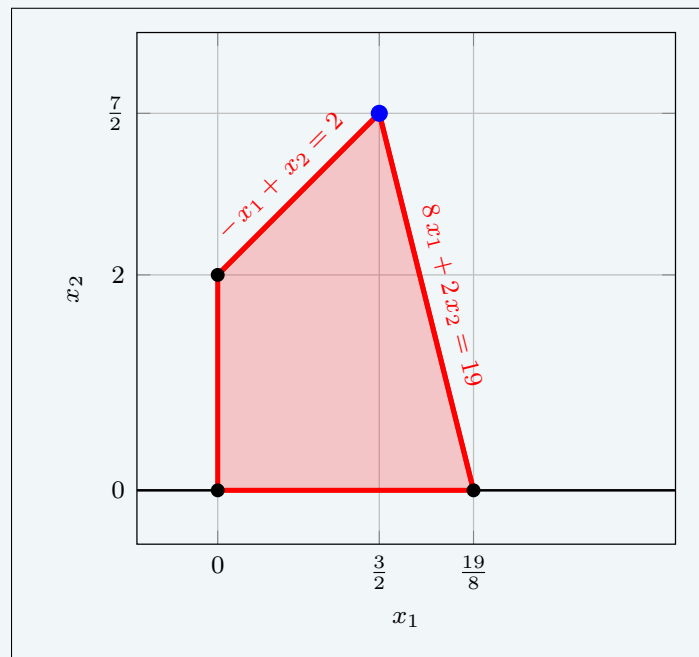
It is a bounded polyhedron, since, for instance, we have:

$$0 \leq x_1 \leq 4, \quad \forall \mathbf{x} \in F$$

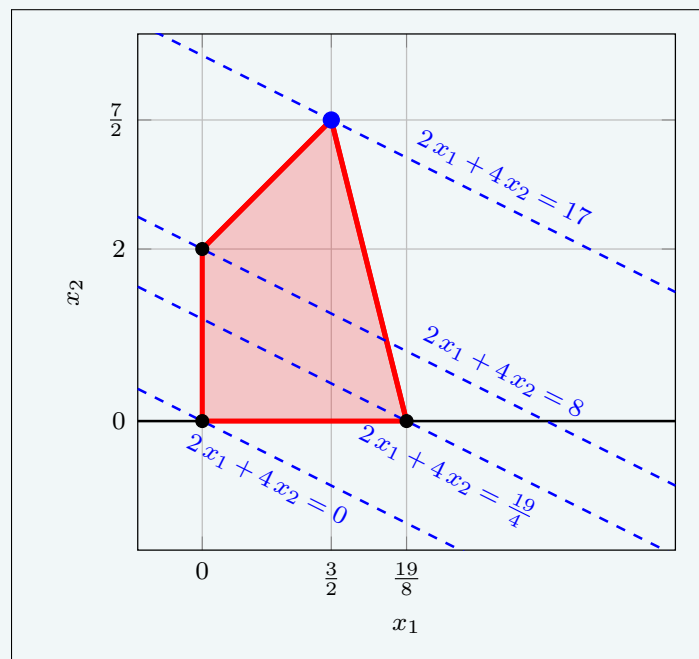
$$0 \leq x_2 \leq 4, \quad \forall \mathbf{x} \in F$$

then it is also a polytope.

- This is the polyhedron of the feasible region:



These are the contour lines at the vertices of the polyhedron:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{3}{2}, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value $z(LP)$ is 17.

Exercise 4.4 — Graphical solution method (P10)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{aligned} \max \quad & 5x_1 + 2x_2 \\ \text{subject to} \quad & -x_1 + 3x_2 \leq 12 \\ & x_1 + 2x_2 \leq 12 \\ & 3x_1 - 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Questions

1. Write the objective function of the LP problem and compute its gradient.
2. Write the feasible region of the LP problem.
3. Find graphically an optimal solution of this LP problem. Determine the optimal value.

Solution

- The objective function of the LP problem is the scalar linear function:

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto 5x_1 + 2x_2 \quad \text{its gradient is} \quad \nabla f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

- The feasible region of the LP problem is the polyhedron:

$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : \begin{array}{l} -x_1 + 3x_2 \leq 12, \quad x_1 + 2x_2 \leq 12, \quad 3x_1 - 2x_2 \leq 8, \\ x_1 \geq 0, \quad x_2 \geq 0 \end{array} \right\}.$$

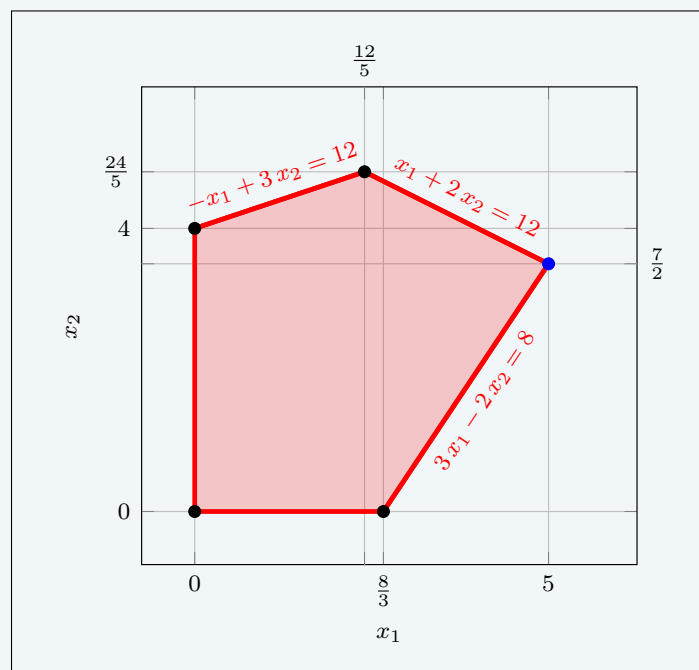
It is a bounded polyhedron, since, for instance, we have:

