



SAPIENZA
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Integer Linear Programming

Volume 1 — Models, bounds and formulations

What an integer linear programming problem is (MIP, ILP, BIP), its solutions and outcomes; primal and dual bounds and the gap; the LP relaxation; ideal formulations, valid inequalities and cover inequalities. Worked examples and exercises with complete solutions.

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Contents

I	Models and bounds	3
1	Integer linear programming	4
1.1	Canonical maximization form	5
1.2	Possible outcomes	9
1.3	Complete enumeration	10
1.4	Typical mistakes	11
1.5	In summary	12
2	Primal and dual bounds	13
2.1	Primal and dual bounds	14
2.2	Primal bounds: heuristic solutions	15
2.3	The LP relaxation	16
2.4	Rounding the solution of the LP relaxation	18
2.5	Upper bounds on the variables	20
2.6	Dual bounds	22
2.7	Typical mistakes	25
2.8	In summary	25
II	Strengthening the formulation	27
3	Ideal formulations	28
3.1	The convex hull	29
3.2	Formulations and ideal formulations	30
3.3	Comparing formulations	32
3.4	Existence of ideal formulations	33
3.5	Typical mistakes	33
3.6	In summary	33
4	Valid inequalities	35
4.1	Valid inequalities	36
4.2	Deriving valid inequalities by rounding	37

4.3	Improving a valid inequality	38
4.4	Families of valid inequalities and cuts	40
4.5	Typical mistakes	41
4.6	In summary	41
5	Cover inequalities	43
5.1	A constraint with positive coefficients and its LP relaxation	44
5.2	Covers and cover inequalities	45
5.3	Geometric interpretation	48
5.4	Separation of cover inequalities	54
5.5	Transforming the CSP into a complementary problem	58
5.6	Extended cover inequalities	60
5.7	Violation and distance from a cover inequality	61
5.8	Typical mistakes	63
5.9	In summary	64

Part I

Models and bounds



Integer linear programming

Prerequisites

- From the module *Linear Programming*: [Chapter 2 «Half-spaces and polyhedra»](#) and [Chapter 4 «Linear Programming problems»](#) (the canonical maximization form, the feasible region, optimal solutions and the optimal value), and [Chapter 5 «Possible outcomes and equivalent formulations»](#) (infeasible, unbounded and optimal problems).

Objectives

At the end of this chapter you will be able to:

- write an ILP problem in canonical maximization form, and distinguish ILP, BIP and MIP problems;
- draw the feasible region of a small ILP problem and find an optimal solution graphically;
- explain why the feasible region is not convex, and what the three outcomes of an ILP problem are;
- explain why complete enumeration fails, and how much it costs;
- say why bounds on the optimal value are needed to avoid enumeration.

1.1 Canonical maximization form

Definition 1.1: Integer Linear Programming problems in canonical maximization form

Given:

1. an objective-function vector $\mathbf{c} \in \mathbb{Z}^n$
2. a constraint matrix $\mathbf{A} \in \mathbb{Z}^{m \times n}$
3. a right-hand-side (RHS) vector $\mathbf{b} \in \mathbb{Z}^m$ with m rows

$$\underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}} \in \mathbb{Z}^n \quad \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_{\mathbf{A}} \in \mathbb{Z}^{m \times n} \quad \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\mathbf{b}} \in \mathbb{Z}^m$$

an **Integer Linear Programming problem** ILP in **canonical maximization form** with n non-negative integer variables

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \in \mathbb{Z}_{\geq 0}^n$$

and m less-than-or-equal-to linear constraints is the following optimization problem:

$$\max \underbrace{\sum_{j=1}^n c_j x_j}_{\mathbf{c}'\mathbf{x}} \tag{1.1a}$$

$$\text{subject to } \underbrace{\sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \in \{1, 2, \dots, m\}}_{\mathbf{A}\mathbf{x} \leq \mathbf{b}} \tag{1.1b}$$

$$\underbrace{x_j \in \mathbb{Z}_{\geq 0}, \quad \forall j \in \{1, 2, \dots, n\}}_{\mathbf{x} \in \mathbb{Z}_{\geq 0}^n} \tag{1.1c}$$

- The data $\mathbf{c}$, $\mathbf{A}$ and $\mathbf{b}$ are **integer**, as in the whole module.
- With a slight abuse of notation, we write the ILP problem (1.1) in compact form as

$$\max \{ \mathbf{c}'\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}_{\geq 0}^n \}.$$

- The set $\mathbb{Z}^n$ of the points of $\mathbb{R}^n$ with integer components is called the **integer lattice**.

An ILP problem (1.1) is an LP problem in canonical maximization form with the additional requirement $\mathbf{x} \in \mathbb{Z}^n$: the linear function $\mathbf{c}'\mathbf{x}$ is maximized over the points of the **integer lattice** that satisfy the constraints.

- If all the variables are restricted to $\{0,1\}$, the problem is a **binary integer linear programming** (BIP) problem. If only some of the variables are integer, the problem is a **mixed integer linear programming** (MIP) problem.
- Constraints $\geq$ and $=$, and minimization problems, are brought to the canonical maximization form as in linear programming.

Example 1.1: ILP problem in canonical maximization form

Consider the following input data of an ILP problem with $n = 2$ and $m = 2$:

$$\mathbf{c} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 10 & 4 \\ 4 & 14 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 45 \\ 49 \end{pmatrix}$$

The numerical entries of the two column vectors and the matrix are

$$c_1 = 3, \quad c_2 = 2, \quad a_{11} = 10, \quad a_{12} = 4, \quad a_{21} = 4, \quad a_{22} = 14, \quad b_1 = 45, \quad b_2 = 49.$$

The associated ILP problem in canonical maximization form with $n = 2$ non-negative integer variables and $m = 2$ less-than-or-equal-to linear constraints reads as follows:

$$\begin{array}{llllll} \max & 3x_1 & + & 2x_2 & & \\ \text{subject to} & 10x_1 & + & 4x_2 & \leq & 45 \\ & 4x_1 & + & 14x_2 & \leq & 49 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

The variables x_1 and x_2 can be read as the numbers of cars of two types to produce: this ILP problem is the **car example**, the guiding example of the module.

1.1.1 Feasible region and objective function

Definition 1.2: feasible region and objective function

- The **feasible region** of an ILP problem (1.1) is the set F defined as follows:

$$F = \left\{ \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \in \mathbb{Z}^n : \underbrace{\sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \in \{1, 2, \dots, m\}}_{\mathbf{A}\mathbf{x} \leq \mathbf{b}}, \right. \quad (1.2)$$

$$\left. \underbrace{x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n\}}_{\mathbf{x} \geq \mathbf{0}} \right\}.$$

- The **objective function** (1.1a) of an ILP problem (1.1) to be maximized is the (scalar) **linear function** f defined as follows:

$$f : \mathbb{Z}^n \rightarrow \mathbb{R}, \quad f : \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \mapsto \sum_{j=1}^n c_j x_j \quad (1.3)$$

$$\underbrace{\qquad\qquad\qquad}_{\mathbf{c}'\mathbf{x}}$$

- Let $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ be the polyhedron of the constraints, that is, the feasible region of the LP problem with the same data. The feasible region of the ILP problem is

$$F = P \cap \mathbb{Z}^n.$$

- The set F is made of isolated points of the lattice, hence it is **not convex**.

Example 1.2: feasible region

Consider the ILP problem of [Example 1.1](#). The feasible region F is the set of the 16 black points of the following figure, the points of the lattice $\mathbb{Z}^2$ in the polyhedron P (red).



Figure 1.1: The car example: the feasible region of the ILP problem is the set of its 16 integer points (black) inside the polyhedron (red). The optimal solution $(4, 1)$ is blue, with the objective line $3x_1 + 2x_2 = 14$ (dashed).

The point $\bar{x} = (2, 2)$ belongs to F , because $10 \cdot 2 + 4 \cdot 2 = 28 \leq 45$ and $4 \cdot 2 + 14 \cdot 2 = 36 \leq 49$. The point $(4, 2)$ does not belong to F , because $10 \cdot 4 + 4 \cdot 2 = 48 > 45$. The point $(\frac{7}{2}, \frac{5}{2})$ belongs to P but not to F , because it is not integer.

1.1.2 Optimal solution and optimal value**Definition 1.3: optimal solution and optimal value**

- A point $\tilde{x}(\text{ILP}) \in F$ is an **optimal solution** of the ILP problem (1.1) if

$$\tilde{x}(\text{ILP}) \in \arg \max \{ c'x : x \in F \}.$$

- The value

$$z(\text{ILP}) = \underbrace{\max \{ c'x : x \in F \}}_{=c'\tilde{x}(\text{ILP})}$$

is the **optimal value** of the ILP problem (1.1).

- A point $\bar{x} \in F$ is a **feasible solution** of the ILP problem. An optimal solution is a feasible solution with the largest value of the objective function.

The goal of an ILP problem (1.1) is to **find an optimal solution** $\tilde{x}(\text{ILP})$, whenever it exists, rather than merely determining the optimal value $z(\text{ILP})$.

1.1.3 Graphical solution method

An optimal solution of an ILP problem (1.1) with two variables can be graphically determined by moving the level line of the objective function in the direction of its gradient, until the last point of the lattice in the polyhedron P is reached.

Example 1.3: Graphical solution method

Consider the ILP problem of [Example 1.1](#). The gradient of the objective function is $\nabla f = (3, 2)$. Moving the level line $3x_1 + 2x_2 = z$ in the direction of ∇f , the last point of F reached is $(4, 1)$, on the dashed line $3x_1 + 2x_2 = 14$ of the figure of [Example 1.2](#). No other point of F lies on this line.

The car example has the **unique** optimal solution $\tilde{x}(\text{ILP}) = (4, 1)$, with optimal value $z(\text{ILP}) = 14$.

1.2 Possible outcomes

- As in linear programming, an ILP problem has exactly one of three **outcomes**: it is **infeasible** ($F = \emptyset$), **unbounded** (the objective function is not bounded from above on F) or it has an **optimal solution**.
- If the polyhedron P is bounded, then F is a **finite set**, and an optimal solution exists as soon as $F \neq \emptyset$.

Example 1.4: an infeasible ILP problem with a feasible LP problem

The ILP problem

$$\max \{ x_1 : 2x_1 \leq 1, -2x_1 \leq -1, x_1 \in \mathbb{Z}_{\geq 0} \}$$

is infeasible, because the constraints require $x_1 = \frac{1}{2}$. The LP problem with the same data is feasible, with the unique feasible solution $x_1 = \frac{1}{2}$.

Example 1.5: an unbounded ILP problem

The ILP problem

$$\max \{ x_1 + x_2 : x_1 - x_2 \leq 1, x_1, x_2 \in \mathbb{Z}_{\geq 0} \}$$

is unbounded: the points (k, k) , $k \in \mathbb{Z}_{\geq 0}$, are feasible, with value $2k$.

1.3 Complete enumeration

- When the feasible region is a finite set we can find an optimal solution by **complete enumeration**: we compute the objective function for every feasible solution and keep the best. For the car example, there are 16 points:

x_1	x_2	z	x_1	x_2	z
0	0	0	3	1	11
1	0	3	4	1	14
2	0	6	0	2	4
3	0	9	1	2	7
4	0	12	2	2	10
0	1	2	3	2	13
1	1	5	0	3	6
2	1	8	1	3	9

- The best value is 14, in (4, 1) (in bold). Here the method works, because the model is tiny.
- Consider now the simplest case, n **binary variables** and no other constraint. Every variable takes 2 values, so there are 2^n solutions: the vertices of an n -dimensional hypercube.

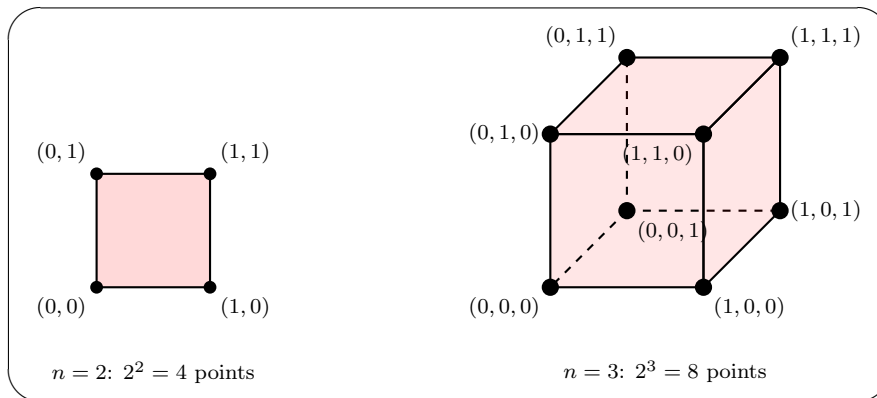


Figure 1.2: The solutions of n binary variables without other constraints: the 2^n vertices of an n -dimensional hypercube, for $n = 2$ (a square) and $n = 3$ (a cube).

- The number 2^n explodes. With $n = 80$ variables there are about $1.2 \cdot 10^{24}$ solutions, comparable to the estimated number of stars in the universe, and with $n = 266$ variables about $1.2 \cdot 10^{80}$, comparable to the estimated number of atoms in the observable universe.
- Suppose that a computer examines 10^9 solutions per second, and then a computer 1000 times faster, 10^{12} solutions per second. The time needed to enumerate 2^n solutions is:

n	2^n	at 10^9 solutions/s	at 10^{12} solutions/s
20	$\approx 1.05 \cdot 10^6$	≈ 1.05 ms	≈ 1.05 μ s
30	$\approx 1.07 \cdot 10^9$	≈ 1.07 s	≈ 1.07 ms
40	$\approx 1.10 \cdot 10^{12}$	≈ 18.3 min	≈ 1.10 s
50	$\approx 1.13 \cdot 10^{15}$	≈ 13.0 days	≈ 18.8 min
60	$\approx 1.15 \cdot 10^{18}$	≈ 36.5 years	≈ 13.3 days
70	$\approx 1.18 \cdot 10^{21}$	≈ 374 centuries	≈ 37.4 years
80	$\approx 1.21 \cdot 10^{24}$	$\approx 3.8 \cdot 10^5$ centuries	≈ 383 centuries

Complete enumeration is impractical already for models of moderate size. A computer 1000 times faster gains only about 10 variables, because $2^{10} = 1024 \approx 1000$.

- The way out is to **avoid enumerating**. If we know a lower bound and an upper bound on the optimal value, we can exclude entire regions of feasible solutions without visiting them. The two bounds come from two different sources, the feasible solutions and the relaxations: they are the subject of the next chapter, «Primal and dual bounds».

1.4 Typical mistakes

Warning

- **Believing that the feasible region is convex.** It is the set of the integer points of a polyhedron, not the polyhedron. The midpoint of two feasible solutions can be infeasible.
- **Believing that the feasible region is always finite.** It is finite only if the polyhedron is bounded; otherwise the ILP problem can be unbounded.
- **Believing that a feasible LP problem gives a feasible ILP problem.** The LP problem can have solutions while no integer point satisfies the constraints.
- **Taking a good feasible solution as optimal.** Without a bound from the other side (next chapter) there is no certificate of optimality.

1.5 In summary

In summary

- An ILP problem maximizes a linear function over the integer points of a polyhedron; its feasible region is not convex, and it is finite when the polyhedron is bounded.
- The outcomes are infeasible, unbounded or optimal, and a feasible LP problem can have an infeasible ILP version.
- Complete enumeration costs 2^n evaluations for n binary variables: it is not an option beyond very small models.
- To avoid enumeration we need bounds on $z(\text{ILP})$ from both sides: feasible solutions and relaxations, the subject of the next chapter.

Dashboard of the car example. No bounds yet: the next chapter gives the first lower and upper bounds.

Primal and dual bounds

Prerequisites

- From this module, the chapter «Integer linear programming»: ILP problems, feasible and optimal solutions, complete enumeration, and the car example.
- From the module *Linear Programming*, [Chapter 9 «Duality theory»](#): the dual problem, weak duality and strong duality.
- From the same module, [Chapter 6 «Simplex algorithm»](#): an LP problem that has an optimal solution has an optimal vertex.

Objectives

At the end of this chapter you will be able to:

- obtain a primal bound (a lower bound) on the optimal value from a feasible solution, also with a simple heuristic algorithm;
- build the LP relaxation of an ILP problem and use its optimal value as a dual bound (an upper bound), rounded down when it is valid;
- compute the optimality gap from a primal bound and a dual bound;
- explain why rounding an optimal LP solution gives a feasible solution at best;
- derive upper bounds on the variables from the constraints, and say when they help;
- obtain a dual bound from non-negative multipliers of the constraints, and recognize the dual problem as the problem of the best multipliers.

Where we were

In the previous chapter we saw that complete enumeration is not an option, and that we need bounds on the optimal value from both sides. We now obtain them: **primal bounds** from feasible solutions, and **dual bounds** from relaxations.

2.1 Primal and dual bounds

Definition 2.1: primal bound, dual bound and optimality gap

Consider an ILP problem in canonical maximization form, with feasible region F and optimal value $z(\text{ILP})$.

- Given a feasible solution $\bar{\mathbf{x}} \in F$, its value $\text{LB}(\bar{\mathbf{x}}) = \sum_{j=1}^n c_j \bar{x}_j$ is a **primal bound** on $z(\text{ILP})$.
- A value UB such that $\sum_{j=1}^n c_j x_j \leq \text{UB}$ for every $\mathbf{x} \in F$ is a **dual bound** on $z(\text{ILP})$.
- Given a primal bound $\text{LB} \neq 0$ and a dual bound UB , the **optimality gap** is

$$\text{optimality gap} = \frac{\text{UB} - \text{LB}}{|\text{LB}|}. \quad (2.1)$$

- In a maximization problem the primal bounds are **lower bounds** and the dual bounds are **upper bounds**. A primal bound comes from one feasible solution; a dual bound is a statement on all the feasible solutions at once.

For every feasible solution $\bar{\mathbf{x}}$ and every dual bound UB :

$$\text{LB}(\bar{\mathbf{x}}) \leq z(\text{ILP}) \leq \text{UB}.$$

A feasible solution is **certified optimal** when the optimality gap is zero.

Definition 2.2: relaxation

Given an ILP problem $\max\{\mathbf{c}'\mathbf{x} : \mathbf{x} \in F\}$, the problem $\max\{\mathbf{c}'\mathbf{x} : \mathbf{x} \in R\}$ is a **relaxation** of the ILP problem if $F \subseteq R$. We denote its optimal value by $z(R)$.

- A maximization over a larger region cannot give a smaller value: the optimal value of a relaxation is a dual bound, $z(\text{ILP}) \leq z(R)$.

2.2 Primal bounds: heuristic solutions

Definition 2.3: heuristic algorithm

A **heuristic algorithm** is a method that finds feasible solutions quickly, with no guarantee of optimality. The value of every solution it finds is a primal bound.

A good feasible solution is a good primal bound.

Example 2.1: primal bound

For the feasible solution $(2, 2)$ of the car example we get $LB = 3 \cdot 2 + 2 \cdot 2 = 10$, so $z(ILP) \geq 10$.

Definition 2.4: of integer part of a number

Given $y \in \mathbb{R}$, the **integer part** of y is $\lfloor y \rfloor = \max\{q \in \mathbb{Z} : q \leq y\}$.

Example 2.2: two heuristic algorithms for the car example

- The **one-variable heuristic** sets a single variable to its maximum value, with the other one equal to 0, and keeps the better of the two solutions.
 - Only x_1 ($x_2 = 0$): $\bar{x}_1 = \min\{\lfloor \frac{45}{10} \rfloor, \lfloor \frac{49}{4} \rfloor\} = \min\{4, 12\} = 4$, so $\bar{x} = (4, 0)$ and $LB = 12$.
 - Only x_2 ($x_1 = 0$): $\bar{x}_2 = \min\{\lfloor \frac{45}{4} \rfloor, \lfloor \frac{49}{14} \rfloor\} = \min\{11, 3\} = 3$, so $\bar{x} = (0, 3)$ and $LB = 6$.
- The **two-step heuristic** sets first one variable to its maximum value, and then the other one to its maximum value with the **residual right-hand sides**. It keeps the better of the two solutions.
 - x_1 first: $\bar{x}_1 = 4$ uses 40 and 16, leaving 5 and 33. Then $\bar{x}_2 = \min\{\lfloor \frac{5}{4} \rfloor, \lfloor \frac{33}{14} \rfloor\} = \min\{1, 2\} = 1$, so $\bar{x} = (4, 1)$ and $LB = 14$.
 - x_2 first: $\bar{x}_2 = 3$ uses 12 and 42, leaving 33 and 7. Then $\bar{x}_1 = \min\{\lfloor \frac{33}{10} \rfloor, \lfloor \frac{7}{4} \rfloor\} = \min\{3, 1\} = 1$, so $\bar{x} = (1, 3)$ and $LB = 9$.

Heuristic	Order	$\bar{x}_1$	$\bar{x}_2$	LB
one-variable	x_1	4	0	12
one-variable	x_2	0	3	6
two-step	x_1 first	4	1	14
two-step	x_2 first	1	3	9

- The two-step heuristic gives a better primal bound, but it needs more computation. This trade-off between the quality of the solution and the computational effort is typical of

heuristic algorithms.

- Here the best primal bound is $LB = 14$, in $(4, 1)$. A primal bound alone is not a certificate of optimality: a dual bound is needed.

2.3 The LP relaxation

Definition 2.5: LP relaxation

The **linear programming (LP) relaxation** of an ILP problem in canonical maximization form is the LP problem obtained by replacing the integrality constraints $x_j \in \mathbb{Z}_{\geq 0}$ with the non-negativity constraints $x_j \geq 0$, for all $j \in \{1, 2, \dots, n\}$:

$$\max \sum_{j=1}^n c_j x_j \quad (2.2a)$$

$$\text{subject to } \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \in \{1, 2, \dots, m\}, \quad (2.2b)$$

$$x_j \geq 0, \quad \forall j \in \{1, 2, \dots, n\}. \quad (2.2c)$$

- We denote by $z(\text{LP})$ the optimal value of the LP relaxation, and by $\tilde{\mathbf{x}}(\text{LP})$ an optimal solution. We always write the argument, because in the same example $\tilde{\mathbf{x}}(\text{ILP})$ and $\tilde{\mathbf{x}}(\text{LP})$ are different points.

Proposition 2.6: upper bound from the LP relaxation

Every feasible solution of the ILP problem is feasible for its LP relaxation, hence

$$z(\text{ILP}) \leq z(\text{LP}). \quad (2.3)$$

Moreover, if an optimal solution of the LP relaxation is integer, then it is an optimal solution of the ILP problem.

Proof. The feasible region F of the ILP problem is contained in the polyhedron of the LP relaxation, and the objective function is the same. The maximum over a larger set is not smaller than the maximum over a subset. If $\tilde{\mathbf{x}}(\text{LP})$ is integer, it belongs to F and has value $z(\text{LP}) \geq z(\text{ILP})$: it is optimal for the ILP problem. $\square$

- The value of a feasible solution can be an integer, and it is at most $z(\text{LP})$: then it is at most the largest integer not exceeding $z(\text{LP})$.

Observation 2.7: rounding the upper bound

If all the variables are integer and all the costs c_j are integer, then

$$z(\text{ILP}) \leq \lfloor z(\text{LP}) \rfloor. \quad (2.4)$$

In all the other cases the bound $\lfloor z(\text{LP}) \rfloor$ is not valid.

- In this module all the ILP problems have integer data, so the bound (2.4) always holds. In a MIP problem with a continuous variable the value of a feasible solution is not necessarily an integer, and the bound cannot be rounded.

Example 2.3: the LP relaxation of the car example

- The LP relaxation of the car example has the optimal solution

$$\tilde{x}_1 = \frac{7}{2}, \quad \tilde{x}_2 = \frac{5}{2}, \quad z(\text{LP}) = \frac{31}{2},$$

and it is fractional: it is not feasible for the ILP problem.

- Since the costs are integer, we have the upper bound $\lfloor z(\text{LP}) \rfloor = 15$. Together with the primal bound 14 of the feasible solution (4, 1) of [Example 2.2](#):

$$14 \leq z(\text{ILP}) \leq 15.$$

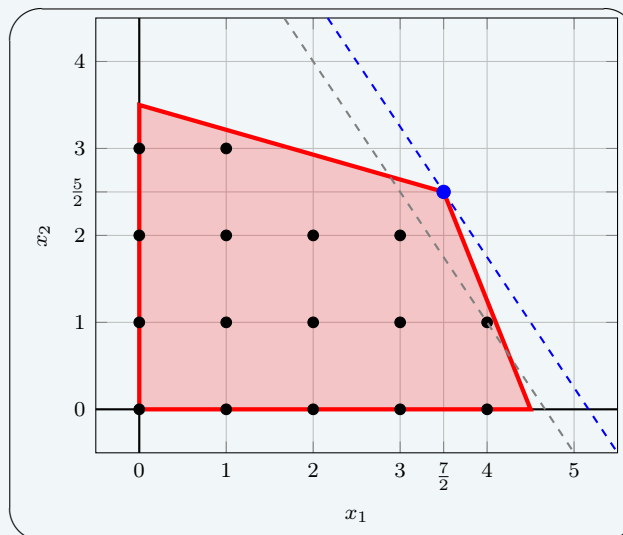


Figure 2.1: The LP relaxation of the car example: the feasible region (red), its 16 integer points (black) and the optimal solution $(\frac{7}{2}, \frac{5}{2})$ of the LP relaxation (blue), with the objective line $3x_1 + 2x_2 = \frac{31}{2}$ through it (dashed). The gray dashed line is the objective line of the optimal ILP solution.

- The optimal value is 14 or 15. With $\text{UB} = 15$ and $\text{LB} = 14$ the optimality gap (2.1) is $\frac{1}{14} \approx 7.1\%$: the solution (4, 1) is not yet certified optimal.

2.4 Rounding the solution of the LP relaxation

Feasible ILP solutions can typically be obtained by rounding an optimal solution of the LP relaxation. But no rounding technique guarantees to find an optimal ILP solution.

Example 2.4: rounding in the car example

- The optimal solution of the LP relaxation is $\tilde{\mathbf{x}}(\text{LP}) = (\frac{7}{2}, \frac{5}{2})$. Rounding to the nearest integer gives $(4, 3)$, which is infeasible: $10 \cdot 4 + 4 \cdot 3 = 52 > 45$.
- Rounding down both components gives the feasible solution $\bar{\mathbf{x}} = (3, 2)$, with $\text{LB} = 3 \cdot 3 + 2 \cdot 2 = 13$. The optimal ILP solution is $\tilde{\mathbf{x}}(\text{ILP}) = (4, 1)$, with $z(\text{ILP}) = 14$: rounding does not give the optimum, and the bounds are $13 \leq z(\text{ILP}) \leq 15$.

Example 2.5: rounding works, but the bound is not tight

Consider the following ILP problem; its LP relaxation has $x_1, x_2 \geq 0$ instead of the integrality constraints:

$$\begin{array}{llllll} \max & 2x_1 & + & 4x_2 & & \\ \text{subject to} & -x_1 & + & x_2 & \leq & 2 \\ & 8x_1 & + & 2x_2 & \leq & 19 \\ & x_1, & & x_2 & \in & \mathbb{Z}_{\geq 0} \end{array}$$

- The optimal solution of the LP relaxation is $\tilde{\mathbf{x}}(\text{LP}) = (\frac{3}{2}, \frac{7}{2})$, with $z(\text{LP}) = 17$ and $\lfloor z(\text{LP}) \rfloor = 17$, so $z(\text{ILP}) \leq 17$.
- Rounding down both components we get the feasible solution $\bar{\mathbf{x}} = (\lfloor \frac{3}{2} \rfloor, \lfloor \frac{7}{2} \rfloor) = (1, 3)$, with $\text{LB} = 2 \cdot 1 + 4 \cdot 3 = 14$.
- The optimal ILP solution is $\tilde{\mathbf{x}}(\text{ILP}) = (1, 3)$, with $z(\text{ILP}) = 14$: the rounded solution is optimal. But the bounds $14 \leq z(\text{ILP}) \leq 17$ do not prove it.

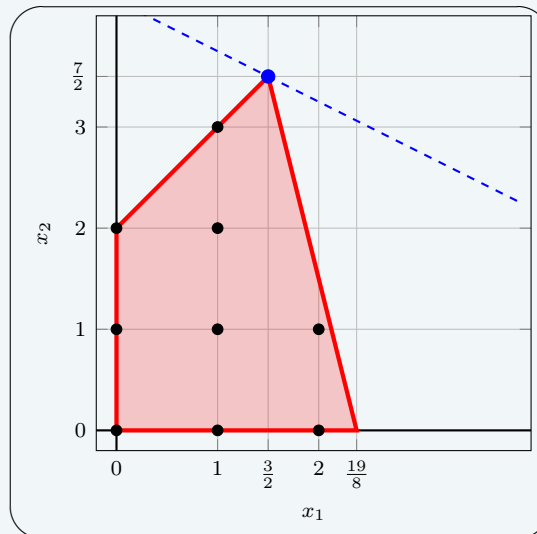


Figure 2.2: The second example: the feasible region of the LP relaxation (red), the integer points (black) and the optimal solution of the LP relaxation (blue), with the objective line $2x_1 + 4x_2 = 17$ (dashed).

Example 2.6: rounding fails

Consider the following ILP problem:

$$\begin{aligned} \max \quad & 25x_1 + 16x_2 \\ \text{subject to} \quad & 50x_1 + 31x_2 \leq 250 \\ & -3x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \in \mathbb{Z}_{\geq 0} \end{aligned}$$

- The optimal solution of the LP relaxation is $\tilde{\mathbf{x}}(\text{LP}) = (\frac{376}{193}, \frac{950}{193}) \approx (1.94, 4.92)$, with $z(\text{LP}) = \frac{24600}{193} \approx 127.46$. Hence $z(\text{ILP}) \leq \lfloor z(\text{LP}) \rfloor = 127$.
- Rounding to the nearest integer gives $(2, 5)$, which is infeasible: $50 \cdot 2 + 31 \cdot 5 = 255 > 250$. Rounding $\tilde{x}_1$ up and $\tilde{x}_2$ down gives the feasible solution $\bar{\mathbf{x}} = (2, 4)$, with $\text{LB} = 25 \cdot 2 + 16 \cdot 4 = 114$.
- The optimal ILP solution is $\tilde{\mathbf{x}}(\text{ILP}) = (5, 0)$, with $z(\text{ILP}) = 125$. It is far from $\tilde{\mathbf{x}}(\text{LP})$, and no rounding of $\tilde{\mathbf{x}}(\text{LP})$ finds it.

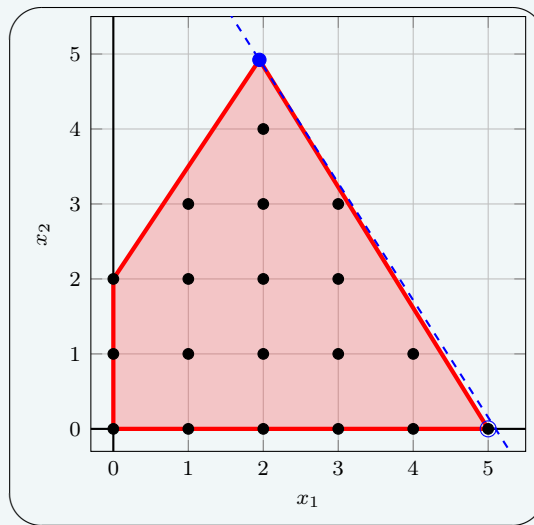


Figure 2.3: The third example: the integer points (black) are far from the optimal solution $(\frac{376}{193}, \frac{950}{193})$ of the LP relaxation (blue). The optimal ILP solution is $(5, 0)$ (blue circle). The dashed line is the objective line $25x_1 + 16x_2 = \frac{24600}{193}$.