$$0 \leq x_1 \leq 5, \quad \forall \mathbf{x} \in F$$

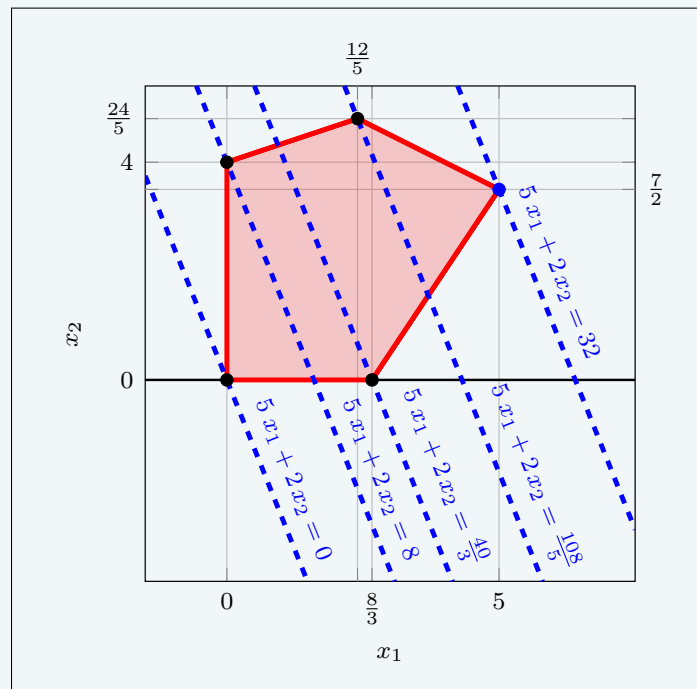
$$0 \leq x_2 \leq 5, \quad \forall \mathbf{x} \in F$$

then it is also a polytope.

- This is the polyhedron of the feasible region:



These are the contour lines at the vertices of the polyhedron:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 5, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value $z(LP)$ is 32.

Exercise 4.5 — Graphical solution method (P11)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & -2x_1 - 4x_2 \\ \text{subject to} \quad & -x_1 - x_2 \leq -2 \\ & x_1 - x_2 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Questions

1. Write the objective function of the LP problem and compute its gradient.
2. Write the feasible region of the LP problem.
3. Find graphically an optimal solution of this LP problem. Determine the optimal value.
4. Verify that $\bar{x} = (1, 2)'$ is a feasible solution and compute the lower bound it gives on the optimal value.

Solution

- The objective function of the LP problem is the scalar linear function:

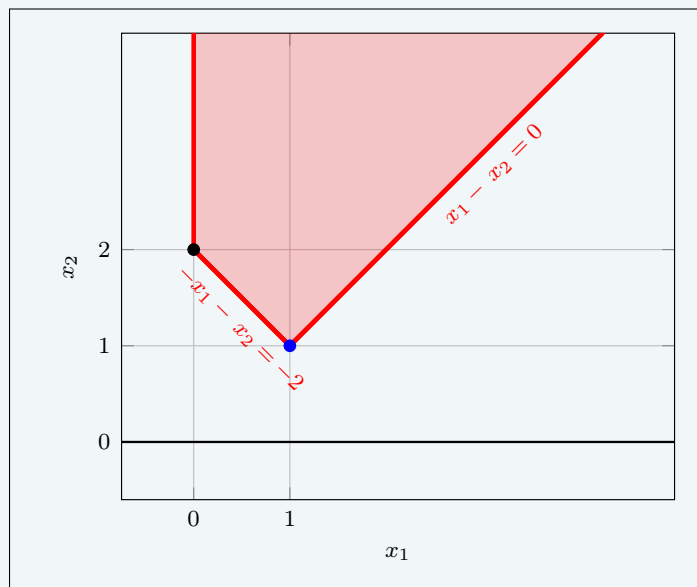
$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto -2x_1 - 4x_2 \quad \text{its gradient is} \quad \nabla f \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} -2 \\ -4 \end{pmatrix}.$$

- The feasible region of the LP problem is the polyhedron:

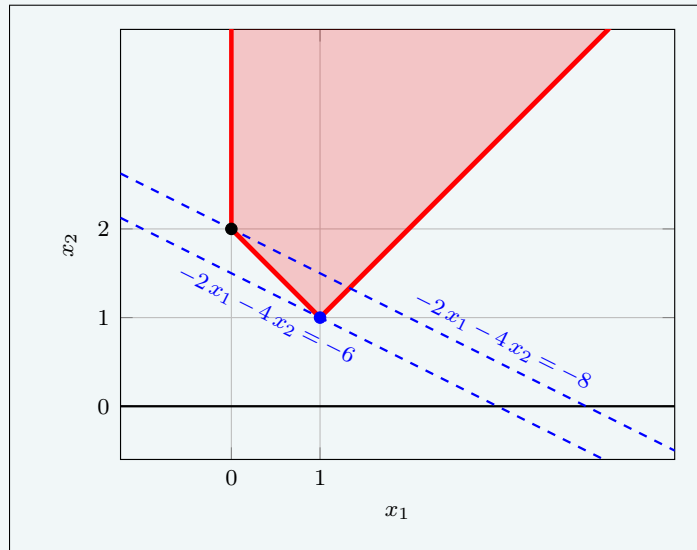
$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -x_1 - x_2 \leq -2, \quad x_1 - x_2 \leq 0, \quad x_1 \geq 0, \quad x_2 \geq 0 \right\}.$$

It is not a bounded polyhedron, then it is not a polytope.

- This is the polyhedron of the feasible region:



These are the contour lines at the vertices of the polyhedron:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1$$

the optimal value $z(LP)$ is -6 .