Optimal ILP solutions can be quite far from optimal solutions of the LP relaxation. The value of the LP relaxation, instead, is often a good bound: here $125 \leq z(\text{LP}) \approx 127.46$.

2.5 Upper bounds on the variables

Proposition 2.8: upper bounds on the variables

Let $\sum_{j=1}^n a_{ij} x_j \leq b_i$ be a constraint with $a_{ij} \geq 0$ for all j . Then every feasible solution of the ILP problem satisfies

$$x_j \leq \left\lfloor \frac{b_i}{a_{ij}} \right\rfloor, \quad \forall j \text{ such that } a_{ij} > 0. \quad (2.5)$$

Proof. Since all the variables are non-negative and all the coefficients are non-negative, $a_{ij} x_j \leq \sum_{k=1}^n a_{ik} x_k \leq b_i$, so $x_j \leq \frac{b_i}{a_{ij}}$. The variable is integer, so it does not exceed the integer part of $\frac{b_i}{a_{ij}}$. $\square$

- If several constraints give a bound on x_j , we keep the smallest. The bounds cut off parts of the polyhedron of the LP relaxation, and no integer point.

The upper bounds on the variables improve the LP relaxation only if they cut off its optimal solution.

Example 2.7: upper bounds on the variables of the car example

Both constraints of the car example have non-negative coefficients:

$$\begin{aligned} x_1 &\leq \min \left\{ \left\lfloor \frac{45}{10} \right\rfloor, \left\lfloor \frac{49}{4} \right\rfloor \right\} = \min\{4, 12\} = 4, \\ x_2 &\leq \min \left\{ \left\lfloor \frac{45}{4} \right\rfloor, \left\lfloor \frac{49}{14} \right\rfloor \right\} = \min\{11, 3\} = 3. \end{aligned}$$

- The inequalities $x_1 \leq 4$ and $x_2 \leq 3$ are satisfied by all the integer points of the car example, and they can be added to the model. They cut off two regions of the polyhedron, and no integer point.

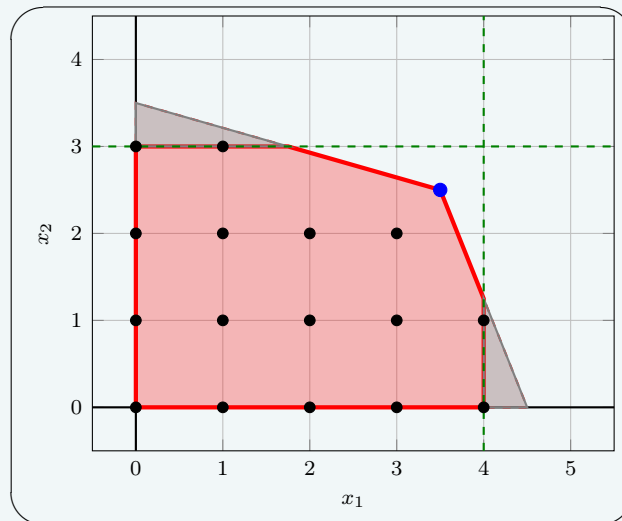


Figure 2.4: The variable upper bounds $x_1 \leq 4$ and $x_2 \leq 3$ (green) cut off the two gray regions of the LP relaxation (red) and no integer point. The optimal solution of the LP relaxation (blue) is not cut off.

- The optimal solution of the LP relaxation, $(\frac{7}{2}, \frac{5}{2})$, satisfies both inequalities. It is not cut off, so the optimal value of the LP relaxation does not change, and the upper bound is still 15.

Example 2.8: upper bounds that cut off the LP optimum

- Consider the objective function $x_1 - x_2$ on the constraints of the car example, whose optimal ILP solution is $(4, 0)$ with $z(\text{ILP}) = 4$. Before the bounds, the optimal solution of the LP relaxation is $(\frac{9}{2}, 0)$ with $z(\text{LP}) = \frac{9}{2}$; after $x_1 \leq 4$ it is $(4, 0)$ with $z(\text{LP}) = 4$, which is integer and therefore optimal.

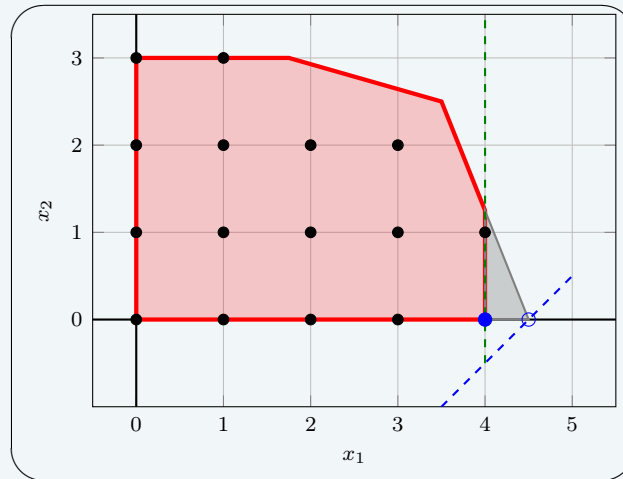


Figure 2.5: The objective $x_1 - x_2$: the variable upper bound $x_1 \leq 4$ (green) cuts off the gray triangle and the optimal solution $(\frac{9}{2}, 0)$ of the LP relaxation (blue circle). The new optimal solution $(4, 0)$ (blue) is integer. The dashed line is $x_1 - x_2 = \frac{9}{2}$.

- The value of the LP relaxation improves from $\frac{9}{2}$ to 4. The rounded bound $\lfloor \frac{9}{2} \rfloor = 4$ was already 4: a better formulation does not always improve the rounded bound.

2.6 Dual bounds

- A dual bound can also be obtained without solving the LP relaxation, by combining the constraints with **non-negative multipliers** π_i , one for each constraint.

Definition 2.9: dual problem of the LP relaxation

The **dual problem** $D(LP)$ of the LP relaxation (2.2) is the following LP problem, with optimal value $z(D(LP))$:

$$\min \sum_{i=1}^m b_i \pi_i \quad (2.6a)$$

$$\text{subject to } \sum_{i=1}^m a_{ij} \pi_i \geq c_j, \quad \forall j \in \{1, 2, \dots, n\}, \quad (2.6b)$$

$$\pi_i \geq 0, \quad \forall i \in \{1, 2, \dots, m\}. \quad (2.6c)$$

- Multiplying the constraint i by $\pi_i \geq 0$ and summing, we get an inequality whose coefficients are at least the costs, by (2.6b). The multipliers must be non-negative: a negative multiplier would reverse the inequality.

Proposition 2.10: dual bounds

For every feasible solution $\bar{\pi}$ of the dual problem (2.6),

$$z(ILP) \leq z(LP) \leq \sum_{i=1}^m b_i \bar{\pi}_i.$$

By strong duality, $z(D(LP)) = z(LP)$: the best multipliers give exactly the value of the LP relaxation.

Proof. Let $\mathbf{x}$ be feasible for the LP relaxation. Multiplying the constraint i by $\bar{\pi}_i \geq 0$ and summing, $\sum_{j=1}^n (\sum_{i=1}^m a_{ij} \bar{\pi}_i) x_j \leq \sum_{i=1}^m b_i \bar{\pi}_i$. By (2.6b) and $x_j \geq 0$, the left-hand side is at least $\sum_{j=1}^n c_j x_j$. Hence $\sum_{j=1}^n c_j x_j \leq \sum_{i=1}^m b_i \bar{\pi}_i$ for every feasible $\mathbf{x}$. $\square$

Every feasible solution of the dual problem is a certificate of an upper bound, found without solving the LP relaxation. The best certificate is an optimal dual solution, and it has the same value as the LP relaxation.

Example 2.9: multipliers for the car example

- A feasible solution of the LP relaxation does not give a dual bound: its value is neither a lower bound nor an upper bound on $z(ILP)$.

(x_1, x_2)	feasible for the LP relaxation	integer	$3x_1 + 2x_2$
$(4, \frac{5}{4})$	yes	no	$\frac{29}{2} > 14$
$(2, \frac{5}{2})$	yes	no	$11 < 14$

- Multiplying the first constraint of the car example by $\frac{3}{4}$ we get

$$\frac{15}{2}x_1 + 3x_2 \leq \frac{135}{4}.$$

Since $x_1, x_2 \geq 0$ and the coefficients $\frac{15}{2}$ and 3 are at least the costs 3 and 2, we have $3x_1 + 2x_2 \leq \frac{15}{2}x_1 + 3x_2 \leq \frac{135}{4}$: an upper bound of $\frac{135}{4}$.

- With multipliers $\pi_1, \pi_2 \geq 0$ of the two constraints, the coefficients of the sum must be at least the costs:

$$\begin{aligned} (10\pi_1 + 4\pi_2)x_1 + (4\pi_1 + 14\pi_2)x_2 &\leq 45\pi_1 + 49\pi_2, \\ 10\pi_1 + 4\pi_2 &\geq 3, \quad 4\pi_1 + 14\pi_2 \geq 2. \end{aligned}$$

(π_1, π_2)	valid multipliers	upper bound $45\pi_1 + 49\pi_2$
$(\frac{3}{4}, 0)$	yes	$\frac{135}{4}$
$(0, \frac{3}{4})$	yes	$\frac{147}{4}$
$(\frac{1}{5}, \frac{1}{4})$	yes	$\frac{85}{4}$
$(\frac{1}{2}, 0)$	yes	$\frac{45}{2}$
$(\frac{1}{5}, 0)$	no: $10 \cdot \frac{1}{5} = 2 < 3$	—

- The best multipliers, with the smallest value $45\pi_1 + 49\pi_2$, solve the dual problem (2.6) of the car example:

$$\begin{aligned} \min \quad & 45\pi_1 + 49\pi_2 \\ \text{subject to} \quad & 10\pi_1 + 4\pi_2 \geq 3 \\ & 4\pi_1 + 14\pi_2 \geq 2 \\ & \pi_1, \pi_2 \geq 0 \end{aligned}$$

Example 2.10: optimal dual solution of the car example

The optimal solution of the dual problem of [Example 2.9](#) is $\tilde{\pi}(\text{D}(\text{LP})) = (\frac{17}{62}, \frac{2}{31})$, and

$$z(\text{D}(\text{LP})) = 45 \cdot \frac{17}{62} + 49 \cdot \frac{2}{31} = \frac{765}{62} + \frac{98}{31} = \frac{31}{2} = z(\text{LP}).$$

The other two vertices of the dual feasible region give the bounds $\frac{45}{2}$ and $\frac{147}{4}$.

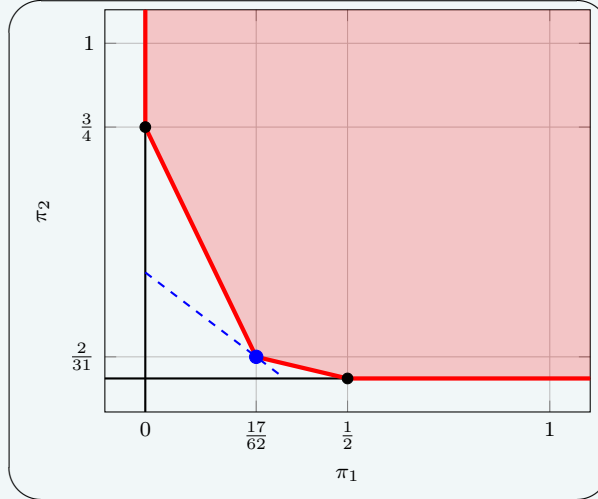


Figure 2.6: The feasible region of the dual LP problem (red, unbounded) and its optimal solution $(\frac{17}{62}, \frac{2}{31})$ (blue), with the objective line $45\pi_1 + 49\pi_2 = \frac{31}{2}$ (dashed). The other vertices (black) give the bounds $\frac{45}{2}$ and $\frac{147}{4}$.

2.6.1 Summary of the notation

- **ILP problem:** optimal solution $\tilde{x}(\text{ILP})$, optimal value $z(\text{ILP})$.
- **Primal bound:** feasible solution $\tilde{x}$ with $\text{LB}(\tilde{x}) = \sum_j c_j \tilde{x}_j \leq z(\text{ILP})$.
- **LP relaxation:** optimal solution $\tilde{x}(\text{LP})$, optimal value $z(\text{LP})$, upper bound $z(\text{ILP}) \leq z(\text{LP})$.
- **Dual problem:** optimal solution $\tilde{\pi}(\text{D}(\text{LP}))$, optimal value $z(\text{D}(\text{LP})) = z(\text{LP})$, and $z(\text{ILP}) \leq \sum_i b_i \tilde{\pi}_i$ for every dual feasible $\tilde{\pi}$.

2.7 Typical mistakes

Warning

- **Taking a heuristic solution as optimal.** A feasible solution gives only a primal bound. Without a dual bound with the same value there is no certificate of optimality.
- **Rounding the bound when it is not valid.** The bound $\lfloor z(\text{LP}) \rfloor$ requires all the variables and all the costs to be integer. In a MIP problem the bound is $z(\text{LP})$.
- **Rounding the solution and calling it optimal.** A rounded solution is feasible at best, and in general not optimal. It may even be infeasible, as the nearest integer point $(2, 5)$ of [Example 2.6](#).
- **Using the value of a feasible LP solution as a bound.** Only the optimal value of the LP relaxation, or the value of a dual feasible solution, is an upper bound.
- **Multipliers with a negative sign.** The multipliers of constraints $\leq$ must be non-negative, and the combined coefficients must be at least the costs.
- **Deriving a bound on a variable from a constraint with negative coefficients.** The proof of (2.5) needs all the coefficients to be non-negative.
- **Believing that a stronger formulation always improves the bound.** The bounds $x_1 \leq 4$ and $x_2 \leq 3$ leave the bound 15 of the car example unchanged.

2.8 In summary

In summary

- Every feasible solution is a primal bound (a lower bound) on $z(\text{ILP})$, and heuristic algorithms find good feasible solutions quickly.
- The LP relaxation drops integrality, so $z(\text{ILP}) \leq z(\text{LP})$; with integer variables and integer costs this dual bound (an upper bound) can be rounded down to $\lfloor z(\text{LP}) \rfloor$.
- Primal and dual bound define the optimality gap; optimality is certified when the gap is zero.
- Rounding an optimal LP solution gives a feasible solution at best, never a certificate.
- The upper bounds on the variables $x_j \leq \lfloor b_i/a_{ij} \rfloor$ come from constraints with non-negative coefficients, and they help only if they cut off the LP optimum.
- Non-negative multipliers of the constraints give dual bounds; the best ones solve the dual problem, whose optimal value is $z(\text{LP})$.

Dashboard of the car example. The bounds on $z(\text{ILP})$ found so far in this module (primal bound: lower; dual bound: upper):

Chapter			Lower bound	Upper bound	From
Primal bounds	and	dual	14	15	heuristic solution $(4, 1)$, and $\lfloor z(\text{LP}) \rfloor$ with $z(\text{LP}) = \frac{31}{2}$

Part II

Strengthening the formulation



Ideal formulations

Prerequisites

- From this module, the chapters «Integer linear programming» and «Primal and dual bounds»: the feasible region of an ILP problem, the LP relaxation and its bound, and the car example.
- From the module *Linear Programming*, [Chapter 15 «The geometry of Linear Programming»](#): vertices of a polyhedron, and the fact that an LP problem with an optimal solution has an optimal vertex.
- From the module *Convex optimization*, [Chapter 1 «Convex sets»](#): convex combinations and convex hulls. A short recap is given in this chapter.

Objectives

At the end of this chapter you will be able to:

- compute the convex hull of the feasible solutions of a small ILP problem, and classify its points as vertices, boundary points or interior points;
- define a formulation of an ILP problem, and decide whether it is ideal;
- explain why an LP problem over an ideal formulation solves the ILP problem;
- compare two formulations of the same problem through their LP relaxations;
- explain why ideal formulations exist but are hard to find.

Where we were

In the car example we have the lower bound 14 and the upper bound $\lfloor z(\text{LP}) \rfloor = 15$. The gap of $\frac{1}{14}$ is due to the polyhedron of the constraints, which contains fractional points outside the convex hull of the integer points. This chapter defines the polyhedron that gives no gap.

3.1 The convex hull

Definition 3.1: of convex hull

Given a set $S \subseteq \mathbb{R}^n$, a **convex combination** of points $s^1, s^2, \dots, s^k \in S$ is a point $\sum_{h=1}^k \lambda_h s^h$ with $\lambda_h \geq 0$ and $\sum_{h=1}^k \lambda_h = 1$. The **convex hull** of S , denoted by $\text{conv}(S)$, is the set of all the convex combinations of points of S .

- The convex hull is the smallest convex set containing S . If S is a finite set of points, the convex hull is a bounded polyhedron (a **polytope**), and its **vertices** are points of S . The other points of S lie either on the boundary of the polytope, or in its interior: they are not needed to describe it.
- For an ILP problem we take $S = F$, the set of the feasible solutions. The vertices of $\text{conv}(F)$ are integer feasible solutions.

Example 3.1: convex hull of the car example

The convex hull of the 16 integer points of the car example has 6 vertices:

$$(0,0), \quad (4,0), \quad (4,1), \quad (3,2), \quad (1,3), \quad (0,3).$$

Of the other 10 points, five lie on its boundary, namely $(0,1)$, $(0,2)$, $(1,0)$, $(2,0)$ and $(3,0)$, and five in its interior, namely $(1,1)$, $(2,1)$, $(3,1)$, $(1,2)$ and $(2,2)$.

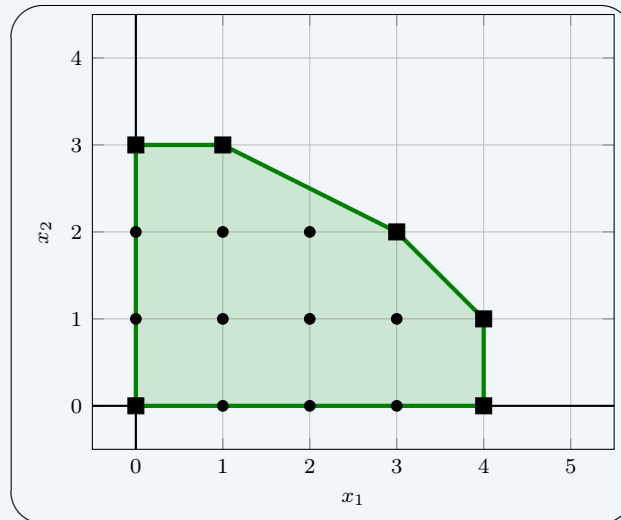


Figure 3.1: The convex hull of the 16 integer points of the car example (green). Its 6 vertices (black squares) are integer points; five other points lie on its boundary and five in its interior.

3.2 Formulations and ideal formulations

Definition 3.2: of formulation

A **formulation** of an ILP problem with feasible region F is a polyhedron $P = \{x \in \mathbb{R}^n : \sum_j a_{ij} x_j \leq b_i, \forall i, x_j \geq 0, \forall j\}$ such that $P \cap \mathbb{Z}^n = F$. The **LP relaxation of the formulation** is the problem of maximizing $\sum_j c_j x_j$ over P .

- An ILP problem has many formulations: they share the integer points and differ in the fractional points. Since P is convex and contains F , every formulation contains the convex hull:

$$\text{conv}(F) \subseteq P.$$

Definition 3.3: of ideal formulation

A formulation P of an ILP problem is **ideal** if $P = \text{conv}(F)$, that is, if the feasible region of its LP relaxation is exactly the convex hull of the integer feasible solutions.

Theorem 3.4: LP over an ideal formulation

Let P be an ideal formulation of an ILP problem with $F \neq \emptyset$ and P bounded. Then the LP relaxation has an optimal vertex, every optimal vertex is an optimal solution of the ILP problem, and $z(\text{LP}) = z(\text{ILP})$.

Proof. The polyhedron $P = \text{conv}(F)$ is non-empty and bounded, so the LP problem has an optimal vertex of P . The vertices of $\text{conv}(F)$ belong to F , so this vertex is a feasible ILP solution, whose value is $z(\text{LP})$. Since $z(\text{ILP}) \leq z(\text{LP})$, the vertex is optimal for the ILP problem. $\square$

- The simplex algorithm returns an optimal vertex, so with an ideal formulation it returns an integer solution. If the LP problem has several optimal solutions, a method that does not return vertices can give a fractional point on the optimal face: this is why the theorem speaks of an optimal vertex.

With an ideal formulation, an ILP problem is solved as an LP problem: an optimal vertex of the LP relaxation is an optimal ILP solution.

Example 3.2: the formulation of the car example is not ideal

The point $(\frac{7}{2}, \frac{5}{2})$ belongs to the polyhedron P of the car example but not to $\text{conv}(F)$, because it violates $x_1 + x_2 \leq 5$, which all the vertices of the hull of [Example 3.1](#) satisfy: $\text{conv}(F) \subsetneq P$.

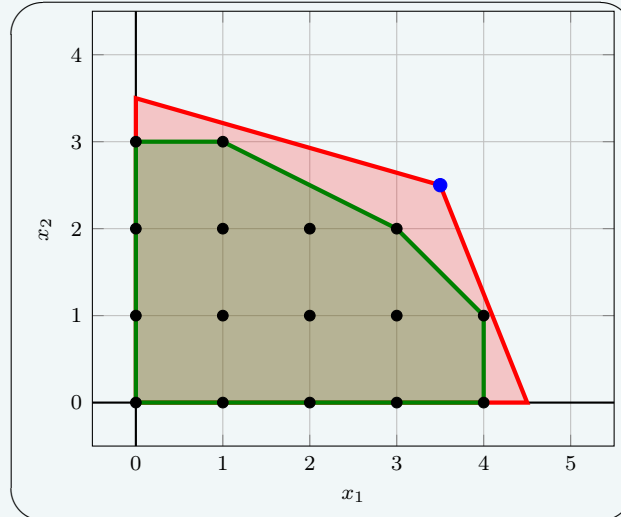


Figure 3.2: The car example: the polyhedron of the LP relaxation (red) contains the convex hull of the integer points (green), and the optimal solution of the LP relaxation (blue) is outside the hull.

Example 3.3: an ideal formulation of the car example

The convex hull of the car example is described by the following four inequalities and the non-negativity constraints:

$$\begin{array}{llll}
 \max & 3x_1 & + & 2x_2 \\
 \text{subject to} & x_1 & + & x_2 \leq 5 \\
 & x_1 & + & 2x_2 \leq 7 \\
 & x_1 & & \leq 4 \\
 & & x_2 & \leq 3 \\
 & x_1, & x_2 & \geq 0
 \end{array}$$

These inequalities define exactly the 6 vertices of the hull, and the same 16 integer points. The LP relaxation of this **ideal formulation** has the optimal solution $(4, 1)$, which is integer, with $z(\text{LP}) = 14 = z(\text{ILP})$.

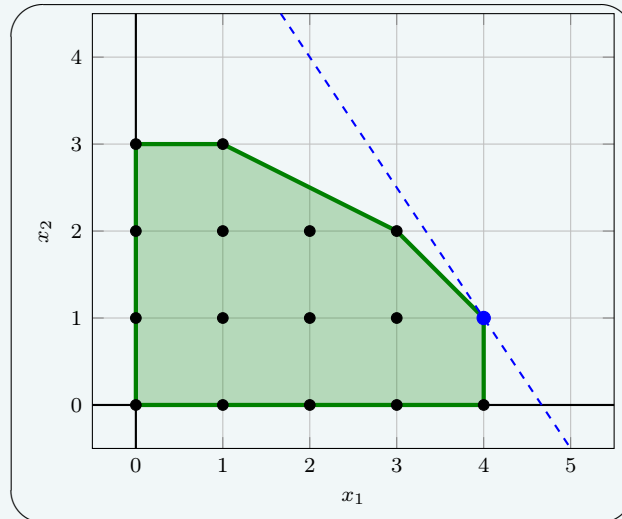


Figure 3.3: The ideal formulation of the car example: its feasible region (green) is the convex hull of the integer points. The optimal solution of the LP problem is the integer point $(4, 1)$ (blue).

3.3 Comparing formulations

Definition 3.5: stronger formulation

Given two formulations P_1 and P_2 of the same ILP problem, P_1 is **stronger** than P_2 if $P_1 \subseteq P_2$.

- A stronger formulation never gives a worse upper bound: the maximum over a smaller set is not larger, so

$$P_1 \subseteq P_2 \implies z(\text{ILP}) \leq z(\text{LP}_1) \leq z(\text{LP}_2).$$

- The ideal formulation is the strongest of all, because it is contained in every formulation. In the car example the original formulation gives the bound $\lfloor \frac{31}{2} \rfloor = 15$, and the ideal one of [Example 3.3](#) gives the bound 14.

Among the formulations of the same ILP problem, we prefer the ones with the smallest polyhedron: their LP relaxations give the best upper bounds, and the ideal formulation gives the optimal value.

3.4 Existence of ideal formulations

- An ideal formulation exists for every ILP problem whose feasible region is bounded or, more in general, whose data are rational (Meyer's theorem; in this module the data are integer). The convex hull of the integer points is a polyhedron, so it is described by a finite number of linear inequalities.
- Finding the ideal formulation is in general very difficult. It can require an exponential number of inequalities, and finding the necessary ones is often as hard as solving the ILP problem itself.
- In practice we keep the formulation we have and add, step by step, inequalities that are satisfied by all the integer points and cut off fractional points: the **valid inequalities** of the next chapter.

3.5 Typical mistakes

Warning

- **Believing that the points that are not vertices lie strictly inside the hull.** They can lie on its boundary, as $(0, 1)$ and $(3, 0)$ in the car example.
- **Checking only the optimal solution.** A formulation can return an integer optimal solution without being ideal: the LP optimum can be integer for one objective function and fractional for another one.
- **Believing that every formulation is ideal.** The formulation of the car example has the right integer points but contains fractional points outside the hull.
- **Adding inequalities that remove integer points.** A formulation must keep all the points of F : the polyhedron can shrink only down to the convex hull, never below it.

3.6 In summary

In summary

- The convex hull of the feasible solutions is the smallest polyhedron containing them, and its vertices are integer feasible solutions.
- A formulation is ideal if its polyhedron is the convex hull; then an optimal vertex of the LP relaxation is an optimal ILP solution.
- A stronger formulation, with a smaller polyhedron, never gives a worse bound.
- The car example has an ideal formulation with 4 inequalities, which gives $z(\text{LP}) = 14 = z(\text{ILP})$.
- Ideal formulations exist for rational data, but in general they need exponentially many inequalities and are hard to find.

Dashboard of the car example. The bounds on $z(\text{ILP})$ found so far in this module:

Chapter	Lower bound	Upper bound	From
Primal and dual bounds	14	15	heuristic solution $(4, 1)$, and $\lfloor z(\text{LP}) \rfloor$ with $z(\text{LP}) = \frac{31}{2}$
Ideal formulations	14	14	ideal formulation, optimal vertex $(4, 1)$

Valid inequalities

Prerequisites

- From this module, the chapter «Primal and dual bounds»: the LP relaxation, the rounded bound $\lfloor z(\text{LP}) \rfloor$, and bounds from non-negative multipliers of the constraints.
- From this module, the chapter «Ideal formulations»: formulations, the convex hull, and the comparison of formulations through their polyhedra.

Objectives

At the end of this chapter you will be able to:

- define a valid inequality, and say when it cuts off a fractional point;
- derive a valid inequality from non-negative multipliers of the constraints by rounding down;
- improve a valid inequality until it touches a feasible integer point;
- measure the effect of a valid inequality on the optimal value of the LP relaxation and on the upper bound;
- explain why valid inequalities are the practical way to approach an ideal formulation.

Where we were

In the car example the ideal formulation gives the upper bound 14, equal to the lower bound, so it certifies that $(4, 1)$ is optimal. But we found its four inequalities by looking at the convex hull, which is not possible in general. We now learn to **derive** such inequalities from the constraints.

4.1 Valid inequalities

Definition 4.1: of valid inequality

An inequality $\sum_{j=1}^n \alpha_j x_j \leq \beta$ is a **valid inequality** for an ILP problem with feasible region F if it is satisfied by all the points of F .

Definition 4.2: cut-off point and tight valid inequality

Let $\sum_{j=1}^n \alpha_j x_j \leq \beta$ be a valid inequality for an ILP problem with feasible region F .

- The inequality **cuts off** a point $\mathbf{x}^* \in \mathbb{R}^n$ if $\sum_{j=1}^n \alpha_j x_j^* > \beta$.
 - The inequality is **tight** if at least one point of F satisfies it with equality.
- A valid inequality cuts off no point of F , but it can cut off fractional points of the LP relaxation. The right-hand side of a tight inequality cannot be decreased without losing a feasible solution.
 - Adding a valid inequality to the formulation does not change F , and it shrinks the polyhedron P . The new formulation is stronger, so its LP relaxation gives an upper bound that is not worse.
 - A valid inequality is satisfied by a point of F if and only if it is satisfied by the convex hull, so it is valid for $\text{conv}(F)$. The inequalities of the ideal formulation are valid inequalities.

A valid inequality removes fractional points and never an integer feasible solution. Adding valid inequalities makes the formulation stronger and the upper bound better.

Example 4.1: valid inequalities of the car example

The inequalities $x_1 \leq 4$ and $x_2 \leq 3$ of the previous chapters are valid inequalities of the car example. They do not cut off the optimal solution $(\frac{7}{2}, \frac{5}{2})$ of the LP relaxation, so they do not improve the upper bound 15.

4.2 Deriving valid inequalities by rounding

- The constraints are combined with non-negative multipliers, as for the dual bounds; since the variables are integer, the right-hand side of the combination can be rounded down.

Proposition 4.3: valid inequalities by rounding

Let $\pi_1, \pi_2, \dots, \pi_m \geq 0$ be multipliers such that $\sum_{i=1}^m \pi_i a_{ij} \geq \alpha_j$, with $\alpha_j \in \mathbb{Z}$, for all $j \in \{1, 2, \dots, n\}$. Then the inequality

$$\sum_{j=1}^n \alpha_j x_j \leq \left\lfloor \sum_{i=1}^m \pi_i b_i \right\rfloor \quad (4.1)$$

is valid for the ILP problem.

Proof. For every feasible solution, $\sum_j \alpha_j x_j \leq \sum_j (\sum_i \pi_i a_{ij}) x_j \leq \sum_i \pi_i b_i$, by $x_j \geq 0$ and by the constraints multiplied by $\pi_i \geq 0$. The left-hand side is an integer, because α_j and x_j are integer, so it is at most the integer part of the right-hand side. $\square$

- The proposition needs all the variables to be integer and non-negative. With a continuous variable the left-hand side is not an integer, and the right-hand side cannot be rounded.