- The point $\bar{\mathbf{x}}(LP) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (where $\bar{x}_1 = 1$ and $\bar{x}_2 = 2$) is a feasible solution, since:

$$-1 - 2 = -3 \leq -2, \quad 1 - 2 = -1 \leq 0, \quad 1 \geq 0 \quad \text{and} \quad 2 \geq 0.$$

Accordingly, a lower bound $LB(\bar{\mathbf{x}})$ on the optimal value $z(LP)$ of the LP problem is:

$$LB(\bar{\mathbf{x}}) = -2 \cdot 1 - 4 \cdot 2 = -10 \leq z(LP) = -6.$$

Exercise 4.6 — Graphical solution method (P12)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{aligned} \max \quad & -x_1 - 2x_2 \\ \text{subject to} \quad & -x_1 - x_2 \leq -3 \\ & x_1 - 2x_2 \leq -1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Questions

1. Write the objective function of the LP problem and compute its gradient.
2. Write the feasible region of the LP problem.
3. Find graphically an optimal solution of this LP problem. Determine the optimal value.

Solution

- The objective function of the LP problem is the scalar linear function:

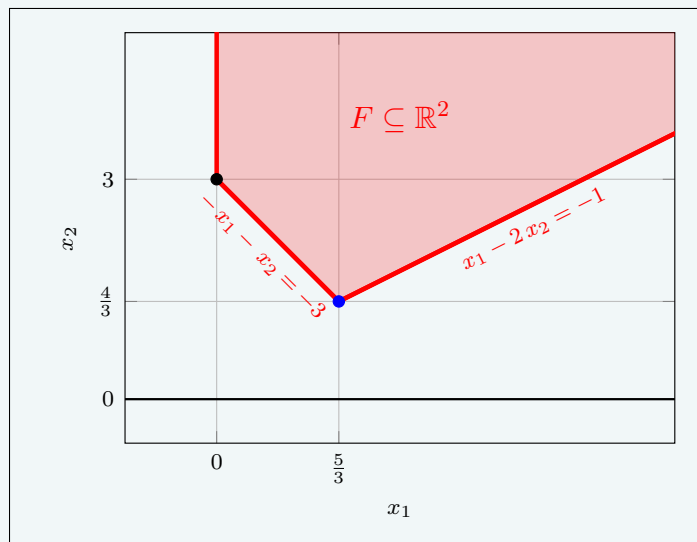
$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto -x_1 - 2x_2 \quad \text{its gradient is} \quad \nabla f \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

- The feasible region of the LP problem is the polyhedron:

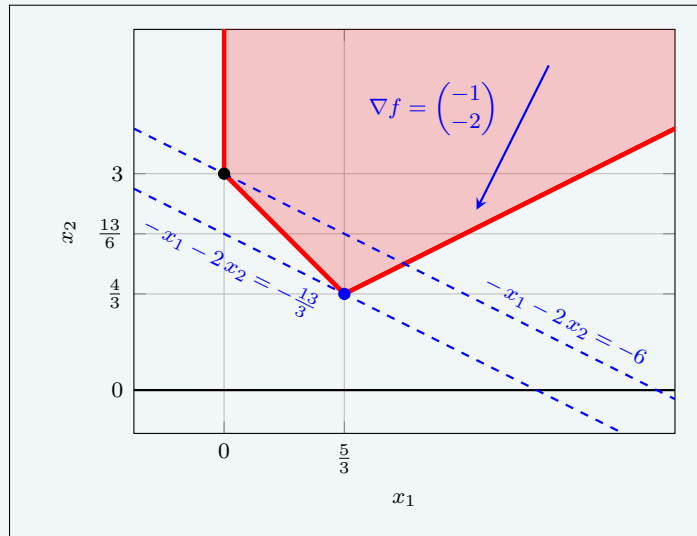
$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -x_1 - x_2 \leq -3, \quad x_1 - 2x_2 \leq -1, \quad x_1 \geq 0, \quad x_2 \geq 0 \right\}.$$

It is not a bounded polyhedron, since, for instance, the points $\begin{pmatrix} t \\ t \end{pmatrix}$ belong to F for every value $t \geq 3$, then it is not a polytope.

- This is the polyhedron of the feasible region:



These are the contour lines at the vertices of the polyhedron:



Since $-x_1 - 2x_2 \leq 0$ along every direction in which F is unbounded, the objective function attains its maximum at a vertex of F . An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = \frac{5}{3}, \quad \tilde{x}_2 = \frac{4}{3}$$

the optimal value $z(LP)$ is:

$$z(LP) = -\frac{5}{3} - 2 \cdot \frac{4}{3} = -\frac{13}{3}.$$

Exercise 4.7 — Graphical solution method (P21)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{array}{llllll} \max & x_1 & + & 2x_2 & & \\ \text{subject to} & -x_1 & - & x_2 & \leq & -2 \\ & 4x_1 & - & 2x_2 & \leq & 7 \\ & x_1 & + & 5x_2 & \leq & 21 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

Questions

1. Write the objective function of the LP problem and compute its gradient.
2. Write the feasible region of the LP problem.
3. Find graphically an optimal solution of this LP problem. Determine the optimal value.

Solution

- The objective function of the LP problem is the scalar linear function:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 + 2x_2 \quad \text{its gradient is} \quad \nabla f \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

- The feasible region of the LP problem is the polyhedron:

$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : \begin{array}{l} -x_1 - x_2 \leq -2, \quad 4x_1 - 2x_2 \leq 7, \quad x_1 + 5x_2 \leq 21, \\ x_1 \geq 0, \quad x_2 \geq 0 \end{array} \right\}.$$