Example 4.2: rounding with multiplier $\frac{1}{2}$

In the car example, multiply the first constraint by $\frac{1}{2}$: we get $5x_1 + 2x_2 \leq \frac{45}{2}$. Since the left-hand side is integer, we obtain the valid inequality

$$5x_1 + 2x_2 \leq 22.$$

Multiplying the second constraint by $\frac{1}{2}$ we get $2x_1 + 7x_2 \leq \frac{49}{2}$, hence

$$2x_1 + 7x_2 \leq 24.$$

Both inequalities cut off the optimal solution $(\frac{7}{2}, \frac{5}{2})$ of the LP relaxation: $5 \cdot \frac{7}{2} + 2 \cdot \frac{5}{2} = \frac{45}{2} > 22$ and $2 \cdot \frac{7}{2} + 7 \cdot \frac{5}{2} = \frac{49}{2} > 24$. The new LP relaxations give:

Added inequality	$\tilde{\mathbf{x}}(\text{LP})$	$z(\text{LP})$	$\lfloor z(\text{LP}) \rfloor$
none	$(\frac{7}{2}, \frac{5}{2})$	$\frac{31}{2}$	15
$5x_1 + 2x_2 \leq 22$	$(\frac{105}{31}, \frac{157}{62})$	$\frac{472}{31}$	15
$2x_1 + 7x_2 \leq 24$	$(\frac{219}{62}, \frac{75}{31})$	$\frac{957}{62}$	15

The LP optimum moves, but the upper bound does not improve: $\lfloor \frac{472}{31} \rfloor = \lfloor \frac{957}{62} \rfloor = 15$.

A valid inequality that cuts off the optimal LP solution does not always improve the upper bound. The choice of the multipliers matters: we look for inequalities that touch the integer points.

4.3 Improving a valid inequality

- Given the coefficients α_j , the strongest valid inequality has the smallest possible right-hand side, which is the maximum of $\sum_j \alpha_j x_j$ over F . We find it by lowering the right-hand side β of (4.1) until a feasible integer point reaches it.

Algorithm

Input: An ILP problem in canonical maximization form with $F \neq \emptyset$, and integer coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$.

Output: A tight valid inequality $\sum_{j=1}^n \alpha_j x_j \leq \beta$.

Step 0: Coarse bound

Find multipliers $\pi_i \geq 0$ with $\sum_i \pi_i a_{ij} \geq \alpha_j$ for all j , and set $\beta \leftarrow \lfloor \sum_i \pi_i b_i \rfloor$.

Step 1: Test the right-hand side

If no feasible integer point satisfies $\sum_j \alpha_j x_j = \beta$, set $\beta \leftarrow \beta - 1$ and repeat Step 1.

Step 2: Tight inequality

A feasible integer point satisfies $\sum_j \alpha_j x_j = \beta$, so $\sum_j \alpha_j x_j \leq \beta$ is valid and tight. STOP.

- In Step 0 the multipliers can be chosen freely: the better they are, the fewer repetitions of Step 1 are needed. In Step 1, for two variables, we substitute one variable using the equation and bound the other one with the constraints: if the bounds contradict each other, no point exists.

Example 4.3: the inequality $x_1 + x_2 \leq 5$

We look for a valid inequality with $\alpha = (1, 1)$.

- Step 0.** With the multipliers $(\frac{1}{14}, \frac{1}{14})$ the sum of the constraints, divided by 14, is $x_1 + \frac{9}{7}x_2 \leq \frac{47}{7}$. The coefficients are at least $(1, 1)$, so $x_1 + x_2 \leq \lfloor \frac{47}{7} \rfloor = 6$, and we set $\beta = 6$.
- Step 1, $\beta = 6$.** Let $x_1 + x_2 = 6$, so $x_2 = 6 - x_1$. The first constraint becomes $6x_1 + 24 \leq 45$, that is $x_1 \leq \frac{7}{2}$, hence $x_1 \leq 3$. The second becomes $84 - 10x_1 \leq 49$, that is $x_1 \geq \frac{7}{2}$, hence $x_1 \geq 4$. The bounds contradict each other: no feasible point has $x_1 + x_2 = 6$, and we set $\beta = 5$.
- Step 1, $\beta = 5$.** Let $x_2 = 5 - x_1$. The constraints become $x_1 \leq \frac{25}{6}$, hence $x_1 \leq 4$, and $x_1 \geq \frac{21}{10}$, hence $x_1 \geq 3$. There is no contradiction, and the points $(3, 2)$ and $(4, 1)$ are feasible with $x_1 + x_2 = 5$.
- Step 2.** The inequality $x_1 + x_2 \leq 5$ is valid and tight.

It cuts off $(\frac{7}{2}, \frac{5}{2})$ (the sum is 6). The new LP relaxation has the optimal solution $(\frac{25}{6}, \frac{5}{6})$ with $z(\text{LP}) = \frac{85}{6}$, so the upper bound is $\lfloor \frac{85}{6} \rfloor = 14$.

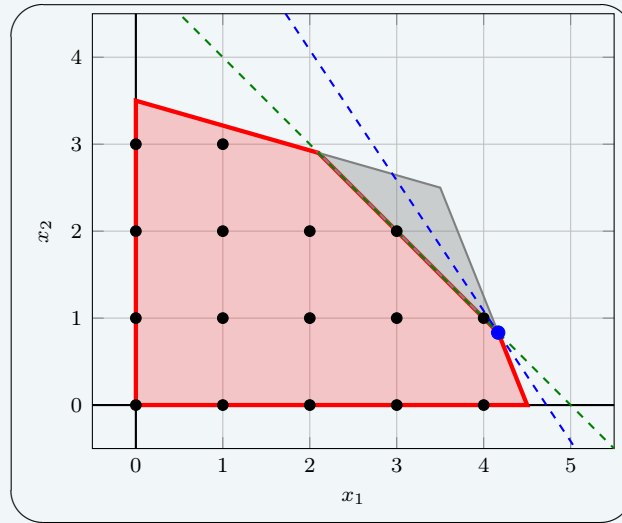


Figure 4.1: The valid inequality $x_1 + x_2 \leq 5$ (green) cuts off the gray triangle, which contains no integer point, and the optimal solution of the previous LP relaxation. The new optimal solution is $(\frac{25}{6}, \frac{5}{6})$ (blue), with $z = \frac{85}{6}$ (dashed line).

Example 4.4: the inequality $x_1 + 2x_2 \leq 7$

We look for a valid inequality with $\alpha = (1, 2)$.

- **Step 0.** With the multipliers $(\frac{3}{62}, \frac{4}{31})$ the combination of the constraints is $x_1 + 2x_2 \leq \frac{17}{2}$, because $\frac{3 \cdot 10}{62} + \frac{4 \cdot 4}{31} = 1$, $\frac{3 \cdot 4}{62} + \frac{4 \cdot 14}{31} = 2$ and $\frac{3 \cdot 45}{62} + \frac{4 \cdot 49}{31} = \frac{17}{2}$. Hence $x_1 + 2x_2 \leq \lfloor \frac{17}{2} \rfloor = 8$, and we set $\beta = 8$.
- **Step 1, $\beta = 8$.** Let $x_1 = 8 - 2x_2$. The first constraint becomes $80 - 16x_2 \leq 45$, hence $x_2 \geq 3$; the second becomes $32 + 6x_2 \leq 49$, hence $x_2 \leq 2$. They contradict each other, and we set $\beta = 7$.
- **Step 1, $\beta = 7$.** Let $x_1 = 7 - 2x_2$: the constraints become $70 - 16x_2 \leq 45$, hence $x_2 \geq 2$, and $28 + 6x_2 \leq 49$, hence $x_2 \leq 3$. The points $(3, 2)$ and $(1, 3)$ are feasible with $x_1 + 2x_2 = 7$.
- **Step 2.** The inequality $x_1 + 2x_2 \leq 7$ is valid and tight.

The new LP relaxation has the optimal solution $(\frac{31}{8}, \frac{25}{16})$ with $z(\text{LP}) = \frac{59}{4}$, so the upper bound is $\lfloor \frac{59}{4} \rfloor = 14$.

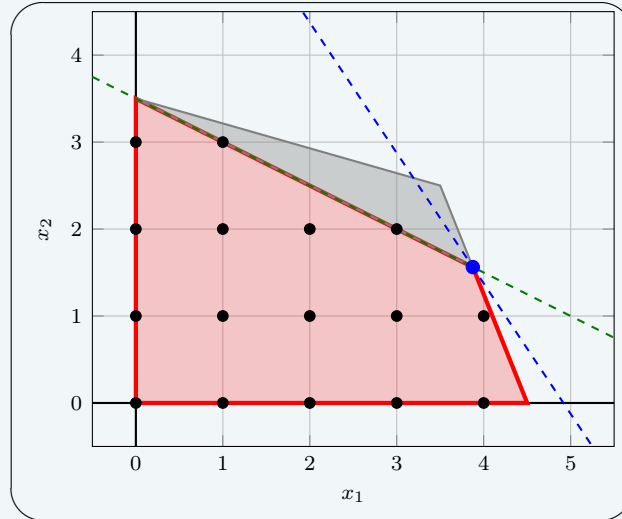


Figure 4.2: The valid inequality $x_1 + 2x_2 \leq 7$ (green) cuts off the gray triangle, which contains no integer point. The new optimal solution is $(\frac{31}{8}, \frac{25}{16})$ (blue), with $z = \frac{59}{4}$ (dashed line).

With the valid inequality $x_1 + x_2 \leq 5$, or with $x_1 + 2x_2 \leq 7$, the upper bound drops from 15 to 14. It equals the lower bound given by $(4, 1)$: **the solution $(4, 1)$ is certified optimal**, and the gap is zero.

- The two inequalities, together with $x_1 \leq 4$ and $x_2 \leq 3$, are exactly the four inequalities of the ideal formulation of the previous chapter. For the car example we have reached the convex hull with valid inequalities derived from the constraints.

4.4 Families of valid inequalities and cuts

- With two variables the directions α are chosen on the picture. In general, valid inequalities come from two sources.
- Families of inequalities for specific structures of the constraints, such as the **cover inequalities** of a constraint with positive coefficients of a binary problem (next chapter).
- The rounding of this chapter, applied systematically to the rows of the optimal dictionary of the LP relaxation: the **Gomory fractional cuts** of the cutting plane algorithm, and then the **branch-and-cut** algorithm.

4.5 Typical mistakes

Warning

- **Rounding when a variable is continuous.** The rounding of (4.1) needs integer variables and integer coefficients α_j .
- **Rounding up instead of down.** The right-hand side is rounded down. Rounding up gives an inequality that is not valid, and it can remove integer solutions.
- **Believing that every valid inequality improves the bound.** The inequalities $5x_1 + 2x_2 \leq 22$ and $2x_1 + 7x_2 \leq 24$ cut off the LP optimum, but leave the bound at 15.
- **Using multipliers that are not large enough.** The combined coefficients must be at least α_j for every j , and the multipliers must be non-negative.
- **Stopping at a coarse right-hand side.** The inequality $x_1 + x_2 \leq 6$ is valid, but it does not cut off the LP optimum of the car example (its sum is 6); the tight inequality $x_1 + x_2 \leq 5$ does.

4.6 In summary

In summary

- A valid inequality is satisfied by all the feasible integer solutions; it can cut off fractional points of the LP relaxation, and it makes the formulation stronger.
- Non-negative multipliers of the constraints, with the right-hand side rounded down, give valid inequalities for ILP problems.
- Lowering the right-hand side until an integer point touches the line gives a tight inequality.
- A valid inequality helps only if it cuts off the LP optimum and improves the rounded bound.
- In the car example, $x_1 + x_2 \leq 5$ or $x_1 + 2x_2 \leq 7$ lowers the upper bound from 15 to 14, and certifies that $(4, 1)$ is optimal.

Dashboard of the car example. The bounds on $z(\text{ILP})$ found so far in this module:

Chapter	Lower bound	Upper bound	From
Primal and dual bounds	14	15	heuristic solution $(4, 1)$, and $\lfloor z(\text{LP}) \rfloor$ with $z(\text{LP}) = \frac{31}{2}$
Ideal formulations	14	14	ideal formulation, optimal vertex $(4, 1)$
Valid inequalities	14	14	$x_1 + x_2 \leq 5$ gives $z(\text{LP}) = \frac{85}{6}$

Cover inequalities

Prerequisites

- From this module, [Chapter 1 «Integer linear programming»](#) and [Chapter 2 «LP relaxation»](#): ILP problems, the LP relaxation, and the upper bound given by the LP relaxation.
- From this module, [Chapter 3 «Ideal formulations»](#) and [Chapter 4 «Valid inequalities»](#): the convex hull of the integer solutions, valid inequalities, and cuts that remove fractional points.
- The LP relaxation of a binary problem with one constraint, solved by inserting the variables in non-increasing order of the ratio c_j/a_j between objective and constraint coefficients (Dantzig's rule).

Objectives

At the end of this chapter you will be able to:

- write the cover inequalities of a $\leq$ constraint with positive coefficients of a binary MIP, and prove that they are valid;
- list the covers and the minimal covers of an instance, and describe the cover closure;
- separate a cover inequality from a fractional point by solving a cover separation problem, exactly or with the greedy heuristic algorithm;
- strengthen a cover inequality into an extended cover inequality;
- compute the violation of a cover inequality and the distance of a point from it.

5.1 A constraint with positive coefficients and its LP relaxation

Definition 5.1: constraint with positive coefficients of a binary MIP

Given binary variables x_j , for $j \in \mathcal{J} = \{1, 2, \dots, n\}$, integer coefficients $a_j \in \mathbb{Z}_{\geq 1}$ and a right-hand side $b \in \mathbb{Z}_{\geq 1}$, consider the constraint

$$\sum_{j \in \mathcal{J}} a_j x_j \leq b, \quad (5.1a)$$

$$x_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (5.1b)$$

Its set of feasible solutions is

$$X = \left\{ \mathbf{x} \in \{0, 1\}^n : \sum_{j \in \mathcal{J}} a_j x_j \leq b \right\}, \quad (5.2)$$

and its LP relaxation is

$$P = \left\{ \mathbf{x} \in [0, 1]^n : \sum_{j \in \mathcal{J}} a_j x_j \leq b \right\}. \quad (5.3)$$

- Constraints of this type are called **knapsack constraints** or **budget constraints**.
- We assume that

$$a_j \leq b \quad \text{for all } j \in \mathcal{J} \quad \text{and} \quad \sum_{j \in \mathcal{J}} a_j > b.$$

The first assumption guarantees that every variable can be set to 1, the second one that they cannot all be set to 1 together.

- X is a finite set of binary points, and an exact description of its convex hull $\text{conv}(X)$ is generally hard to obtain. The aim is to strengthen P by removing fractional points without removing any point of X .
- A MIP can contain several constraints of this type, and further constraints. Valid inequalities for X remain valid when other constraints are imposed.

With objective coefficients c_j , maximizing $\sum_{j \in \mathcal{J}} c_j x_j$ over P gives an upper bound on the integer maximization problem; valid inequalities for X can improve this upper bound.

5.2 Covers and cover inequalities

Definition 5.2: Cover and collection of all covers

A *cover* is a subset $C \subseteq \mathcal{J}$ whose total coefficient $\sum_{j \in C} a_j$ exceeds the right-hand side b . The collection of all covers is

$$\mathcal{C} = \left\{ C \subseteq \mathcal{J} : \sum_{j \in C} a_j > b \right\}. \quad (5.4)$$

Since coefficients and right-hand side are integer, the condition is equivalent to $\sum_{j \in C} a_j \geq b + 1$.

Definition 5.3: Cover inequality

Given $C \in \mathcal{C}$, the associated *cover inequality* is

$$\sum_{j \in C} x_j \leq |C| - 1. \quad (5.5)$$

Proposition 5.4

For every cover $C \in \mathcal{C}$, the cover inequality (5.5) is valid for X .

Proof. Suppose that a point $\mathbf{x} \in X$ violates (5.5). The sum of $|C|$ binary variables is then greater than $|C| - 1$, hence $x_j = 1$ for every $j \in C$. Since all coefficients are positive,

$$\sum_{j \in \mathcal{J}} a_j x_j \geq \sum_{j \in C} a_j > b.$$

This contradicts the constraint (5.1a): at most $|C| - 1$ variables of C can be equal to 1 together. $\square$

5.2.1 Minimal covers

Definition 5.5: Minimal cover

A cover C is *minimal* if no proper subset of C is a cover. Equivalently,

$$\sum_{j \in C} a_j > b, \quad \sum_{j \in C} a_j - a_i \leq b \quad \forall i \in C. \quad (5.6)$$

- Since the coefficients are positive, it suffices to remove one variable at a time: if each of these $|C|$ subsets satisfies the constraint, every smaller subset satisfies it as well.

Proposition 5.6

The inequalities of the minimal covers, together with the upper bounds $x_j \leq 1$ for all $j \in \mathcal{J}$, imply every cover inequality.

Proof. Starting from a cover C , remove variables as long as the remaining set is still a cover: this gives a minimal cover $D \subseteq C$. Adding the cover inequality of D and the upper bounds of the variables in $C \setminus D$ gives

$$\sum_{j \in C} x_j = \sum_{j \in D} x_j + \sum_{j \in C \setminus D} x_j \leq |D| - 1 + |C \setminus D| = |C| - 1. \quad \square$$

5.2.2 The cover closure

Definition 5.7: Cover closure

The *cover closure* of P is the polyhedron

$$P^{\text{cov}} = \left\{ \mathbf{x} \in P : \sum_{j \in C} x_j \leq |C| - 1 \quad \forall C \in \mathcal{C} \right\}. \quad (5.7)$$

- Since all cover inequalities are valid,

$$\text{conv}(X) \subseteq P^{\text{cov}} \subseteq P,$$

and in general the first inclusion is strict.

- By Proposition 5.6, the inequalities of the minimal covers are sufficient to describe P^{cov} .
- The number of covers can be exponential in n . For example, if $a_j = 1$ for all $j \in \mathcal{J}$ and $b = 1$, every subset with at least two indices is a cover, i.e., there are $2^n - n - 1$ covers. Also the minimal covers can be exponentially many: for $a_j = 1$ for all $j \in \mathcal{J}$, $n = 2m$ and $b = m$, they are the $\binom{2m}{m+1}$ subsets of $m + 1$ indices.
- Therefore, cover inequalities are not added all at once: they are added only when they are violated, i.e., they are *separated* (Section 5.4).

Example 5.1: A binary problem with one constraint of this form

Consider $\mathcal{J} = \{1, 2, 3, 4\}$, right-hand side $b = 9$, and the following objective coefficients c_j and constraint coefficients a_j .

j	1	2	3	4
c_j	6	5	8	9
a_j	2	3	6	7

The associated LP model is:

$$\begin{array}{llllll}
\max & 6x_1 & + & 5x_2 & + & 8x_3 & + & 9x_4 \\
\text{subject to} & 2x_1 & + & 3x_2 & + & 6x_3 & + & 7x_4 & \leq & 9 \\
& x_1 & & & & & & & \leq & 1 \\
& & x_2 & & & & & & \leq & 1 \\
& & & x_3 & & & & & \leq & 1 \\
& & & & x_4 & & & & \leq & 1 \\
& x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}$$

The ratios c_j/a_j are $3, 5/3, 4/3, 9/7$, in non-increasing order. Dantzig's rule sets to 1 the variables in this order, as long as the constraint is satisfied, and gives to the first variable that would violate it the largest fractional value allowed: this gives an optimal solution of LP. An optimal solution $\tilde{x}(\text{LP})$ is:

$$\tilde{x}_1 = \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{2}{3}, \quad \tilde{x}_4 = 0.$$

The optimal value $z(\text{LP})$ is $6 + 5 + 8(2/3) = 49/3$.

Example 5.2: The cover closure of the four-variable instance

The instance of Example 5.1 has the following seven covers.

C	$\sum_{j \in C} a_j$	$ C $	Cover inequality	Minimal
$\{2, 4\}$	10	2	$x_2 + x_4 \leq 1$	yes
$\{3, 4\}$	13	2	$x_3 + x_4 \leq 1$	yes
$\{1, 2, 3\}$	11	3	$x_1 + x_2 + x_3 \leq 2$	yes
$\{1, 2, 4\}$	12	3	$x_1 + x_2 + x_4 \leq 2$	no
$\{1, 3, 4\}$	15	3	$x_1 + x_3 + x_4 \leq 2$	no
$\{2, 3, 4\}$	16	3	$x_2 + x_3 + x_4 \leq 2$	no
$\{1, 2, 3, 4\}$	18	4	$x_1 + x_2 + x_3 + x_4 \leq 3$	no

Only the first three covers are minimal; their inequalities already describe the cover closure. Denote by LP^{cov} the LP obtained by adding them to LP. An optimal solution $\tilde{x}(\text{LP}^{\text{cov}})$ is:

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = \frac{3}{5}, \quad \tilde{x}_3 = \frac{2}{5}, \quad \tilde{x}_4 = \frac{2}{5}.$$

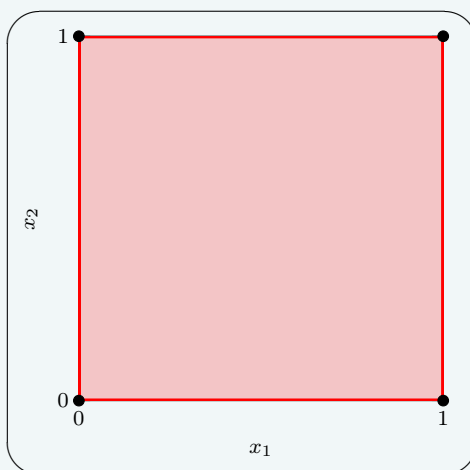
The optimal value is $z(\text{LP}^{\text{cov}}) = 79/5 < z(\text{LP}) = 49/3$.

5.3 Geometric interpretation

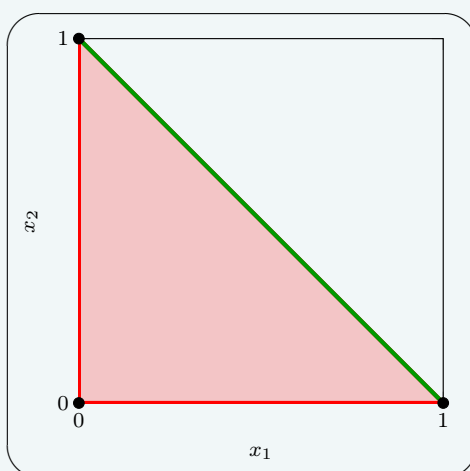
- All coordinates in the figures are the actual variable values. The ticks mark the coordinates of the blue point and of the main vertices; a few vertices have no tick, to keep the axes readable. In 3D, dashed edges are hidden behind the visible faces.
- Every step is illustrated both in two and in three dimensions, with the same colours in all figures: the red region is the continuous region under consideration; green identifies a cover inequality (an edge in 2D, a face in 3D); black points are the feasible binary points; the blue point is the fractional point discussed in the text.
- First consider only the domain, $[0, 1]^2$ or $[0, 1]^3$: no constraint other than the bounds is imposed in the next two examples.

Example 5.3: A cover inequality in the unit square

In $[0, 1]^2$, the inequality $x_1 + x_2 \leq 1$ removes only the binary point $(1, 1)$ and preserves the binary points $(0, 0)$, $(1, 0)$ and $(0, 1)$.



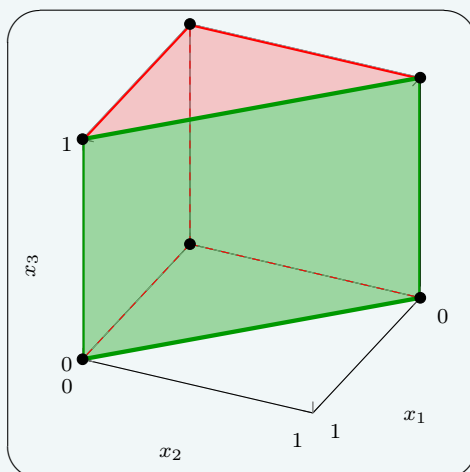
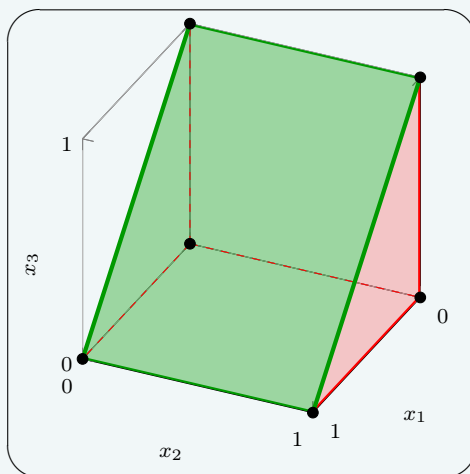
Unit square: domain only.

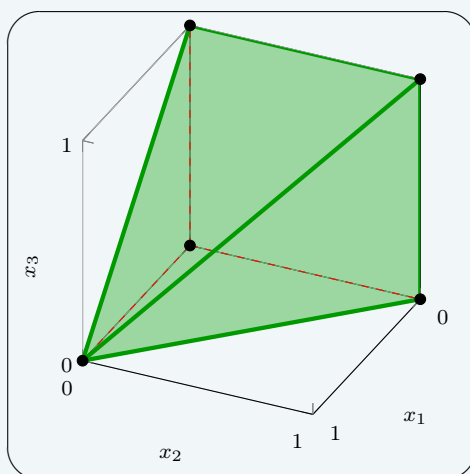


Domain with $x_1 + x_2 \leq 1$.

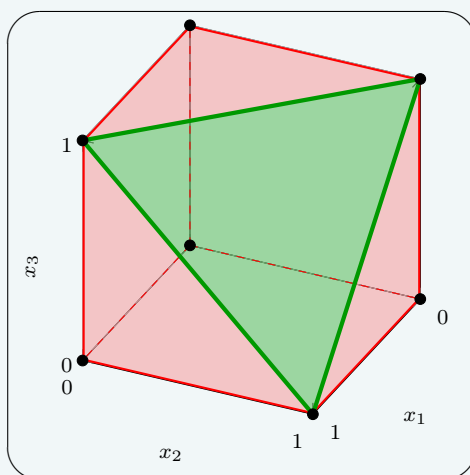
Example 5.4: Cover inequalities in the unit cube

- In $[0, 1]^3$, the inequality $x_1 + x_2 \leq 1$ is independent of x_3 ; its boundary is parallel to the x_3 axis. Similarly, the boundary of $x_1 + x_3 \leq 1$ is parallel to the x_2 axis.
- The two halfspaces remove different portions of the cube. Their intersection preserves exactly the binary points $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, 1, 1)$.
- The inequality $x_1 + x_2 + x_3 \leq 2$ removes only the binary point $x_1 = x_2 = x_3 = 1$.

Domain with $x_1 + x_2 \leq 1$.Domain with $x_1 + x_3 \leq 1$.



Domain with both inequalities.

Domain with $x_1 + x_2 + x_3 \leq 2$.**Example 5.5: A cover in two dimensions**

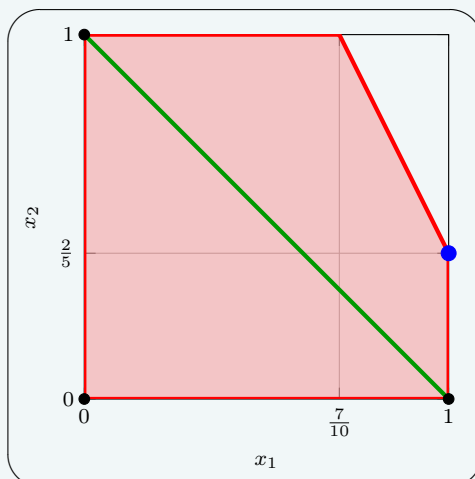
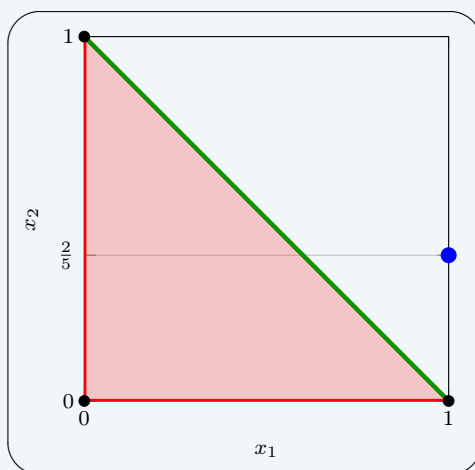
For $\mathcal{J} = \{1, 2\}$, consider

$$P_2 = \{\mathbf{x} \in [0, 1]^2 : 10x_1 + 5x_2 \leq 12\}.$$

The cover $C = \{1, 2\}$ gives $x_1 + x_2 \leq 1$. The point $\tilde{x}_1 = 1$, $\tilde{x}_2 = 2/5$ satisfies the constraint at equality, but violates the cover by $2/5$. The original polygon has vertices

$$(0, 0), (1, 0), (1, 2/5), (7/10, 1), (0, 1).$$

After adding the cut, the region is the triangle with vertices $(0, 0), (1, 0), (0, 1)$. Thus all three feasible binary points are preserved, and the constraint becomes redundant, since $10x_1 + 5x_2 \leq 10 < 12$ on the triangle.

Original P_2 and the boundary $x_1 + x_2 = 1$. $P_2 \cap \{x_1 + x_2 \leq 1\}$: the blue point is removed.**Example 5.6: Successive cover cuts**

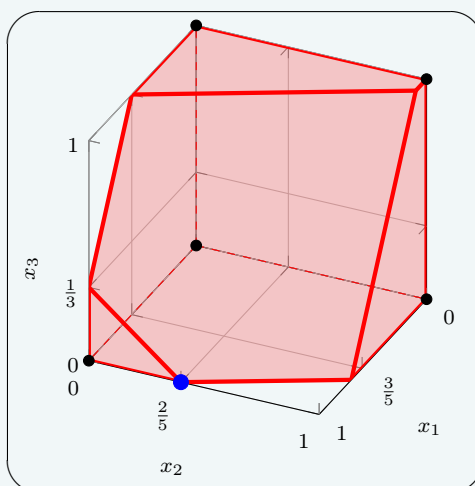
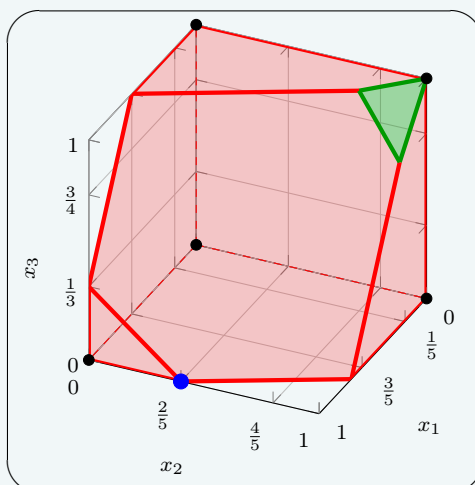
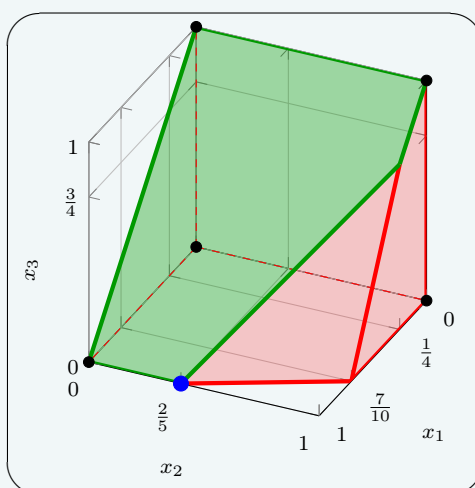
For $\mathcal{J} = \{1, 2, 3\}$, let

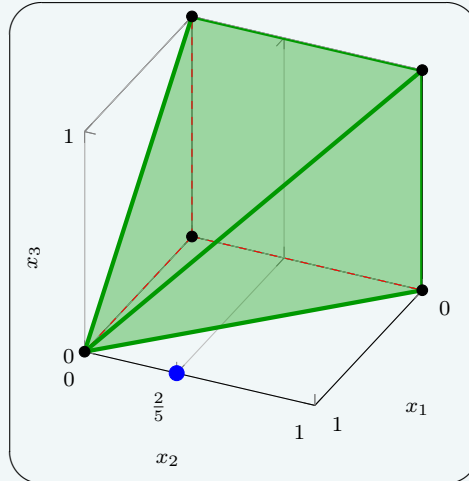
$$P_A = \{\mathbf{x} \in [0, 1]^3 : 10x_1 + 5x_2 + 6x_3 \leq 12\}.$$

There are exactly three covers:

$$\begin{aligned} \{1, 2, 3\} &: x_1 + x_2 + x_3 \leq 2, \\ \{1, 3\} &: x_1 + x_3 \leq 1, \\ \{1, 2\} &: x_1 + x_2 \leq 1. \end{aligned}$$

Only the last two are minimal. The blue point $\tilde{x}_1 = 1$, $\tilde{x}_2 = 2/5$, $\tilde{x}_3 = 0$ lies in P_A . It satisfies the first inequality and is on the boundary of the second; the third is violated by $2/5$, exactly as in Example 5.5. Unlike the previous domain-only panels, every following panel includes the constraint $10x_1 + 5x_2 + 6x_3 \leq 12$.