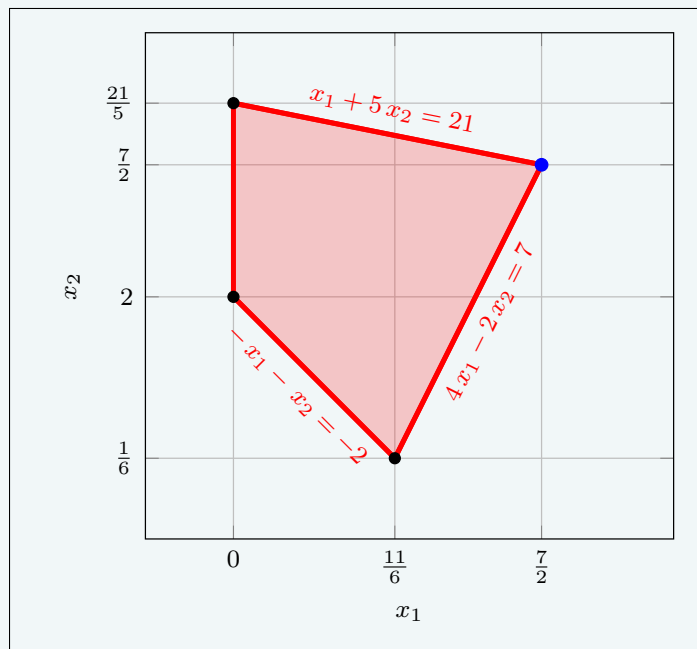
It is a bounded polyhedron, since, for instance, we have:

$$0 \leq x_1 \leq 5, \quad \forall \mathbf{x} \in F$$

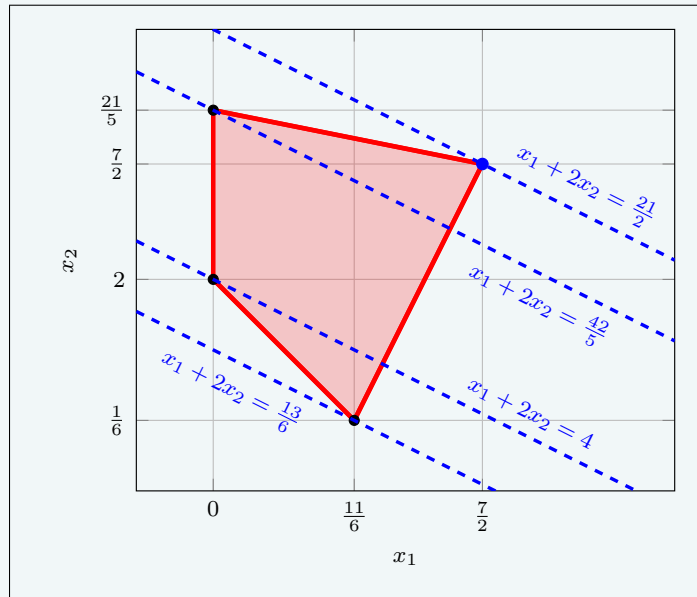
$$0 \leq x_2 \leq 5, \quad \forall \mathbf{x} \in F$$

then it is also a polytope.

- This is the polyhedron of the feasible region:



These are the contour lines at the vertices of the polyhedron:



An optimal solution $\tilde{x}(LP)$ is:

$$\tilde{x}_1 = \frac{7}{2}, \quad \tilde{x}_2 = \frac{7}{2}$$

the optimal value $z(LP)$ is $\frac{21}{2}$.

Exercise 4.8 — Graphical solution method (P22)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 3$ constraints:

$$\begin{array}{llllll} \max & 3x_1 & + & x_2 & & \\ \text{subject to} & x_1 & - & x_2 & \leq & -1 \\ & -x_1 & - & x_2 & \leq & -3 \\ & 2x_1 & + & x_2 & \leq & 4 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

Questions

1. Write the objective function of the LP problem and compute its gradient.

2. Write the feasible region of the LP problem.
3. Find graphically an optimal solution of this LP problem. Determine the optimal value.

Solution

- The objective function of the LP problem is the scalar linear function:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto 3x_1 + x_2 \quad \text{its gradient is} \quad \nabla f \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

- The feasible region of the LP problem is the polyhedron:

$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : \begin{array}{l} x_1 - x_2 \leq -1, \quad -x_1 - x_2 \leq -3, \quad 2x_1 + x_2 \leq 4, \\ x_1 \geq 0, \quad x_2 \geq 0 \end{array} \right\}.$$

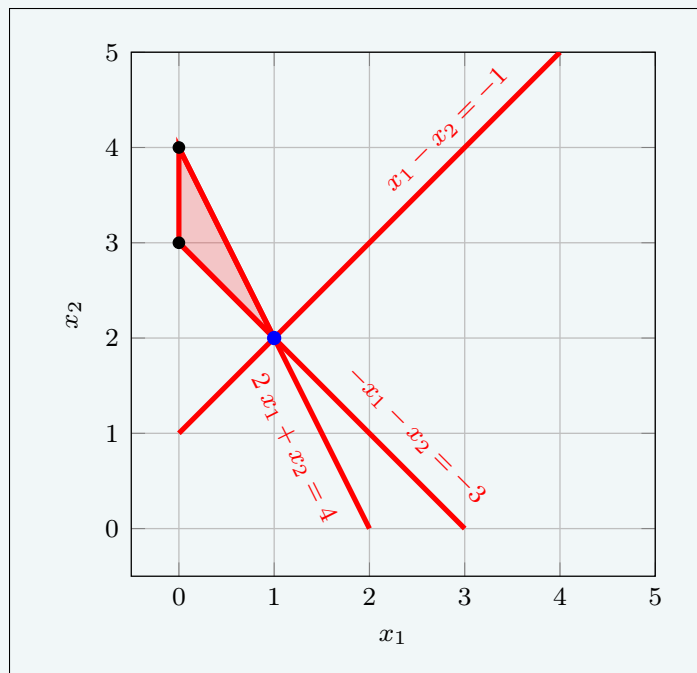
It is a bounded polyhedron, since, for instance, we have:

$$0 \leq x_1 \leq 5, \quad \forall \mathbf{x} \in F$$

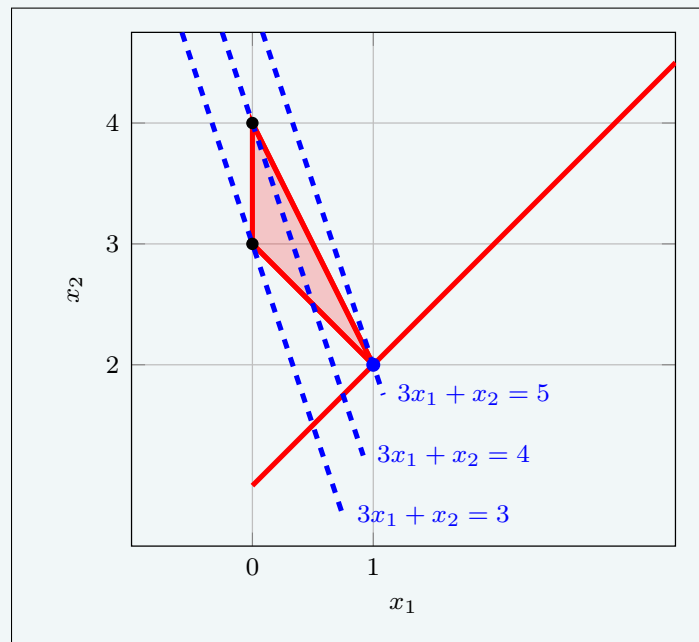
$$0 \leq x_2 \leq 5, \quad \forall \mathbf{x} \in F$$

then it is also a polytope.

- This is the polyhedron of the feasible region:



These are the contour lines at the vertices of the polyhedron:



An optimal solution $\tilde{\mathbf{x}}(LP)$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 2$$

the optimal value $z(LP)$ is 5.

Possible outcomes and equivalent formulations

5.1 Graphical method

Exercise 5.1 — Graphical solution method (*P31*)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & & x_1 & + & x_2 & \\ \text{subject to} & - & x_1 & + & x_2 & \leq 1 \\ & & x_1 & - & x_2 & \leq 1 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

Questions

1. Write the objective function of the LP problem and compute its gradient.
2. Write the feasible region of the LP problem.
3. Determine graphically whether the LP problem admits an optimal solution. If it does, find an optimal solution and the optimal value; otherwise, state whether the LP problem is infeasible or unbounded.

Solution

- The objective function of the LP problem is the scalar linear function:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 + x_2 \quad \text{its gradient is} \quad \nabla f \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

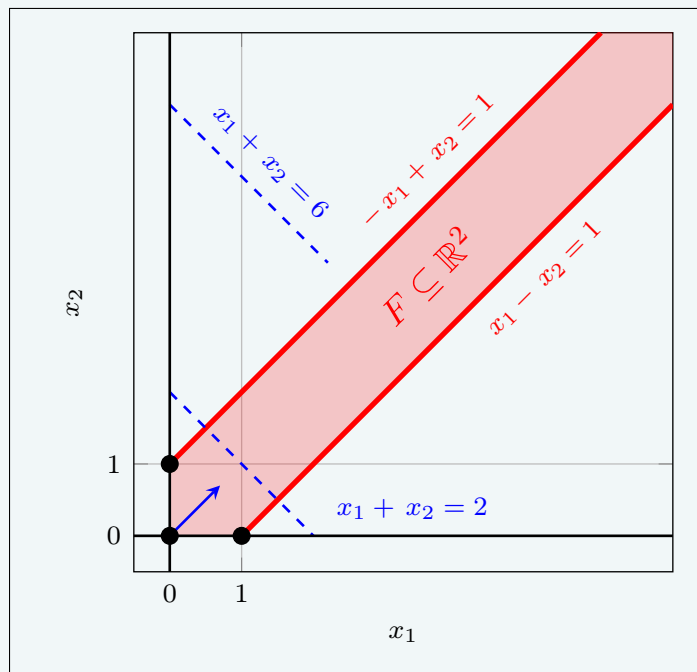
- The feasible region of the LP problem is the polyhedron:

$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -x_1 + x_2 \leq 1, \quad x_1 - x_2 \leq 1, \quad x_1 \geq 0, \quad x_2 \geq 0 \right\}.$$

- For every value $t \geq 0$, the point $\bar{\mathbf{x}}(LP) = \begin{pmatrix} t \\ t \end{pmatrix}$ is a feasible solution, since

$$-t + t = 0 \leq 1, \quad t - t = 0 \leq 1, \quad t \geq 0 \quad \text{and} \quad t \geq 0,$$

and its objective value is $LB(\bar{\mathbf{x}}) = t + t = 2t$. Accordingly, for every value $M \in \mathbb{R}$, the feasible solution obtained with $t > \frac{M}{2}$ has an objective value larger than M , and then this LP problem is **unbounded**.



Exercise 5.2 — Graphical solution method (P33)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & x_1 & + & x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \leq & 2 \\ & -x_1 & - & x_2 & \leq & -4 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

Questions

1. Write the objective function of the LP problem and compute its gradient.
2. Write the feasible region of the LP problem.
3. Determine graphically whether the LP problem admits an optimal solution. If it does, find an optimal solution and the optimal value; otherwise, state whether the LP problem is infeasible or unbounded.

Solution

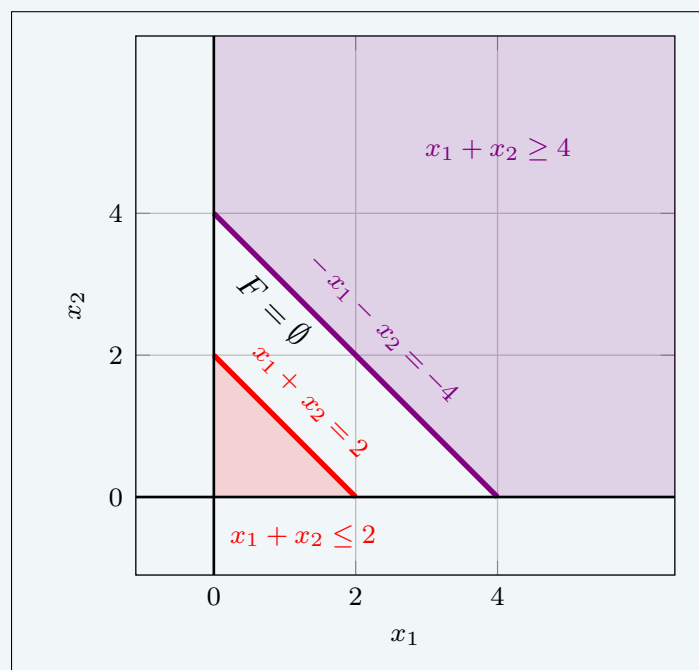
- The objective function of the LP problem is the scalar linear function:

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 + x_2 \quad \text{its gradient is} \quad \nabla f \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- The feasible region of the LP problem is the polyhedron:

$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 + x_2 \leq 2, \quad -x_1 - x_2 \leq -4, \quad x_1 \geq 0, \quad x_2 \geq 0 \right\}.$$

- The first constraint imposes $x_1 + x_2 \leq 2$, while the second one, multiplied by -1 , imposes $x_1 + x_2 \geq 4$. No point of $\mathbb{R}^2$ satisfies both of them, and then we have $F = \emptyset$: this LP problem is **infeasible**.



Exercise 5.3 — Graphical solution method (P36)

Consider the following LP problem in canonical maximization form with $n = 2$ variables and $m = 2$ constraints:

$$\begin{array}{llllll} \max & - & x_1 & - & x_2 & \\ \text{subject to} & - & x_1 & - & x_2 & \leq -2 \\ & & x_1 & - & x_2 & \leq 2 \\ & & x_1, & & x_2 & \geq 0 \end{array}$$

Questions

1. Write the objective function of the LP problem and compute its gradient.
2. Write the feasible region of the LP problem.
3. Determine graphically whether the LP problem admits an optimal solution. If it does, find an optimal solution and the optimal value; otherwise, state whether the LP problem is infeasible or unbounded.

Solution

- The objective function of the LP problem is the scalar linear function:

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto -x_1 - x_2 \quad \text{its gradient is} \quad \nabla f \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

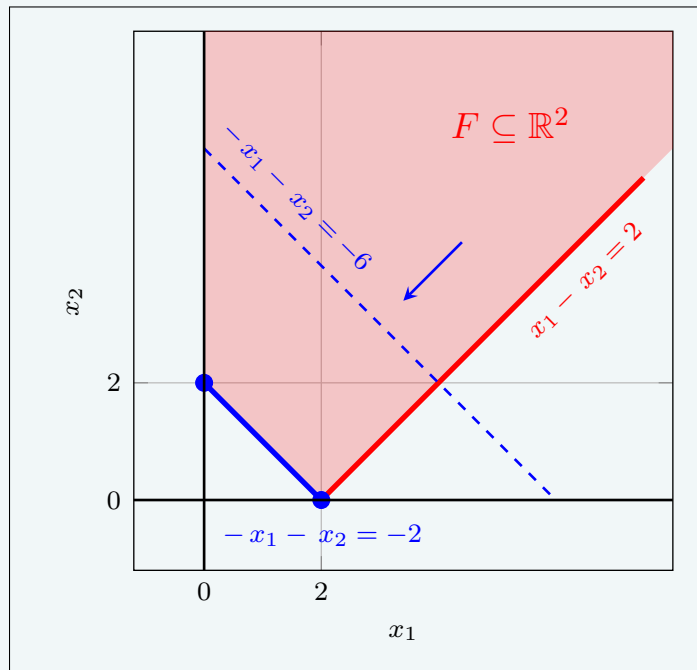
- The feasible region of the LP problem is the polyhedron:

$$F = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : -x_1 - x_2 \leq -2, \quad x_1 - x_2 \leq 2, \quad x_1 \geq 0, \quad x_2 \geq 0 \right\}.$$

- The feasible region F is unbounded, but the LP problem is not unbounded. The first constraint imposes $x_1 + x_2 \geq 2$ for every feasible solution, and then

$$-x_1 - x_2 \leq -2, \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in F.$$

The two feasible solutions $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ attain this value, and then they are two **optimal solutions** and the optimal value is $z(LP) = -2$. Moreover, every point of the segment joining them is an optimal solution as well, and then this LP problem admits **infinitely many optimal solutions**.



5.2 Equivalences

Exercise 5.4 — Equivalent reformulations (E1)

Consider the following LP problem in minimization form with $n = 2$ non-negative variables and $m = 2$ greater-than-or-equal-to linear constraints:

$$\begin{aligned} \min \quad & 2x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 4 \\ & 3x_1 + x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Questions

1. Write an equivalent LP problem in canonical maximization form.
2. Write an equivalent LP problem in standard maximization form.
3. Knowing that an optimal solution of the LP problem in canonical maximization form is $\tilde{\mathbf{x}}(LP) = \begin{pmatrix} 8 \\ 5 \\ 6 \\ 5 \end{pmatrix}$, determine the optimal value of the original LP problem in minimization form.

Solution

- The objective function is minimized, and then we maximize the opposite objective function. The two linear constraints are greater-than-or-equal-to constraints, and then we multiply each of them by -1 . We obtain the following equivalent LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 2$ less-than-or-equal-to linear constraints:

$$\begin{aligned} & - \max \quad -2 \ x_1 \quad - \quad 3 \ x_2 \\ & \text{subject to} \quad - \ x_1 \quad - \quad 2 \ x_2 \leq -4 \\ & \quad \quad \quad -3 \ x_1 \quad - \quad \quad x_2 \leq -6 \\ & \quad \quad \quad x_1, \quad \quad \quad x_2 \geq 0 \end{aligned}$$

- By adding $m = 2$ slack variables x_3 and x_4 , we obtain the following equivalent LP problem in standard maximization form with $n + m = 4$ non-negative variables and $m = 2$ equality linear constraints:

$$\begin{aligned} & - \max \quad -2 \ x_1 \quad - \quad 3 \ x_2 \\ & \text{subject to} \quad - \ x_1 \quad - \quad 2 \ x_2 \quad + \quad x_3 \quad \quad \quad = -4 \\ & \quad \quad \quad -3 \ x_1 \quad - \quad \quad x_2 \quad \quad \quad + \quad x_4 \quad = -6 \\ & \quad \quad \quad x_1, \quad \quad \quad x_2, \quad \quad \quad x_3, \quad \quad \quad x_4 \geq 0 \end{aligned}$$

- The two LP problems have the same optimal solutions, and their optimal values are opposite numbers. The optimal value of the LP problem in canonical maximization form is:

$$z(LP_{\max}) = -2 \cdot \frac{8}{5} - 3 \cdot \frac{6}{5} = -\frac{16}{5} - \frac{18}{5} = -\frac{34}{5},$$

and then the optimal value of the original LP problem in minimization form is:

$$z(LP_{\min}) = -z(LP_{\max}) = \frac{34}{5}.$$

Exercise 5.5 — Equivalent reformulations (E2)

Consider the following LP problem with $n = 2$ non-negative variables and three linear constraints of three different types:

$$\begin{array}{llllll}
 \max & 3 & x_1 & + & x_2 & \\
 \text{subject to} & & x_1 & + & x_2 & \leq 5 \\
 & & x_1 & - & x_2 & \geq 1 \\
 & 2 & x_1 & + & x_2 & = 8 \\
 & & x_1, & & x_2 & \geq 0
 \end{array}$$

Questions

1. Write an equivalent LP problem in canonical maximization form.
2. Write an equivalent LP problem in standard maximization form.
3. Verify that $\bar{x}(LP) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ is a feasible solution of the original LP problem and compute the values of the slack and surplus variables associated with it in the standard maximization form.

Solution

- The objective function is already maximized and the variables are already non-negative. The first constraint is a less-than-or-equal-to constraint and is left unchanged; the second one is a greater-than-or-equal-to constraint and we multiply it by -1 ; the third one is an equality constraint and we split it into two less-than-or-equal-to constraints. We obtain the following equivalent LP problem in canonical maximization form with $n = 2$ non-negative variables and $m = 4$ less-than-or-equal-to linear constraints:

$$\begin{array}{llllll}
 \max & 3 & x_1 & + & x_2 & \\
 \text{subject to} & & x_1 & + & x_2 & \leq 5 \\
 & & - & x_1 & + & x_2 & \leq -1 \\
 & 2 & x_1 & + & x_2 & \leq 8 \\
 & -2 & x_1 & - & x_2 & \leq -8 \\
 & & x_1, & & x_2 & \geq 0
 \end{array}$$

- Starting from the original LP problem, we add one slack variable x_3 to the first constraint, we subtract one surplus variable x_4 from the second constraint, and we leave the third constraint unchanged. We obtain the following equivalent LP problem in standard maximization form with 4 non-negative variables and 3 equality linear constraints:

$$\begin{array}{rcllclclcl}
 \max & 3 & x_1 & + & x_2 & & & & \\
 \text{subject to} & & x_1 & + & x_2 & + & x_3 & = & 5 \\
 & & x_1 & - & x_2 & & - & x_4 & = & 1 \\
 & 2 & x_1 & + & x_2 & & & & = & 8 \\
 & & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
 \end{array}$$

The equality constraint does not require any additional variable: only the inequality constraints do.

- The point $\bar{x}(LP) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ is a feasible solution, since:

$$4 + 0 = 4 \leq 5, \quad 4 - 0 = 4 \geq 1, \quad 2 \cdot 4 + 0 = 8 = 8, \quad 4 \geq 0 \quad \text{and} \quad 0 \geq 0.$$

The values of the slack variable x_3 and of the surplus variable x_4 are obtained from the two equality constraints of the standard maximization form:

$$\bar{x}_3 = 5 - \bar{x}_1 - \bar{x}_2 = 5 - 4 - 0 = 1 \quad \text{and} \quad \bar{x}_4 = \bar{x}_1 - \bar{x}_2 - 1 = 4 - 0 - 1 = 3.$$

Exercise 5.6 — Equivalent reformulations (E3)

Consider the following LP problem with one non-negative variable $x_1 \geq 0$, one non-positive variable $x_2 \leq 0$ and one free variable $x_3 \gtrless 0$:

$$\begin{array}{rcllclclcl}
 \max & x_1 & - & 2 & x_2 & + & x_3 & & \\
 \text{subject to} & x_1 & + & & x_2 & + & x_3 & = & 3 \\
 & x_1 & & & & - & x_3 & \leq & 2 \\
 & x_1 & & & & & & \geq & 0 \\
 & & & & x_2 & & & \leq & 0 \\
 & & & & & & x_3 & \gtrless & 0
 \end{array}$$

Questions

- Write an equivalent LP problem with only non-negative variables.
- Write an equivalent LP problem in standard maximization form.

3. Verify that $\bar{x}_1 = 1$, $\bar{x}_2 = -1$, $\bar{x}_3 = 3$ is a feasible solution of the original LP problem and compute the corresponding feasible solution of the LP problem of the first question.

Solution

- The variable x_2 is non-positive, and then we set $x_2 = -x_2^+$ with $x_2^+ \geq 0$. The variable x_3 is free, and then we set $x_3 = x_3^+ - x_3^-$ with $x_3^+, x_3^- \geq 0$. By performing these substitutions in the objective function and in the two linear constraints, we obtain the following equivalent LP problem with only non-negative variables:

$$\begin{aligned} \max \quad & x_1 + 2x_2^+ + x_3^+ - x_3^- \\ \text{subject to} \quad & x_1 - x_2^+ + x_3^+ - x_3^- = 3 \\ & x_1 - x_3^+ + x_3^- \leq 2 \\ & x_1, x_2^+, x_3^+, x_3^- \geq 0 \end{aligned}$$