Original P_A .Add $x_1 + x_2 + x_3 \leq 2$.Also add $x_1 + x_3 \leq 1$.



Also add $x_1 + x_2 \leq 1$: the cover closure.

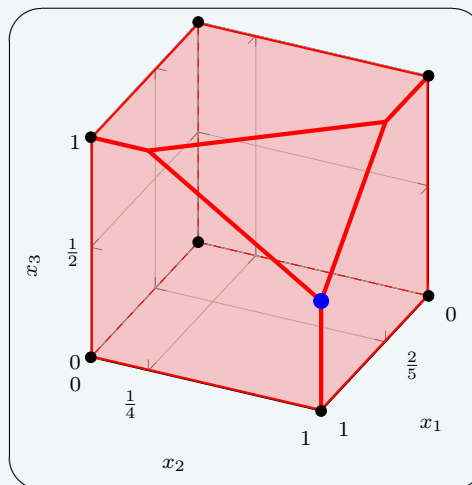
- For the objective $\max 40x_1 + 15x_2 + 12x_3$, the optimal solution over P_A is $\tilde{x} = (1, 2/5, 0)$, with value 46. Over the cover closure, the optimal solution is $(1, 0, 0)$, with value 40, which is also optimal for the binary problem.
- In this instance the cover closure coincides with $\text{conv}(X)$: its vertices, shown in the last panel, are all binary. This is a property of this instance, not of every cover closure.

Example 5.7: A different constraint

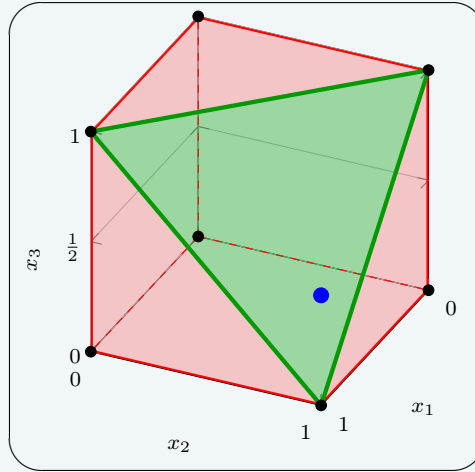
For $\mathcal{J} = \{1, 2, 3\}$, consider now the instance

$$P_B = \{x \in [0, 1]^3 : 5x_1 + 4x_2 + 6x_3 \leq 12\}.$$

Every pair of variables satisfies the constraint, whereas the three coefficients sum to $15 > 12$. Thus the only cover is $\{1, 2, 3\}$ and it is minimal. Its inequality is $x_1 + x_2 + x_3 \leq 2$. The blue point $\tilde{x}_1 = \tilde{x}_2 = 1$, $\tilde{x}_3 = 1/2$ has left-hand side 12 and violates this cut by $1/2$.



Original P_B .



$$P_B \cap \{x_1 + x_2 + x_3 \leq 2\}.$$

- The plane of the constraint intersects the cube edges at $(1, 1, 1/2)$, $(1, 1/4, 1)$ and $(2/5, 1, 1)$.
- The cover removes these fractional vertices while preserving all seven feasible binary points. As in the 2D example, after the cut the constraint is redundant: the sum of the two largest coefficients is $6 + 5 = 11 < 12$.
- The polyhedron $P_B \cap \{x_1 + x_2 + x_3 \leq 2\}$ has only binary vertices (last panel): here a single cover inequality gives $\text{conv}(X)$.

5.4 Separation of cover inequalities

Given a point $\tilde{x} \in P$, the **cover separation problem** (CSP) consists in finding a cover inequality violated by $\tilde{x}$, or in proving that $\tilde{x}$ satisfies all the cover inequalities.

Formally, the CSP asks to find a cover

$$C \in \mathcal{C} : \quad \sum_{j \in C} \tilde{x}_j > |C| - 1,$$

or to prove that

$$\sum_{j \in C} \tilde{x}_j \leq |C| - 1 \quad \forall C \in \mathcal{C}.$$

- Typically, $\tilde{x}$ is an optimal solution of the current LP relaxation, but any point of P can be separated.
- For a cover C , the cover inequality is violated by $\tilde{x}$ if

$$\sum_{j \in C} \tilde{x}_j > |C| - 1. \quad (5.8)$$

Moving $\sum_{j \in C} \tilde{x}_j$ to the right-hand side and 1 to the left-hand side, and writing $|C| =$

$\sum_{j \in C} 1$, we obtain the equivalent conditions:

$$\begin{aligned} |C| - \sum_{j \in C} \tilde{x}_j &< 1, \\ \sum_{j \in C} 1 - \sum_{j \in C} \tilde{x}_j &< 1, \\ \sum_{j \in C} (1 - \tilde{x}_j) &< 1. \end{aligned} \tag{5.9}$$

- Assigning to each index j the cost $q_j = 1 - \tilde{x}_j \geq 0$, the CSP amounts to finding a cover of *minimum cost*: a violated cover inequality exists if and only if this minimum cost is less than 1.

5.4.1 An ILP formulation for the CSP

- Introduce n binary variables z_j : $z_j = 1$ if j belongs to the cover and $z_j = 0$ otherwise.
- The objective minimizes the total cover cost; the single constraint forces the selected set to be a cover.

$$(\text{ILP}_{\text{CSP}}) \quad \min \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j) z_j \tag{5.10a}$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} a_j z_j \geq b + 1, \tag{5.10b}$$

$$z_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \tag{5.10c}$$

Proposition 5.8

A cover inequality violated by $\tilde{\mathbf{x}}$ exists if and only if $z(\text{ILP}_{\text{CSP}}) < 1$.

Proof. Every feasible assignment of the binary variables z_j represents a cover $C = \{j : z_j = 1\}$, and its objective is $\sum_{j \in C} (1 - \tilde{x}_j)$. By (5.9), this is less than 1 exactly when the corresponding cut is violated. Taking the minimum over all covers proves both implications. The CSP is feasible because $\mathcal{J}$ itself is a cover. $\square$

Let $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n$ be an optimal solution of ILP_{CSP} . If $z(\text{ILP}_{\text{CSP}}) < 1$, the cover

$$\tilde{C} = \{j \in \mathcal{J} : \tilde{z}_j = 1\}$$

is associated with the cover inequality $\sum_{j \in \tilde{C}} x_j \leq |\tilde{C}| - 1$ maximally violated by $\tilde{\mathbf{x}}$. Otherwise, no cover inequality is violated by $\tilde{\mathbf{x}}$.

- To find a violated cover inequality, it is not necessary to solve ILP_{CSP} to optimality: any feasible solution $\tilde{z}_1, \dots, \tilde{z}_n$ of cost less than 1 gives the cover $\tilde{C} = \{j \in \mathcal{J} : \tilde{z}_j = 1\}$, whose cover inequality is violated by $\tilde{\mathbf{x}}$.
- Conversely, a feasible solution of cost at least 1 does *not* prove that no violated cover inequality exists: this requires the optimal value (or a lower bound) of ILP_{CSP} .

- If $z(\text{ILP}_{\text{CSP}}) < 1$, the maximum violation of a cover inequality at $\tilde{\mathbf{x}}$ is $1 - z(\text{ILP}_{\text{CSP}})$.

Example 5.8: Separation of the cover inequalities

We consider again Example 5.1. The cover inequalities are separated iteratively: at each iteration, the current LP is solved, the ILP_{CSP} is solved for its optimal solution $\tilde{\mathbf{x}}$, and the maximally violated cover inequality is added, until no cover inequality is violated.

Iteration 1. An optimal solution of LP is

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = \frac{2}{3}, \quad \tilde{x}_4 = 0, \quad z(\text{LP}) = \frac{49}{3}.$$

The associated ILP_{CSP} is:

$$\begin{aligned} \min \quad & \frac{1}{3} z_3 + z_4 \\ \text{subject to} \quad & 2z_1 + 3z_2 + 6z_3 + 7z_4 \geq 10 \\ & z_1, z_2, z_3, z_4 \in \{0, 1\} \end{aligned}$$

An optimal solution of ILP_{CSP} is $\tilde{z}_1 = \tilde{z}_2 = \tilde{z}_3 = 1, \tilde{z}_4 = 0$, with value $z(\text{ILP}_{\text{CSP}}) = \frac{1}{3} < 1$. Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{z}_j = 1\} = \{1, 2, 3\}$ is the cover associated with the cover inequality maximally violated by $\tilde{\mathbf{x}}$:

$$x_1 + x_2 + x_3 \leq 2, \quad \text{since} \quad \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = 1 + 1 + \frac{2}{3} = \frac{8}{3} > 2.$$

Iteration 2. Adding this cover inequality to LP, we obtain the following LP, denoted LP_1 :

$$\begin{aligned} \max \quad & 6x_1 + 5x_2 + 8x_3 + 9x_4 \\ \text{subject to} \quad & 2x_1 + 3x_2 + 6x_3 + 7x_4 \leq 9 \\ & x_1 + x_2 + x_3 \leq 2 \\ & x_1 \leq 1 \\ & x_2 \leq 1 \\ & x_3 \leq 1 \\ & x_4 \leq 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

An optimal solution of LP_1 is

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = 1, \quad \tilde{x}_3 = 0, \quad \tilde{x}_4 = \frac{4}{7}, \quad z(\text{LP}_1) = \frac{113}{7}.$$

The associated ILP_{CSP} is:

$$\begin{aligned} \min \quad & z_3 + \frac{3}{7} z_4 \\ \text{subject to} \quad & 2z_1 + 3z_2 + 6z_3 + 7z_4 \geq 10 \\ & z_1, z_2, z_3, z_4 \in \{0, 1\} \end{aligned}$$

An optimal solution of ILP_{CSP} is $\tilde{z}_1 = 0, \tilde{z}_2 = 1, \tilde{z}_3 = 0, \tilde{z}_4 = 1$, with value $z(\text{ILP}_{\text{CSP}}) = \frac{3}{7} < 1$ (the cover $\{1, 2, 4\}$ has the same cost; we choose the minimal cover). Accordingly, $\tilde{C} = \{j \in \mathcal{J} : \tilde{z}_j = 1\} = \{2, 4\}$ and the maximally violated cover inequality is

$$x_2 + x_4 \leq 1, \quad \text{since} \quad \tilde{x}_2 + \tilde{x}_4 = 1 + \frac{4}{7} = \frac{11}{7} > 1.$$

Iteration 3. Adding this cover inequality to LP_1 , we obtain the following LP, denoted LP_2 :

$$\begin{array}{llllll}
 \max & 6x_1 & + & 5x_2 & + & 8x_3 & + & 9x_4 \\
 \text{subject to} & 2x_1 & + & 3x_2 & + & 6x_3 & + & 7x_4 & \leq & 9 \\
 & x_1 & + & x_2 & + & x_3 & & & \leq & 2 \\
 & & & x_2 & & & + & x_4 & \leq & 1 \\
 & x_1 & & & & & & & \leq & 1 \\
 & & & x_2 & & & & & \leq & 1 \\
 & & & & & x_3 & & & \leq & 1 \\
 & & & & & & & x_4 & \leq & 1 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
 \end{array}$$

An optimal solution of LP_2 is

$$\tilde{x}_1 = 1, \quad \tilde{x}_2 = \frac{3}{5}, \quad \tilde{x}_3 = \frac{2}{5}, \quad \tilde{x}_4 = \frac{2}{5}, \quad z(LP_2) = \frac{79}{5}.$$

The associated ILP_{CSP} is:

$$\begin{array}{llllll}
 \min & & \frac{2}{5}z_2 & + & \frac{3}{5}z_3 & + & \frac{3}{5}z_4 \\
 \text{subject to} & 2z_1 & + & 3z_2 & + & 6z_3 & + & 7z_4 & \geq & 10 \\
 & z_1, & & z_2, & & z_3, & & z_4 & \in & \{0, 1\}
 \end{array}$$

An optimal solution of ILP_{CSP} is $\tilde{z}_1 = 0, \tilde{z}_2 = 1, \tilde{z}_3 = 0, \tilde{z}_4 = 1$, with value $z(ILP_{CSP}) = 1$ (the covers $\{1, 2, 3\}$ and $\{1, 2, 4\}$ have the same cost): **no** cover inequality is violated by $\tilde{\mathbf{x}}$, and the separation stops.

The final value $z(LP_2) = \frac{79}{5}$ coincides with the value $z(LP^{cov})$ of the cover closure (Example 5.2): the separation reaches the bound of the cover closure by adding only 2 of the 7 cover inequalities.

The solution $\tilde{\mathbf{x}}$ is still **fractional**, although it satisfies all the cover inequalities: the cover closure does not coincide, in general, with the convex hull of the integer solutions.

5.5 Transforming the CSP into a complementary problem

- Consider a feasible assignment of the CSP variables z_j . Define the complementary binary variables y_j by

$$y_j = 1 - z_j \quad \forall j \in \mathcal{J}.$$

Thus $y_j = 1$ if j is outside the cover, whereas $y_j = 0$ if j belongs to the cover.

- The constraint becomes

$$\sum_{j \in \mathcal{J}} a_j(1 - y_j) \geq b + 1 \iff \sum_{j \in \mathcal{J}} a_j y_j \leq \sum_{j \in \mathcal{J}} a_j - (b + 1).$$

By the assumption $\sum_{j \in \mathcal{J}} a_j > b$, this right-hand side is non-negative.

- The objective can be written as

$$\sum_{j \in \mathcal{J}} (1 - \tilde{x}_j)(1 - y_j) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j)y_j.$$

Since the first two terms are constant, minimizing it is equivalent to the following problem, called the *complementary problem*, with non-negative (possibly zero) objective coefficients $1 - \tilde{x}_j$.

$$(\text{ILP}_{\text{CSP}}^2) \quad \max \quad \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j)y_j \quad (5.11a)$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} a_j y_j \leq \sum_{j \in \mathcal{J}} a_j - (b + 1), \quad (5.11b)$$

$$y_j \in \{0, 1\}, \quad \forall j \in \mathcal{J}. \quad (5.11c)$$

- Complementation is a bijection between feasible solutions, so

$$z(\text{ILP}_{\text{CSP}}) = n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2). \quad (5.12)$$

Let $\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n$ be an optimal solution of $\text{ILP}_{\text{CSP}}^2$. If $n - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) < 1$, the cover

$$\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\}$$

is associated with the cover inequality $\sum_{j \in \tilde{C}} x_j \leq |\tilde{C}| - 1$ maximally violated by $\tilde{\mathbf{x}}$. Otherwise, no cover inequality is violated by $\tilde{\mathbf{x}}$.

- Any feasible solution $\bar{y}_1, \dots, \bar{y}_n$ of $\text{ILP}_{\text{CSP}}^2$ gives the cover $\bar{C} = \{j \in \mathcal{J} : \bar{y}_j = 0\}$, with cost $n - \sum_{j \in \mathcal{J}} \tilde{x}_j - \sum_{j \in \mathcal{J}} (1 - \tilde{x}_j)\bar{y}_j$: if this cost is less than 1, the cover inequality of $\bar{C}$ is violated by $\tilde{\mathbf{x}}$.

5.5.1 A greedy separation heuristic

The following greedy procedure constructs feasible values $\bar{y}_j$ for the complementary problem.

1. Set $\bar{y}_j = 0$ for all $j \in \mathcal{J}$ and the residual right-hand side $r = \sum_{j \in \mathcal{J}} a_j - (b + 1)$. Sort the indices by non-increasing ratio $q_j/a_j = (1 - \tilde{x}_j)/a_j$, breaking ties by increasing index.
 2. Scan the indices in this order. If $q_j > 0$ and $a_j \leq r$, set $\bar{y}_j = 1$ and $r := r - a_j$; otherwise skip it.
 3. Reconstruct the cover $C = \{j : \bar{y}_j = 0\}$ and evaluate its cost. If desired, reduce it to a minimal cover by removing indices as long as $\sum_{j \in C} a_j$ exceeds b .
- Indices with $q_j = 0$ ($\tilde{x}_j = 1$) are never selected: they remain in the cover, which may then be non-minimal.
 - Removing an index from a cover decreases its cost by $q_j \geq 0$: reducing a violated cover to a minimal one cannot decrease its violation.
 - The greedy algorithm is a *heuristic* for the complementary problem: if it does not find a violated cover inequality, we cannot conclude that none exists.

Example 5.9: Complementary problem and greedy scan

For the point $\tilde{x} = (1, 1, \frac{2}{3}, 0)$ of Iteration 1 in Example 5.8,

$$4 - \sum_{j \in \mathcal{J}} \tilde{x}_j = 4 - 1 - 1 - \frac{2}{3} = \frac{4}{3},$$

$$\sum_{j \in \mathcal{J}} a_j - (b + 1) = 2 + 3 + 6 + 7 - (9 + 1) = 8.$$

The complementary problem is

$$\begin{array}{llllll} \max & & & \frac{1}{3} y_3 & + & y_4 \\ \text{subject to} & 2 y_1 & + & 3 y_2 & + & 6 y_3 & + & 7 y_4 & \leq & 8 \\ & y_1, & & y_2, & & y_3, & & y_4 & \in & \{0, 1\} \end{array}$$

j	q_j	a_j	q_j/a_j	Decision	Residual
4	1	7	$\frac{1}{7}$	$\bar{y}_j = 1$	1
3	$\frac{1}{3}$	6	$\frac{1}{18}$	$\bar{y}_j = 0$	1
1	0	2	0	$q_j = 0, \bar{y}_j = 0$	1
2	0	3	0	$q_j = 0, \bar{y}_j = 0$	1

The scan gives $\bar{y}_1 = \bar{y}_2 = \bar{y}_3 = 0$ and $\bar{y}_4 = 1$, of value 1. Indices 3 and 4 cannot be taken together; the only alternatives with positive objective coefficient have values $1/3$ and 1. Thus this greedy solution is also optimal, and

$$z(\text{ILP}_{\text{CSP}}) = 4 - \sum_{j \in \mathcal{J}} \tilde{x}_j - z(\text{ILP}_{\text{CSP}}^2) = \frac{4}{3} - 1 = \frac{1}{3}.$$

The cover is $\tilde{C} = \{j \in \mathcal{J} : \tilde{y}_j = 0\} = \{1, 2, 3\}$, as found in Iteration 1 of Example 5.8.

5.6 Extended cover inequalities

- A cover inequality can be strengthened by also considering the variables outside the cover whose coefficients are at least as large as every coefficient of the cover. Let

$$\hat{a}(C) = \max_{j \in C} a_j$$

be the maximum coefficient of the variables in C .

Definition 5.9: Extended cover inequality

For a cover C , let

$$E(C) = C \cup \{j \in \mathcal{J} \setminus C : a_j \geq \hat{a}(C)\}.$$

The associated *extended cover inequality* is

$$\sum_{j \in E(C)} x_j \leq |C| - 1. \quad (5.13)$$

The right-hand side depends on $|C|$, not on $|E(C)|$.

Proposition 5.10

For every cover $C \in \mathcal{C}$, the extended cover inequality (5.13) is valid for X .

Proof. Let $k = |C|$ and take any set $S \subseteq E(C)$ with $|S| = k$. The sets $S \setminus C$ and $C \setminus S$ have the same number of variables, and each coefficient a_j with $j \in S \setminus C$ is at least $\hat{a}(C)$, i.e., at least as large as each coefficient with $j \in C \setminus S$. Replacing the indices of $C \setminus S$ with those of $S \setminus C$ gives

$$\sum_{j \in S} a_j \geq \sum_{j \in C} a_j > b.$$

Hence no point $\mathbf{x} \in X$ can have k variables of $E(C)$ equal to 1: at most $k - 1$ of them can be equal to 1 together. $\square$

- Since $C \subseteq E(C)$ and the right-hand side is the same, the extended cover inequality dominates the cover inequality of C .
- It can be seen as a *lifting* of the coefficients of the variables in $E(C) \setminus C$ from 0 to 1. It does not forbid a variable with a large coefficient to be equal to 1 together with variables of C : it limits to $|C| - 1$ the number of variables of the whole set $E(C)$ equal to 1.

Example 5.10: Strengthening the cover of the four-variable instance

In Example 5.1, consider the cover $C = \{1, 2, 3\}$. Then $\hat{a}(C) = 6$, and the coefficient of x_4 is $7 \geq 6$, so

$$E(C) = \{1, 2, 3, 4\}, \quad x_1 + x_2 + x_3 + x_4 \leq 2.$$

The strengthened LP, denoted LP_{ext} , is

$$\begin{array}{llllll}
\max & 6x_1 & + & 5x_2 & + & 8x_3 & + & 9x_4 \\
\text{subject to} & 2x_1 & + & 3x_2 & + & 6x_3 & + & 7x_4 & \leq & 9 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 2 \\
& x_1 & & & & & & & \leq & 1 \\
& & x_2 & & & & & & \leq & 1 \\
& & & x_3 & & & & & \leq & 1 \\
& & & & x_4 & & & & \leq & 1 \\
& x_1, & x_2, & x_3, & x_4 & \geq & 0
\end{array}$$

An optimal solution of LP_{ext} is $\tilde{\mathbf{x}} = (1, 0, 0, 1)$, with value $z(\text{LP}_{\text{ext}}) = 6 + 9 = 15$. Since it is binary, $\tilde{\mathbf{x}}$ is also optimal for the binary problem. We have $z(\text{LP}_{\text{ext}}) = 15 < z(\text{LP}^{\text{cov}}) = 79/5$: in this instance, one extended cover inequality is stronger than all the cover inequalities together.

5.7 Violation and distance from a cover inequality

- Let $C \subseteq \mathcal{J}$ be a cover. Its cover inequality is

$$\sum_{j \in C} x_j \leq |C| - 1,$$

and its boundary hyperplane is

$$H_C = \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{j \in C} x_j = |C| - 1 \right\}.$$

- The violation of the cover inequality at a point $\tilde{\mathbf{x}}$ is

$$v_C(\tilde{\mathbf{x}}) = \max \left\{ 0, \sum_{j \in C} \tilde{x}_j - (|C| - 1) \right\}. \quad (5.14)$$

The point violates the cover inequality precisely when $v_C(\tilde{\mathbf{x}}) > 0$.

- The coefficients of the cover inequality are 1 for $j \in C$ and 0 otherwise, so their Euclidean norm is $\sqrt{|C|}$. Therefore, the Euclidean distance of $\tilde{\mathbf{x}}$ from H_C is

$$d(\tilde{\mathbf{x}}, H_C) = \frac{\left| \sum_{j \in C} \tilde{x}_j - (|C| - 1) \right|}{\sqrt{|C|}}. \quad (5.15)$$

If the cover inequality is violated,

$$d(\tilde{\mathbf{x}}, H_C) = \frac{v_C(\tilde{\mathbf{x}})}{\sqrt{|C|}}.$$

- The orthogonal projection $\mathbf{x}^\perp$ of $\tilde{\mathbf{x}}$ onto H_C has components

$$x_j^\perp = \begin{cases} \tilde{x}_j - \frac{\sum_{k \in C} \tilde{x}_k - (|C| - 1)}{|C|}, & j \in C, \\ \tilde{x}_j, & j \in \mathcal{J} \setminus C. \end{cases} \quad (5.16)$$

Indeed, substitution gives $\sum_{j \in C} x_j^\perp = |C| - 1$.

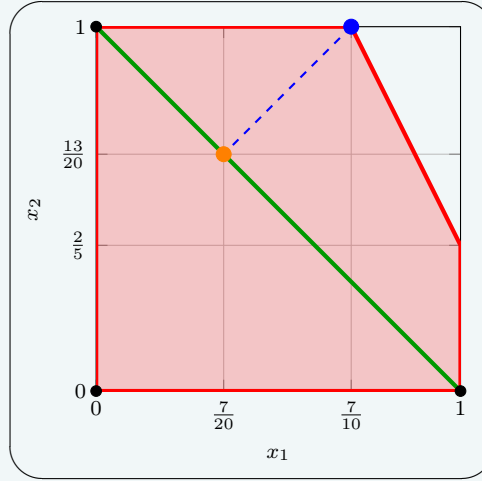
- The CSP maximizes the violation, not the distance: when the covers have different cardinalities, the two criteria can rank them differently. For example, at the point $\tilde{\mathbf{x}} = (1/10, 1, 1)$ of P_A (Example 5.6), the covers $\{1, 2\}$, $\{1, 3\}$ and $\{1, 2, 3\}$ all have violation $1/10$, but their distances are $1/(10\sqrt{2})$, $1/(10\sqrt{2})$ and $1/(10\sqrt{3})$, respectively.

Example 5.11: Orthogonal projection in two dimensions

Take the vertex $\tilde{x}_1 = 7/10$, $\tilde{x}_2 = 1$ of P_2 in Example 5.5 and the cover $C = \{1, 2\}$. Its cover inequality is $x_1 + x_2 \leq 1$ and $|C| = 2$. Then

$$v_C(\tilde{\mathbf{x}}) = \frac{7}{10}, \quad x_1^\perp = \frac{7}{20}, \quad x_2^\perp = \frac{13}{20}, \quad d(\tilde{\mathbf{x}}, H_C) = \frac{7}{10\sqrt{2}}.$$

The red region is P_2 and the green edge is the boundary H_C . The blue dashed segment is perpendicular to H_C and the orange point is the orthogonal projection.

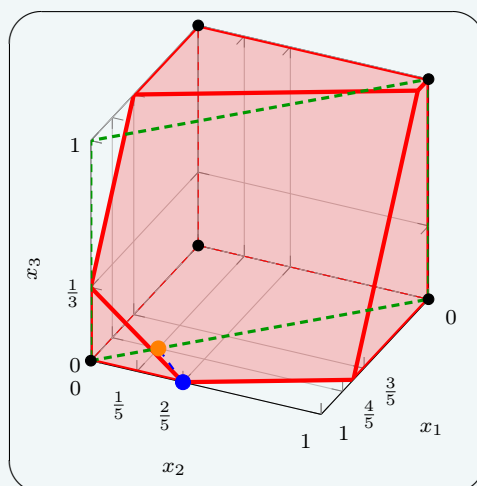


Example 5.12: Orthogonal projection in three dimensions

Take the point $\tilde{x}_1 = 1$, $\tilde{x}_2 = 2/5$, $\tilde{x}_3 = 0$ of P_A in Example 5.6 and the cover $C = \{1, 2\}$. Its cover inequality is $x_1 + x_2 \leq 1$ and $|C| = 2$. Then

$$v_C(\tilde{\mathbf{x}}) = \frac{2}{5}, \quad x_1^\perp = \frac{4}{5}, \quad x_2^\perp = \frac{1}{5}, \quad x_3^\perp = 0, \quad d(\tilde{\mathbf{x}}, H_C) = \frac{2}{5\sqrt{2}}.$$

The red region is P_A and the green dashed rectangle is H_C restricted to the unit cube. Since $3 \notin C$, the projection does not change $\tilde{x}_3$.



5.8 Typical mistakes

Warning

- **Using a set that is not a cover.** The inequality $\sum_{j \in C} x_j \leq |C| - 1$ is valid only if the total coefficient $\sum_{j \in C} a_j$ exceeds the right-hand side b ; check the coefficients before writing it.
- **Using $|E(C)| - 1$ for an extended cover.** The right-hand side of the extended cover inequality is $|C| - 1$, where C is the cover, not $|E(C)| - 1$.
- **Removing all the indices with $q_j = 0$ from a cover.** To obtain a minimal cover, remove indices one at a time as long as the remaining set is still a cover: removing all of them may leave a set that is not a cover.
- **Stopping the separation at a heuristic failure.** If the greedy heuristic algorithm finds no violated cover inequality, we cannot conclude that none exists: only the optimal value of the separation problem proves it.
- **Confusing violation and distance.** The cover separation problem maximizes the violation; covers with the same violation can have different distances from the point.
- **Believing that the cover closure is the convex hull.** A point can satisfy all the cover inequalities and still be fractional (Example 5.8).

5.9 In summary

In summary

- A cover is a set of variables whose total coefficient exceeds the right-hand side; its cover inequality allows at most $|C| - 1$ of its variables to be equal to 1, and it is valid for X .
- The minimal covers are enough: their inequalities imply all the others, but they can be exponentially many, so they are separated only when violated.
- Separating a cover inequality means finding a cover of minimum cost $\sum_{j \in C} (1 - \tilde{x}_j)$: a violated inequality exists if and only if the minimum cost is less than 1; this is a 0–1 problem with one $\leq$ constraint, defined in the chapter.
- An extended cover inequality dominates the cover inequality of the same cover.
- The cover closure is closer to $\text{conv}(X)$ than the LP relaxation, but in general it is not equal to it.

Dashboard of the four-variable instance. The lower and upper bounds on the optimal value of the binary problem of Example 5.1. The objective coefficients are integer, so the upper bound is the rounded-down value of the LP relaxation.

Source	Lower bound	Upper bound	From
LP relaxation	11	16	greedy solution $(1, 1, 0, 0)$, $\lfloor z(\text{LP}) \rfloor$ with $z(\text{LP}) = \frac{49}{3}$
Cover closure	11	15	$\lfloor z(\text{LP}^{\text{cov}}) \rfloor$ with $z(\text{LP}^{\text{cov}}) = \frac{79}{5}$, 2 separated covers
Extended cover	15	15	$z(\text{LP}_{\text{ext}}) = 15$, optimal solution $(1, 0, 0, 1)$