- The first constraint is already an equality constraint. By adding one slack variable x_5 to the second constraint, we obtain the following equivalent LP problem in standard maximization form with 5 non-negative variables and 2 equality linear constraints:

$$\begin{aligned} \max \quad & x_1 + 2x_2^+ + x_3^+ - x_3^- \\ \text{subject to} \quad & x_1 - x_2^+ + x_3^+ - x_3^- = 3 \\ & x_1 - x_3^+ + x_3^- + x_5 = 2 \\ & x_1, x_2^+, x_3^+, x_3^-, x_5 \geq 0 \end{aligned}$$

- The point $\bar{x}_1 = 1$, $\bar{x}_2 = -1$, $\bar{x}_3 = 3$ is a feasible solution of the original LP problem, since:

$$1 - 1 + 3 = 3 = 3, \quad 1 - 3 = -2 \leq 2, \quad 1 \geq 0, \quad -1 \leq 0.$$

The corresponding feasible solution of the LP problem with only non-negative variables is obtained by setting:

$$\bar{x}_1 = 1, \quad \bar{x}_2^+ = -\bar{x}_2 = 1, \quad \bar{x}_3^+ = 3, \quad \bar{x}_3^- = 0.$$

The two solutions have the same objective value:

$$1 - 2 \cdot (-1) + 3 = 6 \quad \text{and} \quad 1 + 2 \cdot 1 + 3 - 0 = 6.$$

The values of x_3^+ and x_3^- are not unique: every pair of non-negative numbers whose difference is $\bar{x}_3 = 3$ gives a feasible solution with the same objective value.

Exercise 5.7 — Equivalent reformulations (E4)

Consider the following generic LP problem with one non-positive variable $x_1 \leq 0$, one non-negative variable $x_2 \geq 0$ and one free variable $x_3 \geq 0$:

$$\begin{array}{llllll}
 \min & 4 & x_1 & - & x_2 & + & 2 & x_3 \\
 \text{subject to} & - & x_1 & + & x_2 & & & \geq & 2 \\
 & 2 & x_1 & & & + & x_3 & = & 5 \\
 & & x_1 & + & x_2 & + & x_3 & \leq & 8 \\
 & & x_1 & & & & & \leq & 0 \\
 & & & & x_2 & & & \geq & 0 \\
 & & & & & & x_3 & \geq & 0
 \end{array}$$

Questions

1. Write an equivalent LP problem in canonical maximization form.
2. Write an equivalent LP problem in standard maximization form.
3. Verify that $\bar{x}_1 = -2$, $\bar{x}_2 = 0$, $\bar{x}_3 = 9$ is a feasible solution of the original LP problem, compute the corresponding feasible solution of the LP problem in canonical maximization form and check that the two objective values are opposite numbers.

Solution

- We apply the transformations one at a time:
 1. the objective function is minimized, and then we maximize the opposite objective function;
 2. the variable x_1 is non-positive, and then we set $x_1 = -x_1^+$ with $x_1^+ \geq 0$;
 3. the variable x_3 is free, and then we set $x_3 = x_3^+ - x_3^-$ with $x_3^+, x_3^- \geq 0$;
 4. the first constraint is a greater-than-or-equal-to constraint, and then we multiply it by -1 ;
 5. the second constraint is an equality constraint, and then we split it into two less-than-or-equal-to constraints;
 6. the third constraint is a less-than-or-equal-to constraint, and it is left unchanged.

We obtain the following equivalent LP problem in canonical maximization form with $n = 4$ non-negative variables and $m = 4$ less-than-or-equal-to linear constraints:

$$\begin{aligned}
 & - \max \quad 4 x_1^+ + x_2 - 2 x_3^+ + 2 x_3^- \\
 \text{subject to} \quad & - x_1^+ - x_2 \leq -2 \\
 & -2 x_1^+ + x_3^+ - x_3^- \leq 5 \\
 & 2 x_1^+ - x_3^+ + x_3^- \leq -5 \\
 & - x_1^+ + x_2 + x_3^+ - x_3^- \leq 8 \\
 & x_1^+, x_2, x_3^+, x_3^- \geq 0
 \end{aligned}$$

- By adding $m = 4$ slack variables x_5, x_6, x_7, x_8 , we obtain an equivalent LP problem in standard maximization form with 8 non-negative variables and 4 equality linear constraints:

$$\begin{aligned}
 & - \max \quad 4 x_1^+ + x_2 - 2 x_3^+ + 2 x_3^- \\
 \text{subject to} \quad & - x_1^+ - x_2 + x_5 = -2 \\
 & -2 x_1^+ + x_3^+ - x_3^- + x_6 = 5 \\
 & 2 x_1^+ - x_3^+ + x_3^- + x_7 = -5 \\
 & - x_1^+ + x_2 + x_3^+ - x_3^- + x_8 = 8 \\
 & x_1^+, x_2, x_3^+, x_3^-, x_5, x_6, x_7, x_8 \geq 0
 \end{aligned}$$

- The point $\bar{x}_1 = -2, \bar{x}_2 = 0, \bar{x}_3 = 9$ is a feasible solution of the original LP problem, since:

$$\begin{aligned}
 -(-2) + 0 &= 2 \geq 2, & 2 \cdot (-2) + 9 &= 5 = 5, & -2 + 0 + 9 &= 7 \leq 8, \\
 -2 &\leq 0, & 0 &\geq 0.
 \end{aligned}$$

The corresponding feasible solution of the LP problem in canonical maximization form is:

$$\bar{x}_1^+ = -\bar{x}_1 = 2, \quad \bar{x}_2 = 0, \quad \bar{x}_3^+ = 9, \quad \bar{x}_3^- = 0,$$

and it is feasible, since:

$$\begin{aligned}
 -2 - 0 &= -2 \leq -2, & -4 + 9 - 0 &= 5 \leq 5, \\
 4 - 9 + 0 &= -5 \leq -5, & -2 + 0 + 9 - 0 &= 7 \leq 8.
 \end{aligned}$$

Finally, the two objective values are opposite numbers:

$$4 \cdot (-2) - 0 + 2 \cdot 9 = 10 \quad \text{and} \quad 4 \cdot 2 + 0 - 2 \cdot 9 + 2 \cdot 0 = -10.$$